SAMPLING EXPANSIONS IN REPRODUCING KERNEL HILBERT AND BANACH SPACES

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Abstract. We investigate the construction of all reproducing kernel Hilbert spaces of functions on a domain $\Omega \subset \mathbb{R}^d$ that have a countable sampling set $\Lambda \subset \Omega$. We also characterize all the reproducing kernel Hilbert spaces that have a prescribed sampling set. Similar problems are considered for reproducing kernel Banach spaces, but now with respect to Λ as a p-sampling set. Unlike the general p-frames, we prove that every p-sampling set for a reproducing kernel Banach space yields a reconstruction formula. Some applications are given to demonstrate the general construction. The results of this paper uncover precisely the affinity between stable sampling expansions and reproducing kernel Hilbert and Banach spaces.

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1. Introduction

The Whittaker-Shannon-Kotel'nikov (WSK) sampling theorem states that if a square-integrable function f is bandlimited to $[-\sigma, \sigma]$, i.e., it is representable as

$$f(t) = \int_{-\sigma}^{\sigma} e^{-ixt} g(x)\, dx, \quad t \in \mathbb{R}$$

for some function $g \in L^2(-\sigma, \sigma)$, then f can be reconstructed from its samples $f(k\pi/\sigma)$ taken at the equally spaced nodes $k\pi/\sigma$ on the time axis $\mathbb{R}$, and

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\sigma}\right) \frac{\sin(\sigma t - k\pi)}{\sigma t - k\pi}, \qquad t \in \mathbb{R} \tag{1.1}$$

where the series is absolutely and uniformly convergent on any compact set of the real line. Moreover

$$\|f\|^2 = \sum_{k\in\mathbb{Z}} \Big| f\Big(\frac{k\pi}{\sigma}\Big)\Big|^2. \tag{1.2}$$

The WSK-sampling theorem has many engineering applications and has been generalized in numerous contexts, the reader may refer to the survey papers and monographs [7, 10, 13, 14, 15, 16, 18, 25, 26]. It is observed that the space of all square-integrable functions bandlimited to $[-\sigma, \sigma]$ is a reproducing kernel Hilbert space on the real line with the reproducing kernel $\frac{\sin(\sigma(t-s))}{\sigma(t-s)}$, although this fact is not used in the classical proof of the WSK-sampling theorem. Recall that a *reproducing kernel Hilbert space* $\mathcal{H}$ (RKHS for short) is a (complex) Hilbert space of functions on a domain Ω such that the evaluation functional is continuous, i.e., for any $x \in \Omega$ there exists a positive constant C_x such that

$$|f(x)| \le C_x \|f\|, \quad f \in \mathcal{H} \tag{1.3}$$

[5]. In this paper, we **always** assume that $\Omega \subset \mathbb{R}^d$. Nashed and his collaborators studied sampling theory for reproducing kernel Hilbert spaces in a series of papers, see [19, 20, 21, 22]. In general, a reproducing kernel Hilbert space $\mathcal{H}$ on a domain Ω may not admit a stable sampling set, that is, there does not exist a countable subset Λ of Ω such that

$$A\|f\| \le \Big(\sum_{\lambda\in\Lambda} |f(\lambda)|^2\Big)^{1/2} \le B\|f\|, \quad f \in \mathcal{H},$$

where A, B are positive constants. This leads to the natural question of characterizing all the reproducing kernel Hilbert spaces that have a sampling set. In this paper we discuss the general construction of all the reproducing kernel Hilbert spaces that have a sampling set, and also we characterize all the reproducing kernel Hilbert spaces on a set Ω that have Λ as a prescribed sampling set. Our results generalize the corresponding work related to Riesz bases [22].

The concept of reproducing kernel Hilbert spaces has a natural generalization in Banach spaces. Like the Hilbert space case, the sampling theory for a Banach space involves Banach space frame (for reconstruction) and p-frames (for determination). However, a p-frame in a Banach space does not necessarily yield a reconstruction formula [9]. In Section 3, we will prove that for every p-sampling set (corresponding to a p-frame) in a reproducing kernel Banach space automatically yields a reconstruction formula. Similar to the reproducing Hilbert space case, for a given set $\Omega \subset \mathbb{R}^d$ and a countable subset $\Lambda \subset \Omega$, we also have a general construction for all the reproducing kernel Banach spaces of functions on Ω with Λ as a p-sampling set. Some applications will be discussed in the last section of this paper.

2. Sampling Sets for Reproducing Kernel Hilbert Spaces

In this section we describe a general construction for all the reproducing kernel Hilbert spaces which admit a sampling set Λ. We first introduce a few definitions and some preliminary results that will be needed in proving our main results.

A *frame* for a separable Hilbert space $\mathcal{H}$ is a sequence $\{x_n\}$ in $\mathcal{H}$ such that there exist two positive constants A, B with the property that

$$A\|x\|^2 \le \sum_n |\langle x, x_n\rangle|^2 \le B\|x\|^2, \quad x \in \mathcal{H}. \tag{2.1}$$

The optimal constants (maximal for A and minimal for B) are called *frame bounds*. When $A = B = 1$, $\{x_n\}$ is called a *normalized tight frame*. A sequence $\{x_n\}$ is called *Bessel* if we only require the right side inequality of (2.1) to hold. In the proof of our main results we also need a concept of *strongly disjoint* or *orthogonal frames* introduced in [6] and [12]: two Bessel sequences $\{x_n\}$ and $\{y_n\}$ are called *strongly disjoint* if $\sum_n \langle x, x_n\rangle y_n = 0$ holds for all $x \in \mathcal{H}$.

