WIENER'S LEMMA FOR LOCALIZED INTEGRAL OPERATORS

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Abstract. In this paper, we introduce two classes of localized integral operators on $L^2(\mathbb{R}^d)$ with the Wiener class $\mathcal{W}$ and the Kurbatov class $\mathcal{K}$ of integral operators as their models. We show that those two classes of localized integral operators are pseudo-inverse closed non-unital subalgebra of $\mathcal{B}^2$, the Banach algebra of all bounded operators on $L^2(\mathbb{R}^d)$ with usual operator norm.

1. Introduction

In this companion paper of [46], we introduce two classes $\mathcal{W}_{p,u}^\alpha$ and $\mathcal{C}_{p,u}^\alpha$ of localized integral operators

$$Tf(x)=\int_{\mathbb{R}^d} K_T(x,y)f(y)dy,\quad f\in L^2:=L^2(\mathbb{R}^d) \tag{1.1}$$

on L^2 (see (2.6) and (4.2) for definitions), and we establish the Wiener's lemma for those two subalgebras of $\mathcal{B}^2$, the Banach algebra of all bounded operators on L^2 with usual operator norm (see Theorems 3.1, 3.5, 4.1 and 4.3 for details). The application of the above Wiener's lemma to the study of stable sampling and reconstruction procedure in a reproducing kernel Hilbert subspace of L^2 will be discussed in the subsequent paper.

Let $\mathcal{W}$ be the Wiener class of integral operators T on L^2,

$$\mathcal{W}:=\Big\{T,\ \|T\|_{\mathcal{W}}:=\|K_T\|_{\mathcal{W}}<\infty\Big\} \tag{1.2}$$

where $\|\cdot\|_p$ $(1\le p\le\infty)$ is the usual $L^p:=L^p(\mathbb{R}^d)$ norm, and

$$\|K\|_{\mathcal{W}}:=\max\Big(\sup_{x\in\mathbb{R}^d}\|K(x,\cdot)\|_1,\ \sup_{y\in\mathbb{R}^d}\|K(\cdot,y)\|_1\Big) \tag{1.3}$$

for a kernel function K on $\mathbb{R}^d\times\mathbb{R}^d$ ([36]). The first class $\mathcal{W}_{p,u}^\alpha$ of integral operators to be introduced in this paper is essentially the Wiener class $\mathcal{W}$ with additional regularity and decay at infinity, see (2.6) for the precise definition. For $p=1,\alpha\in(0,1]$, $u(x,y)=(1+|x-y|)^\gamma$ with $\gamma\ge 0$, $\mathcal{W}_{p,u}^\alpha$ contains all integral operators T on L^2 whose kernels K_T satisfy

$$\|K_Tu\|_{\mathcal{W}}+\sup_{0<\delta\le 1}\delta^{-\alpha}\|\omega_\delta(K_T)u\|_{\mathcal{W}}<\infty, \tag{1.4}$$

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where the modulus of continuity $\omega_\delta(K)$ of a kernel function K on $\mathbb{R}^d\times\mathbb{R}^d$ is defined by

$$\omega_\delta(K)(x,y) := \sup_{|x'|\le\delta, |y'|\le\delta} |K(x+x', y+y') - K(x,y)|. \tag{1.5}$$

Let $\mathcal{K}$ be the Kurbatov class of integral operators T on L^2,

$$\mathcal{K} := \Big\{T,\ \|T\|_{\mathcal{K}} := \big\| \sup_{y\in\mathbb{R}^d} |K_T(y,\cdot+y)|\ \big\|_{\mathcal{K}} < \infty\Big\} \tag{1.6}$$

([37]). The second class $\mathcal{C}^\alpha_{p,u}$ of integral operators to be introduced in this paper is essentially the Kurbatov class $\mathcal{K}$ of integral operators with additional regularity for kernels and decay at infinity for the enveloping kernel, see (4.2) for precise definition. For $1\le p\le\infty, \alpha\in(0,1]$ and $u(x,y)=(1+|x-y|)^\gamma$, $\mathcal{C}^\alpha_{p,u}$ is the set of integral operators T on L^2 with kernels K_T satisfying

$$\big\| \sup_{y\in\mathbb{R}^d} |(K_T u)(y,\cdot+y)|\ \big\|_p + \sup_{0<\delta\le 1}\delta^{-\alpha} \big\| \sup_{y\in\mathbb{R}^d} |(\omega_\delta(K_T)u)(y,\cdot+y)|\ \big\|_p < \infty. \tag{1.7}$$

We are interested in those two classes of integral operators because they include

(i) The projection operator P onto the wavelet space

$$V_2(\Phi) = \Big\{\sum_{\lambda\in\Lambda} c(\lambda)\phi_\lambda, \sum_{\lambda\in\Lambda}|c(\lambda)|^2<\infty\Big\}$$

generated by a family of functions $\phi_\lambda, \lambda\in\Lambda$, on $\mathbb{R}^d$ that have certain regularity and decay at infinity ([13, 14, 15, 17, 35, 47]). The operator P in (3.57) is such an projection operator by (3.58), while the operator Q in (3.59) is a bounded operator from L^2 to $V_2(\Phi)$ that has bounded pseudo-inverse.

(ii) The frame operator

$$\begin{aligned} Sf(x,\omega) &= \sum_{(\lambda^1,\lambda^2)\in\Lambda} \Big(\int_{y,\eta\in\mathbb{R}^d} f(y,\eta)\overline{(Vg)(y-\lambda^1,\eta-\lambda^2)}e^{-i\lambda^1(\eta-\omega)}dyd\eta\Big) \\ &\qquad \times (Vg)(x-\lambda^1,\omega-\lambda^2), \quad f\in VL^2, \end{aligned}$$

associated with a Gabor frame $\{U_{(\lambda^1,\lambda^2)}g\}_{(\lambda^1,\lambda^2)\in\Lambda}$ in the time-frequency plane VL^2, where $g\in L^2$ has certain regularity and fast decay at infinity, $\Lambda\subset \mathbb{R}^{2d}$ is a countable set, and the short-time Fourier transform V of a function $f\in L^2(\mathbb{R}^d)$ with respect to the Gaussian window is defined as

$$Vf(x,\omega) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} f(y)\exp(-|x-y|^2/2)e^{-iy\omega}dy$$

([3, 4, 24, 28]).

(iii) The reconstruction operator

$$Rf(x) = \sum_{\gamma\in\Gamma}\int_{\mathbb{R}^d} f(y)\psi_\gamma(y)dy\tilde{\psi}_\gamma(x), f\in L^2$$

associated with an average sampling and reconstruction procedure, where Γ represents the location of acquirement devices and where for each $\gamma\in\Gamma$, ψ_γ is the impulse response of the acquirement device at location γ and

$\tilde{\psi}_\gamma$ is the displayer for the sampling data at the location γ in a locally finitely-generated space ([1, 47, 48]).

Given a Banach algebra $\mathcal{B}$, we say that a subalgebra $\mathcal{A}$ of $\mathcal{B}$ is *inverse-closed* (respectively *pseudo-inverse closed*) if $T \in \mathcal{A}$ and the inverse T^{-1} of the operator T belongs to $\mathcal{B}$ (respectively the pseudo-inverse of $T^\dagger$ of the operator T belongs to $\mathcal{B}$) implies that $T^{-1} \in \mathcal{A}$ (respectively $T^\dagger \in \mathcal{A}$), see [7, 23, 39, 52]. The inverse-closed subalgebra was first studied for periodic functions with absolutely convergent Fourier series ([53]), which states that *if a periodic function f does not vanish on the real line and has absolutely convergent Fourier series, i.e., $f(x) = \sum_{n=-\infty}^{+\infty} a(n)e^{-inx}$ and $\sum_{n=-\infty}^{+\infty} |a(n)| < \infty$, then f^{-1} has absolutely convergent Fourier series too.* A significant non-commutative extension of the above Wiener's lemma was given in [8] for periodic functions $f(x) = \sum_{n\in-\infty}^{+\infty} a_n e^{-inx}$ such that the Fourier coefficients $a_n, n \in \mathbb{Z}$, belong to a normed ring and satisfy $\sum_{n=-\infty}^{+\infty} \|a_n\| < \infty$, see [23, 39] and references therein for further extensions. There are many equivalent formulations to the classical Wiener lemma. One involving matrix algebras says that the commutative Banach algebra $\tilde{\mathcal{W}} := \big\{\big(a(j-j')\big)_{j,j'\in\mathbb{Z}^d},\ \sum_{j\in\mathbb{Z}^d} |a(j)| < \infty\big\}$ is an inverse-closed (and hence pseudo-inverse closed by the standard holomorphic calculus) Banach subalgebra of $\mathcal{B}^2(\ell^2)$, the algebra of all bounded operators on ℓ^2 ([22, 53]). In the study of spline approximation ([18, 19]), wavelet and affine wavelets ([13, 34]), Gabor frame ([4, 28, 29]) and pseudo-differential operators ([9, 10, 20, 27, 31, 33, 41, 42, 49, 50, 51]), it arises extremely non-commutative matrix of the form $\big(a(\lambda, \lambda')\big)_{\lambda,\lambda'\in\Lambda}$ having certain off-diagonal decay. Moreover the Wiener's lemma for those non-commutative matrices are crucial for the well-localization of dual wavelet frames and dual Gabor frames ([4, 28, 34, 41]) and the robust and finite implementation of the reconstruction procedure ([30, 47]). Therefore there are lots of papers devoted to the Wiener's lemma for infinite matrices with various off-diagonal decay conditions (see [3, 4, 6, 19, 29, 32, 34, 41, 45, 46] and also [30] for a short historical review). The Wiener's lemma for integral (pseudo-differential) operators is also an important chapter of the theory of inverse-closed subalgebras ([5, 6, 9, 10, 11, 20, 28, 29, 32, 33, 38, 41, 42, 49, 50, 51] and also [27, 37] for a historial review). For instance, it was proved that if T is an integral operator in the Kurbatov class and if $I+T$ has bounded inverse $(I+T)^{-1}$ as an operator on L^2, then $(I+T)^{-1}T$ is an integral operator in the Kurbatov class too (see [37, 38]), and that if T is a Weyl transform with symbol in $M^{\infty,1}$, the Sjöstrand class containing the Hörmander class $S^0_{0,0}$ and also some non-smooth symbols, and if T has bounded inverse as an operator on L^2 then T^{-1} is also a Weyl transform with symbol in $M^{\infty,1}$ (see [41, 42]). The objective of this paper is to establish Wiener's lemma for the two classes $\mathcal{W}^\alpha_{p,u}$ and $\mathcal{C}^\alpha_{p,u}$ of localized integral operators (see Theorems 3.1, 3.5, 4.1 and 4.3 for the requirements to the weight u and the regularity exponent α).

