

Phase Retrieval from Multiple-Window Short-Time Fourier Measurements

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Abstract—In this paper, we introduce two symmetric directed graphs depending on supports of signals and windows, and we show that the connectivity of those graphs provides either necessary or sufficient conditions to phase retrieval of a signal from magnitude measurements of its multiple-window short-time Fourier transform. Also we propose an algebraic reconstruction algorithm, and provide an error estimate to our algorithm when magnitude measurements are corrupted by deterministic/random noises.

Keywords: Short-Time Fourier Transform, Phase Retrieval, Graph

I. Introduction

Phase retrieval considers recovering a signal of interest from magnitudes of its (non)linear measurements. It arises in various fields of science and engineering, such as X-ray crystallography, coherent diffractive imaging, optics and more. The underlying recovery is an ill-posed problem inherently. The signal could be reconstructed, in an efficient and robust manner, only if we have additional information about the signal ([1]–[9]). In this paper, we discuss the phase retrieval problem for N -dimensional complex signals

$$\mathbf{x} = (x(0), x(1), \dots, x(N-1))^T \in \mathbb{C}^N \quad (\text{I.1})$$

with some constraints on their supports,

$$V(\mathbf{x}) = \{n : x(n) \neq 0\}. \quad (\text{I.2})$$

Given a nonzero window $\mathbf{w} = (w(0), w(1), \dots, w(N-1))^T$ with period N extension and a separation parameter L between adjacent short-time sections, the short-time Fourier transform (STFT) of a signal \mathbf{x} is given by

$$X_{\mathbf{w}}(Lm, k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)w(Lm-n)e^{-i2\pi kn/N}, \quad (\text{I.3})$$

where $0 \leq m \leq N/L - 1$ and $0 \leq k \leq N-1$. The STFT has been widely used in signal/imaging processing ([10], [11]). In this paper, inspired by applications in microscopy and optical imaging, we consider reconstructing the signal \mathbf{x} from magnitude measurements of its multiple-window STFT,

$$|X_{\mathbf{w}_r}(Lm, k)|, 1 \leq r \leq R, 0 \leq m \leq N/L-1, 0 \leq k \leq N-1, \quad (\text{I.4})$$

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where $\mathbf{W} = \{\mathbf{w}_r\}_{r=1}^R$ is a family of windows with period N extension. The above reconstruction problem has been explored with various approaches ([12]–[19]). The special case with $L = N$ is also known as phase retrieval with structured illuminations and masks ([20]–[23]).

Define the supporting length of a nonzero window \mathbf{w} with period N extension by

$$l(\mathbf{w}) = \min_{0 \leq l \leq N-1} \{l+1 : \text{there exists } n' \text{ so that } w(n) = 0 \text{ for all } n \notin [n', n'+l] + N\mathbb{Z}\}. \quad (\text{I.5})$$

It is observed in [17, Theorem 2] that not all N -dimensional signals can be recovered, up to a global phase, from magnitude measurements (I.4) of their multiple-window STFT if all windows $\mathbf{w}_r, 1 \leq r \leq R$, have supporting length less than $N/2$, cf. [24] for similar phenomenon observed when recovering signals in a shift-invariant space from magnitudes of their sampling data. In Section II of this paper, we introduce a symmetric directed graph $\mathcal{G}(\mathbf{x}, \mathbf{W}, L)$ with $V(\mathbf{x})$ in (I.2) as its vertex set, and we show in Theorem II.1 that connectivity of the above graph is a necessary condition to phase retrievability of the signal \mathbf{x} from magnitude measurements (I.4) of its multiple-window STFT.

A fundamental question in phase retrieval is whether a signal is uniquely determined, up to a global phase, by its noiseless measurements (I.4). For $L = R = 1$, a sufficient condition was proposed in [17, Theorem 1] to recover a signal \mathbf{x} with all components being nonzero from magnitude measurements (I.4) of its STFT, cf. [18, Theorem 2.4]. In Section III of this paper, we introduce a symmetric directed subgraph $\tilde{\mathcal{G}}(\mathbf{x}, \mathbf{W}, L)$ in (III.1), and we prove in Theorem III.1 that, under mild conditions on the window family \mathbf{W} , connectivity of the graph $\tilde{\mathcal{G}}(\mathbf{x}, \mathbf{W}, L)$ is a sufficient condition to reconstruct the signal \mathbf{x} , up to a global phase, from magnitude measurements (I.4) of its multiple-window STFT. Applying Theorem III.1 with $V(\mathbf{x}) = \{0, 1, \dots, N-1\}$ and $L = R = 1$ leads to the result in [17, Theorem 1], see Corollary III.2.

