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An Inhomogeneous Uncertainty Principle for Digital Low-Pass Filters

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ABSTRACT. This article introduces an inhomogeneous uncertainty principle for digital low-pass filters. The measure for uncertainty is a product of two factors evaluating the frequency selectivity in comparison with the ideal filter and the effective length of the filter in the digital domain, respectively. We derive a sharp lower bound for this product in the class of filters with so-called finite effective length and show the absence of minimizers. We find necessary and certain sufficient conditions to identify minimizing sequences. When the class of filters is restricted to a given maximal length, we show the existence of an uncertainty minimizer. The uncertainty product of such minimizing filters approaches the unrestricted infimum as the filter length increases. We examine the asymptotics and explicitly construct a sequence of finite-length filters with the same asymptotics as the sequence of finite-length minimizers.

1. Introduction

Various forms of uncertainty inequalities are central to many aspects of time-frequency analysis and digital signal processing, see [1, 7] or [9, Chapter 2] and references therein. The classical uncertainty inequality in one dimension can be regarded as a lower bound for the product of the "essential length of the support" of a square-integrable function and of its Fourier transform. Thus, it states a restriction on the extent that both a function and its Fourier transform can be concentrated. A bound similar to the classical uncertainty principle has been derived for the discrete Fourier transform in the digital domain [6, Theorem 1].

Instead of considering how *signals* represented by functions or sequences behave under the Fourier transform, we investigate an uncertainty inequality for *filters*. This idea

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was motivated by works discussing certain approximations of low-pass analog filters and their relation with the classical uncertainty principle [11, 12, 2]. Here, we work in the digital domain and introduce a new measure for uncertainty to re-examine a well-known phenomenon in signal processing. It is common knowledge in digital signal processing that due to the discontinuities of an ideal low-pass filter in the frequency domain, it cannot be approximated well without using an increasing filter-length in the time-domain. The uncertainty inequality derived in this article bounds a cost functional for approximations of the ideal half-band low-pass filter by a class of digital filters referred to as *implementable low-pass filters.* Such filters are certain multiplication operators defined on $L^2([-\pi, \pi))$, the space of the Fourier transforms of all square-summable digital signals. They multiply by absolutely continuous, 2π-periodic functions with square-integrable derivatives in the frequency interval $[-\pi, \pi)$. Hereby, the notion of a low-pass filter requires that the content of signals is unchanged at zero frequency and completely suppressed at the frequency $\xi = \pm\pi$. The square-integrability of the derivative is equivalent to a moment condition for the filter taps, see Remark 1 below. We often identify the filter with the associated function in the frequency domain.

The cost functional in the present uncertainty inequality for digital low-pass filters contains two factors: The first one is the mean-square deviation of an implementable low-pass filter from the ideal half-band low-pass filter, the second one evaluates the effective length of the filter taps by the L^2-norm of the derivative of the implementable filter. We call this an inhomogeneous uncertainty principle because of the first factor that is not homogeneous in the implementable filter that is compared to the ideal half-band low-pass filter. The new cost functional is bounded below, which constitutes an uncertainty inequality for 2π-periodic filters: *A sequence of implementable filters that approaches the ideal half-band low-pass filter in $L^2([-\pi, \pi))$ must grow in effective filter-length to observe the uncertainty bound.* Thus, the lower bound for the cost functional has a direct practical relevance for the design of filter banks that are implemented by convolution in the digital domain.

One may ask whether it is possible to use other norms to specify in which sense the implementable filter approaches the ideal one and deduce other uncertainty principles. Indeed, this is a valid question and we point to the remarks in the next section and in the conclusion. For now it suffices to say that the usual approximation of filters in the operator norm would amount to uniform convergence of the associated functions in the frequency domain, which is impossible because implementable filters are continuous and the ideal filter is not. On the other hand, approximating in the least-squares sense is a standard technique in the literature when design restrictions need to be met. Here, it is the requirement of finite effective length of the filter; elsewhere restrictions such as causality or finite length have been considered [20, 3, 18, 19]. Similarly as in [18], we avoid introducing the notion of a transition bandwidth in the filter implementation because it is not implicit in the specification of the ideal low-pass filter and amounts to making an assumption about the typical signal content.

This article is organized as follows. After fixing the notation in Section 2, we derive the claimed uncertainty inequality for digital low-pass filters in Section 3 and exclude the existence of minimizers in the set of low-pass filters with finite essential length. In Section 4, we discuss some necessary conditions that minimizing sequences of filters have to satisfy in order to be asymptotically optimal in the sense of our uncertainty principle. We also prove a necessary and sufficient condition for sequences with a scaling limit and discuss examples. In Section 5 we state properties of the filter having a given finite length that has the lowest uncertainty product. Finally, we construct a minimizing sequence of finite-length filters for which the uncertainty product has the same asymptotics as for the sequence of finite-length minimizers.

2. A Cost Functional for Digital Low-Pass Filters

Definition 1. A complex-valued, essentially bounded 2π-periodic function h defined on $\mathbb{R}$ is called a *low-pass filter* if h has limits $\lim_{\xi\to\pi} h(\xi) = 0$ and $\lim_{\xi\to 0} h(\xi) = 1$. We say h is *implementable,* denoted as $h \in \mathcal{F}$, if it is absolutely continuous, so its derivative h' exists Lebesgue-almost everywhere, and if the restriction of h' to $[-\pi, \pi)$ is square-integrable.

The *ideal (half-band) low-pass filter* I is defined by

$$I(\xi) := \begin{cases} 1\,, & \text{if} \quad \xi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] + 2\pi\mathbb{Z}\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{2.1}$$

Finally, a 2π-periodic low-pass filter h is called *interpolatory* if $h(\xi)+h(\xi+\pi) = 1$ a.e. on $\mathbb{R}$.

Remark 1. Implementable low-pass filters form an affine subspace of the Sobolev-space on the circle, $H^1(\mathbb{T})$, which is a Hilbert space containing all of 2π-periodic functions f with finite Sobolev-norm $\|f\|_{H^1(\mathbb{T})} = (\|f\|^2 + \|f'\|^2)^{1/2}$, where $\|\cdot\|$ denotes the usual L^2-norm on the interval $[-\pi, \pi)$. Interpolatory filters are useful because they give rise to interpolatory refinable or scaling functions via cascade algorithms, see e.g., [15] and references therein.

We remark that for $h \in \mathcal{F}$, the filter taps $\hat{h} : \mathbb{Z} \to \mathbb{C}$ given by the Fourier coefficients $\hat{h}(n) = \int_{-\pi}^{\pi} e^{in\xi} h(\xi)\, d\xi$ observe a moment condition implicit in the square-integrability of the derivative h':

$$\sum_{n\in\mathbb{Z}} n^2 \left|\hat{h}(n)\right|^2 = \left\|h'\right\|^2 \equiv \int_{-\pi}^{\pi} \left|h'(\xi)\right|^2 d\xi < \infty\,. \tag{2.2}$$

Therefore, we also say that the *effective length* $\|h'\|$ of any filter h in $\mathcal{F}$ is finite in the digital domain. This definition of effective length seems less appropriate than the centered form $\min_{n\in\mathbb{Z}} \|(e^{in\xi}h)'\| \le \|h'\|$. We choose the form without minimization for reasons explained in Remark 2. The moment condition implies according to Chebyshev's inequality that the filter taps are concentrated,

$$\left(\sum_{|n|\ge M} \left|\hat{h}(n)\right|^2\right)^{1/2} \le \frac{1}{M}\left\|h'\right\|\,, \quad M > 0\,. \tag{2.3}$$

On the other hand, due to the conditions $h(\pi) = 0$ and $h(0) = 1$, we can estimate

$$1 \le \frac{1}{2}\int_{-\pi}^{\pi} \left|h'(\xi)\right| d\xi \le \sqrt{\frac{\pi}{2}}\left\|h'\right\| \tag{2.4}$$

via the Cauchy-Schwarz inequality, and thus the square norm of h' cannot be arbitrarily small for a low-pass filter $h \in \mathcal{F}$.

Finally, we note that the ideal low-pass filter satisfies the limit conditions in the preceding definition of low-pass filters, but it is not continuous, and thus not implementable.

Now we consider a cost functional that evaluates the efficiency of an implementable low-pass filter in terms of its mean-square deviation from the ideal low-pass filter together with the effective length of its filter taps.

Definition 2. If h is an implementable low-pass filter, we define the value of the cost functional U by

$$U(h) := \|h - I\| \left\|h'\right\| = \left[\left(\int_{-\pi}^{\pi} |h(\xi) - I(\xi)|^2 \, d\xi \right) \left(\sum_{n \in \mathbb{Z}} n^2 \left|\hat{h}(n)\right|^2 \right) \right]^{1/2} . \tag{2.5}$$

In Theorem 1, we will see that the infimum of U is nonzero and is not attained among the implementable filters. However, there are minimizing sequences and according to Theorem 2, they necessarily approach I in the L^2-norm.

Remark 2. Unlike the uncertainty product in the classical case, the factor $\|h - I\|$ used in our definition of U is not homogeneous in h. In addition, we do not bound a product of *centered moments* from below. The inhomogeneity and our definition of $\mathcal{F}$ do not allow a simple, $\mathcal{F}$-preserving translation operation that could be used in a centered version of $\|h - I\|$. However, one could replace $\|h'\|$ with the centered form of effective length,

$$\min_{n_0 \in \mathbb{Z}} \left\| \left(e^{i n_o \xi} h\right)' \right\| = \min_{n_0 \in \mathbb{Z}} \left(\sum_{n \in \mathbb{Z}} (n - n_0)^2 \left|\hat{h}(n)\right|^2 \right)^{1/2} .$$

The following proposition shows that this does not change the infimum of the uncertainty product.

Proposition 1. *The centered cost functional*

$$U_c(h) := \min_{n_0 \in \mathbb{Z}} \|h - I\| \left\| \left(e^{i n_o \xi} h\right)' \right\|$$

satisfies

$$\inf_{h \in \mathcal{F}} U_c(h) = \inf_{h \in \mathcal{F}} U(h) .$$

Proof. From the definition of the centered cost functional, we see $U_c(h) \leq U(h)$ for each fixed $h \in \mathcal{F}$. To see the needed complementary inequality $\inf_{h \in \mathcal{F}} U_c(h) \geq \inf_{h \in \mathcal{F}} U(h)$, we first use simple substitution

$$\inf_{h \in \mathcal{F}} \min_{n_0 \in \mathbb{Z}} \|h - I\| \left\| \left(e^{i n_o \xi} h\right)' \right\| = \inf_{h \in \mathcal{F}} \min_{n_0 \in \mathbb{Z}} \left\| h - e^{i n_o \xi} I \right\| \left\| h' \right\| . \tag{2.6}$$

Then, using the triangle inequality, we obtain $\|h - e^{i n_o \xi} I\| \geq \||h| - I\|$. Combining this with the estimate $\||h|'\| \leq \|h'\|$ [13, Theorem 6.17], we see that passing from $h \in \mathcal{F}$ to $|h| \in \mathcal{F}$ lowers the uncertainty product,

$$\left\| h - e^{i n_o \xi} I \right\| \left\| h' \right\| \geq \||h| - I\| \left\| |h|' \right\| = U(|h|) ,$$

which completes the proof. □

After deriving the lower bound for the uncertainty product in the next section, we turn our attention to minimizing sequences. By the above remark, minimizing sequences for U_c give rise to those for U. For this reason, we ignore the centered version of the uncertainty product in the remainder of this article.

In part of the engineering literature, the extent to which the ideal filter is approximated is usually described by pass-band and stop-band behavior as well as the transition bandwidth. The comparison with our approach motivates the following remark.

Remark 3. Given an implementable filter h and $\epsilon > 0$, we denote

$$\tau := \lambda(\{\xi \in [-\pi, \pi) : |h(\xi) - I(\xi)| > \epsilon\}) , \tag{2.7}$$

where λ is the Lebesgue measure on $\mathbb{R}$. The measure of the set where h and I deviate by more than ϵ can be viewed as the transition bandwidth for a fixed peak error ϵ in the pass and stop bands $\{\xi : |1 - h| < \epsilon\}$ and $\{\xi : |h| < \epsilon\}$, respectively. Then, Chebyshev's inequality implies

$$\int_{-\pi}^{\pi} |h(\xi) - I(\xi)|^2 \, d\xi \geq \epsilon^2 \tau \ . \tag{2.8}$$

The previous inequality shows that convergence of h to I in the L^2-sense implies that the product of the square of the peak error and the transition bandwidth approaches zero.

3. An Inhomogeneous Uncertainty Principle

In this section, we obtain a sharp lower bound for the cost functional U, which evaluates the efficiency of an implementable approximation of the ideal half-band low-pass filter.

To simplify notation, we write the inner product of functions f and g in $L^2([-\pi, \pi))$ as $\langle f, g \rangle$, by convention conjugate linear in the second entry. We also make frequent use of the characteristic function $\chi_{[a,b)}$ of a half-open subinterval $[a, b)$ in $[-\pi, \pi)$.

Theorem 1. *The cost functional $U(h)$ is bounded for all $h \in \mathcal{F}$ by*

$$U(h) > \frac{1}{2} + \left| h\left(-\frac{\pi}{2}\right) - \frac{1}{2} \right|^2 + \left| h\left(\frac{\pi}{2}\right) - \frac{1}{2} \right|^2 \ . \tag{3.1}$$

This bound is sharp in the sense that equality is never achieved and that the infimum of $U(h)$ over all implementable low-pass filters is

$$\inf_{h \in \mathcal{F}} U(h) = \frac{1}{2} \, . \tag{3.2}$$

Proof. First, note that since h' is in $L^2([-\pi, \pi))$ and h is bounded, the function $(I - h)h'$ is integrable. Next, using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} U(h) &= \left(\int_{-\pi}^{\pi} |h(\xi) - I(\xi)|^2 \, d\xi \right)^{1/2} \left(\int_{-\pi}^{\pi} \left| h'(\xi) \right|^2 d\xi \right)^{1/2} \\ &\geq \int_{-\pi}^{\pi} |h(\xi) - I(\xi)| \left| h'(\xi) \right| d\xi \ . \end{aligned} \tag{3.3}$$

We split into four subintervals

$$\left\langle |h - I|, \left|h'\right| \right\rangle = \left\langle (\chi_{[-\pi,\pi/2)} + \chi_{[-\pi/2,0)} + \chi_{[0,\pi/2)} + \chi_{[\pi/2,\pi)})|h - I|, \left|h'\right| \right\rangle \tag{3.4}$$

and examine each term of the sum separately.