Let $\{x_n\}$ be a frame for a Hilbert space $\mathcal{H}$. The associated *analysis operator* T is the bounded linear operator defined by $Tx = \sum_n \langle x, x_n\rangle e_n$, where $\{e_n\}$ is the standard orthonormal basis for $\ell^2(\mathbb{Z})$. It is easy to verify that the range space $T\mathcal{H}$ is closed and T is boundedly invertible operator from $\mathcal{H}$ to $T\mathcal{H}$. Moreover, $T^*e_n = x_n$ for each n and T^* is called the *synthesis operator*. Let $S = T^*T$ (this operator is refereed as the *frame operator* for $\{x_n\}$). Then $\{S^{-1}x_n\}$, which is called *standard dual frame*, provides us the reconstruction formula:

$$x = \sum_n \langle x, S^{-1}x_n\rangle x_n = \sum_n \langle x, x_n\rangle S^{-1}x_n, \quad x \in \mathcal{H}, \tag{2.2}$$

where the convergence is in the Hilbert space norm.

Let $\mathcal{H}$ be a reproducing kernel Hilbert space of functions on a set Ω. A *sampling set* is a countable subset Λ of Ω with the property that there exist two positive constants A and B such that

$$A\|f\|^2 \le \sum_{\lambda\in\Lambda} |f(\lambda)|^2 \le B\|f\|^2, \quad f \in \mathcal{H}. \tag{2.3}$$

By the Riesz representation theorem, for any $x \in \Omega$ there exists $h_x \in \mathcal{H}$ such that

$$f(x) = \langle f, h_x\rangle, \ f \in \mathcal{H}. \tag{2.4}$$

The function

$$k(s,t) = \langle h_s, h_t\rangle \tag{2.5}$$

is known as the *reproducing kernel* of the RKHS $\mathcal{H}$. By (2.1) – (2.4), we conclude that the evaluation linear functional sequence $\{h_\lambda, \lambda \in \Lambda\}$ is a frame for $\mathcal{H}$, and that any function f in $\mathcal{H}$ can be reconstructed from its sampling on Λ by the following reconstruction formula:

$$f = \sum_{\lambda\in\Lambda} f(\lambda) g_\lambda, \quad f \in \mathcal{H}, \tag{2.6}$$

where $\{g_\lambda : \lambda \in \Lambda\}$ is a dual frame of $\{h_\lambda : \lambda \in \Lambda\}$. The series in (2.6) converges in the Hilbert space norm. It is very desirable for our consideration of sampling to have the pointwise and/or uniform convergence for the reconstruction formula. The following well-known result, which can be easily proved, tells us that the norm convergence in a RKHS implies the pointwise convergence, and also the uniform convergence if the reproducing kernel is bounded.

Lemma 2.1. *Suppose that $\mathcal{H}$ is a reproducing kernel Hilbert space on Ω with reproducing kernel $k(s,t)$. If $f_n \to f$ in the Hilbert space norm, then $\{f_n(t)\}$ is convergent to $f(t)$ for each $t \in \Omega$. Moreover, the convergence is uniform on Ω if $k(t,t)$ is bounded.*

By (2.6) and Lemma 2.1, we obtain a reconstruction formula in a reproducing kernel Hilbert space that converges in the Hilbert space norm and also pointwise.

Theorem 2.2. *Assume that Λ is a sampling set for a reproducing kernel Hilbert space $\mathcal{H}$ on a set Ω. Then there exists a frame $\{g_\lambda, \lambda \in \Lambda\}$ for $\mathcal{H}$ such that*

$$f(t) = \sum_{\lambda\in\Lambda} f(\lambda) g_\lambda(t), \quad f \in \mathcal{H}, \tag{2.7}$$

where the convergence is both in norm and pointwise.

Now we introduce a procedure for constructing reproducing kernel Hilbert spaces that have a sampling set, and later we will show that every reproducing kernel Hilbert space that has a sampling set can be constructed in that procedure. Let Ω be a set and let Λ be a countable subset of Ω. Assume that $\{x_n\}$ is a frame for a Hilbert space $\mathcal{G}$, and $S_n : \Omega \to \mathbb{R}$ (or $\mathbb{C}$) is a sequence of functions satisfying the following two conditions:

(H1) $\{S_n(t)\} \in \ell^2(\mathbb{Z})$ for every $t \in \Omega$.

(H2) $\{\eta_\lambda : \lambda \in \Lambda\}$ is a frame for $\ell^2(\mathbb{Z})$, where $\eta_\lambda := \{S_n(\lambda)\}_{n\in\mathbb{Z}}$.

Now we define F by

$$F(t) := \sum_{n\in\mathbb{Z}} S_n(t) x_n, \quad t \in \Omega. \tag{2.8}$$

The function $F(t)$ is well-defined since $\{x_n\}$ is a frame and $\sum_n |S_n(t)|^2 < \infty$. Clearly if $x = 0$ then $\langle x, F(t)\rangle = 0$ for all $t \in \Omega$. Conversely if $\langle x, F(t)\rangle = 0$ for all $t \in \Omega$, then the ℓ^2-sequence $\xi := (\langle x_n, x\rangle)_{n\in\mathbb{Z}}$ satisfies

$$\langle \xi, \eta_\lambda\rangle = \sum_n S_n(\lambda)\langle x, x_n\rangle = 0 \quad \text{for all } \lambda \in \Lambda.$$

This, together with the assumptions that $\{\eta_\lambda : \lambda \in \Lambda\}$ is a frame for $\ell^2(\mathbb{Z})$ and that $\{x_n\}$ is a frame for $\mathcal{G}$, implies that $x = 0$. Using the above function F, we construct a space $\mathcal{H}$ of functions on Ω by

$$\mathcal{H} := \{\langle x, F(t)\rangle : \ x \in \mathcal{G}\}. \tag{2.9}$$

Therefore the linear map

$$\mathcal{G} \ni x \longmapsto \langle x, F(t)\rangle \in \mathcal{H}$$

is one-to-one and onto. So $\mathcal{H}$ becomes a Hilbert space with the inner product on $\mathcal{H}$ defined by $\langle f, g\rangle := \langle x, y\rangle$ if $f(t) = \langle x, F(t)\rangle$ and $g(t) = \langle y, F(t)\rangle$. Moreover the space $\mathcal{H}$ is a reproducing kernel Hilbert space because for any $t \in \Omega$, we have

$$|f(t)| = |\langle x, F(t)\rangle| \le \|F(t)\|\|x\| = \|F(t)\|\|f\| \quad \text{for all } f \in \mathcal{H}. \tag{2.10}$$

For the reproducing kernel space $\mathcal{H}$ in (2.9), we have the following results about its sampling sets:

Theorem 2.3. *(i) Λ is a sampling set for the reproducing kernel Hilbert space $\mathcal{H}$ in (2.9).*

(ii) Every reproducing kernel Hilbert space admitting a sampling set Λ can be constructed in the way described above.

