Cardinal Refinable Functions on Plane

Xianliang Shi[1]
Department of Mathematics, Hangzhou University
Hangzhou, Zhejiang 310028, P. R. China

and

Qiyu Sun[2]
Center for Mathematical Sciences, Zhejiang University
Hangzhou, Zhejiang 310027, P. R. China

Sept. 1995

ABSTRACT In this paper, the construction and regularity of cardinal refinable functions on plane are considered.

AMS Subject Classification:42C15.

[1]Current address: Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, USA

[2]The author is partially supported by National Sciences Foundation of China and Zhejiang Provincial Sciences Foundation of China

1. Introduction

A distribution ϕ on plane is said to be *refinable* if ϕ satisfies a refinement equation

$$\phi(x,y)=\sum_{m,n\in Z}c(m,n)\phi(2x-m,2y-n), \tag{1}$$

where the sequence $\{c(m,n)\}$ satisfies

$$\sum_{m,n\in Z}c(m,n)=1.$$

Let

$$H(\xi,\eta)=\frac{1}{4}\sum_{m,n\in Z}c(m,n)e^{i(m\xi+n\eta)} \tag{2}$$

be the *symbol* of (1). Then the distribution ϕ in (1) is completely determined by its symbol function $H(\xi,\eta)$ up to multiplying a constant. In particaular

$$\hat{\phi}(\xi,\eta)=\prod_{j=1}^{\infty}H(2^{-j}\xi,2^{-j}\eta)\hat{\phi}(0,0). \tag{3}$$

Hereafter the *Fourier transform* of an integrable function f is defined by

$$\hat{f}(\xi,\eta)=\int_R\int_R f(x,y)e^{-i(x\xi+y\eta)}dxdy.$$

In this paper we will deal with compactly suppported refinable distribution ϕ with $\hat{\phi}(0,0)=1$. In this case the symbol function H is assumed to be a trigonometrical polynomial.

A continuous function ϕ is said to be *cardinal* if the restriction of ϕ to $Z\times Z$ satisfies

$$\phi(m,n)=\begin{cases}1, & if \quad (m,n)=(0,0),\\ 0, & if \quad (m,n)\neq(0,0).\end{cases} \tag{4}$$

For cardinal refinable function ϕ, the corresponding symbol H satisfies

$$H(\xi,\eta)+H(\xi+\pi,\eta)+H(\xi,\eta+\pi)+H(\xi+\pi,\eta+\pi)=1. \tag{5}$$

Refinable functions arised in many aspects such as construction of nondifferential functions, subdivision scheme and multiresolution analysis etc. (see [CDM], [C], [D], [M])

The *B-splines* $B_k, k\geq 1$, defined by

$$\widehat{B}_k(\xi)=(\frac{1-e^{-i\xi}}{i\xi})^k$$

and the Daubechies' scaling functions ${}_N\phi$ in [D] are two important classes of compactly supported refinable functions on one dimension. The *box spline* B_Ξ, which is defined by

$$\widehat{B}_\Xi(\xi,\eta) = \prod_{s=1}^{N} \frac{1-e^{-i(a(s1)\xi+a(s2)\eta)}}{i(a(s1)\xi+a(s2)\eta)}, \tag{6}$$

where

$$\Xi = \begin{pmatrix} a(11) & \cdots & a(N1) \\ a(12) & \cdots & a(N2) \end{pmatrix} \tag{7}$$

is a $2\times N$ matrix with integer entries and and with full rank 2, and the scaling functions constructed by tensor product of two scaling functions in one dimension are corresponding important classes of compactly supported refinable functions on plane (see [M]). Here we say that a compactly supported distribution ϕ on plane is a *scaling function* if ϕ is refinable and ϕ is *linearlly independent for its integer translates*, which means that the map $\phi *'$ defined by

$$\phi *' : \{d(m,n)\} \to \sum_{m,n\in Z} d(m,n)\phi(x-m,y-n)$$

is one-to-one.

Cardinal function are very important in approximation of equidistant data and signal processing. The cardinal spline in [S] and $\frac{\sin \pi x}{\pi x}$ in Shannon sampling theorem [Sh] are two widely-used cardinal refinable functions in approximation theory and signal processing, but these functions are not compactly supported. The construction of compactly supported cardinal refinable function in one-dimension is well-studied.(see [L], [BDS], [CS])

The centered tensor product $B_2(x)B_2(y)$ of two B-spline, and the hat function B_Ξ with three direction on plane

$$\Xi = \begin{pmatrix} 1 & 1, & 0, & 0 \\ 0, & 0, & 1, & 1 \end{pmatrix}$$

are two known examples which are cardinal refinable functions with compact support.(see [BHR])

The cadinal refinable function can be constructed from some appropriate refinable function. Until now there are two popular methods to construct cardnial refinable function ϕ from a refinable function ψ. The one is to define ϕ with help of Fourier transform by

$$\hat{\phi}(\xi,\eta) = \frac{\hat{\psi}(\xi,\eta)}{\sum_{m,n\in Z}\hat{\psi}(\xi+2m\pi,\eta+2n\pi)}$$

when $\hat{\psi}$ has appropriate decay at infinity and

$$\sum_{m,n\in Z}\hat{\psi}(\xi+2m\pi,\eta+2n\pi) \neq 0.$$

(see [W])

The other is to define ϕ by

$$\phi(x,y)=\int_R\int_R \psi(s-x,t-y)\psi(s,t)dsdt$$

when ϕ is *orthonormal*, which means

$$\int_R\int_R \phi(s,t)\phi(s-m,t-n)dsdt = \begin{matrix} 1, & if & (m,n)=(0,0) \\ 0, & if & (m,n)\neq(0,0). \end{matrix}$$

The cardinal refinable function constructed by the first method is generally not compactly supported. A problem to construct cardinal refinable function by the second method is that we know few about construction of compactly supported orthonormal refinable functions on plane.

In particaular, study on cardinal refinable function on plane is helpful to understand the construction of compactly supported orthonormal scaling functions on plane. Precisely if the corresponding symbol function $H(\xi,\eta)$ of cardinal refinable function ϕ can be written as

$$H(\xi,\eta)=R(\xi,\eta)R(-\xi,-\eta) \tag{8}$$

for some trigonometrical polynomial R, then the solution of (1) corresponding to symbol function $R(\xi,\eta)$ would be orthonormal. The solvability of (8) in one dimension is shown by Riesz Lemma when $H\geq 0$ is a polynomial of $\cos\xi$.