The Wiener's lemma and its various generalizations and formulations have numerous applications in numerical analysis, time-frequency analysis, wavelet theory, frame theory, and sampling theory. For instance, the classical Wiener's lemma and its weighted variation ([53, 32]) were used to establish the decay property at infinity for dual generators of a shift-invariant space ([1, 35]); the Wiener's lemma for infinite matrices associated with twisted convolution was used in the study the decay properties of the dual Gabor frame for L^2 ([3, 4, 28, 29]); the Jaffard's and Sjöstrand's results ([34, 41]) and their extension ([45, 46]) for infinite matrices with polynomial (exponential) decay were used in numerical analysis ([12, 30, 43, 44]), wavelet analysis ([34]), time-frequency analysis ([24, 25, 26]), pseudo-differential operators ([27, 31, 41]) and sampling ([2, 16, 26, 48]) and will also be used in the proof of Theorems 3.5 and 4.3; and the Kurbatov's result ([37, 38]) and its extension ([20]) was used in solving integral operator equations ([37]). The applications of the Wiener's lemma for the two classes $\mathcal{W}_{p,u}^\alpha$ and $\mathcal{C}_{p,u}^\alpha$ of localized integral operators will be discussed in the subsequent paper.

We finish this section with some notation used later. Let $\mathcal{B}^q := \mathcal{B}(L^q), 1 \le q \le \infty$, be the set of all bounded operators on L^q with its norm denoted by $\|\cdot\|_{\mathcal{B}^q}$. For an operator T on a Hilbert space, we denote its adjoint operator by T^*. For an operator T in a Banach algebra $\mathcal{B}$, we define its spectral radius $\rho_{\mathcal{B}}(T)$ by $\rho_{\mathcal{B}}(T) = \limsup_{n\to\infty} \|A^n\|_{\mathcal{B}}^{1/n}$. For a bounded operator T on L^2, we denote by $N(T) \subset L^2$ the null space of the operator T, and by $N(T)^\perp$ the orthogonal complement of the null space $N(T)$. In this paper, the uppercase letter C denotes an absolute constant which could be different at different occurrences.

2. Localized Integral Operators

In this section, we introduce the first class of localized integral operators on L^2, and discuss some basic algebraic properties of that class of integral operators.

In this paper, we use weights to describe localization of integral operators. Here a *weight* u is a positive continuous function on $\mathbb{R}^d \times \mathbb{R}^d$ that is symmetric, diagonal-normalized and slow-varying, i.e.,

$$u(x,y) = u(y,x) \text{ for all } x,y \in \mathbb{R}^d, \tag{2.1}$$

$$u(x,y) \ge 1 \text{ and } u(x,x) = 1 \text{ for all } x,y \in \mathbb{R}^d, \tag{2.2}$$

and

$$C(u) := \sup_{x,y\in\mathbb{R}^d} \sup_{|x'|\le 1, |y'|\le 1} \frac{u(x+x', y+y')}{u(x,y)} < \infty. \tag{2.3}$$

The model examples of weights are polynomial weights

$$u_\gamma(x,y) = (1+|x-y|)^\gamma \tag{2.4}$$

with $\gamma \ge 0$, and (sub)exponential weights

$$e_{D,\delta}(x,y) = \exp(D|x-y|^\delta) \tag{2.5}$$

with $D > 0$ and $0 < \delta \le 1$.

In this paper, we use modulus of continuity to describe the regularity of integral operators. Given $\alpha \ge 0, 1 \le p \le \infty$ and a weight u on $\mathbb{R}^d \times \mathbb{R}^d$, the first class of localized integral operators to be discussed in this paper is

$$\mathcal{W}_{p,u}^\alpha := \Big\{T, \ \ \|T\|_{\mathcal{W}_{p,u}^\alpha} := \|K_T\|_{\mathcal{W}_{p,u}^\alpha} < \infty\Big\}, \tag{2.6}$$

where K_T is the kernel function of the integral operator

$$Tf(x) := \int_{\mathbb{R}^d} K_T(x,y) f(y) dy \quad f \in L^2(\mathbb{R}^d), \tag{2.7}$$

the norm $\|K\|_{\mathcal{W}^\alpha_{p,u}}$ for a kernel function K on $\mathbb{R}^d\times\mathbb{R}^d$ is defined by

$$\|K\|_{\mathcal{W}^\alpha_{p,u}} := \begin{cases} \max\Big(\sup_{x\in\mathbb{R}^d}\|K(x,\cdot)u(x,\cdot)\|_p, \sup_{y\in\mathbb{R}^d}\|K(\cdot,y)u(\cdot,y)\|_p\Big) \\ \hfill \text{if } \alpha=0 \\ \max_{k,l\in\mathbb{Z}^d,|k|+|l|<\alpha}\|\partial_x^k\partial_y^l K(x,y)\|_{\mathcal{W}^0_{p,u}} \\ \quad +\max_{k,l\in\mathbb{Z}^d,|k|+|l|<\alpha}\ \sup_{0<\delta\le 1}\delta^{-\alpha+\alpha_0}\|\omega_\delta(\partial_x^k\partial_y^l K(x,y))\|_{\mathcal{W}^0_{p,u}} \\ \hfill \text{if } \alpha>0, \end{cases} \tag{2.8}$$

and α_0 is the largest integer strictly smaller than α. Clearly, $\mathcal{W}^\alpha_{p,u}$ becomes the Wiener class $\mathcal{W}$ of integral operators when $\alpha=0, p=1$ and $u\equiv 1$ is the trivial weight.

In order to state our results for the class $\mathcal{W}^\alpha_{p,u}$ of localized integral operators, we recall a technical condition on weights. For $p\ge 1$, we say that a weight u is said to be *p-admissible* if there exists another weight v, and two positive constants $\theta\in(0,1)$ and $C\in(0,\infty)$ such that

$$u(x,y)\le u(x,z)v(z,y)+v(x,z)u(z,y) \tag{2.9}$$

for all $x,y,z\in\mathbb{R}^d$; and

$$\inf_{\tau\ge 1}\|v(x,\cdot)\|_{L^1(B(x,\tau))}+t\|(vu^{-1})(x,\cdot)\|_{L^{p/(p-1)}(\mathbb{R}^d\setminus B(x,\tau))}\le Ct^\theta \tag{2.10}$$

for all $x\in\mathbb{R}^d$ and $t\ge 1$, where $B(x,\tau)=\{y\in\mathbb{R}^d : |x-y|\le\tau\}$ is the ball with center x and radius τ ([46]). The examples of p-admissible weights include polynomial weights u_γ with $\gamma>d(1-1/p)$ and subexponential weights $e_{D,\delta}$ with $\delta\in(0,1)$ and $D>0$. We remark that the exponential weight $e_{D,1}, D>0$ are not p-admissible weights.

Remark 2.1. We say that a weight u on $\mathbb{R}^d\times\mathbb{R}^d$ satisfies the Gelfand-Rykov-Shylov condition if

$$\lim_{n\to\infty}u(nx,ny)^{1/n}=0 \quad \text{for all } x,y\in\mathbb{R}^d \tag{2.11}$$

([21, 29]). One may easily check that the weight $\exp(|x-y|/\ln(1+|x-y|))$ satisfies the Gelfand-Rykov-Shylov condition but it is not a p-admissible weight. For weight u of the form $\exp(\kappa(|x-y|))$ where κ is a concave function on $[0,\infty)$ with $\kappa(0)=0$, it was proved in [46] that if u is a p-admissible weight then it satisfies the Gelfand-Rykov-Shylov condition. But I do not know whether any p-admissible weight satisfies the Gelfand-Rykov-Shylov condition.

Now we state some basic properties of the class $\mathcal{W}^\alpha_{p,u}$ of localized integral operators on L^2.

Theorem 2.2. *Let $\alpha,\beta\ge 0, 1\le p,q\le\infty$ and u,v be weights on $\mathbb{R}^d\times\mathbb{R}^d$. Then the following statements are true.*

(i) *$T\in\mathcal{W}^\alpha_{p,u}$ if and only if $T^*\in\mathcal{W}^\alpha_{p,u}$.*
(ii) *If $\beta\ge\alpha$ and $T\in\mathcal{W}^\beta_{p,u}$, then $T\in\mathcal{W}^\alpha_{p,u}$.*
(iii) *If $p\ge q$, $\sup_{x\in\mathbb{R}^d}\|(vu^{-1})(x,\cdot)\|_{pq/(p-q)}<\infty$ and $T\in\mathcal{W}^\alpha_{p,u}$, then $T\in\mathcal{W}^\alpha_{q,v}$.*

(iv) *If* $T \in \mathcal{W}_{p,u}^\alpha$ *and* $\sup_{x\in\mathbb{R}^d} \|(u(x,\cdot))^{-1}\|_{p/(p-1)} < \infty$, *then* T *is a bounded operator on* L^q *for* $1 \le q \le \infty$ *but* T *does not have bounded inverse on* L^q *for* $1 \le q < \infty$.