To prove Theorem 2.3, we recall a lemma in [12].

Lemma 2.4. *Let $\{x_n\}$ be a frame for a Hilbert space $\mathcal{G}$ and T be its associated analysis operator. Let $\{y_n\}$ be the standard dual of $\{x_n\}$ and P be the orthogonal projection from $\ell^2(\mathbb{Z})$ onto $T\mathcal{G}$. Then*

(i) $\{Ty_n\}$ and $\{P^\perp e_n\}$ are strongly disjoint.

(ii) $\{Ty_n + P^\perp e_n\}$ is a frame for $\ell^2(\mathbb{Z})$.

Proof of Theorem 2.3. (i) Since $\{x_n\}$ and $\{\eta_\lambda\}$ are frames for $\mathcal{G}$ and $\ell^2(\mathbb{Z})$ respectively, there exist two positive constants A and B such that

$$A\|x\|^2 \le \sum_{n\in\mathbb{Z}} |\langle x, x_n\rangle|^2 \le B\|x\|^2, \quad x \in \mathcal{G},$$

and

$$A\|\xi\|^2 \le \sum_{\lambda\in\Lambda} |\langle \xi, \eta_\lambda\rangle|^2 \le B\|\xi\|^2, \quad \xi \in \ell^2(\mathbb{Z}).$$

Take any $f \in \mathcal{H}$, and write $f(t) = \langle x, F(t)\rangle$ for $x \in \mathcal{G}$. Then

$$f(\lambda) = \sum_{n\in\mathbb{Z}} \overline{S_n(\lambda)}\langle x, x_n\rangle = \langle \xi, \eta_\lambda\rangle, \quad \lambda \in \Lambda,$$

where $\xi = \{\langle x, x_n\rangle\} \in \ell^2(\mathbb{Z})$. Therefore

$$\begin{aligned}\sum_{\lambda\in\Lambda} |f(\lambda)|^2 &= \sum_{\lambda\in\Lambda} |\langle \xi, \eta_\lambda\rangle|^2 \le B\|\xi\|^2 \\ &= B\sum_{n\in\mathbb{Z}} |\langle x, x_n\rangle|^2 \le B^2\|x\|^2 = B^2\|f\|^2\end{aligned}$$

and similarly,

$$\sum_{\lambda\in\Lambda} |f(\lambda)|^2 = \sum_{\Lambda\in\Lambda} |\langle \xi, \eta_\lambda\rangle|^2 \ge A\|\xi\|^2 \ge A^2\|x\|^2 = A^2\|f\|^2.$$

This proves that Λ is a sampling set for the reproducing kernel Hilbert space $\mathcal{H}$.

(ii) Now assume that $\mathcal{H}$ is a reproducing kernel Hilbert space which admits a countable sampling set Λ. Then there exists two positive constants A and B such that

$$A\|f\|^2 \le \sum_{\lambda\in\Lambda} |f(\lambda)|^2 \le B\|f\|^2\ , \quad f \in \mathcal{H}. \tag{2.11}$$

By (2.11), $\{x_\lambda : \lambda \in \Lambda\}$ is a frame for $\mathcal{H}$, where $x_\lambda := h_\lambda$, the evaluation functional at the sampling location λ. Let $\{g_\lambda : \lambda \in \Lambda\}$ be the standard dual frame of $\{x_\lambda : \lambda \in \Lambda\}$, and $T : \mathcal{H} \to \ell^2(\Lambda)$ be the analysis operator for $\{g_\lambda\}$. By Lemma 2.4, $\{Tx_\lambda\}$ and $\{P^\perp e_\lambda\}$ are strongly disjoint and $\{Tx_\lambda + P^\perp e_\lambda\}$ is a frame for $\ell^2(\Lambda)$, where P is the orthogonal projection of $T\mathcal{H}$ and $\{e_\lambda\}$ is the standard orthonormal basis for $\ell^2(\Lambda)$.

Now we define

$$S_\lambda(t) := \begin{cases} \overline{g_\lambda(t)} & \text{if } t \notin \Lambda, \\ \overline{g_\lambda(\lambda')} + \overline{\langle P^\perp e_\lambda, e_{\lambda'}\rangle} & \text{if } t = \lambda' \in \Lambda. \end{cases}$$

Then $S_\lambda(t)$ satisfies the two conditions (H1) and (H2). In fact,

(H1): $\sum_{\lambda\in\Lambda} |S_\lambda(t)|^2 < \infty$ follows from

$$\sum_{\lambda\in\Lambda} |\langle P^\perp e_\lambda, e_{\lambda'}\rangle|^2 = \|P^\perp e_{\lambda'}\|^2 < \infty$$

and

$$\sum_{\lambda\in\Lambda} |g_\lambda(t)|^2 \le \sum_{\lambda\in\Lambda} |\langle g_\lambda, h_t\rangle|^2 < \infty$$

by the frame property of $\{g_\lambda\}$, where h_t is the evaluation functional at the location $t \in \Omega$.

(H2): $\{\eta_\lambda := \{S_{\lambda'}(\lambda)\}_{\lambda'\in\Lambda}, \lambda \in \Lambda\}$ is a frame for $\ell^2(\Lambda)$ by Lemma 2.4 and the fact that $\overline{\eta_\lambda} = Tx_\lambda + P^\perp e_\lambda, \lambda \in \Lambda$.