From the construction of refinable function in one dimension, we know that a Hölder continuous refinable solution in one dimension can be written as convolution of a B-spline B_k and a refinable distribution ψ. The Daubechies' scaling function was constructed through the construction of ψ in some sense. But it is not known whether a Hölder continuous solution ϕ of (1) on plane can be written as convolution $B_\Xi * \psi$ of a Box spline B_Ξ and a compactly supported refinable distribution ψ, i.e.,

$$\phi(x,y)=B_\Xi*\psi(x,y)=\langle B_\Xi(x-\cdot,y-\cdot),\psi(\cdot,\cdot)\rangle, \tag{9}$$

where $\langle f,g\rangle$ denotes the inner product of two distribution f and g, which becomes the inner product on the space of square integrable functions when f and g are square integrable. (see [CDM])

In this paper we will construct cardinal refinable functions ϕ which can be written as (9), i.e. the convolution of a class of box-splines B_Ξ and refinable distributions ψ

In particaular there are some restriction on the box-spline B_Ξ such that ϕ in (9) is cardinal. By the definition (4) of cardinal function we know that ϕ is linearly independent for its integer translates. Hence B_Ξ is also linearly independently for its integer translates. By elementary theory on box spline, we obtain that Ξ is essentially one of the following two types:

$$\Xi = \left(\underbrace{\begin{matrix} a, \cdots, a, \\ b, \cdots, b, \end{matrix}}_{r \quad factors} \quad \underbrace{\begin{matrix} c, \cdots, c \\ d, \cdots, d \end{matrix}}_{s \quad factors} \right) \tag{10}$$

where $ad - bc = \pm 1$ and $r, s \geq 1$ and

$$\Xi = \left(\underbrace{\begin{matrix} a, \cdots, a, \\ b, \cdots, b, \end{matrix}}_{r \quad factors} \quad \underbrace{\begin{matrix} c, \cdots, c \\ d, \cdots, d \end{matrix}}_{s \quad factors} \quad \underbrace{\begin{matrix} e, \cdots, e \\ f, \cdots, f \end{matrix}}_{t \quad factors} \right), \tag{11}$$

where $ad - bc = \pm 1, af - be = \pm 1, cf - ed = \pm 1$ and $r, s, t \geq 1$. (see [dBHR])

On the other hand when $ad - bc = \pm 1$ it is easy to check that $\phi(ax + by, cx + dy)$ is also refinable (cardinal, othonormal) when ϕ is. Then we can assume without loss of generality that $a = d = 1, c = b = 0, e = f = 1$ in (10) and (11).

In the case Ξ in (10), the box splines B_Ξ corresponding to Ξ are tensor product of two B-splines in one dimension, and the construction of cardinal refinable functions which has the form (9) can be followed as the one in one dimension.

It seems much more difficult to the case that Ξ has the form (11) with $a = d = e = f = 1, b = c = 0$. In this paper the case for which Ξ has the form (11) and $r = s = t = N$ is considered.

For notational simplity, let B_{Ξ_N} the box spline defined by (7) corresponding to

$$\Xi_N = \left(\underbrace{\begin{matrix} 1, \cdots, 1, \\ 0, \cdots, 0, \end{matrix}}_{N \quad factors} \quad \underbrace{\begin{matrix} 0, \cdots, 0 \\ 1, \cdots, 1 \end{matrix}}_{N \quad factors} \quad \underbrace{\begin{matrix} 1, \cdots, 1 \\ 1, \cdots, 1 \end{matrix}}_{N \quad factors} \right).$$

In this case the symbol function $H(\xi, \eta)$ can be written as

$$H(\xi, \eta) = \left(\frac{1 + e^{i\xi}}{2}\right)^N \left(\frac{1 + e^{i\eta}}{2}\right)^N \left(\frac{1 + e^{i(\xi+\eta)}}{2}\right)^N G(\xi, \eta), \tag{12}$$

where G is a trigonometrical polynomial.

In section 2, we will characterize the solution of (5) when H has the form in (12). Let

$$\begin{aligned} H_1(\xi, \eta) = & \ \tfrac{1+e^{i\xi}}{2} \times \tfrac{1+e^{i\eta}}{2} \times \tfrac{1+e^{i(\xi+\eta)}}{2} \times e^{-i(\xi+\eta)} \\ = & \ \cos \tfrac{\xi}{2} \cos \tfrac{\eta}{2} \cos \tfrac{\xi+\eta}{2} \end{aligned} \tag{13}$$

and

$$\begin{aligned} G_N(\xi, \eta) = & \ \left(\tfrac{1+e^{i\xi}}{2}\right)^{-N} \left(\tfrac{1+e^{i\eta}}{2}\right)^{-N} \left(\tfrac{1+e^{i(\xi+\eta)}}{2}\right)^{-N} \times \\ & \textstyle\sum_{k_1+k_2+k_3+k_4=4N-3} \alpha_N(k_1, k_2, k_3, k_4) \frac{(4N-3)!}{k_1!k_2!k_3!k_4!} \times \\ & H_1(\xi, \eta)^{k_1} H_1(\xi + \pi, \eta)^{k_2} H_1(\xi, \eta + \pi)^{k_3} H_1(\xi, \eta)^{k_4}, \end{aligned} \tag{14}$$

where $\alpha_N(k_1,k_2,k_3,k_4)$ is defined by

$$\alpha_N(k_1,k_2,k_3,k_4) = \begin{array}{ll} 0, & if \quad k_1 \leq N-1, \\ \frac{1}{\#(E)}, & if \quad k_1 \geq N, \end{array} \tag{15}$$

$E = \{1 \leq j \leq 4; k_j \geq N\}$ and $\#(E)$ denotes the cardinality of the set E.

Theorem 1. *Let G be a trigonometrical polynomial and G_N be defined in (14). Then*

$$H(\xi,\eta) = (\frac{1+e^{i\xi}}{2})^N(\frac{1+e^{i\eta}}{2})^N(\frac{1+e^{i(\xi+\eta)}}{2})^N G(\xi,\eta)$$

satisfies (5) if and only if

$$\begin{aligned} G(\xi,\eta) = \quad & G_N(\xi,\eta) + (1-e^{i\xi})^N(1-e^{i(\xi+\eta)})^N G_1(\xi,\eta) + (1-e^{i\eta})^N \times \\ & (1-e^{i(\xi+\eta)})^N G_2(\xi,\eta) + (1-e^{i\xi})^N(1-e^{i\eta})^N G_3(\xi,\eta) + \\ & (1-e^{i\xi})^N(1-e^{i\eta})^N(1-e^{i(\xi+\eta)})^N G_4(\xi,\eta), \end{aligned} \tag{16}$$

where trigonometrical polynomials $G_i, 1 \leq i \leq 4$, satisfy

$$\left\{ \begin{array}{l} G_1(\xi,\eta) + G_1(\xi+\pi,\eta) = 0, \\ G_2(\xi,\eta) + G_2(\xi,\eta+\pi) = 0, \\ G_3(\xi,\eta) + G_3(\xi+\pi,\eta+\pi) = 0, \\ G_4(\xi,\eta) + G_4(\xi,\eta+\pi) + G_4(\xi+\pi,\eta) + G_4(\xi+\pi,\eta+\pi) = 0. \end{array} \right. \tag{17}$$