(v) *If* $T_1, T_2 \in \mathcal{W}_{p,u}^\alpha$ *and if* u *is a* p*-admissible weight, then* $T_1T_2 \in \mathcal{W}_{p,u}^\alpha$.

Proof. (i): The first conclusion holds since

$$\|T^*\|_{\mathcal{W}_{p,u}^\alpha} = \|T\|_{\mathcal{W}_{p,u}^\alpha} \quad \text{for all} \quad T \in \mathcal{W}_{p,u}^\alpha. \tag{2.12}$$

(ii): The second conclusion is true because of

$$\|T\|_{\mathcal{W}_{p,u}^\alpha} \le \|T\|_{\mathcal{W}_{p,u}^\beta} \quad \text{for all} \quad T \in \mathcal{W}_{p,u}^\beta \text{ with } \beta \ge \alpha. \tag{2.13}$$

(iii): The third conclusion follows easily from

$$\|T\|_{\mathcal{W}_{q,v}^\alpha} \le \sup_{x\in\mathbb{R}^d} \|(vu^{-1})(x,\cdot)\|_{pq/(p-q)} \|T\|_{\mathcal{W}_{p,u}^\alpha} \quad \text{for all} \quad T \in \mathcal{W}_{p,u}^\alpha. \tag{2.14}$$

(iv): The boundedness of an operator $T \in \mathcal{W}_{p,u}^\alpha$ on L^p is proved by combining (2.13), (2.14) and

$$\|T\|_{\mathcal{B}^q} \le \|T\|_{\mathcal{W}_{1,u_0}^0} \quad \text{for all } T \in \mathcal{W} := \mathcal{W}_{1,u_0}^0 \text{ and } 1 \le q \le \infty, \tag{2.15}$$

where $u_0 \equiv 1$ is the trivial weight.

To prove the unboundedness of the inverse of the operator $T \in \mathcal{W}_{p,u}^\alpha$, we let $1 \le q < \infty$, $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^d$, ϕ be a nonzero smooth function supported on $[-1/2, 1/2]^d$, and define $\phi_z(x) = \phi(x)e^{izx}$. Then

$$\|\phi_{\alpha\mathbf{1}}\|_q = \|\phi\|_q \quad \text{for all } \alpha \in \mathbb{R} \tag{2.16}$$

by the definition, and

$$\lim_{\alpha\to+\infty} \|T\phi_{\alpha\mathbf{1}}\|_q = 0 \tag{2.17}$$

because

$$\begin{aligned}
\Big\| \sup_{\alpha\in\mathbb{R}} |T\phi_{\alpha\mathbf{1}}| \Big\|_q \quad &\le \quad \Big\| \int_{\mathbb{R}^d} |K(\cdot,y)||\phi(y)|dy \Big\|_q \le \|\phi\|_q \|K\|_{\mathcal{W}_{1,u_0}^0} \\
&\le \quad \|\phi\|_q \|T\|_{\mathcal{W}_{p,u}^\alpha} \sup_{x\in\mathbb{R}^d} \|u^{-1}(x,\cdot)\|_{p/(p-1)} < \infty
\end{aligned}$$

and for almost all $x \in \mathbb{R}^d$

$$\begin{aligned}
|T\phi_{\alpha\mathbf{1}}(x)| \quad &\le \quad \int_{\mathbb{R}^d} \big|K(x, y + (d\alpha)^{-1}\pi\mathbf{1})\phi(y + (d\alpha)^{-1}\pi\mathbf{1}) - K(x,y)\phi(y)\big| dy \\
&\to \quad 0 \quad \text{as} \quad \alpha \to +\infty.
\end{aligned}$$

Therefore the conclusion that the operator T does not have bounded inverse in L^q follows from (2.16) and (2.17).

(v): Take two integral operators T_1, T_2 in $\mathcal{W}_{p,u}^\alpha$, and denote their kernels by K_1, K_2 respectively. Then the kernel of their composition $T := T_1T_2$ is given by

$$K(x,y) = \int_{\mathbb{R}^d} K_1(x,z)K_2(z,y)dz, \tag{2.18}$$

which implies that

$$\partial_x^k \partial_y^l K(x,y) = \int_{\mathbb{R}^d} \partial_x^k K_1(x,z) \partial_y^l K_2(z,y)dz \tag{2.19}$$

and

$$
\begin{aligned}
|\omega_\delta(\partial_x^k\partial_y^l K)(x,y)| \quad \le \quad & \int_{\mathbb{R}^d} \omega_\delta(\partial_x^k K_1)(x,z)|\partial_y^l K_2(z,y)|dz \\
& + \int_{\mathbb{R}^d} |\partial_x^k K_1(x,z)|\omega_\delta(\partial_y^l K_2)(z,y)dz \\
& + \int_{\mathbb{R}^d} \omega_\delta(\partial_x^k K_1)(x,z)\omega_\delta(\partial_y^l K_2)(z,y)dz \qquad (2.20)
\end{aligned}
$$

for all $k,l \in \mathbb{Z}^d$ with $|k|+|l|<\alpha$. Therefore

$$\|T_1T_2\|_{\mathcal{W}^\alpha_{p,u}} \le C\|T_1\|_{\mathcal{W}^\alpha_{p,u}}\|T_2\|_{\mathcal{W}^\alpha_{1,v}} + C\|T_1\|_{\mathcal{W}^\alpha_{1,v}}\|T_2\|_{\mathcal{W}^\alpha_{p,u}} \qquad (2.21)$$

by (2.9) and (2.18) – (2.20), where v is the weight associated with the p-admissible weight u. Applying (2.14) with $q=1$ to the right hand side of the estimate (2.21) yields

$$\|T_1T_2\|_{\mathcal{W}^\alpha_{p,u}} \le C_0\|T_1\|_{\mathcal{W}^\alpha_{p,u}}\|T_2\|_{\mathcal{W}^\alpha_{p,u}} \qquad (2.22)$$

where C_0 is a positive constant independent of $T_1, T_2 \in \mathcal{W}^\alpha_{p,u}$. Hence the conclusion (v) follows. □

By Theorem 2.2, we have

Corollary 2.3. *Let $1 \le p,q \le \infty, \alpha \ge 0$ and u be a p-admissible weight. Then $\mathcal{W}^\alpha_{p,u}$ is a non-unital algebra embedded in the algebra $\mathcal{B}^q$ of bounded operators on L^q with the usual operator norm.*

In order to use the general theory for Banach algebra ([23, 39, 52]), we create a unital Banach subalgebra of $\mathcal{B}^2$ so that $\mathcal{W}^\alpha_{p,u}$ can be imbedded into.

Theorem 2.4. *Let $\alpha \ge 0, 1 \le p \le \infty, u$ be a p-admissible weight, and I be the identity operator on L^2. Define*

$$\mathcal{IW}^\alpha_{p,u} := \{\lambda I + T : \ \lambda \in \mathbf{C} \text{ and } T \in \mathcal{W}^\alpha_{p,u}\} \qquad (2.23)$$

with

$$\|\lambda I + T\|_{\mathcal{IW}^\alpha_{p,u}} := |\lambda| + C_0\|T\|_{\mathcal{W}^\alpha_{p,u}} \qquad (2.24)$$

where C_0 is the constant in (2.22)*. Then $\mathcal{IW}^\alpha_{p,u}$ is a unital Banach subalgebra of $\mathcal{B}^2$.*

Proof. By the conclusions (iv) and (v) of Theorem 2.2, $\mathcal{IW}^\alpha_{p,u}$ is a well-defined algebra with the identity I. For any operators $\lambda_1 I + T_1$ and $\lambda_2 I + T_2$ in $\mathcal{IW}^\alpha_{p,u}$,

$$
\begin{aligned}
& \|(\lambda_1 I + T_1)(\lambda_2 I + T_2)\|_{\mathcal{IW}^\alpha_{p,u}} \\
\le \quad & |\lambda_1\lambda_2| + C_0|\lambda_1|\|T_2\|_{\mathcal{W}^\alpha_{p,u}} + C_0|\lambda_2|\|T_1\|_{\mathcal{W}^\alpha_{p,u}} + C_0\|T_1T_2\|_{\mathcal{W}^\alpha_{p,u}} \\
\le \quad & \|\lambda_1 I + T_1\|_{\mathcal{IW}^\alpha_{p,u}}\|\lambda_2 I + T_2\|_{\mathcal{IW}^\alpha_{p,u}} \qquad (2.25)
\end{aligned}
$$

where the estimate in (2.22) has been used to obtain the last inequality. Hence $\mathcal{IW}^\alpha_{p,u}$ is a unital Banach algebra.

By (2.15), we have

$$\|\lambda I + T\|_{\mathcal{B}^2} \le |\lambda| + \|T\|_{\mathcal{B}^2} \le C\|\lambda I + T\|_{\mathcal{IW}^\alpha_{p,u}}, \qquad (2.26)$$

which implies that $\mathcal{IW}^\alpha_{p,u}$ is a subalgebra of $\mathcal{B}^2$. □

3. Wiener's Lemma for Localized Integral Operators: Part I

In this section, we establish the principal results of this paper, the Wiener's lemma for the unital Banach subalgebra $\mathcal{IW}_{p,u}^{\alpha}$ of $\mathcal{B}^2$ with p-admissible weight u (Theorem 3.1), and a weak version of the Wiener's lemma for the non-unital subalgebra $\mathcal{W}_{p,u}^{\alpha}$ of $\mathcal{B}^2$ with exponential weight u (Theorem 3.5).

Theorem 3.1. *Let $\alpha > 0$, $1 \le p \le \infty$, u be a p-admissible weight, and $\mathcal{IW}_{p,u}^{\alpha}$ be defined as in* (2.23)*. Then $\mathcal{IW}_{p,u}^{\alpha}$ is an inverse-closed Banach subalgebra of $\mathcal{B}^2$.*

Using the standard holomorphic calculus ([40]), we have the following Wiener's lemma (Corollary 3.2) and Wiener-Levy theorem (Corollary 3.3) for the non-unital algebra $\mathcal{W}_{p,u}^{\alpha}$.