Define $F(t) := \sum_{\lambda\in\Lambda} S_\lambda(t) x_\lambda, t \in \Omega$. Then it remains to prove that the operator

$$\mathcal{H} \ni f \longmapsto \langle f, F(t)\rangle \in \mathcal{H}$$

is the identity. Take any $f \in \mathcal{H}$. For $t \notin \Lambda$, we have

$$\langle f, F(t)\rangle = \sum_{\lambda\in\Lambda} \overline{S_\lambda(t)} \langle f, x_\lambda\rangle = \sum_{\lambda\in\Lambda} f(\lambda) g_\lambda(t) = f(t), \tag{2.12}$$

where the last equality follows from the pointwise convergence of the reconstruction formula (2.7). While for $t = \lambda' \in \Lambda$, we obtain

$$\begin{aligned} \langle f, F(t)\rangle &= \sum_{\lambda\in\Lambda} \overline{S_\lambda(\lambda')}\langle f, x_\lambda\rangle \\ &= \sum_{\lambda\in\Lambda} f(\lambda)(g_\lambda(\lambda') + \langle P^\perp e_\lambda, e_{\lambda'}\rangle) \\ &= \sum_{\lambda\in\Lambda} f(\lambda)g_\lambda(\lambda') + \sum_{\lambda\in\Lambda}\langle f, x_\lambda\rangle\langle P^\perp e_\lambda, e_{\lambda'}\rangle \\ &= \sum_{\lambda\in\Lambda} f(\lambda)g_\lambda(\lambda') = f(t), \end{aligned} \tag{2.13}$$

where we have used Theorem 2.2 and Lemma 2.4 to obtain the last two identities. Combining the two cases in (2.12) and (2.13) leads to

$$f(t) = \langle f, F(t)\rangle \text{ for all } f \in \mathcal{H} \text{ and } t \in \Omega,$$

and hence this completes the proof. □

As a special case, we call a sampling set for $\mathcal{H}$ *exact* if the corresponding evaluation functional sequence is a Riesz basis for $\mathcal{H}$. In any non-exact sampling case, we always have a over-sampling for the function space. The following proposition follows immediately from the standard frame theory (cf. [12]):

Proposition 2.5. *A sampling set Λ for a Hilbert space $\mathcal{H}$ of functions is exact if and only if $\{(f(\lambda))_{\lambda\in\Lambda} : f \in \mathcal{H}\} = \ell^2(\Lambda)$.*

In the general construction described before Proposition 2.5, if we further require that both $\{x_n\}$ and $\{\eta_\lambda\}$ be Riesz bases for H and $\ell^2(\Lambda)$, respectively, then we have $\{(f(\lambda)) : f \in \mathcal{H}\} = T_\eta T_x H = \ell^2(\Lambda)$, where T_x and T_η are the analysis operator for $\{x_\lambda\}$ and $\{\eta_\lambda\}$, respectively. Thus, Λ is an exact sampling set for $\mathcal{H}$ by Proposition 2.5. Similar to the proof of Theorem 2.3 (ii), any reproducing kernel Hilbert space with an exact sampling set can be constructed this way. Thus we have

Theorem 2.6. *The following are equivalent:*
(i) $\mathcal{H}$ is a reproducing kernel Hilbert space with an exact sampling set.
(ii) $\mathcal{H}$ can be constructed as in the general construction with $\{x_n\}$ and $\{\eta_\lambda\}$ being Riesz bases for H and $\ell^2(\mathbb{Z})$, respectively.

3. Sampling Sets for Reproducing Kernel Banach spaces

A *reproducing kernel Banach space* is a Banach space $\mathcal{B}$ of functions on a set Ω such that the evaluation functions $f \to f(t)$ is continuous for each $t \in \Omega$. A countable subset Λ of Ω is called a *p-sampling set* ($p \geq 1$) for a

Banach space $\mathcal{B}$ if there exist two positive constants A and B such that

$$A\|f\|_{\mathcal{B}} \le \Big(\sum_{\lambda\in\Lambda}|f(\lambda)|^p\Big)^{1/p} \le B\|f\|_{\mathcal{B}}, \quad f\in\mathcal{B}. \tag{3.1}$$

In this section, we investigate p-sampling in reproducing kernel Banach spaces.

Let X be a Banach space and X^* be its dual. The dual relation $g(x)$ will be still denoted by $\langle x, g\rangle$, where $x \in X$ and $g \in X^*$. A countable set $\{g_k\} \subset X^*$ is called a *p-frame* if there exist positive constants A and B such that

$$A\|x\|_X \le \Big(\sum_{k\in\mathbb{Z}}|\langle x, g_k\rangle|^p\Big)^{1/p} \le B\|x\|_X, \quad x\in X, \tag{3.2}$$

i.e., the *analysis operator* T defined by

$$T : X \ni x \longmapsto \{\langle x, g_k\rangle\} \in \ell^p(\mathbb{Z})$$

is bounded from both above and below. If $\{g_k\}$ is p-frame for a Banach space X, and if, in addition, there exists a bounded linear operator $R : \ell^p \to X$ such that $RT = I_X$, the identity operator on X, then $\{g_k\}$ is called a *Banach frame* with respect to the ℓ^p space. In this case we have the reconstruction formula:

$$x = \sum_{k\in\mathbb{Z}}\langle x, g_k\rangle \tilde{g}_k, \quad x \in X,$$

where $\tilde{g}_k = Re_k$ with $\{e_k, k \in \mathbb{Z}\}$ is the standard basis of ℓ^p.