We do not know whether the refinable function corresponding to the symbol

$$H(\xi,\eta) = (1+e^{i\xi})^N(1+e^{i\eta})^N(1+e^{i(\xi+\eta)})^N G_N(\xi,\eta)$$

is cardinal or not for all $N \geq 2$. Only it is known that the one when $N = 1$ is cardinal since in this case $G_1(\xi,\eta) = e^{-i(\xi+\eta)}$ and the refinable function is the centered box function B_Ξ with

$$\Xi = \left(\begin{array}{ccc} 1, & 0, & 1 \\ 0, & 1, & 1 \end{array} \right)$$

(see [dBHR])

To construct Hölder continuous cardinal refinable functions, we consider the case that $N = 2M$ is an even integer. Let

$$I_2(\xi,\eta) = \cos^2\frac{\xi}{2}\cos^2\frac{\eta}{2}\cos^2\frac{\xi+\eta}{2}(1+\sin^2\frac{\xi}{2}+\sin^2\frac{\eta}{2}+\sin^2\frac{\xi+\eta}{2}) \tag{18}$$

and

$$\begin{aligned} I_{2M}(\xi,\eta) \quad & = \sum_{k_1+k_2+k_3+k_4=4M-3} \tfrac{(4M-3)!}{k_1!k_2!k_3!k_4!}\alpha_M(k_1,k_2,k_3,k_4) \\ & I_2(\xi,\eta)^{k_1} I_2(\xi+\pi,\eta)^{k_2} I_2(\xi,\eta+\pi)^{k_3} I_2(\xi+\pi,\eta+\pi)^{k_4}. \end{aligned} \tag{19}$$

In section 3 we will prove the following

Theorem 2. *Let ϕ_{2M} be the solution of (1) with its coresponding symbol function $I_{2M}(\xi,\eta)$. Then there exists a constant $\lambda > 0$ independent of M such that $\phi_{2M} \in C^{\lambda M}$, where $C^{\lambda M}$ denotes the Hölder class.*

Remark 1. Since each nonzero term in the summation of (19) can divide the term

$$(\cos^2 \frac{\xi}{2} \cos^2 \frac{\eta}{2} \cos^2 \frac{\xi+\eta}{2})^M,$$

we obtain that I_{2M} has the form (12) with $N = 2M$ by (13). Furthermore we have

$$\begin{aligned} & I_{2M}(\xi,\eta) + I_{2M}(\xi,\eta+\pi) + I_{2M}(\xi+\pi,\eta) + I_{2M}(\xi+\pi,\eta+\pi) \\ = & \textstyle\sum_{k_1+k_2+k_3+k_4=4M-3} \big(\alpha_M(k_1,k_2,k_3,k_4) + \alpha_M(k_2,k_3,k_4,k_1) + \\ & \quad \alpha_M(k_3,k_4,k_1,k_2) + \alpha_M(k_4,k_1,k_2,k_3)\big) \times \tfrac{(4M-3)!}{k_1!k_2!k_3!k_4!} \times \\ & \quad I_2(\xi,\eta)^{k_1} I_2(\xi+\pi,\eta)^{k_2} I_2(\xi,\eta+\pi)^{k_3} I_2(\xi+\pi,\eta+\pi)^{k_4} \\ = & (I_2(\xi,\eta) + I_2(\xi,\eta+\pi) + I_2(\xi+\pi,\eta) + I_2(\xi+\pi,\eta+\pi))^{4M-3} \\ = & 1. \end{aligned}$$

Hence $I_{2M}(\xi,\eta)$ satisfies (5).

Remark 2. Let ψ_M be the solution of (1) with its corresponding symbol function

$$e^{i2M(\xi+\eta)} I_{2M}(\xi,\eta)(\cos^2 \frac{\xi}{2} \cos^2 \frac{\eta}{2} \cos^2 \frac{\xi+\eta}{2})^{-M}.$$

Then ϕ_{2M} is the convolution of $B_{\Xi_{2M}}$ and ψ_M, i.e.,

$$\phi = B_{\Xi_{2M}} * \psi_M.$$

By the definition of $I_{2M}(\xi,\eta)$, we obtain that $\hat{\psi}_M(\xi,\eta) > 0$. Then by criterion in [CS], we know that ϕ_{2M} is linearly independent for its integer translates. By standard argument, Remark 1 and Theorem 2, we obtain that ϕ_{2M} is cardinal when M is chosen large enough such that $\lambda M > 2$. (see [D], [BDS], [L])

Remark 3. A multiresolution of $L^2(R^2)$, the space of square integrable functions on plane, is a family of closed subspace of $L^2(R^2)$, which satisfies

i) $V_j \subset V_{j+1}, \quad \forall \quad j \in Z$,
ii) $\overline{\cup_{j\in Z} V_j} = L^2(R^2)$,
iii) $\cap_{j\in Z} V_j = \emptyset$,
iv) $f \in V_j \iff f(2^{-j}\cdot) \in V_0$,
v) there exists a function ϕ in V_0 such that $\{\phi(\cdot - m, \cdot - n)\}_{m,n\in Z}$ is a Riesz basis of V_0, which means that V_0 is spanned by $\{\phi(\cdot - m, \cdot - n)\}_{m,n\in Z}$ and there exist two positive constant A and B such that

$$A\big(\sum_{m,n\in Z} |d(m,n)|^2\big)^{1/2} \le \| \sum_{m,n\in Z} d(m,n)\phi(\cdot - m, \cdot - n)\|_2 \le B\big(\sum_{m,n\in Z} |d(m,n)|^2\big)^{1/2}$$

holds for all square summable sequences $\{d(m,n)\}_{m,n\in Z}$, where $\|\cdot\|_2$ denotes the norm on $L^2(R^2)$. (see [C], [D], [M])

Let $V_j \quad (j \in Z)$ be the closed space of $L^2(R^2)$ spanned by $\{\phi_{2M}(2^j\cdot -m, 2^j\cdot -n)\}_{m,n\in Z}$. Then $\{V_j\}_{j\in Z}$ is a multiresolution when M is chosen large enough such that $\lambda M > 2$. This is because $\{\phi(\cdot - m, \cdot - n)\}_{m,n\in Z}$ is a Riesz basis of V_0 when ϕ is cardinal.