Corollary 3.2. *Let $\alpha > 0$, $1 \le p \le \infty$, u be a p-admissible weight, and $\mathcal{W}_{p,u}^{\alpha}$ be as in* (2.6)*. Assume that $T \in \mathcal{W}_{p,u}^{\alpha}$ has bounded pseudo-inverse, then its pseudo-inverse $T^{\dagger}$ belongs to $\mathcal{W}_{p,u}^{\alpha}$.*

Corollary 3.3. *Let $\alpha > 0$, $1 \le p \le \infty$, u be a p-admissible weight, and $\mathcal{W}_{p,u}^{\alpha}$ be as in* (2.6)*. Assume that $T \in \mathcal{W}_{p,u}^{\alpha}$ and g is an analytic function on a neighborhood of the spectrum of the operator T in $\mathcal{B}^2$. Then $g(T) \in \mathcal{W}_{p,u}^{\alpha}$ if and only if $g(0) = 0$.*

Remark 3.4. For $1 \le p \le \infty, \alpha \ge 0$ and a p-admissible weight u, it follows from Theorem 2.2 that integral operators in $\mathcal{W}_{p,u}^{\alpha}$ do not have bounded inverse on L^2, but those integral operators may or may not have bounded pseudo-inverse. For instance, the projection operators to wavelet spaces, frame operators on the time-frequency plane, and reconstruction operators for sampling mentioned in the Introduction section have bounded pseudo-inverse. On the other side, a convolution operator in the Wiener class $\mathcal{W}$ must not have bounded pseudo-inverse. The reason is the following: Suppose, on the contrary, there is a convolution operator $T \in \mathcal{W}_{p,u}^{\alpha}$ having bounded pseudo-inverse. Define the Fourier transform $\hat{f}$ of an integrable function f by $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx$ and extend that definition to functions in L^2 as usual. Then

$$\|\hat{K}\hat{f}\|_2 \ge C\|\hat{f}\|_2 \quad \text{for all } f \in N(T)^{\perp} \tag{3.1}$$

and

$$\hat{K}\hat{f} = 0 \quad \text{for all } f \in N(T) \tag{3.2}$$

by the bounded pseudo-inverse assumption, where $K(x-y)$ is the kernel of the convolution operator T. Therefore for almost all $\xi \in \mathbb{R}^d$ either $|\hat{K}(\xi)| = 0$ or $|\hat{K}(\xi)| \ge C$, which is a contradiction since $\int_{\mathbb{R}^d} |K(x)| dx < \infty$ by the assumption that $T \in \mathcal{W}_{p,u}^{\alpha} \subset \mathcal{W}$.

The exponential weights $e_{D,1}(x,y) := \exp(D|x-y|), D > 0$, are not p-admissible weights. For integral operators in $\mathcal{W}_{p,u}^{\alpha}$ with $u = e_{D,1}$ for some $D > 0$, we have the following weak version of the Wiener's lemma.

Theorem 3.5. *Let $\alpha, D > 0$, $1 \le p \le \infty$, $e_{D,1}(x,y) := \exp(D|x-y|)$, and let $\mathcal{W}_{p,e_{D,1}}^{\alpha}$ be as in* (2.6)*. Assume that $T \in \mathcal{W}_{p,e_{D,1}}^{\alpha}$ and its pseudo-inverse $T^{\dagger}$ belongs to $\mathcal{B}^2$. Then $T^{\dagger} \in \mathcal{W}_{p,e_{D',1}}^{\alpha}$ for some $D' \in (0, D)$.*

Remark 3.6. The crucial step in the proof of Theorem 3.1 is to establish

$$\rho_{\mathcal{IW}^\alpha_{p,u}}(\lambda I+T)=\rho_{\mathcal{B}^2}(\lambda I+T)\quad \text{for all } \lambda I+T\in \mathcal{IW}^\alpha_{p,u} \tag{3.3}$$

where $\alpha>0, 1\le p\le\infty$ and u is a p-admissible weight. But the above equality about spectral radius on two different algebras is not long true when u is an exponential weight. For instance, we let

$$T_kf(x)=\int_{\mathbb{R}}\phi(x-y-k)f(y)dy,\ \ f\in L^2,$$

where $k\ge 1$ and ϕ is a nonnegative C^∞ function supported on $[-1/2,1/2]$ that satisfies $\int_{\mathbb{R}}\phi(x)=1$. Then for any $\lambda\in\mathbb{C}, n\in\mathbb{N}$ and $D>0$, we have

$$\|(\lambda I+T_k)^n\|_{\mathcal{B}^2}\le(|\lambda|+1)^n, \tag{3.4}$$

and

$$\begin{aligned}\|(\lambda I+T_k)^n\|_{\mathcal{IW}^\alpha_{1,e_{D,1}}} &= |\lambda|^n+\sum_{l=1}^n\frac{n!}{(n-l)!l!}|\lambda|^{n-l}\int_{\mathbb{R}}\phi_l(x-lk)\exp(Dx)dx\\ &\ge |\lambda|^n+\sum_{l=1}^n\frac{n!}{(n-l)!l!}|\lambda|^{n-l}e^{Dl(k-1/2)}\int_{\mathbb{R}}\phi_l(x-lk)dx\\ &= (\lambda+e^{D(k-1/2)})^n,\end{aligned} \tag{3.5}$$

where $\phi_1=\phi$ and for $l\ge 2$, $\phi_l:=\phi_{l-1}*\phi$ is the convolution between ϕ and ϕ_{l-1}. Combining (3.4) and (3.5) shows that the equality (3.3) does not hold for integral operator $\lambda I+T_k$ when u is an exponential weight $e_{D,1}$ with positive D.

Remark 3.7. We see from Remark 3.6 that the standard argument in the Banach algebra ([23, 39, 52]) could not be used to establish Wiener's lemma for the class $\mathcal{W}^\alpha_{p,u}$ of integral operators with exponential weights. Our new approach in the proof of Theorem 3.5 is based on the Wiener's lemma for infinite matrices with exponential decay (see [6, 19, 34, 46] and also Lemma 3.11), and the new observation that for an integral operator $T\in\mathcal{W}^\alpha_{p,u}$ having bounded pseudo-inverse, the orthogonal complement $N(T)^\perp$ of the null space $N(T)$ is a locally finitely-generated space $V_2(\Phi)$ with localized frame generator Φ that was introduced and studied in [47] (see Lemma 3.12 for detailed statement). The above observation for the space $N(T)^\perp$ indicates that signals in a reproducing kernel Hilbert subspace of L^2 that has localized smooth reproducing kernel could be reconstructed from their samples on a discrete set in a stable way, see the subsequent paper for the detailed study.

Remark 3.8. Let $\alpha, D>0$ and $1\le p\le\infty$. I conjecture that a weak version of the Wiener's lemma for integral operators T in $\mathcal{W}^\alpha_{p,e_{D,1}}$ holds, i.e., *If $0\ne\lambda\in\mathbb{C}, T\in\mathcal{W}^\alpha_{p,e_{D,1}}$ and $\lambda I+T$ has bounded inverse in $\mathcal{B}^2$, then there exists $D'\in(0,D)$ such that $(\lambda I+T)^{-1}=\lambda^{-1}I+\tilde T$ for some $\tilde T\in\mathcal{W}^\alpha_{p,e_{D',1}}$, or equivalently $(\lambda I+T)^{-1}T\in\mathcal{W}^\alpha_{p,e_{D',1}}$.*

3.1. **Proof of Theorem 3.1.** By Theorem 2.4, using the standard argument in the Banach algebra ([23, 39, 52]), it suffices to prove (3.3) for any $\lambda I+T\in\mathcal{IW}^\alpha_{p,u}$ where $\alpha>0, 1\le p\le\infty$ and u is a p-admissible weight. In this paper, we will prove a bit stronger result than the equality (3.3).

Theorem 3.9. *Let $\alpha>0, 1\le p\le\infty$, u be a p-admissible weight and $\mathcal{IW}_{p,u}^\alpha$ be as in* (2.23)*. Then there exist positive constants $C\in(0,\infty)$ and $\tilde\theta\in(0,1)$ such that*

$$\|(\lambda I+T)^n\|_{\mathcal{IW}_{p,u}^\alpha}\le C\Big(\frac{C\|\lambda I+T\|_{\mathcal{IW}_{p,u}^\alpha}}{\|\lambda I+T\|_{\mathcal{B}^2}}\Big)^{\frac{3(3+\tilde\theta)}{2+\tilde\theta}n^{\log_4(3+\tilde\theta)}}\Big(\|\lambda I+T\|_{\mathcal{B}^2}\Big)^n \tag{3.6}$$

hold for all $n\ge1$ and $\lambda I+T\in\mathcal{IW}_{p,u}^\alpha$.

To prove Theorem 3.9, other than those properties in Theorem 2.2 for operators in the algebra $\mathcal{W}_{p,u}^\alpha$, we need the following paracompact properties about operators in the algebra $\mathcal{W}_{p,u}^\alpha$.

Lemma 3.10. *Let $\alpha>0, 1\le p\le\infty$, u be a p-admissible weight, and $\mathcal{W}_{p,u}^\alpha$ be as in (2.6). Then the following statements hold.*

(vi) *If $T_1,T_3\in\mathcal{W}_{p,u}^\alpha$ and $T_2\in\mathcal{W}_{p,u}^0$, then $T_1T_2T_3\in\mathcal{W}_{p,u}^\alpha$.*

(vii) *There exist positive constants $C\in(0,\infty)$ and $\tilde\theta\in(0,1)$ such that*

$$\|T^2\|_{\mathcal{W}_{p,u}^0}\le C\|T\|_{\mathcal{W}_{p,u}^\alpha}^{1+\tilde\theta}\|T\|_{\mathcal{B}^2}^{1-\tilde\theta} \tag{3.7}$$

holds for any $T\in\mathcal{W}_{p,u}^\alpha$.