Unlike the Hilbert space case, the reconstruction operator R does not necessarily exist for p-frames. It is observed in [9] that there exists a p-frame $\{g_k\}$ for a Banach space X, for which no family $\{\tilde{g}_k\} \subset X$ satisfies that

$$x = \sum_{\lambda}\langle x, g_k\rangle \tilde{g}_k, \quad x \in X.$$

That is why the existence of a reconstruction operator is often assumed in the definition of Banach frames [1, 3, 8, 9, 11]. However, a reconstruction formula is possible for some special p-frames. For instance, it was proved in [3] that under certain natural conditions on the generators, a translation p-frame for a finitely generated shift-invariant subspace is always a Banach frame. In general, a p-frame $\{g_k\}$ is a Banach frame if and only if the range space TX of the analysis operator T is complemented in ℓ^p [9].

Similarly to sampling in a Hilbert space, p-sampling for a Banach space are related to Banach space frame and p-frame in that space. In particular, for a Banach space $\mathcal{B}$ of functions on a set Ω, a countable subset Λ of Ω is p-sampling set for the space $\mathcal{B}$ if and only if $\{h_\lambda\}$ is p-frame for the space $\mathcal{B}$, where h_λ is the evaluation functional: $x \to x(\lambda)$. In the following result, we show that a reconstruction formula always exists for the p-frame $\{h_\lambda, \lambda \in \Lambda\}$ associated with a p-sampling set Λ on a reproducing kernel Banach space.

Theorem 3.1. *Assume that Λ is a p-sampling set for a reproducing kernel Banach space $\mathcal{B}$ on a set Ω. Then there exists a sequence of functions $\{S_\lambda(t)\}$ on Ω such that the following are satisfied:*

(B1) $\{S_\lambda(t)\} \in \ell^q(\Lambda)$ *for every* $t \in \Omega$, *where* $\frac{1}{p} + \frac{1}{q} = 1$.

(B2) $\{\eta_\lambda : \lambda \in \Lambda\}$ *is a p-frame for* $T\mathcal{B}$, *where* $\eta_\lambda := \{S_{\lambda'}(\lambda) : \lambda' \in \Lambda\}$ *and* $T : \mathcal{B} \to \ell^p(\Lambda)$ *is defined by* $Tf := \{f(\lambda')\}_{\lambda' \in \Lambda}$.

(B3) $f(t) = \sum_{\lambda \in \Lambda} f(\lambda) S_\lambda(t),\ t \in \Omega$, *with the pointwise convergence.*

Proof. Since $\mathcal{B}$ is a reproducing kernel Banach space, it follows that for each $t \in \Omega$ there exists a bounded linear functional h_t on $\mathcal{B}$ such that $f(t) = \langle f, h_t\rangle$ for all $f \in \mathcal{B}$. Note that the analysis operator T associated with the sampling functionals for Λ is both bounded and bounded below. Therefore $T^{-1} : T\mathcal{B} \to \mathcal{B}$ is a bounded linear operator, and so $(T^{-1})^* h_t$ is bounded linear functional on $T\mathcal{B}$ which is closed subspace of ℓ^p. Extend $(T^{-1})^* h_t$ to a bounded linear functional, say S_t, on $(\ell^p)^* = \ell^q$, and write $\eta_t = \{S_\lambda(t)\}_{\lambda\in\Lambda}$. Then, clearly $\{S_\lambda(t)\}$ satisfies $(B1)$. For $(B3)$,

(3.3)
$$f(t) = \langle f, h_t\rangle = \langle T^{-1}Tf, h_t\rangle = \langle Tf, (T^{-1})^* h_t\rangle = \langle Tf, \eta_t\rangle = \sum_{\lambda\in\Lambda} f(\lambda) S_\lambda(t),$$

and the convergence in the summation $f(t) = \sum_{\lambda\in\Lambda} f(\lambda)S_\lambda(t)$ is pointwise. By (3.3), we have

$$f(\lambda) = \sum_{\lambda'\in\Lambda} f(\lambda') S_{\lambda'}(\lambda) = \langle Tf, \eta_\lambda\rangle,$$

and therefore

$$\Big(\sum_{\lambda\in\Lambda} |\langle Tf, \eta_\lambda\rangle|^p\Big)^{1/p} = \Big(\sum_{\lambda\in\Lambda} |f(\lambda)|^p\Big)^{1/p}.$$

This together with the definition of the sampling set Λ implies that

$$\Big(\sum_{\lambda\in\Lambda} |\langle Tf, \eta_\lambda\rangle|^p\Big)^{1/p} \le B\|f\| \le B\|T^{-1}\|\|Tf\|$$

and

$$\Big(\sum_{\lambda\in\Lambda} |\langle Tf, \eta_\lambda\rangle|^p\Big)^{1/p} \ge A\|f\| \ge \frac{A}{\|T\|}\|Tf\|,$$

where A and B are the two constants defining the p-sampling set Ω. Hence $\{\eta_\lambda\}$ is p-frame for $T\mathcal{B}$ from the definition of p-frame, and then (B2) holds. $\square$

Now we consider the general construction of reproducing kernel Banach spaces that have Λ as a p-sampling set. More precisely, we will prove

Theorem 3.2. *Let $\{y_k\} \subset X^*$ be p-frame for a Banach space X and Λ be a countable subset of Ω. Suppose that $S_k(t), k \in \mathbb{Z}$, satisfy*

(B1) $\{S_k(t)\} \in \ell^q(\mathbb{Z})$ *for every* $t \in \Omega$, *where* $\frac{1}{p} + \frac{1}{q} = 1$.

(B2′) $\{\eta_\lambda : \lambda \in \Lambda\}$ *is a p-frame for* $\ell^p(\mathbb{Z})$, *where* $\eta_\lambda := \{S_k(\lambda)\}$.