Remark 4. Define the filter support width $W(\phi)$ of a refinable function ϕ of (1) by the cardinality of integer knots in the least convex set which contains all subindices (m,n) with $c(m,n) \neq 0$. Define the regularity set $R(\phi)$ of a function ϕ by the dimension of polynomials $P(x,y)$ for which $P(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\phi$ is continuous. By elementary argument we know that

$$W(\phi) \geq R(\phi)$$

(see [CDM], [DL], [S]). For ϕ_{2M} we know that $W(\phi_{2M}) \leq C_1 M^2$ and $R(\phi_{2M}) \geq C_2 M^2$ for some constants C_1 and C_2 independent of M by the definition (19) of $I_{2M}(\xi,\eta)$ and Theorem 2. Therefore there exists a constant C independent of M such that

$$W(\phi_{2M}) \leq CR(\phi_{2M}).$$

This shows that the regularity $W(\phi_{2M})$ of the cardinal scaling functions ϕ_{2M} grows propositionally to filter support width $R(\phi_{2M})$.

2. Proof of Theorem 1

To prove Theorem 1, we need some lemmas.

Lemma 1. *Let $N \geq 1, 0 \leq K \leq N-1$. If Laurent polynomials $g_s(z), 0 \leq s \leq K$, satisfies*

$$\sum_{s=0}^{l} \binom{N}{l-s} (-1)^{l-s}\big((1+z)^{N-l+s}g_s(z)+(1-z)^{N-l+s}g_s(-z)\big) = 0, \ \forall\ 0 \leq l \leq K, \tag{20}$$

then there exist Laurent polynomials $g_s^*(z), 0 \leq s \leq K$, such that

$$\sum_{s=0}^{l} \binom{N}{l-s} (-1)^{l-s}(1-z)^{N-l+s}g_s^*(z) = g_l(z), \quad \forall \quad 0 \leq l \leq K. \tag{21}$$

Proof. The above lemma will be proved by induction on $K \leq N-1$. First we prove it when $K = 0$. Therefore we get

$$(1+z)^N g_0(z) + (1-z)^N g_0(-z) = 0$$

by (20) and furthermore

$$g_0(z) = (1-z)^N zR(z^2)$$

for some Laurent polynomial R. Let $g_0^*(z) = zR(z^2)$. Then (21) holds for $l=0$ and Lemma 1 holds when $K=0$.

Inductively we assume that Lemma 1 holds whenl $K=n$. If $n=N-1$, Lemma 1 is proved completely. So we assume that $n \le N-2$. Now we prove Lemma 1 holds when $K=n+1 \le N-1$. By the proof of Lemma 1 when $K=0$, we obtain that

$$g_0(z) = (1-z)^N zR(z^2)$$

for some Laurent polynomial R and $g^*(z)$ defined by

$$g_0^*(z) = zR(z^2)$$

satisfies (21) when $l=0$. Define

$$g_l^1(z) = g_l(z) - \begin{pmatrix} N \\ l \end{pmatrix} (-1)^l g_0^*(z)(1-z)^{N-l}, \quad \forall \quad 0 \le l \le n+1.$$

Then we get

$$g_0^1(z) = 0$$

and

$$\begin{aligned} & \textstyle\sum_{s=0}^{l-1} \begin{pmatrix} N \\ l-s-1 \end{pmatrix} (-1)^{l-1-s}\big((1+z)^{N-l+1+s} g_{s+1}^1(z) + (1-z)^{N-l+1+s} g_{s+1}^1(-z)\big) \\ = & \textstyle\sum_{s=0}^{l} \begin{pmatrix} N \\ l-s \end{pmatrix} (-1)^{l-s}\big((1+z)^{N-l+s} g_s^1(z) + (1-z)^{N-l+s} g_s^1(-z)\big) \\ = & -R(z^2) \textstyle\sum_{s=0}^{l} \begin{pmatrix} N \\ l-s \end{pmatrix} \begin{pmatrix} N \\ s \end{pmatrix} (-1)^l \big(z(1+z)^{N-l+s}(1-z)^{N-s} - z(1-z)^{N-l+s}(1+z)^{N-s}\big) \\ = & 0, \ \forall \ 1 \le l \le n+1. \end{aligned}$$

Therefore $\{g_{s+1}^1, 0 \le s \le n\}$ satisfies (20). By induction there exist Laurent polynomials $g_s^{1*}(z), 0 \le s \le n$, such that

$$\sum_{s=0}^{l} \big(\ N \ l-s \ \big) (-1)^{l-s}(1-z)^{N-l+s} g_s^{1*}(z) = g_{l+1}^1(z), \quad \forall \ 0 \le l \le n.$$

Hence we get

$$\begin{aligned} & \textstyle\sum_{s=1}^{l} \ \begin{matrix} N \\ l-s \end{matrix} \ (-1)^{l-s}(1-z)^{N-l+s} g_{s-1}^{1*}(z) + \begin{pmatrix} N \\ l \end{pmatrix} (-1)^l (1-z)^{N-l} g_0^*(z) \\ = & \textstyle\sum_{s=0}^{l-1} \begin{pmatrix} N \\ l-s-1 \end{pmatrix} (-1)^{l-1-s}(1-z)^{N-l+1+s} g_s^{1*}(z) + \begin{pmatrix} N \\ l \end{pmatrix} (-1)^l (1-z)^{N-l} g_0^*(z) \\ = & g_l^1(z) + \begin{pmatrix} N \\ l \end{pmatrix} (-1)^l (1-z)^{N-l} g_0^*(z) = g_l(z) \quad \forall \quad 1 \le l \le n+1, \end{aligned}$$

and $\{g_0^*(z), g_s^{1*}(z), 0 \leq s \leq n\}$ satisfies (21). Hence Lemma 1 holds when $K = n+1$. This completes the proof of Lemma 1.

Lemma 2. *If $g_l(z), 0 \leq l \leq N-1$, satisfies*

$$\sum_{s=0}^{l} \begin{pmatrix} N \\ l-s \end{pmatrix} (-1)^{l-s}\big((1+z)^{N-l+s} g_s(z) + (1-z)^{N-l+s} g_s(-z)\big) = 0, \ \forall\ 0 \leq l \leq N-1,$$

then there exist Laurent polynomials R_1 and R_2 such that

$$\sum_{s=0}^{N-1} (1-z_1)^s g_s(z_2) = (z_1 - z_2)^N R_1(z_1, z_2) + (1-z_1)^N R_2(z_1, z_2).$$

Proof. By Lemma 1, there exist Laurent polynomials $g_l^*(z_2), 0 \leq l \leq N-1$, such that

$$\sum_{s=0}^{l} \begin{pmatrix} N \\ l-s \end{pmatrix} (-1)^{l-s} (1-z_2)^{N-l+s} g_s^*(z_2) = g_l(z), \quad \forall \quad 0 \leq l \leq N-1.$$

Therefore we get

$$\begin{aligned}
& \textstyle\sum_{s=0}^{N-1} (1-z_1)^s g_s(z_2) \\
= & \textstyle\sum_{l=0}^{N-1} g_l^*(z_2) \sum_{s=0}^{N-1-l} (1-z_1)^{l+s} \begin{pmatrix} N \\ s \end{pmatrix} (-1)^s (1-z_2)^{N-s} \\
= & \textstyle\sum_{l=0}^{N-1} g_l^*(z_2)(1-z_1)^l((1-z_2)-(1-z_1))^N \quad mod \quad (1-z_1)^N \\
= & (z_1 - z_2)^N \textstyle\sum_{l=0}^{N-1} G_l^*(z_2)(1-z_1)^l \quad mod \quad (1-z_1)^N.
\end{aligned}$$

Hereafter we say that

$$A = B \quad mod \quad C$$

for Laurent polynomials A, B and C if $(A-B)/C$ is still a Laurent polynomial. This completes the proof of Lemma 2.