The fact that h' is in $L^2([-\pi,\pi))$ and h is bounded implies that $(|h|^2)'$ is integrable and that $|h|^2$ is absolutely continuous on $[-\pi,\pi]$. By the same token, $|1-h|^2$ is also absolutely continuous on $[-\pi,\pi]$. Therefore,

$$\begin{aligned}\left\langle \chi_{[\pi/2,\pi)}|h-I|, \left|h'\right|,\right\rangle &= \left\langle \chi_{[\pi/2,\pi)}|h|, \left|h'\right|\right\rangle \geq -\mathrm{Re}\left\langle \chi_{[\pi/2,\pi)}h, \overline{h'}\right\rangle \\ &= -\mathrm{Re}\int_{\pi/2}^{\pi} \frac{1}{2}\frac{d}{d\xi}|h(\xi)|^2\,d\xi = \frac{1}{2}\left|h\left(\frac{\pi}{2}\right)\right|^2 .\end{aligned}$$

Similarly, we obtain

$$\left\langle \chi_{[-\pi,-\pi/2)}|h-I|, \left|h'\right|\right\rangle \geq \mathrm{Re}\left\langle \chi_{[-\pi,-\pi/2)}h, \overline{h'}\right\rangle = \frac{1}{2}\left|h\left(-\frac{\pi}{2}\right)\right|^2$$

and

$$\begin{aligned}\left\langle \chi_{[-\pi/2,0)}|h-I|, \left|h'\right|,\right\rangle &\geq -\mathrm{Re}\left\langle \chi_{[-\pi/2,0)}(h-I), \overline{h'}\right\rangle \\ &= -\mathrm{Re}\int_{-\pi/2}^{0} \frac{1}{2}\frac{d}{d\xi}|h(\xi)-1|^2\,d\xi = \frac{1}{2}\left|h\left(-\frac{\pi}{2}\right)-1\right|^2,\end{aligned}$$

as well as

$$\left\langle \chi_{[0,\pi/2)}|h-I|, \left|h'\right|\right\rangle \geq \mathrm{Re}\left\langle \chi_{[0,\pi/2)}(h-1), \overline{h'}\right\rangle = \frac{1}{2}\left|h\left(\frac{\pi}{2}\right)-1\right|^2 .$$

The preceding inequalities imply

$$\begin{aligned}U(h) &\geq \int_{-\pi}^{\pi} |h(\xi)-I(\xi)|\left|h'(\xi)\right|d\xi \qquad (3.5)\\ &\geq \frac{1}{2}\left[\left|h\left(\frac{\pi}{2}\right)\right|^2 + \left|h\left(-\frac{\pi}{2}\right)\right|^2 + \left|h\left(\frac{\pi}{2}\right)-1\right|^2 + \left|h\left(-\frac{\pi}{2}\right)-1\right|^2\right] .\end{aligned}$$

Together with the parallelogram law applied twice to the right-hand side of this inequality, we conclude

$$U(h) \geq \frac{1}{2} + \left|h\left(-\frac{\pi}{2}\right)-\frac{1}{2}\right|^2 + \left|h\left(\frac{\pi}{2}\right)-\frac{1}{2}\right|^2 . \qquad (3.6)$$

Thus, inequality (3.6) implies $U(h)\geq 1/2$ for every $h\in\mathcal{F}$.

Now we exclude possible cases of equality. Let us assume that there is $h_0\in\mathcal{F}$ giving equality in (3.6). Then, inequality (3.3) together with the inequalities for each subinterval imply

$$\mathrm{Re}\left\langle h-I, sh'\right\rangle = \left|\left\langle h-I, sh'\right\rangle\right| = \|h-I\|\left\|h'\right\| ,$$

with a function s given by

$$s(\xi) := \begin{cases} 1, & \xi\in[-\pi,-\pi/2)\cup[0,\pi/2) \\ -1, & \xi\in[-\pi/2,0)\cup[\pi/2,\pi) \end{cases} .$$

The last two equalities can only be true if there exists $\lambda\in\mathbb{R}\setminus\{0\}$, such that $h-I=\lambda sh'$ almost everywhere, so

$$h'(\xi) = -\lambda h(\xi), \quad \frac{\pi}{2} < \xi \leq \pi \qquad (3.7)$$

and h is seen to be continuously differentiable on this subinterval. In addition, the boundary value $h(\pi) = 0$ implies that the solution h of this ordinary differential equation satisfies $h(\xi) = 0,\ \pi/2 < \xi \leq \pi$ and analogously $h(\xi) = e^{\lambda\xi}, 0 \leq \xi \leq \pi/2$, thus contradicting the continuity of h at $\xi = \pi/2$.

Moreover, the same argument excludes that there is a minimizer h_0 of U giving $U(h_0) = \frac{1}{2}$. This establishes the first assertion of the theorem.

Let us now prove (3.2) by constructing a minimizing sequence. This construction is motivated by Theorem 4 below. We consider a sequence $\{h_n\}$ of 2π-periodic functions whose derivatives are of the form

$$h_n'(\xi) = c_n e^{-n\left||\xi|-\frac{\pi}{2}\right|} \sin\xi\,, \quad -\pi \leq \xi \leq \pi\,, \tag{3.8}$$

where c_n is appropriately chosen so that h_n' can be integrated to an implementable low-pass filter. To see that this is possible, we observe that $h_n'(\xi) = -h_n'(-\xi)$, for all $|\xi| < \pi$, so every h_n is even. Furthermore, $h_n'(\xi) + h_n'(\xi + \pi) = 0$, for all $|\xi| < \pi$ and h_n' is square integrable. Integrating both sides of (3.8), we choose the constant of integration so that $h_n(\pi) = 0$ and obtain

$$h_n(\xi) = \frac{-c_n}{n^2+1}\left(e^{-n\pi/2} + e^{-n(\xi-\frac{\pi}{2})}(n\sin\xi + \cos\xi)\right)\,, \qquad \pi/2 \leq \xi \leq \pi\,.$$

Setting

$$c_n := -\frac{1+n^2}{2n + 2e^{-n\pi/2}}\,,$$

so that $h_n(\pm\pi/2) = 1/2$, we obtain $h_n(\xi) + h_n(\xi+\pi) = 1$, for all $|\xi| \leq \pi$, and $h_n(0) = 1$, thus concluding that h_n is an interpolatory low-pass filter in $\mathcal{F}$.

We proceed by showing $\{h_n\}$ is a minimizing sequence. Since I and each h_n are even, we obtain

$$\begin{aligned}\int_{-\pi}^{\pi} |I(\xi) - h_n(\xi)|^2\, d\xi &= 2\int_{-\pi}^{-\pi/2} |h_n(\xi)|^2\, d\xi + 2\int_{\pi/2}^{\pi} |h_n(\xi)|^2\, d\xi \\ &= 4\int_{\pi/2}^{\pi} |h_n(\xi)|^2\, d\xi\,.\end{aligned}$$

Similarly, from $h_n'(\xi) = -h_n'(\xi+\pi)$ and the fact that h' is odd follows

$$\int_{-\pi}^{\pi} \left|h_n'(\xi)\right|^2 d\xi = 4\int_{\pi/2}^{\pi} \left|h_n'(\xi)\right|^2 d\xi\,.$$

Thus, we can simplify

$$(U(h_n))^2 = 16\int_{\pi/2}^{\pi} |h_n(\xi)|^2\, d\xi \times \int_{\pi/2}^{\pi} \left|h_n'(\xi)\right|^2 d\xi\,, \tag{3.9}$$

and observe that $U(h_n) \to \frac{1}{2}$ is equivalent to

$$\limsup_{n\to\infty} \int_{\pi/2}^{\pi} |h_n(\xi)|^2\, d\xi \times \int_{\pi/2}^{\pi} \left|h_n'(\xi)\right|^2 d\xi \leq \frac{1}{64}\,. \tag{3.10}$$

Now we estimate

$$\int_{\pi/2}^{\pi} \left|h_n'(\xi)\right|^2 d\xi \leq c_n^2 e^{n\pi} \int_{\pi/2}^{\pi} e^{-2n\xi}\, d\xi = \frac{c_n^2}{2n}\left(1 - e^{-n\pi}\right) \leq \frac{c_n^2}{2n}\,. \tag{3.11}$$

On the other hand, for $\pi/2 \le \xi \le \pi$ we have $e^{-n\pi} \le e^{-n\xi}$, so

$$|h_n(\xi)| \le \frac{c_n}{n^2+1} e^{n\pi/2}\big(e^{-n\xi}|n\sin\xi + \cos\xi| + e^{-n\pi}\big) \le \frac{c_n}{n^2+1} e^{n\pi/2}(n+2)e^{-n\xi} .$$

Therefore,

$$\int_{\pi/2}^{\pi} |h_n(\xi)|^2\, d\xi \le \frac{c_n^2}{\big(n^2+1\big)^2}\frac{(n+2)^2}{2n}\big(1-e^{-n\pi}\big) \le \frac{c_n^2}{\big(n^2+1\big)^2}\frac{(n+2)^2}{2n} . \tag{3.12}$$

Inequalities (3.11) and (3.12) imply

$$\begin{aligned}\limsup_{n\to\infty} \int_{\pi/2}^{\pi} |h_n(\xi)|^2\, d\xi \times \int_{\pi/2}^{\pi} \big|h_n'(\xi)\big|^2 d\xi &\le \limsup_{n\to\infty} \frac{c_n^4}{4n^2\big(n^2+1\big)^2}(n+2)^2 \\ &\le \lim_{n\to\infty} \frac{\big(n^2+1\big)^2}{16n^4}\frac{(n+2)^2}{4n^2} = \frac{1}{64} . \qquad \square\end{aligned}$$

4. Some Necessary and Sufficient Conditions for Minimizing Sequences

Since U measures the effective length of the filter as well as its frequency selectivity, it can be considered as a cost functional that evaluates the effectiveness of approximations of the ideal low-pass filter by implementable low-pass filters. So far, we have established (1) that such approximations impose a positive cost which cannot be less than $1/2$, (2) that we *cannot* construct a minimizer of U within the class of implementable low-pass filters, but (3) that the lower bound $1/2$ is sharp because it is the limiting value of U for a minimizing sequence of implementable low-pass filters. The goal of the present section is to find some necessary and sufficient conditions for minimizing sequences.

Theorem 2. *Let $\{h_n\}_{n\in\mathbb{N}}$ be a minimizing sequence of implementable low-pass filters, that is, $\lim_{n\to\infty} U(h_n) = \frac{1}{2}$. Then the following properties hold:*

(i) *Pointwise convergence at the cut-off frequency*

$$\lim_{n\to\infty} h_n\Big(\frac{\pi}{2}\Big) = \lim_{n\to\infty} h_n\Big(-\frac{\pi}{2}\Big) = \frac{1}{2} , \tag{4.1}$$

(ii) *convergence in the square mean*

$$\lim_{n\to\infty} \|h_n - I\| = 0 , \tag{4.2}$$

(iii) *almost-uniform convergence. More precisely, for every $0 < \delta < \pi/2$ we have*

$$\lim_{n\to\infty} (\sup\{|h_n(\xi) - I(\xi)| : |\xi| \in [0, \pi/2-\delta] \cup [\pi/2+\delta, \pi]\}) = 0 . \tag{4.3}$$

Proof.

(i) By (3.3), we have

$$0 \le \Big|h_n\Big(\frac{\pi}{2}\Big) - \frac{1}{2}\Big|^2 + \Big|h_n\Big(-\frac{\pi}{2}\Big) - \frac{1}{2}\Big|^2 \le U(h_n) - \frac{1}{2} ,$$

and (i) follows because $\lim_{n\to\infty} U(h_n) = \frac{1}{2}$.

(ii) Suppose, on the contrary, that (ii) is not true. Then, we have $\lim_{n\to\infty} U(h_n) = 1/2$ and the limit superior of $\{\|h_n - I\|\}_{n\in\mathbb{N}}$ is nonzero. Consequently, there exists a subsequence $\{h_k\}_{k\in\mathbb{K}}$, $\mathbb{K} \subset \mathbb{N}$ of implementable low-pass filters such that $\lim_{k\to\infty} \int_{-\pi}^{\pi} |h_k(\xi) - I(\xi)|^2\, d\xi = M_0 \in (0, +\infty]$. Let us from now on for the sake of simplicity assume that the subsequence $\{h_k\}_{k\in\mathbb{K}}$ is identical with $\{h_n\}_{n\in\mathbb{N}}$. If $M_0 = +\infty$, then necessarily $||h_n'|| \to 0$, which contradicts the lower bound $\|h'\| \geq \sqrt{2/\pi}$ in Remark 1. If $M_0 \in (0, \infty)$, then

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \left|h_n'(\xi)\right|^2 d\xi = \frac{1}{4M_0}\ , \tag{4.4}$$

and so the sequence $\{h_n\}$ is bounded in the Sobolev space $H^1(\mathbb{T})$ and we can pass to a weakly convergent subsequence. Using compact Sobolev embedding [10, Theorem 7.26], we see the convergence of this subsequence $\{h_n\}$ is uniform on $[-\pi, \pi]$. By weak convergence in the Sobolev space, taking an appropriate subsequence gives $h(\xi) = 1 + \lim_{p\to\infty} \int_0^{\xi} h_{l_p}'(\eta)\, d\eta = 1 + \int_0^{\xi} g(\eta)\, d\eta$ for almost every ξ, thus h is absolutely continuous and the derivative $g = h'$ is square integrable. Consequently, h is a minimizer of U in $\mathcal{F}$, which contradicts Theorem 1.