Consider the scenario that magnitude measurements (I.4) of the multiple-window STFT are corrupted by deterministic/random noises $\boldsymbol{\epsilon} = (\boldsymbol{\epsilon}(r, m, k))$ with level $|\boldsymbol{\epsilon}|$,

$$Y(r, m, k) := |X_{\mathbf{w}_r}(Lm, k)|^2 + \boldsymbol{\epsilon}(r, m, k), \quad (\text{I.6})$$

where $|\boldsymbol{\epsilon}| = \max\{|\boldsymbol{\epsilon}(r, m, k)| : 1 \leq r \leq R, 0 \leq m \leq N/L-1, 0 \leq k \leq N-1\}$. Another fundamental issue in phase retrieval is to design efficient and robust algorithms so that a good approximation \mathbf{x}_ϵ to the original signal \mathbf{x} , up to a

global phase, could be found when only noisy measurements (I.6) are available. Designing such reconstruction algorithms is a great challenge in general, and several algorithms have been proposed in the literature, see [3], [19], [25], [26] and references therein. In Section IV of this paper, we propose an algebraic reconstruction algorithm from noisy measurements (I.6), and we establish an error estimate in Theorem IV.1 when the graph $\tilde{\mathcal{G}}(\mathbf{x}, \mathbf{W}, L)$ in (III.1) is connected.

Notation: \mathbf{A}^H is Hermitian of a matrix \mathbf{A} ; $\mathbf{a} \circ \mathbf{b}$ is the componentwise (Hadamard) product of vectors \mathbf{a} and \mathbf{b} ; $\lfloor t \rfloor$ is the largest integer less than or equal to t , and $k \bmod N$ is the remainder of the Euclidean division of an integer k by N .

II. NECESSARY CONDITIONS ON PHASE RETRIEVAL

Given an N -dimensional complex signal \mathbf{x} , a family $\mathbf{W} = \{\mathbf{w}_r\}_{r=1}^R$ of window functions with period N extension, and a separation parameter L with $N/L \in \mathbb{Z}$, we define a graph

$$\mathcal{G}(\mathbf{x}, \mathbf{W}, L) := (V(\mathbf{x}), E(\mathbf{W}, L)) \quad (\text{II.1})$$

with

$$E(\mathbf{W}, L) := \left\{ (n, n') \in V(\mathbf{x}) \times V(\mathbf{x}) : n \neq n' \text{ and } \sum_{r=1}^R \sum_{m=0}^{N/L-1} |w_r(Lm - n')w_r(Lm - n)|^2 \neq 0 \right\}, \quad (\text{II.2})$$

where $V(\mathbf{x})$ is given in (I.2) and $\mathbf{w}_r = ((w_r(0), \dots, w_r(N-1)))^\top$, $1 \leq r \leq R$. The symmetric directed graph $\mathcal{G}(\mathbf{x}, \mathbf{W}, L)$ has indices of nonzero components of the signal \mathbf{x} as its vertices, and it has edges between two distinct vertices n and n' only if $w_r(Lm - n')w_r(Lm - n) \neq 0$ for some $1 \leq r \leq R$ and $0 \leq m \leq N/L - 1$.