Proof. (vi): Take integral operators T_1,T_3 in $\mathcal{W}_{p,u}^\alpha$ and T_2 in $\mathcal{W}_{p,u}^0$, and denote the kernels of the integral operators T_1,T_2,T_3 and their composition $T:=T_1T_2T_3$ by K_1,K_2,K_3 and K respectively. One may easily verify that those integral kernels are related by

$$K(x,y)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K_1(x,z_1)K_2(z_1,z_2)K_3(z_2,y)dz_1dz_2. \tag{3.8}$$

Thus

$$\partial_x^k\partial_y^lK(x,y)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\partial_x^kK_1(x,z_1)K_2(z_1,z_2)\partial_y^lK_3(z_2,y)dz_1dz_2 \tag{3.9}$$

and

$$\begin{aligned}&|\omega_\delta(\partial_x^k\partial_y^lK)(x,y)|\\ \le\ & \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\omega_\delta(\partial_x^kK_1)(x,z_1)|K_2(z_1,z_2)||\partial_y^lK_3(z_2,y)|dz_1dz_2\\ &+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\partial_x^kK_1(x,z_1)||K_2(z_1,z_2)|\omega_\delta(\partial_y^lK_3)(z_2,y)dz_1dz_2\\ &+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\omega_\delta(\partial_x^kK_1)(x,z_1)||K_2(z_1,z_2)|\omega_\delta(\partial_y^lK_3)(z_2,y)dz_1dz_2\end{aligned} \tag{3.10}$$

where $k,l\in\mathbb{Z}^d$ with $|k|+|l|<\alpha$. Let α_0 be the largest integer strictly smaller than α. Applying (2.22) twice, we obtain from (3.9) and (3.10) that

$$\begin{aligned}\|\partial_x^k\partial_y^lK(x,y)\|_{\mathcal{W}_{p,u}^0}&\le C\big(\sup_{x\in\mathbb{R}^d}\|(vu^{-1})(x,\cdot)\|_{p/(p-1)}\big)^2\|\partial_x^kK_1(x,y)\|_{\mathcal{W}_{p,u}^0}\\ &\quad\times\|K_2(x,y)\|_{\mathcal{W}_{p,u}^0}\|\partial_y^lK_3(x,y)\|_{\mathcal{W}_{p,u}^0}\\ &\le C\|T_1\|_{\mathcal{W}_{p,u}^\alpha}\|T_2\|_{\mathcal{W}_{p,u}^0}\|T_3\|_{\mathcal{W}_{p,u}^\alpha}\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
\|\omega_\delta(\partial_x^k\partial_y^l K)(x,y)\|_{\mathcal{W}^0_{p,u}} &\le C\Big(\sup_{x\in\mathbb{R}^d}\|(vu^{-1})(x,\cdot)\|_{p/(p-1)}\Big)^2\\
&\quad\times\Big(\|\omega_\delta(\partial_x^k K_1)(x,y)\|_{\mathcal{W}^0_{p,u}}\|K_2(x,y)\|_{\mathcal{W}^0_{p,u}}\\
&\qquad\times\|\partial_y^l K_3(x,y)\|_{\mathcal{W}^0_{p,u}}+\|\partial_x^k K_1(x,y)\|_{\mathcal{W}^0_{p,u}}\\
&\qquad\times\|K_2(x,y)\|_{\mathcal{W}^0_{p,u}}\|\omega_\delta(\partial_y^l K_3)(x,y)\|_{\mathcal{W}^0_{p,u}}\\
&\quad+\|\omega_\delta(\partial_x^k K_1)(x,y)\|_{\mathcal{W}^0_{p,u}}\|K_2(x,y)\|_{\mathcal{W}^0_{p,u}}\\
&\qquad\times\|\omega_\delta(\partial_y^l K_3)(x,y)\|_{\mathcal{W}^0_{p,u}}\Big)\\
&\le C\delta^{\alpha-\alpha_0}\|T_1\|_{\mathcal{W}^\alpha_{p,u}}\|T_2\|_{\mathcal{W}^0_{p,u}}\|T_3\|_{\mathcal{W}^\alpha_{p,u}} \qquad (3.12)
\end{aligned}$$

for all $\delta\in(0,1)$ and $k,l\in\mathbb{Z}^d$ with $|k|+|l|<\alpha$. Therefore

$$\|T_1T_2T_3\|_{\mathcal{W}^\alpha_{p,u}}\le C\|T_1\|_{\mathcal{W}^\alpha_{p,u}}\|T_2\|_{\mathcal{W}^0_{p,u}}\|T_3\|_{\mathcal{W}^\alpha_{p,u}} \qquad (3.13)$$

by (3.11) and (3.12), and the conclusion (vi) is proved.

(vii): Take $T\in\mathcal{W}^\alpha_{p,u}$, and let K be the kernel of the integral operator T in $\mathcal{W}^\alpha_{p,u}$. Then for any $x,y\in\mathbb{R}^d$ and $\delta\in(0,1)$,

$$\begin{aligned}
|K(x,y)| &\le (\omega_\delta K)(x,y)+\Big|\delta^{-2d}\int_{t,t'\in\delta[-1/2,1/2]^d}K(x+t,y+t')dtdt'\Big|\\
&\le (\omega_\delta K)(x,y)+\delta^{-3d/2}\|T\chi_{y+\delta[-1/2,1/2]^d}\|_2\\
&\le (\omega_\delta K)(x,y)+\delta^{-d}\|T\|_{\mathcal{B}^2} \qquad (3.14)
\end{aligned}$$

where χ_E is the characteristic function on a set E. Let α_0 be the largest integer strictly smaller than α. Then

$$\delta_0:=\min\Big(1,\Big(\frac{\|T\|_{\mathcal{B}^2}}{\|T\|_{\mathcal{W}^\alpha_{p,u}}}\Big)^{1/(\alpha-\alpha_0+d)}\Big)\le C\Big(\frac{\|T\|_{\mathcal{B}^2}}{\|T\|_{\mathcal{W}^\alpha_{p,u}}}\Big)^{1/(\alpha-\alpha_0+d)} \qquad (3.15)$$

by (2.13) and (2.15). Let θ be the exponent in the definition of the p-admissible weight u, v be the weight associated with the p-admissible weight u, $\theta_1=\frac{d}{\alpha-\alpha_0+d}$, and $\tilde\theta=\theta_1+(1-\theta_1)\theta$. Then it follows from (2.9), (2.10), (3.14) and (3.15) that

$$\begin{aligned}
&\int_{\mathbb{R}^d}|K(x,y)|v(x,y)dy\\
\le\ & \inf_{\tau\ge 1}\Big(\int_{|x-y|\le\tau}\Big((\omega_{\delta_0}K)(x,y)+\delta_0^{-d}\|T\|_{\mathcal{B}^2}\Big)v(x,y)dy\\
&\qquad +\int_{|x-y|>\tau}|K(x,y)|v(x,y)dy\Big)\\
\le\ & C\inf_{\tau\ge 1}\Big(\|\omega_{\delta_0}(K)\|_{\mathcal{W}^0_{p,u}}\Big(\int_{|x-y|\le\tau}|(vu^{-1})(x,y)|^{p/(p-1)}dy\Big)^{(p-1)/p}\\
&\qquad +\delta_0^{-d}\|T\|_{\mathcal{B}^2}\int_{|x-y|\le\tau}v(x,y)dy\\
&\qquad +\|T\|_{\mathcal{W}^0_{p,u}}\Big(\int_{|x-y|>\tau}|(vu^{-1})(x,y)|^{p/(p-1)}dy\Big)^{(p-1)/p}\Big)\\
\le\ & C\inf_{\tau\ge 1}\Big(\big(\delta_0^{\alpha-\alpha_0}\|T\|_{\mathcal{W}^\alpha_{p,u}}+\delta_0^{-d}\|T\|_{\mathcal{B}^2}\big)\|v(x,\cdot)\|_{L^1(B(x,\tau))}\\
&\qquad +\|T\|_{\mathcal{W}^\alpha_{p,u}}\|(vu^{-1})(x,\cdot)\|_{L^{p/(p-1)}(\mathbb{R}^d\backslash B(x,\tau))}\Big)\\
\le\ & C\|T\|^{\theta_1}_{\mathcal{W}^\alpha_{p,u}}\inf_{\tau\ge 1}\Big(\|T\|^{1-\theta_1}_{\mathcal{B}^2}\|v(x,\cdot)\|_{L^1(B(x,\tau))}\\
&\qquad +\|T\|^{1-\theta_1}_{\mathcal{W}^\alpha_{p,u}}\|(vu^{-1})(x,\cdot)\|_{L^{p/(p-1)}(\mathbb{R}^d\backslash B(x,\tau))}\Big)\\
\le\ & C\|T\|^{\tilde\theta}_{\mathcal{W}^\alpha_{p,u}}\|T\|^{1-\tilde\theta}_{\mathcal{B}^2}\quad \text{for all } x\in\mathbb{R}^d.
\end{aligned}$$

Thus

$$\|T\|_{\mathcal{W}^0_{1,v}}\le C\|T\|^{\tilde\theta}_{\mathcal{W}^\alpha_{p,u}}\|T\|^{1-\tilde\theta}_{\mathcal{B}^2} \tag{3.16}$$

where v is the weight associated with the p-admissible weight u, and $C\in(0,\infty)$ and $\tilde\theta\in(0,1)$ are constants independent of $T\in\mathcal{W}^\alpha_{p,u}$. The conclusion (vii) then follows from (2.21) and (3.16). □

Now we start to prove Theorem 3.9.