(iii) Let $J_1 = [-\pi, -\pi/2 - \delta]$, $J_2 = [-\pi/2 + \delta, 0]$, $J_3 = [0, \pi/2 - \delta]$ and $J_4 = [\pi/2 + \delta, \pi]$. The continuity of every h_n implies that there exist $\xi_n^{(i)} \in J_i$, $i \in \{1, 2, 3, 4\}$, such that

$$\left|I\big(\xi_n^{(i)}\big) - h_n\big(\xi_n^{(i)}\big)\right| = \max_{\xi\in J_i} |I(\xi) - h_n(\xi)|, \quad i \in \{1, 2, 3, 4\}\,. \tag{4.5}$$

By appropriately adjusting the values of I at $\pm\pi/2$ so that every $I - h_n$ is continuous on each of the intervals $\tilde{J}_i$, where $\tilde{J}_1 = [-\pi/2 - \delta, -\pi/2]$, $\tilde{J}_2 = [-\pi/2, -\pi/2 + \delta]$, $\tilde{J}_3 = [\pi/2 - \delta, \pi/2]$ and $\tilde{J}_4 = [\pi/2, \pi/2 + \delta]$, we get that there exist $\eta_n^{(i)}$ in each $\tilde{J}_i$ satisfying

$$\left|I\big(\eta_n^{(i)}\big) - h_n\big(\eta_n^{(i)}\big)\right| = \min_{\xi\in \tilde{J}_i} |I(\xi) - h_n(\xi)|, \quad i \in \{1, 2, 3, 4\}\,. \tag{4.6}$$

Starting from

$$\delta\left|I\big(\eta_n^{(i)}\big) - h_n\big(\eta_n^{(i)}\big)\right| \leq \int_{\tilde{J}_i} |I(\xi) - h_n(\xi)|\, d\xi$$

the Cauchy-Schwarz inequality implies

$$0 \leq \left|I\big(\eta_n^{(i)}\big) - h_n\big(\eta_n^{(i)}\big)\right| \leq \delta^{-1/2}\, \|I - h_n\| \quad n \in \mathbb{N},\ i \in \{1, 2, 3, 4\}\,.$$

Using (4.2) we get that for every $i \in \{1, 2, 3, 4\}$,

$$\lim_{n\to\infty} \left|I\big(\eta_n^{(i)}\big) - h_n\big(\eta_n^{(i)}\big)\right| = 0\,. \tag{4.7}$$

Similarly as in the proof of Theorem 1, we first split into subintervals and use the Cauchy-Schwarz inequality,

$$\begin{aligned} U(h_n) &= \left(\int_{-\pi}^{\pi} |I(\xi) - h_n(\xi)|^2\, d\xi\right)^{1/2} \left(\int_{-\pi}^{\pi} \left|h_n'(\xi)\right|^2 d\xi\right)^{1/2} \\ &\geq \left(\int_{-\pi}^{\xi_n^{(1)}} + \int_{\xi_n^{(2)}}^{0} + \int_{0}^{\xi_n^{(3)}} + \int_{\xi_n^{(4)}}^{\pi} + \int_{\eta_n^{(1)}}^{-\pi/2} + \int_{-\pi/2}^{\eta_n^{(2)}} + \int_{\eta_n^{(3)}}^{\pi/2} + \int_{\pi/2}^{\eta_n^{(4)}}\right) \\ &\quad \left|(I(\xi) - h_n(\xi))\overline{h_n'(\xi)}\right| d\xi\ , \end{aligned}$$

and then estimate term by term as in inequality (3.5),

$$
\begin{aligned}
U(h_n) \geq \mathrm{Re}\Bigg(&-\int_{-\pi}^{\xi_n^{(1)}} + \int_{\xi_n^{(2)}}^{0} - \int_{0}^{\xi_n^{(3)}} + \int_{\xi_n^{(4)}}^{\pi} - \int_{\eta_n^{(1)}}^{-\pi/2} + \int_{-\pi/2}^{\eta_n^{(2)}} \\
&- \int_{\eta_n^{(3)}}^{\pi/2} + \int_{\pi/2}^{\eta_n^{(4)}}\Bigg) (I(\xi) - h_n(\xi))\overline{h_n'(\xi)}\, d\xi \\
\geq{}& \frac{1}{2}\sum_{i=1}^{4} \left|I\big(\xi_n^{(i)}\big) - h_n\big(\xi_n^{(i)}\big)\right|^2 \\
&+ \mathrm{Re}\Bigg(-\int_{\eta_n^{(1)}}^{-\pi/2} + \int_{-\pi/2}^{\eta_n^{(2)}} - \int_{\eta_n^{(3)}}^{\pi/2} + \int_{\pi/2}^{\eta_n^{(4)}}\Bigg) (I(\xi) - h_n(\xi))\overline{h_n'(\xi)}\, d\xi \\
\geq{}& \frac{1}{2}\sum_{i=1}^{4} \left|I\big(\xi_n^{(i)}\big) - h_n\big(\xi_n^{(i)}\big)\right|^2 \\
&+ \frac{1}{2}\Big(\Big|h_n\Big(-\frac{\pi}{2}\Big)\Big|^2 - \big|h_n\big(\eta_n^{(1)}\big)\big|^2\Big) + \frac{1}{2}\Big(\Big|1 - h_n\Big(-\frac{\pi}{2}\Big)\Big|^2 - \big|1 - h_n\big(\eta_n^{(2)}\big)\big|^2\Big) \\
&+ \frac{1}{2}\Big(\Big|1 - h_n\Big(\frac{\pi}{2}\Big)\Big|^2 - \big|1 - h_n\big(\eta_n^{(3)}\big)\big|^2\Big) + \frac{1}{2}\Big(\Big|h_n\Big(\frac{\pi}{2}\Big)\Big|^2 - \big|h_n\big(\eta_n^{(4)}\big)\big|^2\Big).
\end{aligned}
$$

Using the parallelogram law as before, we obtain

$$
\begin{aligned}
U(h_n) \geq{}& \frac{1}{2} + \frac{1}{2}\sum_{i=1}^{4} \left|I\big(\xi_n^{(i)}\big) - h_n\big(\xi_n^{(i)}\big)\right|^2 \\
&- \frac{1}{2}\Big(\big|h_n\big(\eta_n^{(1)}\big)\big|^2 + \big|1 - h_n\big(\eta_n^{(2)}\big)\big|^2 + \big|1 - h_n\big(\eta_n^{(3)}\big)\big|^2 + \big|h_n\big(\eta_n^{(4)}\big)\big|^2\Big).
\end{aligned}
$$

Since by Equation (4.7), the terms $h_n(\eta_n^{(2)}) - 1 \to 0$, $h_n(\eta_n^{(3)}) - 1 \to 0$, $h_n(\eta_n^{(1)}) \to 0$ and $h_n(\eta_n^{(4)}) \to 0$, and by the assumption $U(h_n) \to \frac{1}{2}$, the remaining sum must also converge to zero. □

Remark 4. The almost-uniform convergence stated in the preceding theorem is commonly used in filter design to measure how well the ideal filter is approximated [16]. Since this condition is necessary for minimizing sequences of our uncertainty inequality, we have a more refined tool to distinguish between various approximating sequences. In fact, we will see examples of filters that fulfill all of the above necessary conditions but fail to be minimizing for our uncertainty inequality.

Now we turn our attention to a class of sequences that covers examples of practical relevance and enables us to formulate necessary and sufficient conditions to characterize minimizing sequences in this class. As a first step, we discuss uncertainty-lowering operations.

Proposition 2. *Symmetrization lowers the uncertainty product. More explicitly, let* $S_\pm f(\xi) := \frac{1}{2}(f(\xi) \pm f(-\xi))$ *for any* $f \in L^2([-\pi, \pi))$*. Then for* $h \in \mathcal{F}$*,* $S_+h \in \mathcal{F}$ *and* $U(h) \geq U(S_+h)$*. Equality holds if and only if* $S_-h = 0$*.*

Proof. It is straightforward to verify that $S_+h \in \mathcal{F}$ for any $h \in \mathcal{F}$. Using the parallelogram equality and the symmetry of the ideal filter, we have

$$
|h(\xi) - I(\xi)|^2 + |h(-\xi) - I(\xi)|^2 = 2\,|S_+h(\xi) - I(\xi)|^2 + 2\,|S_-h(\xi)|^2 . \tag{4.8}
$$

Similarly,

$$\left|h'(\xi)\right|^2 + \left|h'(-\xi)\right|^2 = 2\left|S_+h'(\xi)\right|^2 + 2\left|S_-h'(\xi)\right|^2 \ . \tag{4.9}$$

Inserting this identity in the definition of the uncertainty product yields

$$\begin{aligned}(U(h))^2 = &\int_0^\pi \left(|h(\xi) - I(\xi)|^2 + |h(-\xi) - I(\xi)|^2\right) d\xi \\ &\times \int_0^\pi \left(\left|h'(\xi)\right|^2 + \left|h'(-\xi)\right|^2\right) d\xi \\ &\geq 4\int_0^\pi \left|S_+h(\xi) - I(\xi)\right|^2 d\xi \times \int_0^\pi \left|S_+h'(\xi)\right|^2 d\xi = (U(S_+h))^2 .\end{aligned} \tag{4.10}$$

If S_-h is nonzero, then the above inequality is strict. □

Proposition 3. *Let P be the idempotent map defined on $f \in L^2([-\pi, \pi))$ by*

$$Pf(\xi) = \frac{1}{2}(1 + f(\xi) - f(\xi + \pi)) \ . \tag{4.11}$$

We have for all $h \in \mathcal{F}$ that $Ph \in \mathcal{F}$ and $U(h) \geq U(Ph)$. Equality holds if and only if $Ph = h$.

Proof. From the definition it follows that $Ph \in \mathcal{F}$ if $h \in \mathcal{F}$, in particular $Ph(0) = 1$ and $Ph(\pm\pi) = 0$, and that Ph is interpolatory. The filter taps of Ph observe

$$(Ph)^\wedge(n) = \begin{cases} \pi, & n = 0 \\ 0, & n \in 2\mathbb{Z} \setminus \{0\} \\ \hat{h}(n), & \text{else} \end{cases} \ . \tag{4.12}$$

So all even Fourier coefficients of Ph are zero except the zeroth one. This property is also true for the ideal filter I. Therefore, unless these Fourier coefficients are unchanged, the sum $\sum_n n^2|\hat{h}(n)|^2$ strictly decreases, and so does $\|I - h\|^2 = \sum_n |\hat{I}(n) - \hat{h}(n)|^2$. □

Corollary 1. *Symmetrizing and applying P are commuting operations, $PS_+h = S_+Ph$ for any filter $h \in \mathcal{F}$. Consequently, the composition of both operations maps $\mathcal{F}$ into the affine subspace of symmetric, interpolatory filters in $\mathcal{F}$, while lowering U.*

Proof. This follows from $(S_+h)^\wedge(n) = \frac{1}{2}(\hat{h}(n) + \overline{(\overline{h})^\wedge(n)})$ and from the definition of the mapping P. □

Theorem 3. *If the sequence $\{h_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ satisfies $U(h_n) \to \frac{1}{2}$ then necessarily both $\|S_-h_n\|\|h_n'\| \to 0$ and $\|Ph_n - h_n\|\|h_n'\| \to 0$.*

Proof. This follows from inspecting the nonnegative terms that get dropped in the proofs of the preceding uncertainty-lowering operations. The term discarded in inequality (4.10) must converge,

$$\int_0^\pi |S_-h_n(\xi)|^2 d\xi \times \int_{-\pi}^\pi \left|h_n'(\xi)\right|^2 d\xi \to 0 , \tag{4.13}$$

and using the orthogonality $\langle h - Ph, Ph - I\rangle = 0$ that is seen from the Fourier series representation of $h - Ph$ and $Ph - I$ as derived from Equation 4.12 yields

$$\sum_k k^2\left|\hat{h}_n(k)\right|^2 \times \sum_k \left|\hat{h}_n(k) - (Ph_n)^\wedge(k)\right|^2 \to 0 , \tag{4.14}$$

otherwise it would be possible to lower either $U(Ph_n)$ or $U(S_+h_n)$ below the infimum value of U for sufficiently large n. □

Because of the preceding propositions and corollaries, it is sufficient to concentrate on symmetric, interpolatory filters for the practical purpose of designing minimizing sequences. In the following theorem, we study a necessary and sufficient condition for minimizing sequences in the presence of an asymptotic scaling behavior for h_n near the cut-off frequency.

Theorem 4. *If a sequence of filters $\{h_n\} \subset \mathcal{F}$ satisfies $\|S_-h_n\|\|h_n'\| \to 0$ as well as $\|Ph_n - h_n\|\|h_n'\| \to 0$ and there exists a sequence of nonnegative scaling factors $\{s_n\}_{n\in\mathbb{N}}$, $s_n \to 0$, such that $\{PS_+h_n\}$ can be rescaled to a sequence*

$$\left\{f_n(\eta) = (PS_+h_n)\left(\frac{\pi}{2}(1+s_n\eta)\right), \eta \geq 0\right\}_{n\in\mathbb{N}}, \tag{4.15}$$

that converges in Sobolev norm

$$\int_0^{1/s_n} |f_n - f|^2\, d\eta + \int_0^{1/s_n} \left|f_n' - f'\right|^2 d\eta \to 0 \tag{4.16}$$

to some absolutely continuous function f observing $f, f' \in L^2(\mathbb{R}^+)$ and $f(0) = \frac{1}{2}$, then $U(h_n) \to \frac{1}{2}$ if and only if $f(\xi) = \frac{1}{2}e^{\lambda\xi}$ for any fixed $\lambda < 0$.

Proof. First we note that passing from h_n to S_+h_n does not change the limit of $U(h_n)$. In addition, the assumptions on the antisymmetric and noninterpolatory parts imply that $\|PS_+h_n - S_+h_n\|\,\|S_+h_n'\| \to 0$, so we have $\lim_{n\to\infty} U(h_n) = \lim_{n\to\infty} U(PS_+h_n)$ if the limit on the right-hand side exists.