Theorem II.1. *Let $\mathbf{W} = \{\mathbf{w}_r\}_{r=1}^R$ be a family of window functions with period N extension, and $L \geq 1$ be a separation parameter with $N/L \in \mathbb{Z}$. If $\mathbf{x} \in \mathbb{C}^N$ can be determined, up to a global phase, from magnitude measurements (I.4) of its multiple-window STFT, then the graph $\mathcal{G}(\mathbf{x}, \mathbf{W}, L)$ in (II.1) is connected.*

Proof: Suppose, on the contrary, that $\mathcal{G}(\mathbf{x}, \mathbf{W}, L)$ in (II.1) is disconnected. Then there exists a subset $V_1 \subset V(\mathbf{x})$ such that $V_1 \neq \emptyset$, $V(\mathbf{x}) \setminus V_1 \neq \emptyset$, and there are no edges between vertices in V_1 and $V(\mathbf{x}) \setminus V_1$. Let $\mathbf{x}_{V_1} \in \mathbb{C}^N$ be the signal which coincides with \mathbf{x} on the indices in V_1 and is extended to zeros in $\{0, 1, \dots, N-1\} \setminus V_1$. Observe from (II.1) that for any $1 \leq r \leq R$ and $0 \leq m \leq N/L - 1$, there is an edge between two indices of nonzero components of $\mathbf{x}_\theta \circ \mathbf{w}_{r,Lm}$, where

$$\mathbf{w}_{r,Lm} = (w_r(Lm), \dots, w_r(Lm - N + 1))^\top.$$

Therefore either $\mathbf{x}_\theta \circ \mathbf{w}_{r,Lm} = e^{-2\pi i \theta} \mathbf{x}_{V_1} \circ \mathbf{w}_{r,Lm}$ or $\mathbf{x}_\theta \circ \mathbf{w}_{r,Lm} = (\mathbf{x} - \mathbf{x}_{V_1}) \circ \mathbf{w}_{r,Lm}$ by the construction of V_1 , where

$$\mathbf{x}_\theta = e^{-2\pi i \theta} \mathbf{x}_{V_1} + (\mathbf{x} - \mathbf{x}_{V_1}), \quad \theta \in \mathbb{R}.$$

Hence magnitude measurements of the multiple-window STFT of the signal \mathbf{x}_θ are independent on $\theta \in \mathbb{R}$. This, together with $\mathbf{x}_0 = \mathbf{x}$ and the phase retrievability assumption, implies that $\mathbf{x}_{1/2} = e^{-2\pi i \beta} \mathbf{x}$ for some $\beta \in \mathbb{R}$. Thus

$$(1 + e^{-2\pi i \beta}) \mathbf{x}_{V_1} = (1 - e^{-2\pi i \beta})(\mathbf{x} - \mathbf{x}_{V_1}). \quad (\text{II.3})$$

For the case that $e^{-4\pi i \beta} = 1$, either \mathbf{x}_{V_1} or $\mathbf{x} - \mathbf{x}_{V_1}$ is a zero signal, which is a contradiction. For the remaining case that $e^{-4\pi i \beta} \neq 1$, it follows from (II.3) that \mathbf{x}_{V_1} and $\mathbf{x} - \mathbf{x}_{V_1}$ have the same support, which contradicts to the construction of \mathbf{x}_{V_1} . \blacksquare

Given a window family \mathbf{W} , the graph $\mathcal{G}(\mathbf{x}, \mathbf{W}, L)$ in (II.1) could be disconnected for some signals \mathbf{x} . As an application of Theorem II.1, we have the following result on phase retrievability, cf. [17, Theorem 2], [18, Proposition 2.3] and [13].

Corollary II.2. *Let $\mathbf{W} = \{\mathbf{w}_r\}_{r=1}^R$ be a family of window functions with period N extension such that $l(\mathbf{w}_r) \leq N/2$ for all $1 \leq r \leq R$. Then not all N -dimensional signals can be determined, up to a global phase, from magnitude measurements (I.4) of their multiple-window STFT.*

Proof: Let \mathbf{x}_0 be the signal having 0-th and $\lfloor N/2 \rfloor$ -th components as one and other components zero. Then the corresponding graph $\mathcal{G}(\mathbf{x}_0, \mathbf{W}, L)$ in (II.1) has two vertices 0 and $\lfloor N/2 \rfloor$. From the supporting length assumption for \mathbf{w}_r , $1 \leq r \leq R$, it follows that $w_r(n)w_r(n - \lfloor N/2 \rfloor) = 0$ for all $0 \leq n \leq N - 1$. Hence $\mathcal{G}(\mathbf{x}_0, \mathbf{W}, L)$ is disconnected. This together with Theorem II.1 completes the proof. \blacksquare