Then

$$\mathcal{B} := \Big\{ \sum_{k\in\mathbb{Z}} \langle x, y_k\rangle S_k(t), x \in X \Big\}$$

is a reproducing kernel Banach space with Λ *as a* p*-sampling set of* $\mathcal{B}$*, where the norm of* $f \in \mathcal{B}$ *is defined to be* $\|x\|$ *when* $f(t) := \sum_{k\in\mathbb{Z}} \langle x, y_k\rangle S_k(t)$*.*

Proof. Let A, B and C, D be the p-frame bounds corresponding to the p-frames $\{y_k\}$ and $\{\eta_\lambda\}$. To see that $\mathcal{B}$ is well defined (and so it is Banach space isometric to X), we only need to check that if $\sum_k \langle x, y_k\rangle S_k(t) = 0$ for all t, then $x = 0$. In fact, if $\sum_k \langle x, y_k\rangle S_k(t) = 0$ for all t, then $\sum_k \langle x, y_k\rangle S_k(\lambda) = 0$ for all $\lambda \in \Lambda$, which implies that $\langle \xi, \eta_\lambda\rangle = 0$ for each $\lambda \in \Lambda$, where $\xi = \{\langle x, y_k\rangle\}$. Since $\{\eta_\lambda\}$ is a p-frame for ℓ^p, it follows that $\xi = 0$, and therefore $x = 0$ since $\{y_k\}$ is a p-frame for X.

Next we show that $\mathcal{B}$ is a reproducing kernel Banach space. For each $t \in \Omega$, we have

$$\begin{aligned} |f(t)| &\le \Big(\sum_k |\langle x, y_k\rangle|^p \Big)^{1/p} \Big(\sum_k |S_k(t)|^q \Big)^{1/q} \\ &\le B\|x\| \Big(\sum_k |S_k(t)|^q \Big)^{1/q} = B\Big(\sum_k |S_k(t)|^q \Big)^{1/q} \|f\|, \end{aligned}$$

where $1/p + 1/q = 1$. Thus $f \to f(t)$ is continuous, and so $\mathcal{B}$ is a reproducing kernel Banach space.

Finally we prove that Λ is a p-sampling set. Note that

$$f(\lambda) = \sum_k \langle x, y_k\rangle S_k(\lambda) = \langle Tx, \eta_\lambda\rangle$$

where $Tx := \{\langle x, y_k\rangle\}$. Thus

$$\begin{aligned} \Big(\sum_{\lambda\in\Lambda} |f(\lambda)|^p \Big)^{1/p} &= \Big(\sum_{\lambda\in\Lambda} |\langle Tx, \eta_\lambda\rangle|^p \Big)^{1/p} \le D\|Tx\| \\ &= D\Big(\sum_k |\langle x, y_k\rangle|^p \Big)^{1/p} \le DB\|x\| = DB\|f\|, \end{aligned}$$

and similarly

$$\Big(\sum_{\lambda\in\Lambda} |f(\lambda)|^p \Big)^{1/p} \ge CA\|f\|.$$

Therefore Λ is a p-sampling set for $\mathcal{B}$, as claimed. □

Remark 3.2. When the sampling set Λ induces a bounded unconditional basis, i.e, when $T\mathcal{B} = \ell^p$, it is clear from the construction that $\{S_\lambda\} \subset \mathcal{B}$ (in fact, $S_\lambda = T^{-1}e_\lambda$, where $e_\lambda = \delta_\lambda \in \ell^p$). However, unlike the Hilbert space sampling case, we don't know whether $\{S_\lambda\}$ can be always chosen in such a way that each S_λ is in $\mathcal{B}$. Therefore we ask:

Question. Is it true that we can always choose $\{S_\lambda\}$ in such a way that each $S_\lambda \in \mathcal{B}$? If so, is the convergence in

$$f(t) = \sum_k f(\lambda) S_\lambda(t)$$

always in the Banach space norm?

4. Applications

We give a few applications of Theorem 3.2 in this section.

Denote by $W(L^1)$ the space of all measurable functions f such that

$$\|f\|_{W(L^1)} := \sum_{k\in\mathbb{Z}^d} \sup_{x\in k+[0,1)^d} |f(x)| < \infty.$$

For $\phi_1, \ldots, \phi_N$ in $W(L^1)$, we define the shift-invariant space $V_p(\phi_1, \ldots, \phi_N)$ by

$$V_p(\phi_1, \ldots, \phi_N) := \Big\{ \sum_{n=1}^N \sum_{k\in\mathbb{Z}^d} c_n(k)\phi_n(t-k) : \ \{c_n(k)\} \in \ell^p \Big\},$$

and equip $V_p(\phi_1, \ldots, \phi_N)$ by usual L^p norm, where $1 \le p \le \infty$. For sampling and reconstruction of signals in a shift-invariant space, the reader may refer to [2, 4, 23, 25] and references therein.

Theorem 4.1. *Let $1 \le p \le \infty$, Λ be a countable subset of $\mathbb{R}^d$, and $\phi_1, \ldots, \phi_N$ be continuous functions in $W(L^1)$. Assume that $V_p(\phi_1, \ldots, \phi_N)$ is a closed subspace of L^p. Then*

(i) *$V_p(\phi_1, \ldots, \phi_N)$ is a reproducing kernel Banach space.*
(ii) *Λ is a p-sampling set for $V_p(\phi_1, \ldots, \phi_N)$ provided that $\{\eta_\lambda : \lambda \in \Lambda\}$ is a p-frame for $\ell^p(\mathbb{Z}^d \times \mathbb{Z}_N)$ with $\eta_\lambda = \{\phi_n(\lambda - k) : (k, n) \in \mathbb{Z}^d \times \mathbb{Z}_N\}$, where $\mathbb{Z}_N = \{1, \ldots, N\}$.*

Proof. (i) By the assumption on the closedeness condition of $V_p(\phi_1, \ldots, \phi_N)$, it is shown in [3] that there exists a positive constant B such that