Thus, we may from now on assume that h_n is a sequence of interpolatory, symmetric filters. We note that using these properties of each h_n together with the rescaling

$$\int_{\pi/2}^{\pi} |h_n(\xi)|^2\, d\xi = \frac{\pi}{2} s_n \int_0^{1/s_n} |f_n(\eta)|^2\, d\eta \tag{4.17}$$

and

$$\int_{\pi/2}^{\pi} \left|h_n'(\xi)\right|^2 d\xi = \frac{2}{\pi} s_n^{-1} \int_0^{1/s_n} \left|f_n'(\eta)\right|^2 d\eta \tag{4.18}$$

give

$$(U(h_n))^2 = 16 \int_0^{1/s_n} |f_n(\eta)|^2\, d\eta \int_0^{1/s_n} \left|f_n'(\eta)\right|^2 d\eta\,, \tag{4.19}$$

converging by assumption to

$$\lim_{n\to\infty} (U(h_n))^2 = 16 \int_0^{\infty} |f(\eta)|^2\, d\eta \int_0^{\infty} \left|f'(\eta)\right|^2 d\eta\,. \tag{4.20}$$

Finding the minimum of the right-hand side expression over all $f, f' \in L^2(\mathbb{R}^+)$ that observe $f(0) = \frac{1}{2}$ yields again via an argument involving the Cauchy-Schwarz inequality and cases of equality as in Theorem 1 the condition

$$f' = \lambda f \tag{4.21}$$

with some $\lambda < 0$. Thus, $f(\eta) = \frac{1}{2}e^{\lambda\eta}$ and by inserting this in the right-hand side of Equation (4.20) gives $\lim_n (U(h_n))^2 = \frac{1}{4}$, independent of $\lambda < 0$. □

The following two examples satisfy the necessary conditions in Theorem 2, but they fail to be minimizing sequences. These examples have been chosen because they are for other reasons desirable as approximations of the ideal low-pass filter. Either of the two represents an example in the unified class of filters with maximum flatness described in [17]. For more background on the Butterworth interpolatory filter, see [8].

Example 1 (Digital Butterworth interpolatory filter). If we take the square modulus of the digital Butterworth filter with cut-off frequency $\pi/2$, we obtain a family $\{h_n\}_{n\in\mathbb{N}}$ of symmetric and interpolatory filters given by

$$h_n(\xi) := \frac{(1+\cos\xi)^n}{(1+\cos\xi)^n + (1-\cos\xi)^n}\,. \tag{4.22}$$

The sequence observes $h_n(0) = 1$ for each n, and because of the monotonicity of the cosine it approaches the ideal filter uniformly almost everywhere, but we note that

$$\lim_{n\to\infty} h_n\left(\frac{\pi}{2} + \frac{\eta}{n}\right) = \frac{1}{1+e^{2\eta}} \tag{4.23}$$

and thus this sequence cannot be minimizing for our uncertainty principle. Instead, by the scaling limit (4.23) and the explicit evaluation of (4.20) in this case, we have

$$\begin{aligned}\lim_{n\to\infty} U(h_n) &= 4\left[\int_0^\infty \frac{d\eta}{\left(1+e^{2\eta}\right)^2}\int_0^\infty \frac{4e^{4\eta}}{\left(1+e^{2\eta}\right)^4}\,d\eta\right]^{1/2}\\ &= \frac{1}{3}\sqrt{6(\ln 4 - 1)} \approx 0.5075\,.\end{aligned}$$

Example 2 (Daubechies interpolatory filter). We consider the filter sequence

$$h_n(\xi) = \cos^{2n}\frac{\xi}{2}\sum_{k=0}^{n-1}\binom{n+k}{k}\sin^{2k}\frac{\xi}{2}$$

used by Ingrid Daubechies in the construction of compactly supported wavelets [4, 5]. Yves Meyer [14] expresses these nonnegative, interpolating filters as

$$h_n(\xi) = c_n\int_\xi^\pi (\sin\eta)^{2n+1}\,d\eta \tag{4.24}$$

$$= 1 - c_n\int_0^\xi (\sin\eta)^{2n+1}\,d\eta \tag{4.25}$$

with $c_n^{-1} = \int_0^\pi (\sin\eta)^{2n+1}\,d\eta$, similarly as in the definition of the minimizing sequence in the proof of Theorem 1. Again, each h_n is symmetric, interpolatory, and contained in $\mathcal{F}$. One may show that h_n approaches the ideal filter almost uniformly. However,

$$\lim_{n\to\infty} h_n\left(\frac{\pi}{2} + \frac{\eta}{\sqrt{2n+1}}\right) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\int_0^\eta e^{-y^2/2}\,dy\,, \tag{4.26}$$

and again this cannot constitute a minimizing sequence for U. Calculating the integrals in this case, gives

$$\lim_{n\to\infty} U(h_n) = \left[\frac{2}{\pi^{3/2}}\int_0^\infty \left(\int_\xi^\infty e^{-\eta^2/2}\,d\eta\right)^2 d\xi\right]^{1/2} = \sqrt{\frac{2\sqrt{2}-2}{\pi}} \approx 0.5135\,. \quad (4.27)$$

Example 3. The minimizing sequence constructed in the proof of Theorem 1, given by the even, interpolatory filters that observe $h_n(0) = 1$ and

$$h_n'(\xi) = -\frac{1+n^2}{2n+2e^{-n\pi/2}} e^{-n||\xi|-\frac{\pi}{2}|}\sin\xi\,, \quad -\pi \le \xi \le \pi\,, \quad (4.28)$$

satisfies the condition of the preceding theorem, because with $f_n(\eta) = h_n(\frac{\pi}{2}(1+\frac{\eta}{n}))$ we obtain the convergence

$$\begin{aligned} f_n(\eta) &= \frac{\chi_{[0,n)}(\eta)}{2n+2e^{-n\pi/2}}\left(e^{-n\pi/2} + e^{-\pi\eta/2}\left(n\sin\left(\frac{\pi}{2}+\frac{\pi\eta}{2n}\right) - \cos\left(\frac{\pi}{2}+\frac{\pi\eta}{2n}\right)\right)\right) \\ &\to \frac{1}{2}e^{-\pi\eta/2} \end{aligned}$$

that is for $n \in \mathbb{N}$ dominated by the square-integrable function $\eta \mapsto \frac{3}{2}e^{-\eta\pi/2}$. Similarly, the derivative

$$f_n'(\eta) = -\frac{\pi}{2n}\chi_{[0,n)}(\eta)\frac{1+n^2}{2n+2e^{-n\pi/2}}\sin\left(\frac{\pi}{2}+\frac{\pi\eta}{2n}\right)e^{-\pi\eta/2} \to -\frac{\pi}{4}e^{-\pi\eta/2} \quad (4.29)$$

is dominated by $\eta \mapsto \frac{\pi}{2}e^{-\pi\eta/2}$. Consequently, $\{h_n\}$ is a minimizing sequence, which has already been demonstrated by explicit calculation of $U(h_n)$ in the proof of Theorem 1.

5. Uncertainty Minimizers Among Low-Pass Filters of a Given Length

For practical purposes, digital filters with infinite impulse response are implemented by truncating the filter taps. Clearly, this would apply to our example of a minimizing sequence, and one could now study how any such minimizing sequence is affected by a truncation operation. Instead, we choose to specialize the problem of finding minimizing sequences for U by restricting $\mathcal{F}$ to trigonometric polynomials, that is, filters of finite length. It turns out that in this restricted affine space there is a minimizer for U. The crucial property derived in this section is the rate of convergence of the minimal value of U as the filter length increases, see Theorem 5 below.

Definition 3. For $n \ge 0$, let $\mathcal{F}_n$ be the set of all implementable low-pass filters that are trigonometric polynomials of degree at most $2n+1$, that is,

$$\mathcal{F}_n := \left\{ \sum_{k=-2n-1}^{2n+1} a_k e^{-ik\xi} \in \mathcal{F} \right\}\,.$$

The set $\mathcal{F}_n$ is an affine subspace of $\mathcal{F}$ and hence the set of all differences of two filters in $\mathcal{F}_n$ is a linear space, denoted as

$$\mathcal{Q}_n = \{h_1(\xi) - h_2(\xi) : h_1, h_2 \in \mathcal{F}_n\}\,.$$

Lemma 1. *Let $n \geq 0$. Then there exists $h_n \in \mathcal{F}_n$ so that $U(h_n) = \inf_{h\in\mathcal{F}_n} U(h)$. This minimizer is a trigonometric polynomial of the form*

$$h_n(\xi) = \frac{1}{2} + \sum_{k=0}^{n} c_{n,k} \cos(2k+1)\xi \tag{5.1}$$

with coefficients $c_{n,k} \in \mathbb{R}$, $0 \leq k \leq n$. The coefficients satisfy

$$c_{n,k} = \frac{1}{1+\gamma_n^2(2k+1)^2}\left(c_{\infty,k} + \frac{1}{2\sigma_n} - \frac{1}{\sigma_n}\sum_{j=0}^{n} \frac{c_{\infty,j}}{1+\gamma_n^2(2j+1)^2}\right), \tag{5.2}$$

where $c_{\infty,j} = \frac{2(-1)^j}{(2j+1)\pi}$, $\sigma_n = \sum_{j=0}^{n} \frac{1}{1+\gamma_n^2(2j+1)^2}$, and the only remaining unknown γ_n in the expression for $c_{n,k}$ observes

$$\gamma_n^2 = \frac{\int_{-\pi}^{\pi} |I(\xi) - h_n(\xi)|^2\, d\xi}{\int_{-\pi}^{\pi} \left|h_n'(\xi)\right|^2 d\xi}\,. \tag{5.3}$$

Proof. Choosing $h_0(\xi) = \frac{1}{2} + \frac{1}{2}\cos\xi$ gives $U(h_0) = \sqrt{\pi}\left(\frac{3\pi}{16} - \frac{1}{2}\right)^{1/2} \approx 0.529$. Since $h_0 \in \mathcal{F}_n$ for all $n \geq 0$, we know $\inf_{h\in\mathcal{F}_n} U(h) \leq U(h_0)$.

By the inequality $\|h'\| \geq \sqrt{2/\pi}$ stated in Remark 1 for any $h(\xi) = \sum_{k=-2n-1}^{2n+1} a_k e^{-ik\xi} \in \mathcal{F}_n \subset \mathcal{F}$, when h is close to optimal, $\|h - I\|$ cannot be arbitrarily large. In addition, by Minkowski's inequality we have

$$\|h\| \leq \|h - I\| + \|I\| = \|h - I\| + \sqrt{\pi}\,.$$

Therefore we can restrict $\inf_{h\in\mathcal{F}_n} U(h) = \inf_{h\in\mathcal{F}_n, \|h\|\leq K} U(h)$ with some sufficiently large K. The existence of $h_n \in \mathcal{F}_n$ so that $\inf_{h\in\mathcal{F}_n} U(h) = U(h_n)$ now follows because U is continuous on the compact set $\{h \in \mathcal{F}_n : \|h\| \leq K\}$.

Let $h_n \in \mathcal{F}_n$ satisfy $U(h_n) = \inf_{h\in\mathcal{F}_n} U(h)$. We note that the previously defined operators S_+ and P leave $\mathcal{F}_n$ invariant, therefore the minimizer h_n must satisfy $S_+ h_n = h_n$ and $P h_n = h_n$. In addition, h_n must be real-valued, because otherwise taking the real part would lower $U(h_n)$. Thus,

$$h_n(\xi) = \frac{1}{2} + \sum_{k=0}^{n} c_{n,k} \cos(2k+1)\xi \tag{5.4}$$

for coefficients $c_{n,k} \in \mathbb{R}$, $0 \leq k \leq n$.

By the definition of h_n, $U(h_n + tq) \geq U(h_n)$ for all $t \in \mathbb{R}$ and all $q \in \mathcal{Q}_n$. If we specialize to real-valued q, then $\frac{d}{dt}|_{t=0}(U(h_n + tq))^2 = 0$ implies

$$\int_{-\pi}^{\pi} |I(\xi) - h_n(\xi)|^2\, d\xi \times \int_{-\pi}^{\pi} h_n'(\xi) q'(\xi)\, d\xi - \int_{-\pi}^{\pi} \left|h_n'(\xi)\right|^2 d\xi \times \int_{-\pi}^{\pi} q(\xi)(I(\xi) - h_n(\xi))\, d\xi = 0\,.$$

Integrating by parts, and setting $\alpha_n := \int_{-\pi}^{\pi} |h_n'(\xi)|^2\, d\xi$, $\beta_n := \int_{-\pi}^{\pi} |I(\xi) - h_n(\xi)|^2 d\xi$ we obtain

$$\int_{-\pi}^{\pi} \Big(\beta_n h_n''(\xi) + \alpha_n (I_n(\xi) - h_n(\xi))\Big) q(\xi)\, d\xi = 0 \tag{5.5}$$

for all real-valued $q \in \mathcal{Q}_n$, where $I_n(\xi) = \frac{1}{2} + \sum_{k=0}^{n} c_{\infty,k} \cos(2k+1)\xi$, $n \geq 1$, denotes the projection of the ideal filter onto the subspace of trigonometric polynomials of maximal degree $2n+1$ in $L^2([-\pi, \pi))$.