Lemma 3. *Let $N \geq 1$ and let $H(z_1, z_2) = (1+z_1)^N (1+z_2)^N (z_1 + z_2)^N G(z_1, z_2)$, where G is a Laurent polynomial. If H satisfies*

$$H(z_1, z_2) + H(z_1, -z_2) + H(-z_1, z_2) + H(-z_1, -z_2) = 0, \tag{22}$$

then there exist Laurent polynomials R and G_2 such that

$$G(z_1, z_2) = (1-z_1)^N R(z_1, z_2) + (1-z_2)^N (z_1 - z_2)^N G_2(z_1, z_2) \tag{23}$$

and

$$G_2(z_1, z_2) + G_2(z_1, -z_2) = 0.$$

Proof. Write

$$G(z_1,z_2) = (1-z_1)^N h_N(z_1,z_2) + \sum_{l=0}^{N-1}(1-z_1)^l h_l(z_2).$$

Observe that

$$(z_1+z_2)^N = \sum_{l=0}^{n-1}\left(\begin{array}{c} N \\ l \end{array}\right)(1+z_2)^{N-l}(-1)^l(1-z_1)^l.$$

Therefore we get

$$(z_1+z_2)^N G(z_1,z_2) = \sum_{l=0}^{N-1}(1-z_1)^l \tilde{h}_l(z_2) \quad mod \quad (1-z_1)^N,$$

where

$$\tilde{h}_l(z_2) = \sum_{s=0}^{l}\left(\begin{array}{c} N \\ l-s \end{array}\right)(-1)^{l-s}(1+z_2)^{N-l+s}h_s(z_2). \tag{24}$$

By (22) we get

$$\begin{array}{l}(1+z_1)^N \sum_{l=0}^{N-1}(1-z_1)^l\big((1+z_2)^N\tilde{h}_l(z_2)+(1-z_2)^N\tilde{h}_l(-z_2)\big)+(1-z_1)^N\times \\ \quad \sum_{l=0}^{N-1}(1+z_1)^l\big((1+z_2)^N\tilde{h}_l(z_2)+(1-z_2)^N\tilde{h}_l(-z_2)\big)=0 \quad mod \quad (1-z_1^2)^N.\end{array}$$

and

$$(1+z_2)^N\tilde{h}_l(z_2)+(1-z_2)^N\tilde{h}_l(-z_2)=0, \quad \forall \quad 0\le l\le N-1. \tag{25}$$

By the definition (24) of $\tilde{h}_l$, we obtain that

$$h_s(z_2) = 0 \quad mod \quad (1-z_2)^N$$

and

$$h_s(z_2) = (1-z_2)^N g_s(z_2)$$

for some Laurent polynomial $g_s, 0\le s\le N-1$. Then we have

$$\begin{array}{l}\sum_{s=0}^{l}\left(\begin{array}{c} N \\ l-s \end{array}\right)(-1)^{l-s}\big((1+z_2)^{N-l+s}g_s(z_2)+ \\ \qquad (1-z_2)^{N-l+s}g_s(-z_2)\big)=0 \quad \forall \quad 0\le l\le N-1,\end{array}$$

and

$$G(z_1,z_2) = (1-z_1)^N\tilde{R}_1(z_1,z_2)+(1-z_2)^N(z_1-z_2)^N R_2(z_1,z_2)$$

for Laurent polynomials R_1 and R_2 by Lemma 2. To use the formula in (25), we get

$$(1-z_2^2)^N(z_1^2-z_2^2)^N\big\{(1+z_1)^N(R_2(z_1,z_2)+R_2(z_1,-z_2)) \\ +(1-z_1)^N(R_2(-z_1,z_2)+R_2(-z_1,-z_2))\big\}=0 \quad mod \quad (1-z_1^2)^N$$

and

$$R_2(z_1,z_2)+R_2(z_1,-z_2)=0 \quad mod \quad (1-z_1)^N.$$

Let $G_2(z_1,z_2) = (R_2(z_1,z_2)-R_2(z_1,-z_2))/2$ and $R(z_1,z_2) = R_1(z_1,z_2) + 1/2(1-z_2)^N(z_1-z_2)^N(1-z_1)^{-N}(R_2(z_1,z_2)+R_2(z_1,-z_2))$. Then we get

$$G_2(z_1,z_2)+G_2(z_1,-z_2)=0$$

and

$$G(z_1,z_2)=(1-z_1)^N R(z_1,z_2)+(1-z_2)^N(z_1-z_2)^N G_2(z_1,z_2).$$

Lemma 3 is proved.

Lemma 4. *Let G be a Laurent polynomial and $H(z_1,z_2)=(1+z_1)^N(1+z_2)^N(z_1+z_2)^N G(z_1,z_2)$. If H satisfies (22), then G can be written as*

$$G(z_1,z_2)=(1-z_1)^N(z_1-z_2)^N G_1(z_1,z_2)+(1-z_2)^N(z_1-z_2)^N G_2(z_1,z_2) \\ (1-z_1)^N(1-z_2)^N G_3(z_1,z_2)+(1-z_1)^N(1-z_2)^N(z_1-z_2)^N G_4(z_1,z_2),$$

where Laurent polynomials $G_j, 1\le j\le 4$, satisfies

$$\begin{cases} G_1(z_1,z_2)+G_1(-z_1,z_2)=0, \\ G_2(z_1,z_2)+G_2(z_1,-z_2)=0, \\ G_3(z_1,z_2)+(-1)^N G_3(-z_1,-z_2)=0, \\ G_4(Z-1,z_2)+G_4(-z_1,z_2)+G_4(z_1,z_2)+G_4(-z_1,-z_2)=0. \end{cases} \tag{26}$$

Proof. By Lemma 3, there exists Laurent polynomial R and G_2 such that

$$G_2(z_1,z_2)+G_2(z_1,-z_2)=0$$

and

$$G(z_1,z_2)=(1-z_1)^N R(z_1,z_2)+(1-z_2)^N(z_1-z_2)^N G_2(z_1,z_2).$$

Therefore it suffices to prove

$$R(z_1,z_2)=(z_1-z_2)^N G_1(z_1,z_2)+(1-z_2)^N G_3(z_1,z_2)+(1-z_2)^N(z_1-z_2)^N G_4(z_1,z_2) \tag{27}$$

where G_1, G_3, G_4 satsifies (26).