III. SUFFICIENT CONDITIONS FOR PHASE RETRIEVAL

Given an N -dimensional complex signal \mathbf{x} , a family $\mathbf{W} = \{\mathbf{w}_r\}_{r=1}^R$ of window functions with period N extension, and a separation parameter L with $N/L \in \mathbb{Z}$, we define a graph

$$\tilde{\mathcal{G}}(\mathbf{x}, \mathbf{W}, L) := (V(\mathbf{x}), \tilde{E}(\mathbf{W}, L)) \quad (\text{III.1})$$

with

$$\tilde{E}(\mathbf{W}, L) := \left\{ (n, n') \in V(\mathbf{x}) \times V(\mathbf{x}) : n \neq n' \text{ and } n, n' \in Lm - a(\mathbf{w}_r) - \{0, l(\mathbf{w}_r) - 1\} + N\mathbb{Z} \text{ for some } 0 \leq m \leq N/L - 1 \text{ and } 1 \leq r \leq R \right\}, \quad (\text{III.2})$$

where $V(\mathbf{x})$ is given in (I.2) and supporting intervals $[a(\mathbf{w}_r), a(\mathbf{w}_r) + l(\mathbf{w}_r) - 1] + N\mathbb{Z}$ of windows \mathbf{w}_r , $1 \leq r \leq R$, are so chosen that $0 \leq a(\mathbf{w}_r) \leq N - 1$,

$$w_r(a(\mathbf{w}_r))\overline{w_r(a(\mathbf{w}_r) + l(\mathbf{w}_r) - 1)} \neq 0, \quad (\text{III.3})$$

and

$$w_r(n) = 0 \text{ for all } n \notin [a(\mathbf{w}_r), a(\mathbf{w}_r) + l(\mathbf{w}_r) - 1] + N\mathbb{Z}. \quad (\text{III.4})$$

The existence and uniqueness of $a(\mathbf{w}_r)$ follow from (I.5). By (II.2) and (III.3), we see that $\tilde{\mathcal{G}}(\mathbf{x}, \mathbf{W}, L)$ is a symmetric directed subgraph of the graph $\mathcal{G}(\mathbf{x}, \mathbf{W}, L)$ in (II.1).

Theorem III.1. *Let $\mathbf{x} \in \mathbb{C}^N$, L be a separation parameter with $N/L \in \mathbb{Z}$, and let $\mathbf{W} = \{\mathbf{w}_r\}_{r=1}^R$ be a family of window functions with period N extension such that*

$$l(\mathbf{w}_r) \leq N/2 \quad (\text{III.5})$$

for all $1 \leq r \leq R$, and

$$\mathbf{A}_m = \left(\beta_r(m + jN/L) \right)_{1 \leq r \leq R, 0 \leq j \leq L-1} \quad (\text{III.6})$$

have rank L for all $0 \leq m \leq N/L - 1$, where

$$\beta_r(k) = \frac{1}{N} \sum_{n=0}^{N-1} |w_r(n)|^2 e^{-2\pi i kn/N}, \quad 0 \leq k \leq N-1. \quad (\text{III.7})$$

If the graph $\tilde{\mathcal{G}}(\mathbf{x}, \mathbf{W}, L)$ in (III.1) is connected, then \mathbf{x} can be recovered, up to a global phase, from magnitude measurements (I.4) of its multiple-window STFT.

For $L = 1$, the full rank requirement (III.6) becomes

$$\sum_{r=1}^R |\beta_r(m)|^2 \neq 0 \text{ for all } 0 \leq m \leq N-1, \quad (\text{III.8})$$

and the set $\tilde{E}(\mathbf{W}, L)$ of edges can be rewritten as

$$\tilde{E}(\mathbf{W}, 1) := \left\{ (n, n') : n - n' = \pm(l(\mathbf{w}_r) - 1) + N\mathbb{Z} \text{ for some } 1 \leq r \leq R \right\}. \quad (\text{III.9})$$

If we further assume that the signal \mathbf{x} has its all components being nonzero (i.e., $V(\mathbf{x}) = \{0, \dots, N-1\}$), one may verify from (III.9) that $\tilde{\mathcal{G}}(\mathbf{x}, \mathbf{W}, 1)$ is connected if and only if

$$l(\mathbf{w}_1) - 1, \dots, l(\mathbf{w}_R) - 1 \text{ and } N \text{ are coprime.} \quad (\text{III.10})$$

Therefore applying Theorem III.1 with $L = 1$, we obtain the following result, which is given in [17, Theorem 1] for $R = 1$, cf. [18, Theorems 2.4].