Choosing q in (5.5) by $q_k(\xi) := \cos(2k+1)\xi - \cos\xi \in \mathcal{Q}_n$, $1 \leq k \leq n$, and using the usual orthogonality relations for trigonometric polynomials in the inner product with

$$\beta_n h_n''(\xi) + \alpha_n (I_n(\xi) - h_n(\xi)) = \sum_{k=0}^{n} \left(\alpha_n c_{\infty,k} - \left(\alpha_n + (2k+1)^2 \beta_n\right) c_{n,k}\right) \cos(2k+1)\xi\,,$$

we obtain

$$\alpha_n c_{\infty,k} - \left(\alpha_n + (2k+1)^2 \beta_n\right) c_{n,k} = \alpha_n c_{\infty,0} - (\alpha_n + \beta_n) c_{n,0},\ 1 \leq k \leq n\,. \tag{5.6}$$

Solving this for $c_{n,k}$ leads to the expression

$$c_{n,k} = \frac{c_{\infty,k} - \left(c_{\infty,0} - \left(1 + \gamma_n^2\right) c_{n,0}\right)}{1 + \gamma_n^2 (2k+1)^2} \tag{5.7}$$

in terms of the unknowns $\gamma_n = \sqrt{\beta_n/\alpha_n}$ and $c_{n,0}$. The latter can then be eliminated using $\sum_{k=0}^{n} c_{n,k} = 1/2$ which follows from $h_n(0) = 1$. This gives the claimed expression for $c_{n,k}$ in Equation (5.2). □

Remark 5. Using the last lemma, we can reduce the problem of finding the minimizers among the finite-length filters to a problem of minimizing the uncertainty product in terms of the unknown parameter γ_n. To this end, we insert the expression (5.2) for $c_{n,k}$ into Equation (5.4) and compute

$$\alpha_n = \pi \sum_{k=0}^{n} (2k+1)^2 |c_{n,k}|^2$$

and

$$\begin{aligned}\beta_n &= \pi \left(\sum_{k=0}^{n} |c_{\infty,k} - c_{n,k}|^2 + \sum_{k=n+1}^{\infty} |c_{\infty,k}|^2 \right) \\ &= \pi \left(\sum_{k=0}^{n} |c_{\infty,k} - c_{n,k}|^2 + \frac{1}{2} - \sum_{k=0}^{n} |c_{\infty,k}|^2 \right)\end{aligned}$$

in terms of the unknown γ_n. The minimization of the resulting uncertainty product $U(h_n) = (\alpha_n \beta_n)^{1/2}$ can then be accomplished numerically with standard software packages such as Mathematica. The results suggest that there is a unique minimizer for each n. However, at this time we have no analytical proof of uniqueness. We have listed the numerical values of the coefficients $c_{n,k}$ for h_n up to $n = 10$ in Table 1 and plotted the filters for $n \in \{1, 3, 5, 7, 9\}$ in Figure 1. Approximating the ideal filter by finite-length minimizers does not seem to give rise to a Gibbs-like phenomenon. Intuitively, this can be attributed to the presence of the factor $\|h'\|$ in $U(h)$, which imposes smoothness. Moreover, numerical evidence suggests that all the finite-length minimizers are decreasing on $[0, \pi]$. It would be nice to have a proof of this property.

The numerically constructed finite-length minimizers exhibit slow decay of $\|I - h_n\|$, see Figure 1, while $U(h_n)$ approaches $\frac{1}{2}$ rather rapidly according to Table 1. The following theorem calculates the asymptotics of $U(h_n)$, $\|I - h_n\|^2$, and $\|h_n'\|^2$.

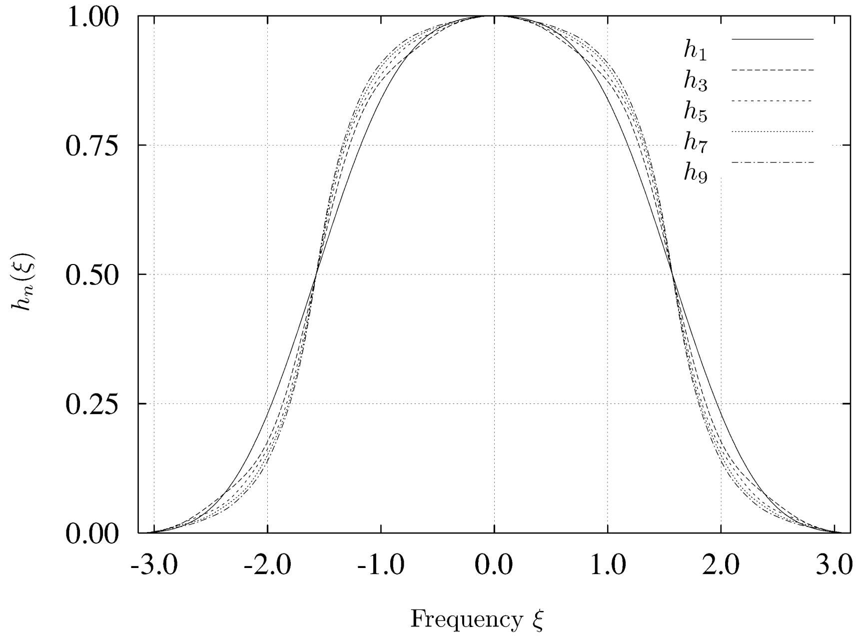

FIGURE 1 Finite-length minimizers for $n = 1, 3, 5, 7, 9$.

Theorem 5. *Let $n \geq 0$ and $h_n \in \mathcal{F}_n$ be chosen as a minimizer, $U(h_n) = \inf_{h \in \mathcal{F}_n} U(h)$. Then*

$$U(h_n) - \frac{1}{2} = \frac{9(\ln n)^3}{8\pi^4 n^3}(1 + o(1)) \,, \tag{5.8}$$

$$\int_{-\pi}^{\pi} |I(\xi) - h_n(\xi)|^2 \, d\xi = \frac{\pi}{6 \ln n}(1 + o(1)) \,, \tag{5.9}$$

and

$$\int_{-\pi}^{\pi} \left|h_n'(\xi)\right|^2 d\xi = \frac{3 \ln n}{2\pi}(1 + o(1)) \,. \tag{5.10}$$

The lemmas below prepare the proof of Theorem 5. They include the use of some sum formulas for series appearing in Appendix A.

The general strategy we pursue in these lemmas is controlling the decay of the unknown γ_n as the index n of the finite-length uncertainty minimizer h_n increases. The proof of Theorem 5 is a consequence of using the estimates for γ_n to bound α_n.

Lemma 2. *Let the sequence $\{h_n\}_{n \in \mathbb{N}}$ be chosen as in Theorem 5. For each $n \in \mathbb{N}$, we denote $\alpha_n := \int_{-\pi}^{\pi} |h_n'(\xi)|^2 \, d\xi$, $\beta_n := \int_{-\pi}^{\pi} |I(\xi) - h_n(\xi)|^2 \, d\xi$, $\gamma_n := (\beta_n/\alpha_n)^{1/2}$, and $\delta_n := ((n+1)\gamma_n)^{-1}$. Then*

$$\lim_{n \to \infty} \gamma_n = 0 \,, \tag{5.11}$$

but it converges sufficiently slowly such that

$$\lim_{n \to \infty} \delta_n = 0 \,. \tag{5.12}$$

TABLE 1

The Coefficients for the Finite-Length Minimizers up to $n = 10$

n	$U(h_n)$	$c_{n,0}$	$c_{n,1}$	$c_{n,2}$	$c_{n,3}$
0	0.528918	0.5			
1	0.517389	0.544726	-0.0447260		
2	0.505252	0.538903	-0.0616654	0.0227624	
3	0.503762	0.553877	-0.0728444	0.0261662	-0.00719880
4	0.501931	0.555770	-0.0798107	0.0269070	-0.00856115
5	0.501501	0.563675	-0.0868949	0.0293444	-0.00983698
6	0.500951	0.565695	-0.0910562	0.0301887	-0.0107038
7	0.500775	0.570723	-0.0960616	0.0320831	-0.0117058
8	0.500549	0.572397	-0.0989599	0.0328662	-0.0123470
9	0.500463	0.575938	-0.1027480	0.0344095	-0.0131691
10	0.500351	0.577304	-0.1049410	0.0351125	-0.0136816

n	$c_{n,4}$	$c_{n,5}$	$c_{n,6}$	$c_{n,7}$	$c_{n,8}$
4	0.00569485				
5	0.00625008	-0.00253760			
6	0.00639742	-0.00283263	0.00231151		
7	0.00685014	-0.00314338	0.00247207	-0.00121753	
8	0.00701549	-0.00335921	0.00252005	-0.00131591	0.00118328
9	0.00739941	-0.00361843	0.00265801	-0.00142683	0.00124577
10	0.00756222	-0.00379042	0.00271027	-0.00150460	0.00126632

n	$c_{n,9}$	$c_{n,10}$
9	-6.88330×10^{-4}	
10	-7.31237×10^{-4}	6.93547×10^{-4}

Proof. First, we show that if $U(h_n) = \inf_{h\in\mathcal{F}_n} U(h)$, then $\{h_n\}$ forms a minimizing sequence for U on $\mathcal{F}$. This is true because U is continuous on $\mathcal{F}$ equipped with the Sobolev norm and $\cup_n \mathcal{F}_n$ is dense in $\mathcal{F}$. So with the existence of a minimizing sequence in $\mathcal{F}$ there is one in $\cup_n \mathcal{F}_n$. Consequently, the sequence of finite-length minimizers gives

$$\lim_{n\to\infty} \alpha_n \beta_n = \frac{1}{4}\,. \tag{5.13}$$

Now (5.11) follows easily from the necessary condition $\beta_n \to 0$ for uncertainty minimizing sequences in Theorem 2 (ii) and the limit in (5.13).

We begin the proof of estimate (5.12) by showing a weaker version,

$$\limsup_{n\to\infty} \delta_n < \infty\,. \tag{5.14}$$

Suppose on the contrary that (5.14) is not true. Then there exists an increasing sequence $n_l, l \geq 1$, so that $\lim_{l\to\infty} \delta_{n_l} = +\infty$. Without loss of generality we may then assume that

$$\gamma_{n_l} \leq \frac{1}{n_l} \tag{5.15}$$

for all $l \geq 1$.

We recall that taking $\xi = 0$ in (5.1) gives $\sum_{k=0}^{n} c_{n,k} = \frac{1}{2}$ by $h_n \in \mathcal{F}_n$. Henceforth, we abbreviate

$$\tau_n := \big(1 + \gamma_n^2\big) c_{n,0} - c_{\infty,0} \,.$$

Inserting the expression for $c_{n,k}$ in (5.7) in this sum, we have

$$\left(\sum_{k=0}^{n} \frac{1}{1 + \gamma_n^2 (2k+1)^2}\right) \tau_n = \frac{1}{2} - \sum_{k=0}^{n} \frac{c_{\infty,k}}{1 + \gamma_n^2 (2k+1)^2}$$
$$= \sum_{k=0}^{\infty} \frac{\gamma_n^2 (2k+1)^2 c_{\infty,k}}{1 + \gamma_n^2 (2k+1)^2} + \sum_{k=n+1}^{\infty} \frac{c_{\infty,k}}{1 + \gamma_n^2 (2k+1)^2} \,. \tag{5.16}$$

The prefactor of τ_n in (5.16) is bounded below by $\sum_{k=0}^{n} \frac{1}{1+\gamma_n^2 (2n+1)^2}$. With the assumption (5.15) we can estimate the corresponding prefactor of τ_{n_l} by $\frac{n_l+1}{1+\gamma_{n_l}^2 (2n_l+1)^2} \geq (n_l + 1)(1 + \frac{(2n_l+1)^2}{n_l^2})^{-1} \geq n_l/10$. Using series summation formula (A.3) for the first term and inserting the explicit values of $c_{\infty,k} = 2(-1)^k/(2k+1)\pi$ then implies

$$|\tau_{n_l}| \leq \frac{20}{\pi n_l} \left(\left| \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_{n_l}^2 (2k+1)}{1 + \gamma_{n_l}^2 (2k+1)^2} \right| + \left| \sum_{k=n_l+1}^{\infty} \frac{(-1)^k}{(2k+1)\big(1 + \gamma_{n_l}^2 (2k+1)^2\big)} \right| \right)$$
$$\leq \frac{20}{\pi n_l} \left(\frac{\pi e^{-\pi/(2\gamma_{n_l})}}{2\big(1 + e^{-\pi/\gamma_{n_l}}\big)} + \frac{1}{(2n_l+3)\big(1 + \gamma_{n_l}^2 (2n_l+3)^2\big)} \right) .$$

Hereby, we have estimated the magnitude of the remaining alternating series by that of its first term. The exponential $e^{-\pi/(2\gamma_{n_l})}$ decreases faster than any polynomial as $\gamma_{n_l} \leq \frac{1}{n_l} \to 0$, so we conclude

$$|\tau_{n_l}| \leq C \left(\gamma_{n_l}^2 + \frac{1}{n_l^2} \right) \leq C/n_l^2 \,.$$

Here and hereafter, C denotes a positive absolute constant that may change from one inequality to the next. Using the bound on τ_{n_l} and the explicit values of $c_{\infty,k}$ in the expression for $c_{n,k}$ in (5.7) gives the inequality

$$|c_{n_l,k}| = \frac{|c_{\infty,k} + \tau_{n_l}|}{1 + \gamma_{n_l}^2 (2k+1)^2} \leq \frac{C}{2k+1} \quad \text{for all} \quad l \in \mathbb{N} \quad \text{and} \quad 0 \leq k \leq n_l \,.$$

Substituting the above estimate into $\alpha_{n_l} = \pi \sum_{k=0}^{n_l} (2k+1)^2 c_{n_l,k}^2$, we obtain

$$\alpha_{n_l} \leq C(n_l + 1) \,,$$

which by $\delta_{n_l} = \frac{\alpha_{n_l}}{(n_l+1)\sqrt{\alpha_{n_l}\beta_{n_l}}}$ and the limit in (5.13) contradicts the assumption that $\lim_{l\to\infty} \delta_{n_l} = +\infty$.