(4.1)
$$B^{-1}\|f\|_{L^p} \le \inf\Big\{ \|\{c_n(k)\}\|_{\ell^p} : f(t) = \sum_{n=1}^N \sum_{k\in\mathbb{Z}^d} c_n(k)\phi_n(t-k) \Big\} \le B\|f\|_{L^p}$$

holds for all $f \in V_p(\phi_1, \ldots, \phi_N)$. For every $t \in \mathbb{R}^d$ and

$$f(t) = \sum_{n=1}^N \sum_{k\in\mathbb{Z}^d} c_n(k)\phi_n(t-k) \in V_p(\phi_1, \ldots, \phi_N)$$

we have

$$\begin{aligned} |f(t)| &= \Big|\sum_{n=1}^{N}\sum_{k\in\mathbb{Z}^d} c_n(k)\phi_n(t-k)\Big| \\ &\le \|\{c_n(k)\}\|_{\ell^\infty}\sum_{n=1}^{N}\sum_{k\in\mathbb{Z}^d}|\phi_n(t-k)| \\ &\le \|\{c_n(k)\}\|_{\ell^p}\Big(\sum_{n=1}^{N}\|\phi_n\|_{W(L^1)}\Big). \end{aligned} \tag{4.2}$$

Therefore, by (4.1) and (4.2), we obtain

$$\begin{aligned} |f(t)| &\le \inf\Big\{\|\{c_n(k)\}\|_{\ell^p} : f(t)=\sum_{n=1}^{N}\sum_{k\in\mathbb{Z}^d} c_n(k)\phi_n(t-k)\Big\}\times\sum_{n=1}^{N}\|\phi_n\|_{W(L^1)} \\ &\le B\|f\|_{L^p}\Big(\sum_{n=1}^{N}\|\phi_n\|_{W(L^1)}\Big), \quad t\in\mathbb{R}^d, \end{aligned}$$

and hence the evaluation functional $f\to f(t)$ is a continuous on $V_p(\phi_1,\ldots,\phi_N)$. This proves that $V_p(\phi_1,\ldots,\phi_N)$ is a reproducing kernel Banach space. Recalling from the Theorem 1 in [3] that there exist functions $\psi_1,...,\psi_N$ in $V_1(\phi_1,\ldots,\phi_N)$ with the property that

$$f(t)=\sum_{n=1}^{N}\sum_{k\in\mathbb{Z}^d}\phi_n(t-k)\times\int_{\mathbb{R}^d} f(s)\overline{\psi_n(s-k)}ds, \quad f\in V_p(\phi_1,\ldots,\phi_N).$$

Therefore the function k_t defined by

$$k_t(s):=\sum_{n=1}^{N}\sum_{k\in\mathbb{Z}^d}\overline{\phi_n(t-k)}\psi_n(s-k)$$

belongs to $V_1(\phi_1,\ldots,\phi_N)\subset V_p(\phi_1,\ldots,\phi_N)^*$ for every $t\in\mathbb{R}$, and is the reproducing kernel k_t for the reproducing kernel Banach space $V_p(\phi_1,\ldots,\phi_N)$ in the sense that

$$f(t)=\int_{\mathbb{R}^d} f(s)\overline{k_t(s)}ds, \quad f\in V_p(\phi_1,\ldots,\phi_N).$$

(ii) If $f(t)=\sum_{n=1}^{N}\sum_{k\in\mathbb{Z}^d} c_n(k)\phi_n(t-k)=\sum_{n=1}^{N}\sum_{k\in\mathbb{Z}^d} d_n(k)\phi_n(t-k)$ for some $\{c_n(k)\},\{d_n(k)\}\in\ell^p(\mathbb{Z}^d\times\mathbb{Z}_N)$, then $\sum_{n=1}^{N}\sum_{k\in\mathbb{Z}^d}(c_n(k)-d_n(k))\phi_n(t-k)=0$. In particular we have $\{c_n(k)-d_n(k)\}$ is orthogonal to η_λ for all $\lambda\in\Lambda$. Thus $c_n(k)=d_n(k)$ for all $1\le n\le N$ and $k\in\mathbb{Z}^d$ by the p-frame assumption, which implies that any function f in $V_p(\phi_1,\ldots,\phi_N)$ has a unique representation $f(t)=\sum_{n=1}^{N}\sum_{k\in\mathbb{Z}^d} c_n(k)\phi_n(t-k)$. By (4.1) and the above unique representation of a function in $V_p(\phi_1,\ldots,\phi_N)$, $\phi_1,\ldots,\phi_N$ is a

p-Riesz basis for $V_p(\phi_1, \dots, \phi_N)$, i.e., there exist positive constants $A, B > 0$ such that

$$A\|f\|_{L^p} \le \|\{c_n(k)\}\|_{\ell^p} \le B\|f\|_{L^p} \tag{4.3}$$

whenever $f(t) = \sum_{n=1}^N \sum_{k\in\mathbb{Z}^d} c_n(k)\phi_n(t-k) \in V_p(\phi_1, \dots, \phi_N)$. Let $e_{k,n}(m, j) = \delta_{(k,n),(m,j)}$. Then clearly $\{e_{k,n} : (k, n) \in \mathbb{Z}^d \times \mathbb{Z}_N\}$ is a p-frame for $\ell^p(\mathbb{Z}^d \times \mathbb{Z}_N)$. Let

$$\mathcal{B} := \Big\{ \sum_{n=1}^N \sum_{k\in\mathbb{Z}^d} \langle x, e_{k,n}\rangle \phi_n(t-k), x \in \ell^p(\mathbb{Z}^d \times \mathbb{Z}_N)\Big\}$$

be as defined in the proof of Theorem 3.2. Then we have that $V_p(\phi_1, \dots, \phi_N) = \mathcal{B}$ and that the norm $\|f\|_{\mathcal{B}} = \|x\|_{\ell^p} = \|\{\langle x, e_{k,n}\rangle\}\|_{\ell^p}$ is equivalent the L^p-norm of $V_p(\phi_1, \dots, \phi_N)$ by (4.3). This together with Theorem 3.2 implies that Λ is a p-sampling set for $V_p(\phi_1, \dots, \phi_N)$. $\square$

Now we consider the second application.