By (22), we get

$$\begin{aligned}&(1+z_2)^N\big((z_1+z_2)^N R(z_1,z_2)+(-z_1+z_2)^N R(-z_1,z_2)\big)+\\&\quad(1-z_2)^N\big((z_1-z_2)^N R(z_1,-z_2)+(-z_1-z_2)^N R(-z_1,-z_2)\big)=0\end{aligned} \tag{28}$$

and

$$(z_1+z_2)^N R(z_1,z_2)+(-z_1+z_2)^N R(-z_1,z_2)=0quadmod \quad (1-z_2)^N.$$

Therefore we can written R as

$$R(z_1,z_2)=(1-z_2)^N R_1(z_1,z_2)+(z_1-z_2)^N R_2(z_1,z_2)$$

for some Laurent polynomials R_1 and R_2. To use the above formula in (28), we get

$$\begin{aligned}&(1-z_2^2)^N\{(z_1+z_2)^N(R_1(z_1,z_2)+(-1)^N R_1(-z_1,-z_2))+\\&\quad(z_1-z_2)^N(R(z_1,-z_2)+(-1)^N R_1(-z_1,z_2))\}=0 \quad mod \quad (z_1^2-z_2^2)^N\end{aligned}$$

and

$$R_1(z_1,z_2)+(-1)^N R_1(-z_1,-z_2)=0 \quad mod \quad (z_1^2-z_2^2)^N.$$

Let

$$G_3(z_1,z_2)=(R_1(z_1,z_2)-(-1)^N R_1(-z_1,-z_2))/2$$

and

$$R_3(z_1,z_2)=R_2(z_1,z-2)+(1-z_2)^N(z_1-z_2)^{-N}((R_1(z_1,z_2)+(-1)^N R_1(-z_1,-z_2))/2.$$

Then we obtain that

$$G_3(z_1,z_2)+G_3(-z_1,-z_2)=0$$

and

$$R(z_1,z_2)=(z_1-z_2)^N R_3(z_1,z_2)+(1-z_2)^N G_3(z_1,z_2).$$

To use the formula in (28), we get

$$(1+z_2)^N(R_3(z_1,z_2)+R_3(-z_1,z_2))+(1-z_2)^N(R_3(z_1,-z_2)+R_3(-z_1,-z_2))=0.$$

Hence there exist Laurent polynomials G_1 and G_4 such that

$$\begin{cases}R_3(z_1,z_2)=G_1(z_1,z_2)+(1-z_2)^N G_4(z_1,z_2)\\G_1(z_1,z_2)+G_1(-z_1,z_2)=0\\G_4(z_1,z_2)+G_4(z_1,-z_2)+G_4(-z_1,z_2)+G_4(-z_1,-z_2)=0.\end{cases}$$

This completes the proof of Lemma 4.

Now we start to prove Theorem 1.

Proof of Theorem 1. To prove the necessity, we only need to prove that $H_N(\xi,\eta)$ satisfies (5).

By the definition (15) of $\alpha_N(k_1,k_2,k_3,k_4)$ we get

$$\alpha_N(k_1,k_2,k_3,k_4)+\alpha_N(k_2,k_3,k_4,k_1)+\alpha_N(k_3,k_4,k_1,k_2)+\alpha_N(k_4,k_1,k_2,k_3)=1.$$

Hence

$$\begin{array}{rl} & H_N(\xi,\eta)+H_N(\xi+\pi,\eta)+H_N(\xi,\eta+\pi)+H_N(\xi+\pi,\eta+\pi) \\ = & \sum_{k_1+k_2+k_3+k_4=4N-3}\frac{(4N-3)!}{k_1!k_2!k_3!k_4!}\times\big(\alpha_N(k_1,k_2,k_3,k_4)+\alpha_N(k_2,k_3,k_4,k_1)+ \\ & \quad \alpha_N(k_3,k_4,k_1,k_2)+\alpha_N(k_4,k_1,k_2,k_3)\big)\times H_1(\xi,\eta)^{k_1} \\ & \qquad H_1(\xi+\pi,\eta)^{k_2}H_1(\xi,\eta+\pi)^{k_3}H_1(\xi+\pi,\eta+\pi)^{k_4} \\ = & \big(G_0(\xi,\eta)+G_0(\xi+\pi,\eta)+G_0(\xi,\eta+\pi)+G_0(\xi+\pi,\eta+\pi)\big)^{4N-3}=1. \end{array}$$

The necessity is proved.

To prove the sufficiency, we need to prove all trigonometric polynomial $G(\xi,\eta)$ such that $H(\xi,\eta)=(1+e^{i\xi})^N(1+e^{i\eta})^N(1+e^{i(\xi+\eta)})^N G(\xi,\eta)$ satisfies

$$H(\xi,\eta)+H(\xi+\pi,\eta)+H(\xi,\eta+\pi)+H(\xi+\pi,\eta+\pi)=0 \qquad (29)$$

can be written as

$$\begin{array}{l} G(\xi,\eta)=G_N(\xi,\eta)+(1-e^{i\xi})^N(1-e^{i(\xi+\eta)})^N G_1(\xi,\eta)+(1-e^{i\eta})^N(1-e^{i(\xi+\eta)})^N G_2(\xi,\eta) \\ \quad +(1-e^{i\xi})^N(1-e^{i\eta})^N G_3(\xi,\eta)+(1-e^{i\xi})^N(1-e^{i\eta})^N(1-e^{i(\xi+\eta)}G_4(\xi,\eta), \end{array}$$

where trigonometrical polynomials $G_j, 1\le j\le 4$, satisfies (16).

Let $z_1=e^{-i\xi}, z_2=e^{i\eta}$. Define

$$\tilde{G}(z_1,z_2)=e^{2Ni\xi}G(\xi,\eta)$$

and

$$\tilde{H}(z_1,z_2)=(1+z_1)^N(1+z_2)^N(z_1+z_2)^N\tilde{G}(z_1,z_2).$$

Then we can write (29) as

$$\tilde{H}(z_1,z_2)+\tilde{H}(-z_1,z_2)+\tilde{H}(z_1,-z_2)+\tilde{H}(-z_1,-z_2)=0.$$

Hence there exists $G_j, 1\le j\le 4$, such that G_j satisfy (16) and

$$\begin{array}{l} \tilde{G}(z_1,z_2)=(1-z_1)^N(z_1-z_2)^N G_1(z_1,z_2)+(1-z_2)^N(z_1-z_2)^N G_2(z_1,z_2) \\ \quad (1-z_1)^N(1-z_2)^N G_3(z_1,z_2)+(1-z_1)^N(1-z_2)^N(z_1-z_2)^N G_4(z_1,z_2). \end{array}$$

Therefore Theorem 1 follows when $z_1=e^{-i\xi}$ and $z_2=e^{i\eta}$.