Now we prove (5.12). We first derive a more precise lower bound for the prefactor of τ_n in (5.16). By the monotonicity of the function $(1 + 4t^2)^{-1}$ on $\mathbb{R}^+$, we can use an integral comparison estimate

$$\int_{\gamma_n/2}^{(n+1/2)\gamma_n} \frac{dt}{1 + 4t^2} \geq \sum_{k=0}^{n} \frac{\gamma_n}{1 + \gamma_n^2 (2k+1)^2} \geq \int_{3\gamma_n/2}^{(n+3/2)\gamma_n} \frac{dt}{1 + 4t^2}$$

to obtain

$$\left|\sum_{k=0}^{n} \frac{1}{1+\gamma_n^2(2k+1)^2} - \frac{1}{2\gamma_n}\arctan((2n+1)\gamma_n) + \frac{1}{2\gamma_n}\arctan\gamma_n\right| \le 2\,. \tag{5.17}$$

Combining this estimate with $\gamma_n \to 0$ and the boundedness of δ_n, the inverse of the prefactor of τ_n in (5.16) is seen to be $O(\gamma_n)$. Applying series summation formula (A.3) in (5.16) and estimating the second alternating series as for the case of τ_{n_l}, we have

$$\begin{aligned} |\tau_n| &\le C\gamma_n\left(\frac{1}{2} - \sum_{k=0}^{n}\frac{c_{\infty,k}}{1+\gamma_n^2(2k+1)^2}\right) \\ &\le C\gamma_n\left(\left|\sum_{k=0}^{\infty}\frac{(-1)^k\gamma_n^2(2k+1)}{1+\gamma_n^2(2k+1)^2}\right| + \left|\sum_{k=n+1}^{\infty}\frac{c_{\infty,k}}{1+\gamma_n^2(2k+1)^2}\right|\right) \\ &\le C\gamma_n\left(\frac{e^{-\pi/(2\gamma_n)}}{1+e^{-\pi/\gamma_n}} + \frac{1}{(2n+3)+\gamma_n^2(2n+3)^3}\right). \end{aligned} \tag{5.18}$$

The second term observes

$$\frac{1}{2n+3+\gamma_n^2(2n+3)^3} = \frac{\gamma_n}{(\gamma_n(2n+3))^3\left(1+\frac{1}{\gamma_n^2(2n+3)^2}\right)} \le C\gamma_n\delta_n^3\,, \tag{5.19}$$

so we conclude

$$|\tau_n| \le C\gamma_n\left(e^{-\pi/(2\gamma_n)} + \gamma_n\delta_n^3\right)\,.$$

Inserting the bound for $|\tau_n|$ in $\alpha_n = \pi\sum_k(2k+1)^2|c_{n,k}|^2$, with the expression (5.7) for the coefficients $c_{n,k}$, gives

$$\begin{aligned} \alpha_n &= \pi\sum_{k=0}^{n}\frac{(2k+1)^2(c_{\infty,k}+\tau_n)^2}{\left(1+\gamma_n^2(2k+1)^2\right)^2} \\ &= O\left(\gamma_n^4\right)\sum_{k=0}^{n}\frac{(2k+1)^2}{\left(1+\gamma_n^2(2k+1)^2\right)^2} + O\left(\gamma_n^2\right)\left|\sum_{k=0}^{n}\frac{(-1)^k(2k+1)}{\left(1+\gamma_n^2(2k+1)^2\right)^2}\right| \\ &\quad + \frac{4}{\pi}\sum_{k=0}^{n}\frac{1}{\left(1+\gamma_n^2(2k+1)^2\right)^2}\,. \end{aligned} \tag{5.20}$$

All terms but the last one are $O(\gamma_n)$ by the usual integral comparison argument. The last term is estimated using the Taylor formula for the function $(1+4t^2)^{-2}$ and $t \in [k\gamma_n, (k+1)\gamma_n]$, $0 \le k \le n$, we know that there exists $\theta \in [k\gamma_n, (k+1)\gamma_n]$ such that

$$\begin{aligned} &\left|\frac{1}{\left(1+4t^2\right)^2} - \frac{1}{\left(1+\gamma_n^2(2k+1)^2\right)^2} + 16\Big(k+\frac{1}{2}\Big)\gamma_n\frac{(t-(k+1/2)\gamma_n)}{\left(1+\gamma_n^2(2k+1)^2\right)^3}\right| \\ &\quad \le \frac{16\left|20\theta^2-1\right|}{\left(1+4\theta^2\right)^4}(t-(k+1/2)\gamma_n)^2 \le C\gamma_n^2\left(1+4\theta^2\right)^{-3} \le C\gamma_n^2\left(1+4t^2\right)^{-3}, \end{aligned}$$

where replacing θ by t involves changing C by a factor that is independent of n. We obtain

$$\left|\sum_{k=0}^{n} \frac{\gamma_n}{\left(1+\gamma_n^2(2k+1)^2\right)^2} - \int_0^{(n+1)\gamma_n} \frac{dt}{\left(1+4t^2\right)^2}\right| \leq C\gamma_n^3 \int_0^{(n+1)\gamma_n} \frac{dt}{\left(1+4t^2\right)^3} \leq C\gamma_n^3 \,. \tag{5.21}$$

Inserting this estimate in (5.20) gives

$$\alpha_n = \frac{4}{\pi\gamma_n} \int_0^{(n+1)\gamma_n} \frac{dt}{\left(1+4t^2\right)^2} + O(\gamma_n) \,. \tag{5.22}$$

Multiplying both sides of the above estimate by $\pi\gamma_n/4$, and then taking limit, we have

$$\lim_{n\to\infty} \int_0^{(n+1)\gamma_n} \frac{dt}{\left(1+4t^2\right)^2} = \frac{\pi}{8} \,.$$

On the other hand,

$$\int_0^{\infty} \frac{dt}{\left(1+4t^2\right)^2} = \frac{1}{2}\int_0^{\pi/2} \cos^2\theta \, d\theta = \frac{\pi}{8} \,.$$

Comparing the upper limits of integration, (5.12) follows. □

We can now use the result $\gamma_n(n+1) \to \infty$ in a repetition of some of the preceding arguments to make the estimate of γ_n more precise.

Lemma 3. *Let the sequence $\{h_n\}_{n\in\mathbb{N}}$ be as in Theorem 5. Using the same notation as in the preceding lemma, we have the asymptotic estimate*

$$\gamma_n^{-1} = \frac{3\ln n}{\pi}(1+o(1)) \,. \tag{5.23}$$

Proof. We begin by showing that $\tau_n = (1+\gamma_n^2)c_{n,0} - c_{\infty,0}$ is given by

$$\tau_n = \frac{4}{\pi}\gamma_n e^{-\pi/(2\gamma_n)}(1+o(1)) + O\left(\gamma_n^2\delta_n^3\right) . \tag{5.24}$$

Revisiting the proof of the preceding theorem, we use $\gamma_n \to 0$ and $\gamma_n(n+1) \to \infty$ to deduce from (5.17) that

$$\sum_{k=0}^{n} \frac{\gamma_n}{1+\gamma_n^2(2k+1)^2} = \frac{1}{2}\arctan(2(n+1)\gamma_n) + O(\gamma_n) = \frac{\pi}{4}(1+o(1)) \,.$$

Therefore, we have instead of (5.18) the more precise expression

$$\tau_n = \frac{4\gamma_n}{\pi}(1+o(1))\left(\frac{e^{-\pi/(2\gamma_n)}}{1+e^{-\pi/\gamma_n}} + \frac{O(1)}{(2n+3)+\gamma_n^2(2n+3)^3}\right) . \tag{5.25}$$

The second term within the parentheses is $O(\gamma_n\delta_n^3)$ as demonstrated in (5.19) with the help of $\delta_n \to 0$. Further estimating $(1+e^{-\pi/\gamma_n})^{-1} = 1+o(1)$ completes the derivation of (5.24).

We proceed by showing

$$\gamma_n^2 e^{-\pi/\gamma_n} = \frac{1}{8\pi^2 n^3}(1+o(1)) . \tag{5.26}$$

Recall that

$$\alpha_n = \pi \sum_{k=0}^{n} (2k+1)^2 |c_{n,k}|^2 , \tag{5.27}$$

and

$$\beta_n = \pi \sum_{k=n+1}^{\infty} |c_{\infty,k}|^2 + \pi \sum_{k=0}^{n} |c_{\infty,k} - c_{n,k}|^2 . \tag{5.28}$$

Multiplying both sides of (5.27) with γ_n^2, subtracting (5.28), and using (5.7), we obtain

$$\begin{aligned} 0 &= \tau_n^2 \sum_{k=0}^{n} \frac{1-\gamma_n^2(2k+1)^2}{\left(1+\gamma_n^2(2k+1)^2\right)^2} - 4\tau_n \sum_{k=0}^{n} \frac{\gamma_n^2(2k+1)^2 c_{\infty,k}}{\left(1+\gamma_n^2(2k+1)^2\right)^2} \\ &\quad + \sum_{k=0}^{\infty} \frac{\gamma_n^4(2k+1)^4 c_{\infty,k}^2 - \gamma_n^2(2k+1)^2 c_{\infty,k}^2}{\left(1+\gamma_n^2(2k+1)^2\right)^2} \\ &\quad + \sum_{k=n+1}^{\infty} \frac{c_{\infty,k}^2 + 3\gamma_n^2(2k+1)^2 c_{\infty,k}^2}{\left(1+\gamma_n^2(2k+1)^2\right)^2} \\ &=: S_1 + S_2 + S_3 + S_4 . \end{aligned} \tag{5.29}$$

To bound S_1, we note that $\gamma_n \sum_{k=0}^{n} \frac{1-\gamma_n^2(2k+1)^2}{(1+\gamma_n^2(2k+1)^2)^2}$ is the Riemann sum approximation of the integral $\int_0^{(n+1)\gamma_n} (1-4t^2)(1+4t^2)^{-2}\, dt$ with the nodes at $(k+1/2)\gamma_n, 0 \le k \le n$. The bound $\gamma_n \sum_{k=n+1}^{\infty} \frac{1-\gamma_n^2(2k+1)^2}{(1+\gamma_n^2(2k+1)^2)^2} \le \sum_{k=n+1}^{\infty} (1+\gamma_n^2(2k+1)^2)^{-1} \le \int_{\gamma_n(n+1/2)}^{\infty} (1+4t^2)^{-1}\, dt$, with $(n+1)\gamma_n \to \infty$ by Lemma 3 and $\int_0^{\infty} (1-4t^2)(1+4t^2)^{-2}\, dt = 0$ then gives

$$\sum_{k=0}^{n} \frac{1-\gamma_n^2(2k+1)^2}{\left(1+\gamma_n^2(2k+1)^2\right)^2} = \gamma_n^{-1} \left(\int_0^{\infty} \frac{1-4t^2}{\left(1+4t^2\right)^2}\, dt + o(1) \right) = o\big(\gamma_n^{-1}\big) . \tag{5.30}$$

Together with the refined estimate of τ_n in (5.24), this yields

$$\begin{aligned} S_1 &= \Big(O\big(\gamma_n^2 e^{-\pi/\gamma_n}\big) + O\big(\gamma_n^3 \delta_n^3 e^{-\pi/2\gamma_n}\big) + O\big(\gamma_n^4 \delta_n^6\big) \Big) o\big(\gamma_n^{-1}\big) \\ &= o\big(\gamma_n e^{-\pi/\gamma_n}\big) + o\big(\gamma_n \delta_n^3\big) . \end{aligned} \tag{5.31}$$

Before combining the Landau symbols, we have dropped powers of γ_n and δ_n that are not needed in the final estimate, as well as the bounded term $e^{-\pi/2\gamma_n}$.

Inserting the value of $c_{\infty,k}$ in S_2, we may write this sum as a series of type (A.4) with a remainder, and then estimate the alternating series remainder by its first term, since $t \mapsto \frac{2t}{(1+4t^2)^2}$ is decreasing on $t > 1/\sqrt{12}$ and we know $\gamma_n(n+1) \to \infty$. In so doing we have

$$\begin{aligned} S_2 &= -4\tau_n \left(\left(\sum_{k=0}^{\infty} - \sum_{k=n+1}^{\infty} \right) \frac{\gamma_n^2 (2k+1)^2 c_{\infty,k}}{\left(1+\gamma_n^2(2k+1)^2\right)^2} \right) \\ &= -4\Big(\frac{4}{\pi}\gamma_n e^{-\pi/(2\gamma_n)}(1+o(1)) + O\big(\gamma_n\delta_n^3\big)\Big) \\ &\quad \times \Big(\frac{\pi e^{-\pi/(2\gamma_n)}}{4\gamma_n}(1+o(1)) + O\big(\gamma_n\delta_n^3\big)\Big) \\ &= -4e^{-\pi/\gamma_n}(1+o(1)) + o\big(\gamma_n\delta_n^3\big)\,. \end{aligned} \tag{5.32}$$

Using the summation formula (A.2) for the third term gives

$$S_3 = \frac{4\gamma_n^2}{\pi^2} \sum_{k=0}^{\infty} \frac{\gamma_n^2(2k+1)^2 - 1}{\left(1+\gamma_n^2(2k+1)^2\right)^2} = 2e^{-\pi/\gamma_n}(1+o(1))\,. \tag{5.33}$$

Moreover,

$$\begin{aligned} S_4 &= \frac{4}{\pi^2} \sum_{k=n+1}^{\infty} \frac{1+3\gamma_n^2(2k+1)^2}{\left(1+\gamma_n^2(2k+1)^2\right)^2 (2k+1)^2} \\ &= (1+o(1))\frac{12}{\pi^2\gamma_n^2} \sum_{k=n+1}^{\infty} (2k+1)^{-4} = \frac{\gamma_n\delta_n^3}{4\pi^2}(1+o(1))\,. \end{aligned} \tag{5.34}$$

Since the terms (5.31), (5.32), (5.33), and (5.34) sum to zero according to (5.29), we conclude

$$-2e^{-\pi/\gamma_n}(1+o(1)) + \frac{\gamma_n\delta_n^3}{4\pi^2}(1+o(1)) = 0\,. \tag{5.35}$$

Then (5.26) follows because by definition $\gamma_n^3\delta_n^3 = \frac{1}{(n+1)^3}$.

Now writing the exponent on the left-hand side as $(2\gamma_n \ln\gamma_n - \pi)/\gamma_n$ and using $\gamma_n \to 0$ yields the claimed estimate

$$\gamma_n^{-1} = \frac{3\ln n}{\pi}(1+o(1))\,. \qquad \square \tag{5.36}$$

Now we are ready to prove Theorem 5.