Proof of Theorem 3.9. Take an operator $B:=\lambda I+T\in\mathcal{IW}^u_{p,\alpha}$. For $n\ge 1$, we write $B^n=\lambda^n I+B_n$ and define

$$b_n=\|B\|^n_{\mathcal{B}^2}+\|B^n\|_{\mathcal{IW}^\alpha_{p,u}}. \tag{3.17}$$

Then

$$b_{n+1}\le\|B\|^{n+1}_{\mathcal{B}^2}+\|B\|_{\mathcal{IW}^\alpha_{p,u}}\|B^n\|_{\mathcal{IW}^\alpha_{p,u}}\le b_1b_n \tag{3.18}$$

by Theorem 2.4. Let $\tilde\theta\in(0,1)$ and $C_0\in(0,\infty)$ be the constants in the conclusion (vii) of Lemma 3.10 and in the equation (2.22) respectively. Then by Theorem 2.2

and Lemma 3.10, we obtain

$$\begin{aligned} b_{4n} &\le \|B\|_{\mathcal{B}^2}^{4n} + |\lambda|^{4n} + C_0\|B_n^4\|_{\mathcal{W}_{p,u}^\alpha} + 4C_0|\lambda|^n\|B_n^3\|_{\mathcal{W}_{p,u}^\alpha} \\ &\quad +6C_0|\lambda|^{2n}\|B_n^2\|_{\mathcal{W}_{p,u}^\alpha} + 4|\lambda|^{3n}C_0\|B_n\|_{\mathcal{W}_{p,u}^\alpha} \\ &\le 4(|\lambda|^n + \|B\|_{\mathcal{B}^2}^n)(b_n)^3 + C\|B_n\|_{\mathcal{W}_{p,u}^\alpha}^2\|B_n^2\|_{\mathcal{W}_{p,u}^0} \\ &\le 4(|\lambda|^n + \|B\|_{\mathcal{B}^2}^n)(b_n)^3 + C(b_n)^{3+\tilde\theta}\|B_n\|_{\mathcal{B}^2}^{1-\tilde\theta} \\ &\le C(b_n)^{3+\tilde\theta}\big(\max(|\lambda|, \|B\|_{\mathcal{B}^2})\big)^{n(1-\tilde\theta)}. \end{aligned} \tag{3.19}$$

Therefore for any $n = \sum_{i=0}^k \epsilon_i 4^i$ with $\epsilon_i \in \{0, 1, 2, 3\}$,

$$\begin{aligned} b_n &\le b_1^{\epsilon_0} C^{1-\tilde\theta}(\max|\lambda|, \|B\|_{\mathcal{B}^2})^{n_1(1-\tilde\theta)}(b_{n_1})^{3+\tilde\theta} \\ &\le \cdots \\ &\le \prod_{t=0}^k (b_1)^{\epsilon_t(3+\tilde\theta)^t} \times \prod_{t=1}^k C^{(1-\tilde\theta)(3+\tilde\theta)^{t-1}} \\ &\qquad \times \prod_{t=1}^k \big(\max|\lambda|, \|B\|_{\mathcal{B}^2}\big)^{n_t(1-\tilde\theta)(3+\tilde\theta)^{t-1}} \\ &\le \Big(\frac{Cb_1}{\max(|\lambda|, \|B\|_{\mathcal{B}^2})}\Big)^{\frac{3(3+\tilde\theta)}{2+\tilde\theta} n^{\log_4(3+\tilde\theta)}} \Big(\max(|\lambda|, \|B\|_{\mathcal{B}^2})\Big)^n, \end{aligned} \tag{3.20}$$

where $n_t = \sum_{i=t}^k \epsilon_i 4^{i-t}$. Combining the above estimate of b_n with the fact that

$$|\lambda| \le \|B\|_{\mathcal{B}^2} \tag{3.21}$$

by the conclusion (iv) of Theorem 2.2 proves the desired conclusion (3.6). □

3.2. **Proof of Theorem 3.5.** To prove Theorem 3.5, we first recall the Schur class $\mathcal{A}_{p,u}(\Lambda)$ of infinite matrices in [46], see [3, 4, 6, 19, 29, 32, 34, 41] for other classes of infinite matrices and [30] for a historical review. We say that Λ is a relatively-separated subset of $\mathbb{R}^d$ if $\sup_{x\in\mathbb{R}^d}\sum_{\lambda\in\Lambda}\chi_{\lambda+[-1/2,1/2]^d}(x) < \infty$ where χ_E is the characteristic function on a set E. For $1 \le p \le \infty$, a relatively-separated set Λ and a weight u on $\mathbb{R}^d\times\mathbb{R}^d$, the Schur class $\mathcal{A}_{p,u}(\Lambda)$ of infinite matrices is defined as follows:

$$\mathcal{A}_{p,u}(\Lambda) = \Big\{A := (a(\lambda, \lambda'))_{\lambda,\lambda'\in\Lambda}, \quad \|A\|_{\mathcal{A}_{p,u}} < \infty\Big\} \tag{3.22}$$

where

$$\|A\|_{\mathcal{A}_{p,u}(\Lambda)} = \max\Big(\sup_{\lambda\in\Lambda}\|a(\lambda,\cdot)u(\lambda,\cdot)\|_{\ell^p}, \ \sup_{\lambda'\in\Lambda}\|a(\cdot,\lambda')u(\cdot,\lambda')\|_{\ell^p}\Big) \tag{3.23}$$

and $\|\cdot\|_{\ell^p}$ denotes the usual ℓ^p norm (see [6, 19, 34] for $p = \infty$ and [46] for $1 \le p \le \infty$). Also we define

$$\tilde{\mathcal{A}}_{p,u}(\Lambda) = \mathcal{A}_{p,u}(\Lambda) \cap \mathcal{A}_{1,u}(\Lambda) \tag{3.24}$$

with

$$\|A\|_{\tilde{\mathcal{A}}_{p,u}(\Lambda)} = \max(\|A\|_{\mathcal{A}_{p,u}(\Lambda)}, \|A\|_{\mathcal{A}_{1,u}(\Lambda)}), \quad A \in \tilde{\mathcal{A}}_{p,u}(\Lambda) \tag{3.25}$$

see [5, 29] for similar setting. For infinite matrices in $\tilde{\mathcal{A}}_{p,u}(\Lambda)$ with exponential weight, we have the following properties:

Lemma 3.11. *Let $1 \le p \le \infty$, $D > 0$ and Λ be a relatively-separated subset of $\mathbb{R}^d$. Then the following statements are true.*

(i) $\mathcal{A}_{p,e_{\tilde D,1}}(\Lambda) \subset \tilde{\mathcal{A}}_{p,e_{D,1}}(\Lambda)$ *for all* $\tilde D > D$.

(ii) $A \in \tilde{\mathcal{A}}_{p,e_{D,1}}(\Lambda)$ *if and only if* $A^* \in \tilde{\mathcal{A}}_{p,e_{D,1}}(\Lambda)$.

(iii) *If* $A, B \in \tilde{\mathcal{A}}_{p,e_{D,1}}(\Lambda)$, *then* $AB \in \tilde{\mathcal{A}}_{p,e_{D,1}}(\Lambda)$.

(iv) *If* $A \in \tilde{\mathcal{A}}_{p,e_{D,1}}(\Lambda)$ *and the pseudo-inverse* $A^\dagger$ *of the operator A is a bounded operator on* ℓ^2, *then there exists a positive constant* $D' \in (0, D)$ *such that* $A^\dagger \in \tilde{\mathcal{A}}_{p,e_{D',1}}(\Lambda)$.

Proof. We follow the same argument as the one used in [34, 46]. For the completeness of the paper, we include a concise proof here.

(i): The first conclusion holds because $\|\exp(-\delta|\cdot|)\|_{\ell^q} < \infty$ for all $\delta > 0$ and $1 \le q \le \infty$.

(ii): The second statement follows directly from the definition of the Schur class $\tilde{A}_{p,e_{D,1}}(\Lambda)$.

(iii): Note that

$$e_{D,1}(x,y) \le e_{D,1}(x,z)e_{D,1}(z,y),\ \ x,y,z \in \mathbb{R}^d. \tag{3.26}$$

Therefore for any $A, B \in \tilde{\mathcal{A}}_{p,e_{D,1}}$, we obtain

$$\begin{aligned}\|AB\|_{\tilde{\mathcal{A}}_{p,e_{D,1}}} &\le \max\Big(\|A\|_{\tilde{\mathcal{A}}_{p,e_{D,1}}}\|B\|_{\mathcal{A}_{1,e_{D,1}}}, \|A\|_{\mathcal{A}_{1,e_{D,1}}}\|B\|_{\tilde{\mathcal{A}}_{p,e_{D',1}}}\Big)\\ &\le \|A\|_{\tilde{\mathcal{A}}_{p,e_{D,1}}}\|B\|_{\tilde{\mathcal{A}}_{p,e_{D,1}}},\end{aligned} \tag{3.27}$$

which proves the third conclusion.