3. Proof of Theorem 2

To prove Theorem 2, we need some lemmas.

Lemma 5. *Define* $D = \cup_{j=0}^{8} D_j$, *where*

$$\begin{array}{rcl} D_0 & = & \{(\xi,\eta); 2\pi/3 \le \xi \le 4\pi/3, 2\pi/3 \le \eta \le 4\pi/3\}, \\ D_1 & = & \{(\xi,\eta); 2\pi/3 \le \xi \le 6\pi/5, 0 \le \eta \le 2\pi/3\}, \\ D_2 & = & \{(\xi,\eta); (\eta,\xi) \in D_1\}, \\ D_3 & = & \{(\xi,\eta); 6\pi/5 \le \xi \le 23\pi/18, 0 \le \eta \le 2\pi/3\}, \\ D_4 & = & \{(\xi,\eta); (\eta,\xi) \in D_3\}, \\ D_5 & = & \{(\xi,\eta); 23\pi/18\xi \le 4\pi/3, 23\pi/36 \le \eta \le 2\pi/3\}, \\ D_6 & = & \{(\xi,\eta); (\eta,\xi) \in D_5\}, \\ D_7 & = & \{(\xi,\eta); 23\pi/36\xi \le 2\pi/3, 0 \le \eta \le \pi/3\}, \\ D_8 & = & \{(\xi,\eta); (\eta,\xi) \in D_7\}. \end{array}$$

Then there exists a constant $\delta_1 < 1$ *such that*

$$0 \le I_2(\xi,\eta) \le \frac{1}{4}\delta_1, \quad \forall \quad (\xi,\eta) \in D.$$

Proof. By the symmetry and continuity of I_2, it suffices to prove

$$0 \le I_2(\xi,\eta) < 1/4 \quad \forall \quad (\xi,\eta) \in D_0 \cup D_1 \cup D_3 \cup D_5 \cup D_7.$$

We divide five cases to prove it.

Case 1. $(\xi,\eta) \in D_0$.

In this case $\cos^2\xi/2 \le 1/4$ and $\cos^2\eta/2 \le 1/4$. Then we obtain

$$I_2(\xi,\eta) \le 1/16\cos^2(\xi+\eta)/2(1+\sin^2\xi/2+\sin^2\eta/2+\sin^2(\xi+\eta)/2) < 1/4. \quad (30)$$

Case 2. $(\xi,\eta) \in D_1$.

In this case, we have

$$\cos^2\eta/2\cos^2(\xi+\eta)/2 \le max(\cos^2\xi/2, 1/4\cos^2(\xi/2+2\pi/3), \sin^4\xi/4).$$

Then we have

$$I_2(\xi,\eta) \quad < 4\max(\cos^4\xi/2, 1/4\cos^2\xi/2\cos^2(\xi/2+2\pi/3), \cos^2(\xi/2)\sin^4\xi/4) \le 1/4 \quad (31)$$

Case 3.a. $(\xi,\eta) \in D_3$ and $0 \le \eta \le \pi/3$.

In this case, $\sin^2\eta/2 + \sin^2(\xi+\eta)/2 \le \sin^2\xi/2$ and $\cos^2\eta/2\cos^2(\xi+\eta)/2 \le \cos^2\pi/6\cos^2(\xi/2+\pi/6)$. Then we have

$$\begin{array}{rl} I_2(\xi,\eta) & \le \cos^2\xi/2\cos^2\pi/6\cos^2(\xi/2+\pi/6)(1+2\sin^2\xi/2) \\ & \le \cos^2 23\pi/36\cos^2\pi/6\cos^2 29\pi/36(1+2\sin^2 23\pi/36) < 1/4. \end{array} \quad (32)$$

Case 3.b. $(\xi,\eta) \in D_3$ and $\pi/3 \le \eta \le 2\pi/3$.

In this case, $\sin^2\eta/2+\sin^2(\xi+\eta)/2 \le \sin^2\pi/3+\sin^2(\xi/2+\pi/3)$ and $\cos^2\eta/2\cos^2(\xi+\eta)/2 \le \sin^4\xi/4$. Then we have

$$\begin{aligned} I_2(\xi,\eta) &\le \cos^2\xi/2\sin^4\xi/4(7/4+\sin^2 3\pi/5+\sin^2 14\pi/15) \\ &\le \cos^2 23\pi/36\sin^4 23\pi/72(7/4+\sin^2 3\pi/5+\sin^2 14\pi/15) < 1/4. \end{aligned} \tag{33}$$

Case 4. $(\xi,\eta)\in D_5$.

In this case, $1+\sin^2\xi/2+\sin^2\eta/2+\sin^2(\xi+\eta)/2\le 3$ and $\cos^2\eta/2\cos^2(\xi+\eta)/2 \le \cos^2 23\pi/72\cos^2(\xi/2+23\pi/72)\le\cos^2 23\pi/72$. Then we have

$$I_2(\xi,\eta)\le 3\cos^2\pi/3\cos^2 23\pi/72<1/4. \tag{34}$$

Case 5. $(\xi,\eta)\in D_7$.

In this case $\sin^2\eta/2+\sin^2(\xi+\eta)/2\le\sin^2\pi/6+\sin^2(\xi/2+\pi/6)\le 5/4$ and $\cos^2\eta/2\cos^2(\xi+\eta)/2\le\cos^2\xi/2$. Then we have

$$I_2(\xi,\eta)\le\cos^4\xi/2(9/4+\sin^2\xi/2)\le\cos^2 23\pi/72(9/4+\sin^2 23\pi/72)<1/4. \tag{35}$$

Combining (31)-(36), we get (30) and Lemma 5 is proved.

Lemma 6. *Let $\delta<1$ and C be two constants. If nonnegative function f satisfies*

i) $f(\xi+2m\pi,\eta+2n\pi)=f(\xi,\eta)$ *holds for all* $m,n\in Z$*;*

ii) $0\le f(\xi,\eta)\le 1$*;*

iii) $|f(\xi,\eta)|\le C\min(|\xi-\pi|,|\eta-\pi|)$ *when* $0\le\xi,\eta\le 2\pi$*;*

iv) $0\le f(\xi,\eta)\le\delta<1$ *when* $(\xi,\eta)\in[2\pi/3,4\pi/3]\times[0,2\pi]\cup[0,2\pi]\times[2\pi/3,4\pi/3]$.

then there exists $\lambda>0$ independent of k such that

$$\prod_{j=1}^{k} f(2^j\xi,2^j\eta)\le 2^{-\lambda k}$$

holds for all $k\ge 2$ when $(\xi,\eta)\in[\pi/2,3\pi/2]\times[0,2\pi]\cup[0,2\pi]\times[\pi/2,3\pi/2]$. $\pi/2\le\xi\le 3\pi/2$ or $\pi/4\le\eta\le 3\pi/4$.