Theorem 4.2. *Let φ be a real function that satisfies*

$$\int_{\mathbb{R}} \varphi(t)^2(1 + |t|)^{2\gamma} dt < \infty \tag{4.4}$$

for some $\gamma > 1/2$, and

(4.5) the Fourier transform $\widehat{\varphi}(\xi)$ does not vanish for all $\xi \in \mathbb{R}$.

Write

$$\kappa_\varphi(u, v) = \int_{\mathbb{R}} \varphi(t-u)\varphi(t-v)dt, \tag{4.6}$$

and let $U := \{u_j\}_{j\in\mathbb{Z}}$ be such that

$$\infty > \sup_{j\ne j'} |u_j - u_{j'}| \ge \inf_{j\ne j'} |u_j - u_{j'}| \ge \epsilon > 0. \tag{4.7}$$

Define

$$X_p := \Big\{ \sum_{j\in\mathbb{Z}} c(j)\kappa_\varphi(u, u_j) : \{c(j)\} \in \ell^p(\mathbb{Z})\Big\}, \tag{4.8}$$

where $1 \le p \le \infty$ and $\|f\| := \|\{c(j)\}\|_{\ell^p}$ for $f \in X_p$. Then the set U is a p-sampling set for X_p.

Proof. For any $u\in\mathbb{R}$, we have

$$\begin{aligned}
&\sum_j |\kappa_\varphi(u,u_j)|^2(1+|u-u_j|)^{2\gamma}\\
\le\ & \sum_j \Big(2^\gamma\int_{\mathbb{R}} |\varphi(t-u)||\varphi(t-u_j)|\\
&\times\Big((1+|t-u|)^\gamma+(1+|t-u_j|)^\gamma\Big)dt\Big)^2\\
\le\ & 2^{2\gamma+1}\sum_j\Big(\int_{\mathbb{R}}|\varphi(t-u)||\varphi(t-u_j)|(1+|t-u|)^\gamma dt\Big)^2\\
&+2^{2\gamma+1}\sum_j\Big(\int_{\mathbb{R}}|\varphi(t-u)||\varphi(t-u_j)|(1+|t-u_j|)^\gamma dt\Big)^2\\
=\ & 2^{2\gamma+1}(I+II).
\end{aligned}\tag{4.9}$$

Noting that there exists an absolute constant C such that

$$\begin{aligned}
&\sum_{l\in\mathbb{Z}}\Big(\int_l^{l+1}|\varphi(t-u)|^2dt\Big)^{1/2}\\
\le\ & \Big(\sum_{l\in\mathbb{Z}}\int_l^{l+1}|\varphi(t-u)|^2dt\times(1+|l-u|)^{2\gamma}\Big)^{1/2}\\
&\times\Big(\sum_{l\in\mathbb{Z}}(1+|l-u|)^{-2\gamma}\Big)^{1/2}\le C\quad\text{for all } u\in\mathbb{R},
\end{aligned}\tag{4.10}$$

and

$$\begin{aligned}
&\sum_j\Big(\int_l^{l+1}|\varphi(t-u_j)|^2dt\Big)^{1/2}\\
\le\ & \Big(\sum_j\int_l^{l+1}|\varphi(t-u_j)|^2dt\times(1+|l-u_j|)^{2\gamma}\Big)^{1/2}\\
&\times\Big(\sum_j(1+|l-u_j|)^{-2\gamma}\Big)^{1/2}\le C\quad\text{for all } l\in\mathbb{Z},
\end{aligned}\tag{4.11}$$

we then have the following estimate for the item I in (4.9):

$$\begin{aligned} I \ \le\ & \sum_j \Big(\sum_{l\in\mathbb{Z}} \Big(\int_l^{l+1} |\varphi(t-u)|^2 (1+|t-u|)^{2\gamma} dt \Big)^{1/2} \\ & \times \Big(\int_l^{l+1} |\varphi(t-u_j)|^2 dt \Big)^{1/2} \Big)^2 \\ \le\ & \sum_j \Big(\sum_{l\in\mathbb{Z}} \int_l^{l+1} |\varphi(t-u)|^2 (1+|t-u|)^{2\gamma} dt \\ & \times \Big(\int_l^{l+1} |\varphi(t-u_j)|^2 dt \Big)^{1/2} \Big) \\ & \times \sum_{l'\in\mathbb{Z}} \Big(\int_{l'}^{l'+1} |\varphi(t-u_j)|^2 dt \Big)^{1/2} \le C_1 \qquad (4.12) \end{aligned}$$

for some positive constant C_1 independent of u. Similarly, we have the following estimate for the item II in (4.9):

$$II \le C_2 \qquad (4.13)$$

for some positive constant C_2 independent of u. Combining (4.12) and (4.13), and recalling $\kappa_\varphi(u,v) = \kappa_\varphi(v,u)$, we conclude that

$$\sup_i \sum_j |\kappa_\varphi(u_i,u_j)|^2 (1+|u_i-u_j|)^{2\gamma} + \sup_j \sum_i |\kappa_\varphi(u_i,u_j)|^2 (1+|u_i-u_j|)^{2\gamma} < \infty. \qquad (4.14)$$

By Theorem 3.2 in [19], the matrix $A = [\kappa_\varphi(u_i,u_j)]$ is an invertible matrix on $\ell^2(\mathbb{Z})$. This, together with the estimate (4.14) and the Wiener lemma for infinite matrices in [24], implies that $A^{-1} = (b(i,j))$ satisfies

$$\sup_i \sum_j |b(i,j)|^2 (1+|u_i-u_j|)^{2\gamma} + \sup_j \sum_i |b(i,j)|^2 (1+|u_i-u_j|)^{2\gamma} < \infty. \qquad (4.15)$$

Therefore $\{\eta_i : i \in \mathbb{Z}\}$ is a frame for $\ell^q(\mathbb{Z})$, where $\eta_i = \{\kappa_\varphi(u_i,u_j)\}$ by (4.15). From Theorem 3.2, it then follows that $\{u_j\}$ is a sampling set for X_p. □

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