(iv): By (3.27), we have

$$\|A^n\|_{\tilde{\mathcal{A}}_{p,e_{D,1}}} \le \big(\|A\|_{\tilde{\mathcal{A}}_{p,e_{D,1}}}\big)^n,\ \ 1 \le n \in I\!N. \tag{3.28}$$

By the definition of an infinite matrix, we obtain

$$\|A^n\|_{\mathcal{A}_{\infty,e_{0,1}}} \le \|A^n\|_{\mathcal{B}^2} \le \|A\|^n_{\mathcal{B}^2},\ \ 1 \le n \in I\!N. \tag{3.29}$$

Combining (3.28) and (3.29) leads to

$$\begin{aligned}\|A^n\|_{\mathcal{A}_{1,e_{D',1}}} &\le \inf_{\tau\ge 1}\|A\|^n_{\mathcal{B}^2}\sum_{|k|\le\tau}\exp(D'|k|) + \|A\|_{\mathcal{A}_{1,e_{D,1}}}\exp(-(D-D')\tau)\\ &\le C\inf_{\tau\ge 1}\|A\|^n_{\mathcal{B}^2}\exp(2D'\tau) + \|A\|_{\mathcal{A}_{1,e_{D,1}}}\exp(-(D-D')\tau)\\ &\le C(\|A\|_{\tilde{\mathcal{A}}_{p,e_{D,1}}})^{nD'/(D+D')}(\|A\|_{\mathcal{B}^2})^{nD/(D+D')},\end{aligned}$$

and

$$\begin{aligned}\|A^n\|_{\mathcal{A}_{p,e_{D',1}}} &\le \inf_{\tau\ge 1}\|A\|^n_{\mathcal{B}^2}\big(\sum_{|k|\le\tau}e^{D'p|k|}\big)^{1/p} + \|A\|_{\mathcal{A}_{p,e_{D,1}}}e^{-(D-D')\tau}\\ &\le C(\|A\|_{\tilde{\mathcal{A}}_{p,e_{D,1}}})^{nD'/(D+D')}(\|A\|_{\mathcal{B}^2})^{nD/(D+D')}.\end{aligned}$$

Combining the above two estimates leads to

$$\|A^n\|_{\tilde{\mathcal{A}}_{p,e_{D',1}}} \le C(\|A\|_{\tilde{\mathcal{A}}_{p,e_{D,1}}})^{nD'/(D+D')}(\|A\|_{\mathcal{B}^2})^{nD/(D+D')}. \tag{3.30}$$

Therefore for any positive matrix $A \in \tilde{\mathcal{A}}_{p,e_{D,1}}$ with $C_1 I \le A \le C_2$,

$$\|A^{-1}\|_{\tilde{\mathcal{A}}_{p,e_{D',1}}} \le C_2^{-1} \sum_{n=0}^{\infty} \|B^n\|_{\tilde{\mathcal{A}}_{p,e_{D',1}}}$$

$$\le C \sum_{n=0}^{\infty} \Big(\frac{C_2 + \|A\|_{\tilde{\mathcal{A}}_{p,e_{D,1}}}}{C_2}\Big)^{nD'/(D+D')} \Big(\frac{C_2 - C_1}{C_2}\Big)^{nD/(D+D')} < \infty \tag{3.31}$$

by (3.30), where $B = I - C_2^{-1} A$ and $D' \in (0, D)$ is chosen so that

$$\Big(\frac{C_2 + \|A\|_{\tilde{\mathcal{A}}_{p,e_{D,1}}}}{C_2}\Big)^{D'} \Big(\frac{C_2 - C_1}{C_2}\Big)^{D} < 1.$$

By the standard holomorphic calculus ([40]),

$$A^\dagger = \int_{\mathcal{C}} \big((\bar{z} I - A^*)(zI - A)\big)^{-1} (\bar{z} I - A) \frac{dz}{z} \tag{3.32}$$

for any matrix A with $A^\dagger \in \mathcal{B}^2$, where $\mathcal{C}$ is a smooth curve on the complex plane that contains the nonzero spectrum of the operator A. Therefore the fourth conclusion follows from (3.31) and (3.32). $\square$

To prove Theorem 3.5, we need the following crucial result about the orthogonal complement of the null space $N(T)$.

Lemma 3.12. *Let $\alpha > 0$ and $u_0 \equiv 1$ be the trivial weight. Assume that $T \in \mathcal{W}^\alpha_{1,u_0}$ and its pseudo-inverse $T^\dagger$ belongs to $\mathcal{B}^2$. Then there exists $\delta_0 > 0$ such that $\Phi = \{\phi_\lambda,\ \lambda \in \delta_0 \mathbb{Z}^d\}$ is a frame generator of the locally finitely-generated space*

$$V_2(\Phi) := \Big\{ \sum_{\lambda \in \delta_0 \mathbb{Z}^d} c(\lambda) \phi_\lambda,\ \sum_{\lambda \in \delta_0 \mathbb{Z}^d} |c(\lambda)|^2 < \infty \Big\} \tag{3.33}$$

and

$$N(T)^\perp = V_2(\Phi), \tag{3.34}$$

where

$$\phi_\lambda(x) = \int_{[-\delta_0/2, \delta_0/2]^d} K_{T^*T}(\lambda + t, x) dt \in N(T)^\perp,\ \lambda \in \delta_0 \mathbb{Z}^d \tag{3.35}$$

and K_{T^*T} is the integral kernel of T^*T.

Proof. By Theorem 2.2 and the assumption on the integral operator T, we have

$$T^*T \in \mathcal{W}^\alpha_{1,u_0}, \tag{3.36}$$

$$T^*T f = 0 \quad \text{for all } f \in N(T) \tag{3.37}$$

and

$$A\|f\|_2 \le \|T^*T f\|_2 \le B\|f\|_2 \quad \text{for all } f \in N(T)^\perp \tag{3.38}$$

where A and B are positive constants.

We note that

$$\begin{aligned}\|T^*Tf\|_2 &\le \Big(\sum_{\lambda\in\delta\mathbb{Z}^d}\delta^{-d}\Big|\int_{[-\delta/2,\delta/2]^d}(T^*Tf)(\lambda+t)dt\Big|^2\Big)^{1/2}\\ &\quad+\Big(\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}\omega_\delta(K_{T^*T})(x,y)|f(y)|dy\Big|^2dx\Big)^{1/2}\\ &\le \Big(\sum_{\lambda\in\delta\mathbb{Z}^d}\delta^{-d}\Big|\int_{[-\delta/2,\delta/2]^d}(T^*Tf)(\lambda+t)dt\Big|^2\Big)^{1/2}\\ &\quad+C\delta^{\alpha-\alpha_0}\|T^*T\|_{\mathcal{W}^\alpha_{1,u_0}}\|f\|_2\quad\forall f\in L^2,\end{aligned}\tag{3.39}$$

where $\delta\in(0,1)$. Therefore by (3.36) – (3.39) there exists a sufficiently small $\delta_0\in(0,1)$ such that

$$\frac{1}{2}A\delta_0^{d/2}\|f\|_2\le\Big(\sum_{\lambda\in\delta_0\mathbb{Z}^d}\Big|\int_{[-\delta_0/2,\delta_0/2]^d}(T^*Tf)(\lambda+t)dt\Big|^2\Big)^{1/2}\le 2\delta_0^{d/2}B\|f\|_2$$

for all $f\in N(T)^\perp$, or equivalently

$$\frac{1}{2}A\delta_0^{d/2}\|f\|_2\le\Big(\sum_{\lambda\in\delta_0\mathbb{Z}^d}|\langle f,\phi_\lambda\rangle|_2\Big)^{1/2}\le 2\delta_0^{d/2}B\|f\|_2\tag{3.40}$$

for all $f\in N(T)^\perp$. Then the conclusion follows from (3.40) and the fact that $\phi_\lambda\in N(T)^\perp$ for all $\lambda\in\delta_0\mathbb{Z}^d$. □

Now we start to prove Theorem 3.5.

Proof of Theorem 3.5. Let δ_0 and $\Phi=\{\phi_\lambda\}_{\lambda\in\delta_0\mathbb{Z}^d}$ be as in Lemma 3.12. Define the auto-correlation matrices $A_{\Phi,\Phi}$ and $A_{T\Phi,T\Phi}$ for the frame generators Φ and $T\Phi$ of the space $V_2(\Phi)$ in (3.33) by

$$A_{\Phi,\Phi}=\big(\langle\phi_\lambda,\phi_\mu\rangle\big)_{\lambda,\mu\in\delta_0\mathbb{Z}^d}\tag{3.41}$$

and

$$A_{T\Phi,T\Phi}=\big(\langle T\phi_\lambda,T\phi_\mu\rangle\big)_{\lambda,\mu\in\delta_0\mathbb{Z}^d},\tag{3.42}$$

and let

$$H:=A_{\Phi,\Phi}\ell^2(\delta_0\mathbb{Z}^d)=\big\{(\langle f,\phi_\lambda\rangle)_{\lambda\in\delta_0\mathbb{Z}^d},\ f\in V_2(\Phi)\big\}.\tag{3.43}$$

Then by (3.41) – (3.43) and Theorem 2.2

$$A_{\Phi,\Phi}c=A_{T\Phi,T\Phi}c=0\tag{3.44}$$

for all $c\in H^\perp$,

$$\begin{cases}\|c\|^2_{\ell^2}=\langle S_\Phi f,f\rangle\\ \langle A_{\Phi,\Phi}c,c\rangle=\|S_\Phi f\|_2^2\\ \langle A_{T\Phi,T\Phi}c,c\rangle=\|TS_\Phi f\|_2^2\end{cases}\tag{3.45}$$

for all $c=(\langle f,\phi_\lambda\rangle)_{\lambda\in\delta_0\mathbb{Z}^d}\in H$, where $S_\Phi f=\sum_{\lambda\in\delta_0\mathbb{Z}^d}\langle f,\phi_\lambda\rangle\phi_\lambda$ is the frame operator on $V_2(\Phi)$.