Corollary III.2. *Let $\mathbf{x} \in \mathbb{C}^N$ have its all components being nonzero, and \mathbf{W} be a family of window functions having period N extension and satisfying (III.5), (III.8) and (III.10). Then \mathbf{x} can be recovered, up to a global phase, from magnitude measurements (I.4) of its multiple-window STFT.*

For $L = N$, the full rank requirement (III.6) can be rewritten as

$$\text{the matrix } (|w_r(n)|^2)_{1 \leq r \leq R, 0 \leq n \leq N-1} \text{ has rank } N. \quad (\text{III.11})$$

Then applying Theorem III.1 with $L = N$ yields the following result on phase retrievability with structured illuminations and masks, cf. [20]–[23].

Corollary III.3. *Let \mathbf{W} be a family of window functions having period N extension and satisfying (III.5) and (III.11). Then any signal $\mathbf{x} \in \mathbb{C}^N$ with $\tilde{\mathcal{G}}(\mathbf{x}, \mathbf{W}, N)$ being connected can be recovered, up to a global phase, from magnitude measurements (I.4) of its multiple-window STFT.*

To prove Theorem III.1, we need a technical lemma.

Lemma III.4. *Let L be a separation parameter with $N/L \in \mathbb{Z}$ and $\mathbf{W} = \{\mathbf{w}_r\}_{r=1}^R$ be a family of window functions having period N extension and satisfying (III.6). Then magnitudes of any signal $\mathbf{x} = (x(0), x(1), x(2), \dots, x(N-1))^T \in \mathbb{C}^N$ can be recovered from magnitude measurements (I.4) of its multiple-window STFT. Moreover,*

$$|x(n)|^2 = \frac{L}{N} \sum_{m, m'=0}^{N/L-1} \sum_{j, j'=0}^{L-1} e^{-2\pi i(m(m'L-n)/N - jn/L)} \times a_m(j, j') \left(\sum_{r=1}^R \overline{\beta_r(m + j'N/L)} Z(\mathbf{w}_r, m') \right) \quad (\text{III.12})$$

for all $0 \leq n \leq N-1$, where

$$(\mathbf{A}_m^H \mathbf{A}_m)^{-1} = (a_m(j, j'))_{0 \leq j, j' \leq L-1}$$

and

$$Z(\mathbf{w}_r, m) = \sum_{k=0}^{N-1} |X_{\mathbf{w}_r}(Lm, k)|^2, \quad 0 \leq m \leq N/L - 1.$$

We postpone the proof of Lemma III.4 to the end of this section and start the proof of Theorem III.1.

Proof of Theorem III.1: By Lemma III.4, $|x(n)|^2, 0 \leq n \leq N-1$, are determined from $|X_{\mathbf{w}_r}(Lm, k)|^2, 1 \leq r \leq R, 0 \leq m \leq L/N - 1, 0 \leq k \leq N-1$. Therefore it remains to find $x(n)/|x(n)|, n \in V(\mathbf{x})$, up to a global phase. From connectivity of the graph $\tilde{\mathcal{G}}(\mathbf{x}, \mathbf{W}, L)$, it suffices to show that for endpoints n_1, n_2 of any edge, the phase difference between $x(n_1)/|x(n_1)|$ and $x(n_2)/|x(n_2)|$ is determined from magnitude measurements (I.4) of the multiple-window STFT.