Proof of Theorem 5. We recall that Lemma 3 states

$$\gamma_n^{-1} = \frac{3\ln n}{\pi}(1+o(1))\,. \tag{5.37}$$

The asymptotics (5.9) and (5.10) follow from $\alpha_n\beta_n \to \frac{1}{4}$ and (5.23). It remains to prove (5.8). By (5.1) and $\alpha_n = \pi \sum_{k=0}^n (2k+1)^2 |c_{n,k}|^2$, we have

$$\begin{aligned}
\alpha_n &= \pi \sum_{k=0}^{n} \frac{(2k+1)^2 |c_{\infty,k} + \tau_n|^2}{\left(1+\gamma_n^2(2k+1)^2\right)^2} \\
&= \frac{4}{\pi} \sum_{k=0}^{n} \frac{1}{\left(1+\gamma_n^2(2k+1)^2\right)^2} + 4\tau_n \sum_{k=0}^{n} \frac{(-1)^k(2k+1)}{\left(1+\gamma_n^2(2k+1)^2\right)^2} \\
&\quad + \pi \tau_n^2 \sum_{k=0}^{n} \frac{(2k+1)^2}{\left(1+\gamma_n^2(2k+1)^2\right)^2} \\
&= S_1' + S_2' + S_3' . \qquad (5.38)
\end{aligned}$$

Eliminating $\gamma_n^2 \delta_n^3$ in (5.24) with (5.35) gives

$$\tau_n = \frac{4}{\pi} \gamma_n e^{-\pi/(2\gamma_n)} (1 + o(1)) . \qquad (5.39)$$

Writing S_1' as a sum of series appearing in (A.1) and (A.2) with a remainder, and estimating with the help of (5.26), we get

$$\begin{aligned}
S_1' &= \frac{4}{\pi} \left(\frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{1-(2k+1)^2\gamma_n^2}{\left(1+(2k+1)^2\gamma_n^2\right)^2} + \frac{1}{1+\gamma_n^2(2k+1)^2} \right) \right. \\
&\quad \left. - \sum_{k=n+1}^{\infty} \frac{1}{\left(1+\gamma_n^2(2k+1)^2\right)^2} \right) \\
&= \frac{4}{\pi} \left\{ \frac{\pi}{8\gamma_n} - \frac{\pi^2 e^{-\pi/\gamma_n}}{4\gamma_n^2}(1+o(1)) - \sum_{k=n+1}^{\infty} 2^{-4}\gamma_n^{-4}(k+1/2)^{-4}(1+o(1)) \right\} \\
&= \frac{4}{\pi} \left\{ \frac{\pi}{8\gamma_n} - \frac{\pi^2 e^{-\pi/\gamma_n}}{4\gamma_n^2}(1+o(1)) - \frac{1}{48 n^3 \gamma_n^4}(1+o(1)) \right\} \\
&= \frac{4}{\pi} \left\{ \frac{\pi}{8\gamma_n} - \frac{\pi^2 e^{-\pi/\gamma_n}}{4\gamma_n^2}(1+o(1)) - \frac{\pi^2 e^{-\pi/\gamma_n}}{6\gamma_n^2}(1+o(1)) \right\} \\
&= \frac{1}{2\gamma_n} - \frac{5\pi e^{-\pi/\gamma_n}}{3\gamma_n^2}(1+o(1)) . \qquad (5.40)
\end{aligned}$$

Using the series formula (A.4) as for (5.32) with the estimate (5.39) for τ_n, we obtain

$$\begin{aligned}
S_2' &= 4\Big(\frac{4}{\pi}\gamma_n e^{-\pi/(2\gamma_n)}\Big) \times \left(\left(\sum_{k=0}^{\infty} - \sum_{k=n+1}^{\infty} \right) \frac{(-1)^k(2k+1)}{\left(1+\gamma_n^2(2k+1)^2\right)^2} \right) (1+o(1)) \\
&= \frac{16}{\pi}\gamma_n e^{-\pi/(2\gamma_n)} \left(\frac{\pi^2 e^{-\pi/(2\gamma_n)}}{8\gamma_n^3}(1+o(1)) + O\Big(\frac{1}{n^3\gamma_n^4}\Big) \right) \\
&= 2\pi \gamma_n^{-2} e^{-\pi/\gamma_n} (1+o(1)) . \qquad (5.41)
\end{aligned}$$

To bound S_3' we apply again a Riemann sum argument analogous to (5.30),

$$\sum_{k=0}^{n} \frac{(2k+1)^2}{\left(1+\gamma_n^2(2k+1)^2\right)^2} = \gamma_n^{-3} \Big(\int_0^\infty \frac{4t^2}{(1+4t^2)^2}\, dt + o(1) \Big) = O\big(\gamma_n^{-3}\big) ,$$

which yields together with (5.39) that

$$S_3' = O\left(\gamma_n^{-1} e^{-\pi/\gamma_n}\right) = o\left(\gamma_n^{-2} e^{-\pi/\gamma_n}\right) . \tag{5.42}$$

Combining the estimates for (5.42), (5.41), and (5.40) in (5.38), we get

$$\alpha_n = \frac{1}{2\gamma_n} + \frac{\pi e^{-\pi/\gamma_n}}{3\gamma_n^2}(1 + o(1)) .$$

This, together with the asymptotics of γ_n in (5.26) and (5.23), gives

$$\begin{aligned} \alpha_n \beta_n &= \alpha_n^2 \gamma_n^2 = \frac{1}{4} + \frac{\pi e^{-\pi/\gamma_n}}{3\gamma_n}(1 + o(1)) \\ &= \frac{1}{4} + \frac{1}{24\pi n^3 \gamma_n^3}(1 + o(1)) \\ &= \frac{1}{4} + \frac{9(\ln n)^3}{8\pi^4 n^3}(1 + o(1)) . \end{aligned}$$

This proves the remaining estimate in Theorem 5,

$$U(h_n) = (\alpha_n \beta_n)^{1/2} = \frac{1}{2} + \frac{9(\ln n)^3}{8\pi^4 n^3}(1 + o(1)) . \qquad \square \tag{5.43}$$

6. Explicit Construction of Finite-Length Low-Pass Filters with Optimal Uncertainty Asymptotics

In this section, we consider a sequence of interpolatory, symmetric finite-length filters that behave asymptotically like the optimal finite-length approximation of the ideal low-pass filter in the sense of the uncertainty product. Let $\gamma_n^\star$ be given by

$$\left(\gamma_n^\star\right)^2 e^{-\pi/\gamma_n^\star} = \frac{1}{8\pi n^3} , \tag{6.1}$$

and define

$$\tau_n^\star := \left(\sum_{k=0}^{n} \frac{1}{1 + \left(\gamma_n^\star\right)^2 (2k+1)^2} \right)^{-1} \left(\frac{1}{2} - \sum_{k=0}^{n} \frac{c_{\infty,k}}{1 + \left(\gamma_n^\star\right)^2 (2k+1)^2} \right) . \tag{6.2}$$

These quantities are defined for all $n \in \mathbb{N}$ because of the monotonicity of the function $t^{-2} e^{-\pi t}$ in $\{t \geq 1\}$. We note that $\gamma_n^\star \to 0$ as $n \to \infty$. Moreover, taking the logarithm on both sides of (6.1) yields as in (5.23) that

$$\frac{1}{\gamma_n^\star} = \frac{3 \ln n}{\pi}(1 + o(1)) , \tag{6.3}$$

so $\delta_n^\star := ((n+1)\gamma_n^\star)^{-1} = o(1)$, similarly as before. Let

$$h_n^\star(\xi) := \frac{1}{2} + \sum_{k=0}^{n} \frac{c_{\infty,k} + \tau_n^\star}{1 + \left(\gamma_n^\star\right)^2 (2k+1)^2} \cos(2k+1)\xi . \tag{6.4}$$

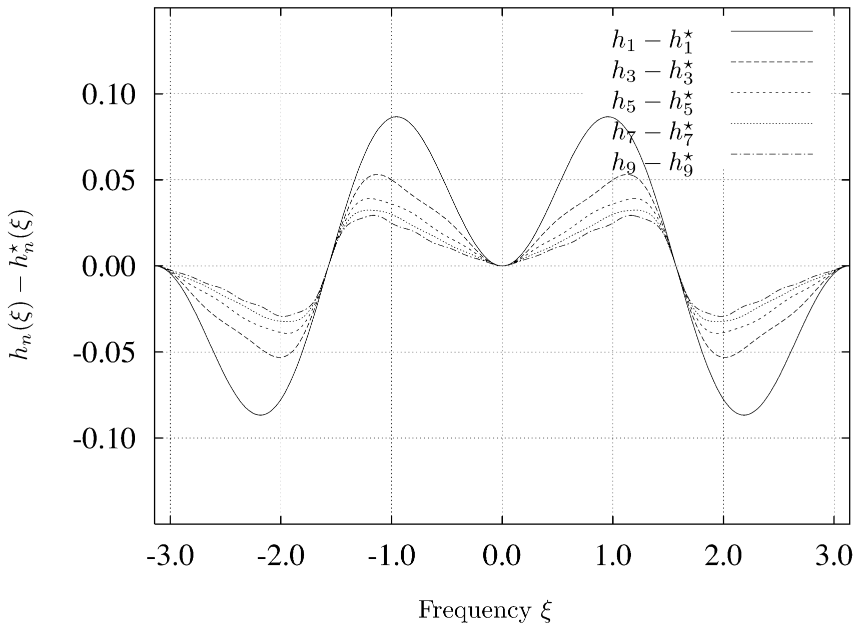

FIGURE 2 Difference between h_n and $h_n^\star$.

Clearly $h_n^\star$ belongs to $\mathcal{F}_n$ and is a symmetric interpolatory low-pass filter. Moreover, $h_n^\star$ is close to the optimal implementable low-pass filter in $\mathcal{F}_n$ in the sense of the following theorem. We have plotted the difference between h_n and $h_n^\star$ for $n = 1, 3, 5, 7, 9$ in Figure 2.

Theorem 6. *Let $\{h_n\}_{n\in\mathbb{N}}$ be the sequence of finite-length minimizers and $\{h_n^\star\}_{n\in\mathbb{N}}$ be defined as in* (6.4)*. Then both sequences have the same asymptotic behavior,*

$$\lim_{n\to\infty} \frac{U(h_n^\star) - 1/2}{\inf_{h\in\mathcal{F}_n} U(h) - 1/2} = 1\,. \tag{6.5}$$

Proof. By setting $\xi = 0$ in Equation (6.4), we have that

$$\tau_n^\star = \big(1 + (\gamma_n^\star)^2\big)c_{n,0}^\star - c_{\infty,0}\,.$$

Similar to the proof of the estimate (5.26), we deduce that the sequence $\{\tau_n^\star\}$ has the asymptotics

$$\tau_n^\star = \frac{4}{\pi}\gamma_n^\star e^{-\pi/2\gamma_n^\star}(1 + o(1))\,. \tag{6.6}$$

Set $\alpha_n^\star := \int_{-\pi}^{\pi} |h_n^{\star\prime}(\xi)|^2\, d\xi$, $\beta_n^\star := \int_{-\pi}^{\pi} |I(\xi) - h_n^\star(\xi)|^2\, d\xi$. By expressing $c_{n,k}^\star$ in terms of $\gamma_n^\star$ and $\tau_n^\star$, we obtain

$$\alpha_n^\star = \pi \sum_{k=0}^{n} (2k+1)^2 \frac{\big(c_{\infty,k} + \tau_n^\star\big)^2}{\big(1 + (\gamma_n^\star)^2 (2k+1)^2\big)^2}\,, \tag{6.7}$$

and

$$\beta_n^\star = \pi \sum_{k=n+1}^{\infty} |c_{\infty,k}|^2 + \pi \sum_{k=0}^{n} \frac{\big((2k+1)^2 \gamma_n^{\star 2} c_{\infty,k} - \tau_n^\star\big)^2}{\big(1 + (\gamma_n^\star)^2 (2k+1)^2\big)^2}\,. \tag{6.8}$$

Replacing γ_n in (5.38) by $\gamma_n^\star$ in (6.1), and using the estimate (6.6) for $\tau_n^\star$ instead of (5.39) for τ_n, we get

$$\alpha_n^\star = \frac{1}{2\gamma_n^\star} + \frac{\pi e^{-\pi/\gamma_n^\star}}{3\gamma_n^{\star 2}}(1+o(1)) \,. \tag{6.9}$$

Multiplying both sides of (6.7) with $\gamma_n^{\star 2}$ and then subtracting (6.8), we obtain

$$\begin{aligned}\beta_n^\star - \alpha_n^\star \gamma_n^{\star 2} \quad = \quad & \tau_n^{\star 2} \sum_{k=0}^{n} \frac{1-\gamma_n^{\star 2}(2k+1)^2}{\left(1+\gamma_n^{\star 2}(2k+1)^2\right)^2} - 4\tau_n \sum_{k=0}^{n} \frac{\gamma_n^{\star 2}(2k+1)^2 c_{\infty,k}}{\left(1+\gamma_n^{\star 2}(2k+1)^2\right)^2} \\ & + \sum_{k=0}^{\infty} \frac{\gamma_n^{\star 4}(2k+1)^4 c_{\infty,k}^2 - \gamma_n^{\star 2}(2k+1)^2 c_{\infty,k}^2}{\left(1+\gamma_n^{\star 2}(2k+1)^2\right)^2} \\ & + \sum_{k=n+1}^{\infty} \frac{c_{\infty,k}^2 + 3\gamma_n^{\star 2}(2k+1)^2 c_{\infty,k}^2}{\left(1+\gamma_n^{\star 2}(2k+1)^2\right)^2} \,. \end{aligned} \tag{6.10}$$

Similar to the estimates in (5.31), (5.32), (5.33), and (5.34), we then have

$$\beta_n^\star - \alpha_n^\star \gamma_n^{\star 2} = -2e^{-\pi/\gamma_n^\star}(1+o(1)) + \frac{1}{4\pi^2\gamma_n^{\star 2} n^3}(1+o(1)) = o\big(e^{-\pi/\gamma_n^\star}\big) \,. \tag{6.11}$$

Combining the estimates in (6.9) and (6.11) yields

$$\begin{aligned}\alpha_n^\star \beta_n^\star \quad &= \quad \alpha_n^\star \Big(\alpha_n^\star \gamma_n^{\star 2} + o\big(e^{-\pi/\gamma_n^\star}\big)\Big) \\ &= \quad \alpha_n^{\star 2} \gamma_n^{\star 2} + o\Big(\big(\gamma_n^\star\big)^{-1} e^{-\pi/\gamma_n^\star}\Big) \\ &= \quad \Big(\frac{1}{2} + \frac{\pi e^{-\pi/\gamma_n^\star}}{3\gamma_n^\star}(1+o(1))\Big)^2 + o\Big(\big(\gamma_n^\star\big)^{-1} e^{-\pi/\gamma_n^\star}\Big) \\ &= \quad \frac{1}{4} + \frac{\pi e^{-\pi/\gamma_n^\star}}{3\gamma_n^\star}(1+o(1)) \,. \end{aligned}$$

Thus,

$$U\big(h_n^\star\big) - \frac{1}{2} = \big(\alpha_n^\star \beta_n^\star\big)^{1/2} - \frac{1}{2} = \frac{\pi e^{-\pi/\gamma_n^\star}}{3\gamma_n^\star}(1+o(1)) \,. \tag{6.12}$$

Therefore the estimate in (6.5) follows from applying l'Hôpital's rule to the quotient of (6.12) and (5.8), together with the asymptotics of $\gamma_n^\star$ according to (6.3). □

7. Conclusion

In this work, we derived a lower bound for the cost functional U that evaluates the efficiency of digital low-pass filters. The motivation for this cost functional was to search for an optimal compromise in the trade-off between the frequency selectivity of a filter in comparison with the ideal filter and the effective length of a filter when implemented in the time domain. The fact that in the affine space of implementable low-pass filters, U never assumes its infimum lead to the study of minimizing sequences for U. The last part of the article was dedicated to filters that minimize U for a given maximal filter length, and to the construction of a sequence that has the same asymptotics as the sequence of finite-length minimizers. Perhaps the most surprising part of our results is that this sequence of finite-length minimizers converges only very slowly to the ideal filter in the mean-square norm of the frequency interval. Unfortunately, we cannot offer any intuitive reason for this behavior, other than that the cost functional U imposes smoothness on the finite-length minimizers. A related phenomenon is the absence of Gibbs-like oscillations in the plots of the minimizers.