Lemma 6 can be proved by the argument similar to the one of Lemma 10 of Chapter 3 in [M]. We omit the details here.

Lemma 7. *Let I_{2M} be defined in (19). Then there exists a constant $\lambda>0$ such that*

$$\prod_{j=1}^{k} I_{2M}(2^j\xi,2^j\eta)\le 2^{-\lambda kM}$$

holds for all $k\ge 4$ when $\pi/2\le\xi\le 3\pi/2$ or $\pi/2\le\eta\le 3\pi/2$.

Proof. By (15), (19) we get $0 \le \alpha_M(k_1,k_2,k_3,k_4) \le 1$, $\alpha_M(k_1,k_2,k_3,k_4)$ is not zero only if $k_1 \ge M$ and $I_2 \ge 0$. Therefore we get

$$\begin{array}{rl} I_{2M}(\xi,\eta) \le & \sum_{k_1=M}^{4M-3} \frac{(4M-3)!}{k_1!(4M-3-k_1)!} I_2(\xi,\eta)^{k_1}(1-I_2(\xi,\eta)^{4M-3-k_1} \\ \le & (I_2(\xi,\eta)+1-I_2(\xi,\eta))^{4M-3} = 1. \end{array}$$

when $I_2(\xi,\eta) \ge 1/4$ and

$$\begin{array}{rl} I_{2M}(\xi,\eta) \le & I_2(\xi,\eta)^M(1-I_2(\xi,\eta))^{4M-3}\sum_{k_1=M}^{4M-3}\frac{(4M-3)!}{k_1!(4M-3-k_1)!}(1/3)^{k_1} \\ & \quad (by the monotony of \quad \frac{t}{1-t} \quad when \quad 0 \le t \le 1/4) \\ \le & I_2(\xi,\eta)^M(1-I_2(\xi,\eta))^{4M-3}3^M(4/3)^{4M-3}\times \\ & \quad \sum_{k_1=M}^{4M-3}\frac{(4M-3)!}{k_1!(4M-3-k_1)!}(\frac{1}{4})^{k_1}(\frac{3}{4})^{4M-3-k_1} \\ \le & \left(\frac{256}{27}I_2(\xi,\eta)(1-I_2(\xi,\eta))^3\right)^M, \end{array}$$

when $I_2(\xi,\eta) \le 1/4$. By the monotony of $t(1-t)^3$ when $0 \le t \le 1/4$ and Lemma 5, there exists a constant $0<\delta<1$ independent of M such that

$$I_{2M}(\xi,\eta) \le \delta^M$$

when $(\xi,\eta) \in D$, where D is defined in Lemma 5. Observe that $(\xi,\eta) \in D_7 \cup D_8$ or $(2\xi,2\eta) \in D$ when $(\xi,\eta) \in [\pi/3,2\pi/3]\times[0,2\pi/3]\cup[0,2\pi/3]\times[\pi/3,2\pi/3]$. Therefore we get

$$I_{2M}(\xi,\eta)I_{2M}(2\xi,2\eta) \le \delta^M$$

holds for all $(\xi,\eta) \in D\cup[\pi/3,2\pi/3]\times[0,2\pi/3]\cup[0,2\pi/3]\times[\pi/3,2\pi/3]$. Observe that $(2\xi-2\pi,2\eta) \in [\pi/3,2\pi/3]\times[0,23\pi/18]$ when $(\xi,\eta) \in [23\pi/18,4\pi/3]\times[0,23\pi/36]$ and $(2\xi,2\eta-2\pi) \in [0,23\pi/18]\times[\pi/3,2\pi/3]$ when $(\xi,\eta) \in [0,23\pi/36]\times[23\pi/18,4\pi/3]$. Therefore we get

$$I_{2M}(\xi,\eta)(I_{2M}(2\xi,2\eta))^2 I_{2M}(4\xi,4\eta) \le \delta^M$$

holds when

$$(\xi,\eta) \in [0,4\pi/3]\times[2\pi/3,4\pi/3]\cup[2\pi/3,4\pi/3]\times[0,4\pi/3].$$

By the definition of I_{2M} we have

$$I_{2M}(\xi,\eta) = I_{2M}(-\xi,-\eta).$$

Therefore we get

$$I_{2M}(\xi,\eta)(I_{2M}(2\xi,2\eta))^2 I_{2M}(4\xi,4\eta) \le \delta^M$$

when

$$(\xi,\eta) \in [0,2\pi]\times[2\pi/3,4\pi/3]\cup[2\pi/3,4\pi/3]\times[0,2\pi].$$

Then Lemma 7 follows from Lemma 6.

Now
we start to prove Theorem 2.

Proof of Theorem 2. By (3) and the definition of ϕ_{2M}, we have

$$\hat{\phi}_{2M}(\xi,\eta) = I_{2M}(\xi/2,\eta/2)\hat{\phi}_{2M}(\xi/2,\eta/2).$$

Let k_0 be the unique integer such that $\pi \geq \max(|2^{-k_0}\xi|, |2^{-k_0}\eta|) > \pi/2$ when $\max(|\xi|,|\eta|) \geq 8\pi$, Let $(\xi_{k_0},\eta_{k_0}) \in [0,2\pi]\times[\pi/2,3\pi/2]\cup[\pi/2,3\pi/2]\times[0,2\pi]$ be the unique point such that $(\xi_{k_0},\eta_{k_0}) - (2^{-k_0}\xi, 2^{-k_0}\eta) \in 2\pi Z\times 2\pi Z$. Observe that

$$|\hat{\phi}_{2M}(\xi,\eta)| \leq 1$$

since $0 \leq I_{2M}(\xi,\eta) \leq 1$. Then by Lemma 7 we get

$$\begin{aligned} |\hat{\phi}_{2M}(\xi,\eta)| &= \prod_{j=1}^{k} I_{2M}(2^{-j}\xi, 2^{-j}\eta)\hat{\phi}_{2M}(2^{-k}\xi, 2^{-k}\eta) \\ &\leq \prod_{j=0}^{k_0-1} I_N(2^j\xi_{k_0}, 2^j\eta_{k_0}) \\ &\leq \delta^{(k_0-1)N} \leq \delta^{-M}(\max(|\xi|/\pi, |\eta|/\pi))^{-M\ln\delta/\ln 2}, \end{aligned}$$

when $\max(|\xi|,|\eta|) \geq 8\pi$. Theorem 2 is proved since

$$|\hat{\phi}(\xi,\eta)| \leq C(1+|\xi|+|\eta|)^{-\beta}$$

implies $\phi \in C^{\beta-2}$ when $\beta > 2$.

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