By the assumption on vertices n_1 and n_2 , there exist $1 \leq r \leq R$ and $0 \leq m \leq N/L - 1$ such that $l(\mathbf{w}_r) \geq 2$ and

$$n_1, n_2 \in Lm - a(\mathbf{w}_r) - \{0, l(\mathbf{w}_r) - 1\} + N\mathbb{Z}. \quad (\text{III.13})$$

Without loss of generality, we assume that

$$n_1 \in Lm - a(\mathbf{w}_r) + N\mathbb{Z} \text{ and } n_2 \in Lm - a(\mathbf{w}_r) - l(\mathbf{w}_r) + 1 + N\mathbb{Z}. \quad (\text{III.14})$$

By (III.3), (III.4) and (III.5), we have

$$w_r(n) \overline{w_r(n + l(\mathbf{w}_r) - 1)} \neq 0 \text{ if and only if } n \in a(\mathbf{w}_r) + N\mathbb{Z}. \quad (\text{III.15})$$

From (III.15) we obtain

$$\begin{aligned} & N \sum_{k=0}^{N-1} |X_{\mathbf{w}_r}(Lm, k)|^2 e^{i2\pi k(l(\mathbf{w}_r)-1)/N} \\ &= \sum_{n=0}^{N-1} x(n + l(\mathbf{w}_r) - 1 \bmod N) \overline{x(n)} \\ & \quad \times w_r(Lm - n - l(\mathbf{w}_r) + 1) \overline{w_r(Lm - n)} \\ &= x(n_1) \overline{x(n_2)} w_r(a(\mathbf{w}_r)) \overline{w_r(a(\mathbf{w}_r) + l(\mathbf{w}_r) - 1)}. \end{aligned} \quad (\text{III.16})$$

Therefore

$$\begin{aligned} \frac{x(n_1)}{|x(n_1)|} \cdot \frac{\overline{x(n_2)}}{|x(n_2)|} &= \frac{w_r(a(\mathbf{w}_r) + l(\mathbf{w}_r) - 1) \overline{w_r(a(\mathbf{w}_r))}}{|w_r(a(\mathbf{w}_r) + l(\mathbf{w}_r) - 1) \overline{w_r(a(\mathbf{w}_r))}|} \\ & \times \frac{\sum_{k=0}^{N-1} |X_{\mathbf{w}_r}(Lm, k)|^2 e^{i2\pi k(l(\mathbf{w}_r)-1)/N}}{\left| \sum_{k=0}^{N-1} |X_{\mathbf{w}_r}(Lm, k)|^2 e^{i2\pi k(l(\mathbf{w}_r)-1)/N} \right|}. \end{aligned} \quad (\text{III.17})$$

Combining (III.14), (III.15) and (III.17) shows that the phase difference between $x(n_1)/|x(n_1)|$ and $x(n_2)/|x(n_2)|$ is determined from magnitude measurements (I.4) of the multiple-window STFT. This completes the proof. \blacksquare

We finish this section with the proof of Lemma III.4.

Proof of Lemma III.4: For $0 \leq m' \leq N/L - 1$ and $1 \leq r \leq R$, we have

$$\begin{aligned} Z(\mathbf{w}_r, m') &= \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 |w_r(Lm' - n)|^2 \\ &= \sum_{k=0}^{N-1} \alpha(k) \beta_r(k) e^{2\pi i m' k L/N}, \end{aligned}$$

where

$$\alpha(k) = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 e^{-2\pi i k n / N}, \quad 0 \leq k \leq N-1.$$

Hence for $0 \leq m \leq N/L - 1$ and $1 \leq r \leq R$, we obtain

$$\begin{aligned} & \frac{L}{N} \sum_{m'=0}^{N/L-1} Z(\mathbf{w}_r, m') e^{-2\pi i m m' L / N} \\ &= \sum_{j'=0}^{L-1} \alpha(m + j' N / L) \beta_r(m + j' N / L). \end{aligned} \quad (\text{III.18})$$