We conclude by remarking that generalizations of the uncertainty principle to band-pass filters with several disjoint pass bands are straightforward. In addition, one may adjust the amplification factor of each pass band separately. All that is needed is the requirement that the filter given by an absolutely continuous function with square-integrable derivative attains the desired value in each pass and stop band at least once. The resulting uncertainty bound is then half of the ℓ^2-norm of the sequence of amplification factors. Another generalization of the uncertainty inequality presented here would be to use an L^p-norm instead of the mean-square deviation $\|h - I\|$ and replace the Cauchy-Schwarz inequality by Hölder's in the proof of the lower bound for U. However, this requires using the L^q-norm of h' with the index q conjugate to p, and in order to relate this norm to localization properties of the filter in the time domain one needs the Hausdorff-Young inequality which only applies when $q \geq 2$ or equivalently $1 \leq p \leq 2$. Another version of the uncertainty inequality can be obtained by replacing the interval $[-\pi, \pi)$ with the real line and reformulating the limiting conditions for low-pass filters in the analog domain. Finally, one may investigate in which sense this uncertainty inequality generalizes to low-pass filters in higher dimensions.

Appendix

A. Series Summation Formulas

Lemma A.1. *Let $s > 0$. Then*

$$\sum_{k=0}^{\infty} \frac{1}{1+(2k+1)^2 s^2} = \frac{\pi}{4s} - \frac{\pi e^{-\pi/s}}{2s\left(1+e^{-\pi/s}\right)}, \tag{A.1}$$

$$\sum_{k=0}^{\infty} \frac{(2k+1)^2 s^2 - 1}{\left(1+(2k+1)^2 s^2\right)^2} = \frac{\pi^2 e^{-\pi/s}}{2s^2\left(1+e^{-\pi/s}\right)^2}, \tag{A.2}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k (2k+1) s^2}{1+(2k+1)^2 s^2} = \frac{\pi e^{-\pi/(2s)}}{2\left(1+e^{-\pi/s}\right)}, \tag{A.3}$$

and

$$\sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)}{\left(1+(2k+1)^2 s^2\right)^2} = \frac{\pi^2 e^{-\pi/(2s)}\left(1-e^{-\pi/s}\right)}{8s^3\left(1+e^{-\pi/s}\right)^2}. \tag{A.4}$$

Proof. First, we show these series summation formulas for fixed $s > 0$. By the elementary identity $(1+a^2)^{-1} = \int_0^\infty \sin t e^{-at}\, dt$ valid for all $a > 0$, we have

$$\sum_{k=0}^\infty \frac{1}{1+(2k+1)^2 s^2} = \sum_{k=0}^\infty \int_0^\infty \sin t e^{-(2k+1)st}\, dt$$
$$= \int_0^\infty \frac{e^{-st}\sin t}{1-e^{-2st}}\, dt = \frac{1}{2si} \lim_{\epsilon\to 0+} \int_{|t|\ge\epsilon} \frac{e^{-t(1-i/s)}}{1-e^{-2t}}\, dt\ . \tag{A.5}$$

For any fixed $1 \le L \in \mathbb{Z}$,

$$\int_0^{(L+1/2)\pi} \left|\frac{e^{-(K+it)(1-i/s)}}{1-e^{-2(K+it)}}\right| + \left|\frac{e^{-(-K+it)(1-i/s)}}{1-e^{-2(-K+it)}}\right| dt$$
$$\le 4\int_0^{(L+1/2)\pi} e^{-K-t/s}\, dt \to 0 \tag{A.6}$$

as $K \to +\infty$, and

$$\int_{\mathbb{R}} \left|\frac{e^{-(t+i(L+1/2)\pi)(1-i/s)}}{1-e^{-2(t+i(L+1/2)\pi)}}\right| dt = e^{-(L+1/2)\pi/s} \int_{\mathbb{R}} \left|\frac{e^{-t}}{1+e^{-2t}}\right| dt \to 0 \tag{A.7}$$

as the integer L tends to positive infinity. Combining the above two limits and the fact that the function $e^{-z(1-i/s)}(1-e^{-2z})^{-1}$ is an analytic function on the whole complex plane except at $\pi i\mathbb{Z}$, we obtain

$$\lim_{\epsilon\to 0+} \int_{|t|\ge\epsilon} \frac{e^{-t(1-i/s)}}{1-e^{-2t}}\, dt = \lim_{\epsilon\to 0+} \int_{|z|=\epsilon, \mathrm{Re} z>0} \frac{e^{-z(1-i/s)}}{1-e^{-2z}}\, dz$$
$$+ \sum_{l=1}^\infty \lim_{\epsilon\to 0+} \int_{|z-l\pi i|=\epsilon} \frac{e^{-z(1-i/s)}}{1-e^{-2z}}\, dz$$
$$= \frac{\pi i}{2} + \pi i \sum_{l=1}^\infty (-1)^l e^{-l\pi/s} = \frac{\pi i}{2} - \frac{\pi i e^{-\pi/s}}{1+e^{-\pi/s}}\ . \tag{A.8}$$

Therefore (A.1) follows from (A.5) and (A.8). Similar to the proof of the assertion (A.1), we have

$$\sum_{k=0}^\infty \frac{(2k+1)^2 s^2 - 1}{\left(1+(2k+1)^2 s^2\right)^2} = \sum_{k=0}^\infty \int_0^\infty t \cos t e^{-(2k+1)st}\, dt$$
$$= \frac{1}{2s^2} \int_{\mathbb{R}} \frac{t e^{-(1-i/s)t}}{1-e^{-2t}}\, dt = -\frac{\pi^2}{2s^2} \sum_{l=1}^\infty l(-1)^l e^{-l\pi/s} = \frac{\pi^2 e^{-\pi/s}}{2s^2\left(1+e^{-\pi/s}\right)^2}\ ,$$

and (A.2) follows.

Using the identity $\frac{a}{1+a^2} = \int_0^\infty \cos t e^{-at}\, dt$ for $a > 0$, we have

$$\sum_{k=0}^\infty \frac{(-1)^k (2k+1)s^2}{1+(2k+1)^2 s^2} = \sum_{k=0}^\infty (-1)^k \int_0^\infty s \cos t e^{-(2k+1)st}\, dt$$
$$= \frac{1}{2} \int_{\mathbb{R}} \frac{e^{-(1-i/s)t}}{1+e^{-2t}}\, dt\ .$$

One may easily verify that the function $f(z) := e^{-(1-i/s)z}(1+e^{-2z})^{-1}$ is analytic on the whole complex plane except at $\pi i(\mathbb{Z}+1/2)$, that the integral $\int_{|\mathrm{Re} z|=K, 0\le \mathrm{Im} z\le L\pi} |f(z)||dz|$ has zero limit as $K\to+\infty$ for any positive constant, and that $\int_{\mathrm{Im} z=L\pi} |f(z)||dz|$ has zero limit as the positive integer L tends to infinity. Therefore

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}} \frac{e^{-(1-i/s)t}}{1+e^{-2t}}\,dt &= \frac{1}{2}\sum_{l=0}^{\infty} \lim_{\epsilon\to 0+} \int_{|z-(l+1/2)\pi i|=\epsilon} \frac{s^2 e^{-(1-i/s)z}}{1+e^{-2z}}\,dz \\ &= \frac{\pi}{2}\sum_{l=0}^{\infty} (-1)^l e^{-(l+1/2)\pi/s} = \frac{\pi e^{-\pi/(2s)}}{2\big(1+e^{-\pi/s}\big)},\end{aligned}$$

which yields the identity (A.3).

Analogous to the proof of the identity (A.3), we obtain (A.4) by

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)}{\big(1+(2k+1)^2 s^2\big)^2} = \frac{1}{2s}\sum_{k=0}^{\infty}(-1)^k \int_0^{\infty} t\sin t e^{-(2k+1)st}\,dt \\ &= \frac{1}{4is^3}\int_{\mathbb{R}} \frac{te^{-(1-i/s)t}}{1+e^{-2t}}\,dt = \frac{1}{4is^3}\sum_{l=0}^{\infty} \lim_{\epsilon\to 0+}\int_{|z-(l+1/2)\pi i|=\epsilon} \frac{ze^{-(1-i/s)z}}{1+e^{-2z}}\,dz \\ &= \frac{\pi^2}{4s^3}\sum_{l=0}^{\infty}(-1)^l\Big(l+\frac{1}{2}\Big)e^{-(l+1/2)\pi/s} = \frac{\pi^2 e^{-\pi/(2s)}\big(1-e^{-\pi/s}\big)}{8s^3\big(1+e^{-\pi/s}\big)^2}.\end{aligned}$$

□

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References

[1] Benedetto, J. J. (1994). Frame decompositions, sampling, and uncertainty principle inequalities, in *Wavelets: Mathematics and Applications,* CRC, Boca Raton, FL, 247–304.

[2] Bodmann, B. G., Hoffman, D. K., Kouri, D. J., and Papadakis, M. (2006). Hermite distributed approximating functionals as almost-ideal low-pass filters, *Sampl. Theory Image Signal Process.* to appear.

[3] Burrus, C. S. (1995). Multiband least squares FIR filter design, *IEEE Trans. Signal Process.* **43**, 412–421.

[4] Daubechies, I. (1988). Orthonormal bases of compactly supported wavelets, *Comm. Pure Appl. Math.* **41**, 909–999.

[5] Daubechies, I. (1992). *Ten Lectures on Wavelets,* SIAM, Philadelphia.

[6] Donoho, D. L. and Stark, P. B. (1989). Uncertainty principles and signal recovery, *SIAM J. Appl. Math.* **49**, 906–931.

[7] Folland, G. B. and Sitaram, A. (1997). The uncertainty principle: A mathematical survey, *J. Fourier Anal. Appl.* **3**(3), 207–238.

[8] Fan, A. and Sun, Q. (2001). Regularity of Butterworth refinable functions, *Asian J. Math.* **5**, 433–440.

[9] Gröchenig, K. (2001). *Foundations of Time-Frequency Analysis,* Birkhäuser, Boston.

[10] Gilbarg, D. and Trudinger, N. S. (1983). *Elliptic Partial Differential Equations of Second Order,* 2nd ed., Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), **224**, Springer-Verlag, Berlin.

[11] Hoffman, D. K. and Kouri, D. J. (2002). Hierarchy of local minimum solutions of Heisenberg's uncertainty principle, *Phys. Rev. A* **65**, 052106(1–13).

[12] Kouri, D. J., Papadakis, M., Kakadiaris, I. A., and Hoffman, D. K. (2003). Properties of minimum uncertainty wavelets and their relations to the harmonic oscillator and the coherent states, *J. Phys. Chem. A* **107**, 7318–7327.

[13] Lieb, E. H. and Loss, M. (2001). *Analysis,* 2nd ed., Graduate Studies in Mathematics, **14**, American Mathematical Society, Providence, RI.

[14] Meyer, Y. (1992). *Wavelets and Operators,* Cambridge University Press, Cambridge, UK.

[15] Micchelli, C. A. and Sun, Q. (2006). Interpolating filters with prescribed zeros and their refinable functions, *Comm. Pure Appl. Anal.* to appear.

[16] Oppenheim, A. V. and Schafer, R. W. (1989). *Discrete-Time Signal Processing,* Prentice-Hall, Englewood Cliffs, NJ.

[17] Selesnick, I. W. (1998). Formulas for orthogonal IIR wavelet filters, *IEEE Trans. Signal Process.* **46**, 1138–1141.

[18] Selesnick, I. W., Lang, M., and Burrus, C. S. (1995). Constrained least square design of FIR filters without specified transition bands, *IEEE Proc. Int. Conf. Acoust. Speech, Signal Processing* (Detroit, MI), **2**, 1260–1263.

[19] Spurbeck, M. S. and Mullis, C. T. (1998). Least squares approximation of perfect reconstruction filter banks, *IEEE Trans. Signal Process.* **46**, 968–978.

[20] Tufts, D. W. and Francis, J. T. (1970). Designing digital lowpass filters: Comparison of some methods and criteria, *IEEE Trans. Audio Electroacoust.* **AU-18**, 487–494.

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