Then we get

$$\begin{aligned} \alpha(m + j N / L) &= \frac{L}{N} \sum_{j'=0}^{L-1} \sum_{m'=0}^{N/L-1} a_m(j, j') e^{-2\pi i m m' L / N} \\ &\quad \times \left(\sum_{r=1}^R \overline{\beta_r(m + j' N / L)} Z(\mathbf{w}_r, m') \right), \end{aligned}$$

where $0 \leq m \leq N/L - 1$ and $0 \leq j \leq L - 1$. This together with

$$|x(n)|^2 = \sum_{k=0}^{N-1} \alpha(k) e^{2\pi i k n / N}, \quad 0 \leq n \leq N-1, \quad (\text{III.19})$$

proves (III.12). \blacksquare

IV. RECONSTRUCTION ALGORITHM AND ERROR ESTIMATES

Consider the family $\mathbf{W} = \{\mathbf{w}_r\}_{r=1}^R$ of window functions having period N extension and satisfying (III.5) and (III.6). From Theorem III.1, it follows that any signal $\mathbf{x} = (x(0), \dots, x(N-1))^\top$ with a connected graph $\tilde{\mathcal{G}}(\mathbf{x}, \mathbf{W}, L)$ can be reconstructed, up to a global phase, from magnitude measurements $|X_{\mathbf{w}_r}(Lm, k)|$, $1 \leq r \leq R$, $0 \leq m \leq N/L - 1$, $0 \leq k \leq N-1$, of its multiple-window STFT. From the constructive proof of Theorem III.1, we propose the following reconstruction algorithm:

1. Apply (III.12) to find magnitudes $|x(n)|$, $0 \leq n \leq N-1$.
2. Create the graph $\tilde{G}(\mathbf{x}, \mathbf{W}, L)$ in (III.1) and verify its connectivity.
3. Apply (III.17) to find phase difference between $\frac{x(n_1)}{|x(n_1)|}$ and $\frac{x(n_2)}{|x(n_2)|}$, where n_1, n_2 are endpoints of an edge of the graph $\tilde{G}(\mathbf{x}, \mathbf{W}, L)$, provided that $\tilde{G}(\mathbf{x}, \mathbf{W}, L)$ is connected.

The reconstruction algorithm proposed above indicates that \mathbf{x} can be recovered, up to a global phase, from its $2NR/L$ measurements $\sum_{k=0}^{N-1} |X_{\mathbf{w}_r}(Lm, k)|^2 e^{i2\pi k(l(\mathbf{w}_r)-1)/N}$ and $\sum_{k=0}^{N-1} |X_{\mathbf{w}_r}(Lm, k)|^2$, where $1 \leq r \leq R$, $0 \leq m \leq N/L - 1$.

For a window family $\mathbf{W} = \{\mathbf{w}_r\}_{r=1}^R$ with period N extension, we set $\|\mathbf{W}\|_2 = \left(\sum_{r=1}^R \sum_{n=0}^{N-1} |w_r(n)|^2 \right)^{1/2}$, $\|\mathbf{W}\|_* = \min_{1 \leq r \leq R} |w_r(a(\mathbf{w}_r)) w_r(a(\mathbf{w}_r) + l(\mathbf{w}_r) - 1)|$, and $\|\mathbf{A}\|_1 = \sum_{m=0}^{N/L-1} \sum_{j,j'=0}^{L-1} |a_m(j, j')|$. For the scenario that magnitude measurements (I.4) of the multiple-window STFT are corrupted, we have the following error estimate between the original signal \mathbf{x} and the approximation \mathbf{x}_ϵ obtained

from the proposed reconstructed algorithm with the corrupted magnitude measurements (I.6).

Theorem IV.1. *Let L, \mathbf{W} and $\mathbf{x} = (x(0), \dots, x(N-1))^\top$ be as in Theorem III.1, and $\mathbf{x}_\epsilon = (x_\epsilon(0), \dots, x_\epsilon(N-1))^\top$ be the approximation obtained from the proposed reconstructed algorithm with the corrupted magnitude data (I.6). If*

$$|\epsilon| \leq \frac{\min_{n \in V(\mathbf{x})} |x(n)|^2}{4 \|\mathbf{A}\|_1 \|\mathbf{W}\|_2^2}, \quad (\text{IV.1})$$

then there exists $\beta \in \mathbb{R}$ such that

$$||x_\epsilon(n)|^2 - |x(n)|^2| \leq \|\mathbf{A}\|_1 \|\mathbf{W}\|_2^2 |\epsilon| \quad (\text{IV.2})$$

for all $0 \leq n \leq N-1$, and

$$\left| \frac{x_\epsilon(n)}{|x_\epsilon(n)|} - e^{2\pi i \beta} \frac{x(n)}{|x(n)|} \right| \leq \frac{2N^3 |\epsilon|}{\|\mathbf{W}\|_* \min_{n \in V(\mathbf{x})} |x(n)|^2} \quad (\text{IV.3})$$

for all $n \in V(\mathbf{x})$.

Proof: The estimate (IV.2) follows from (III.12) and

$$\sum_{r=1}^R |\beta_r(k)| \leq \frac{\|\mathbf{W}\|_2^2}{N}, \quad 0 \leq k \leq N-1.$$

Take endpoints n_1, n_2 of an edge of the graph $\tilde{\mathcal{G}}(\mathbf{x}, \mathbf{W}, L)$, and select $1 \leq r \leq R$ and $0 \leq m \leq N/L - 1$ so that (III.13) holds. Set $c := \sum_{k=0}^{N-1} |X_{\mathbf{w}_r}(Lm, k)|^2 e^{i2\pi k(l(\mathbf{w}_r)-1)/N}$ and $c_\epsilon := \sum_{k=0}^{N-1} (|X_{\mathbf{w}_r}(Lm, k)|^2 + \epsilon(r, m, k)) e^{i2\pi k(l(\mathbf{w}_r)-1)/N}$. Then

$$|c| \geq \frac{\|\mathbf{W}\|_*}{N} \min_{n \in V(\mathbf{x})} |x(n)|^2 \quad (\text{IV.4})$$

by (III.16), and

$$|c_\epsilon - c| \leq N |\epsilon|. \quad (\text{IV.5})$$

Combining (III.17), (IV.4) and (IV.5), we obtain

$$\begin{aligned} & \left| \frac{x_\epsilon(n_1)}{|x_\epsilon(n_1)|} \frac{x_\epsilon(n_2)}{|x_\epsilon(n_2)|} - \frac{x(n_1)}{|x(n_1)|} \frac{x(n_2)}{|x(n_2)|} \right| \\ &= \left| \frac{c_\epsilon}{|c_\epsilon|} - \frac{c}{|c|} \right| \leq \frac{|c_\epsilon - c| + ||c_\epsilon| - |c||}{|c|} \\ &\leq \frac{2N^2 |\epsilon|}{\|\mathbf{W}\|_* \min_{n \in V(\mathbf{x})} |x(n)|^2}. \end{aligned}$$

This, together with the fact that the diameter of the graph $\tilde{\mathcal{G}}(\mathbf{x}, \mathbf{W}, L)$ is at most N , proves (IV.3). \blacksquare

Remark IV.2. Let $\tilde{\mathbf{x}}_\epsilon = (\tilde{x}_\epsilon(0), \dots, \tilde{x}_\epsilon(N-1))^\top$, where

$$|\tilde{x}_\epsilon(n)| = \begin{cases} 0, & \text{if } |x_\epsilon(n)| \leq \frac{1}{2} \min_{n \in V(\mathbf{x})} |x(n)| \\ |x_\epsilon(n)|, & \text{otherwise} \end{cases}$$

for $0 \leq n \leq N-1$. Then it follows from (IV.1) and (IV.2) that $V(\mathbf{x}) = V(\tilde{\mathbf{x}}_\epsilon)$ and $\tilde{G}(\mathbf{x}, \mathbf{W}, L) = \tilde{G}(\tilde{\mathbf{x}}_\epsilon, \mathbf{W}, L)$. This indicates that the graph $\tilde{G}(\mathbf{x}, \mathbf{W}, L)$ associated with the original signal \mathbf{x} could be found in the noisy environment.

V. CONCLUSIONS

For multiple windows with small supporting lengths, certain constraints on the support of a signal could be crucial for its phase retrievability from magnitude measurements of its multiple-window STFT. The proposed reconstruction algorithm from corrupted magnitude measurements yields a good approximation to the original signal if we have some priori knowledge on the noise level and the minimal magnitude of nonzero components of the original signal.

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