For $1\le p\le\infty$ and $0\le\alpha,D_0<\infty$, we define

$$\tilde{\mathcal{W}}^\alpha_{p,e_{D_0},1}:=\mathcal{W}^\alpha_{p,e_{D_0},1}\cap\mathcal{W}^\alpha_{1,e_{D_0},1},\tag{3.46}$$

and

$$\|T\|_{\tilde{\mathcal W}^\alpha_{p,e_{D_0,1}}} = \max\left(\|T\|_{\mathcal W^\alpha_{p,e_{D_0,1}}}, \|T\|_{\mathcal W^\alpha_{1,e_{D_0,1}}}\right) \quad \text{for } T\in \tilde{\mathcal W}^\alpha_{p,e_{D_0,1}}. \tag{3.47}$$

Applying the similar argument used in the proof of Theorem 2.2 and recalling the fact that

$$e_{D_0,1}(x,y)\le e_{D_0,1}(x,z)e_{D_0,1}(z,y) \quad \text{for all } x,y,z\in\mathbb R^d \text{ and } D_0\ge 0, \tag{3.48}$$

we have

$$\|T_1T_2\|_{\tilde{\mathcal W}^\alpha_{p,e_{D_0,1}}} \le C\|T_1\|_{\tilde{\mathcal W}^\alpha_{p,e_{D_0,1}}}\|T_2\|_{\tilde{\mathcal W}^\alpha_{p,e_{D_0,1}}} \quad \text{for all } T_1,T_2\in \tilde{\mathcal W}^\alpha_{p,e_{D_0,1}}, \tag{3.49}$$

and

$$\|T_3^*\|_{\tilde{\mathcal W}^\alpha_{p,e_{D_0,1}}} = \|T_3\|_{\tilde{\mathcal W}^\alpha_{p,e_{D_0,1}}} \quad \text{for all } T_3\in \tilde{\mathcal W}^\alpha_{p,e_{D_0,1}}. \tag{3.50}$$

Also we notice that

$$\mathcal W^\alpha_{p,e_{\tilde D_0,1}} \subset \tilde{\mathcal W}^\alpha_{p,e_{D_0,1}} \subset \mathcal W^\alpha_{p,e_{D_0,1}} \tag{3.51}$$

for any $1\le p\le\infty$ and $0\le D_0<\tilde D_0$. Therefore

$$T\in \tilde{\mathcal W}^\alpha_{p,e_{\tilde D,1}} \tag{3.52}$$

by (3.51) and the assumption on the operator T, where $\tilde D=D/2$.

For any $\lambda,\mu\in\delta_0\mathbb Z^d$, we have

$$\langle\phi_\lambda,\phi_\mu\rangle=\int_{\delta_0[-1/2,1/2]^d}\int_{\delta_0[-1/2,1/2]^d}\tilde K_2(\lambda+t,\mu+s)dtds$$

and

$$\langle T\phi_\lambda,T\phi_\mu\rangle=\int_{\delta_0[-1/2,1/2]^d}\int_{\delta_0[-1/2,1/2]^d}\tilde K_3(\lambda+t,\mu+s)dtds$$

where $\tilde K_2$ and $\tilde K_3$ are the kernel of the integral operators $(T^*T)^2$ and $(T^*T)^3$ in $\tilde{\mathcal W}^\alpha_{p,e_{\tilde D,1}}$ respectively. The above two expressions together with (3.49), (3.50) and (3.52) implies that the correlation matrices $A_{\Phi,\Phi}$ and $A_{T\Phi,T\Phi}$ belong to the Schur class $\tilde{\mathcal A}_{p,e_{\tilde D,1}}(\delta_0\mathbb Z^d)$ of infinite matrices, i.e.,

$$A_{\Phi,\Phi}, A_{T\Phi,T\Phi}\in \tilde{\mathcal A}_{p,e_{\tilde D,1}}(\delta_0\mathbb Z^d). \tag{3.53}$$

By (3.40), (3.44), (3.45), (3.53), and Lemma 3.11, there exists $D'\in(0,D/2)$ such that the pseudo-inverses $A^\dagger_{\Phi,\Phi}$ and $A^\dagger_{T\Phi,T\Phi}$ of the auto-correlation matrices $A_{\Phi,\Phi}$ and $A_{T\Phi,T\Phi}$ belong to $\tilde{\mathcal A}_{p,e_{D',1}}(\delta_0\mathbb Z^d)$, that is,

$$A^\dagger_{\Phi,\Phi}:=(b(\lambda,\mu))_{\lambda,\mu\in\delta_0\mathbb Z^d}\in \tilde{\mathcal A}_{p,e_{D',1}}(\delta_0\mathbb Z^d) \tag{3.54}$$

and

$$A^\dagger_{T\Phi,T\Phi}:=(t(\lambda,\mu))_{\lambda,\mu\in\delta_0\mathbb Z^d}\in \tilde{\mathcal A}_{p,e_{D',1}}(\delta_0\mathbb Z^d). \tag{3.55}$$

Define

$$P(x,y)=\sum_{\lambda,\mu\in\delta_0\mathbb Z^d} b(\lambda,\mu)\phi_\lambda(x)\phi_\mu(y) \tag{3.56}$$

and

$$Pf(x) = \int_{\mathbb{R}^d} P(x,y)f(y)dy \quad \forall\, f \in L^2. \tag{3.57}$$

Then

$$Pf(x) = \sum_{\lambda,\mu\in\delta_0\mathbb{Z}^d} b(\lambda,\mu)\phi_\lambda(x)\int_{[-\delta_0/2,\delta_0/2]^d} Sf(\mu+t)dt = 0$$

for all $f \in N(T)$ by (3.37), and

$$\begin{aligned} Pf(x) &= \sum_{\lambda,\mu,\mu'\in\delta_0\mathbb{Z}^d} b(\lambda,\mu)\phi_\lambda(x)\langle\phi_\mu,\phi_{\mu'}\rangle c(\mu') \\ &= \sum_{\lambda\in\delta_0\mathbb{Z}^d} c(\lambda)\phi_\lambda =: f \quad \forall\, f \in N(T)^\perp = V_2(\Phi) \end{aligned}$$

by (3.34), (3.43) – (3.45), (3.54), and Lemma 3.12, where $c = (c(\lambda))_{\lambda\in\delta_0\mathbb{Z}^d} \in H$. Thus P is the projection operator onto $N(T)^\perp$, that is,

$$Pf = f \text{ if } f \in N(T)^\perp, \quad \text{and} \quad Pf = 0 \text{ if } f \in N(T). \tag{3.58}$$

Similarly we define

$$Q(x,y) = \sum_{\lambda,\mu\in\delta_0\mathbb{Z}^d} t(\lambda,\mu)\phi_\lambda(x)\phi_\mu(y) \tag{3.59}$$

and

$$Qf(x) = \int_{\mathbb{R}^d} Q(x,y)f(y)dy \quad \forall\, f \in L^2. \tag{3.60}$$

Clearly we have

$$QP = PQ = Q. \tag{3.61}$$

Also for any $\lambda \in \delta_0\mathbb{Z}^d$,

$$\begin{aligned} T^*TQ\phi_\lambda &= \sum_{\mu_1,\mu_2\in\delta_0\mathbb{Z}^d} t(\mu_1,\mu_2)\langle\phi_{\mu_2},\phi_\lambda\rangle T^*T\phi_{\mu_1} \\ &= \sum_{\mu_1,\mu_2,\mu_3,\mu_4\in\delta_0\mathbb{Z}^d} t(\mu_1,\mu_2)\langle\phi_{\mu_2},\phi_\lambda\rangle\langle T\phi_{\mu_1},T\phi_{\mu_3}\rangle b(\mu_3,\mu_4)\phi_{\mu_4} \\ &= \phi_\lambda, \end{aligned} \tag{3.62}$$

where we have used (3.58) to obtain the second equality. Therefore Q is the pseudo-inverse of the operator T^*T by (3.61) and (3.62).

Now we prove that $Q \in \tilde{\mathcal{W}}^{\alpha}_{p,e_{D',1}}$. Let $\varphi(x) = \chi_{[-\delta_0/2,\delta_0/2]^d}(x)$ and K be the kernel of the integral operator T^*T. By (3.49) and (3.55), we obtain

$$\begin{aligned}
&\big\|\partial_x^k\partial_y^l Q(x,y)\big\|_{\tilde{\mathcal{W}}^0_{p,e_{D',1}}}\\
= & \Big\|\int_{[-\delta_0/2,\delta_0/2]^d}\int_{[-\delta_0/2,\delta_0/2]^d}\sum_{\lambda,\mu\in\delta_0\mathbb{Z}^d} t(\lambda,\mu)\\
&\qquad \times(\partial_x^k K)(x,\lambda+t)(\partial_y^l K)(\mu+t',y)dtdt'\Big\|_{\tilde{\mathcal{W}}^0_{p,e_{D',1}}}\\
\le & C\Big\|\sum_{\lambda,\mu\in\delta_0\mathbb{Z}^d}|t(\lambda,\mu)|\varphi(x-\lambda)\varphi(y-\mu)\Big\|_{\tilde{\mathcal{W}}^0_{p,e_{D',1}}}\\
&\qquad \times\|(\partial_x^k K)(x,y)\|_{\tilde{\mathcal{W}}^0_{p,e_{D',1}}}\|(\partial_y^l K)(x,y)\|_{\tilde{\mathcal{W}}^0_{p,e_{D',1}}}\\
\le & C\|A^\dagger_{T\Phi,T\Phi}\|_{\tilde{\mathcal{A}}_{p,e_{D',1}}(\delta_0\mathbb{Z}^d)}\|T\|^4_{\tilde{\mathcal{W}}^\alpha_{p,e_{D',1}}},
\end{aligned}$$

and

$$\begin{aligned}
&\big\|\omega_\delta(\partial_x^k\partial_x^l Q)(x,y)\big\|_{\tilde{\mathcal{W}}^0_{p,e_{D',1}}}\\
\le & \Big\|\int_{[-\delta_0/2,\delta_0/2]^d}\int_{[-\delta_0/2,\delta_0/2]^d}\sum_{\lambda,\mu\in\delta_0\mathbb{Z}^d} |t(\lambda,\mu)|\\
&\qquad \times\omega_\delta(\partial_x^k K)(x,\lambda+t)|\partial_y^l K(y,\mu+t')|dtdt'\Big\|_{\tilde{\mathcal{W}}^0_{p,e_{D',1}}}\\
&+\Big\|\int_{[-\delta_0/2,\delta_0/2]^d}\int_{[-\delta_0/2,\delta_0/2]^d}\sum_{\lambda,\mu\in\delta_0\mathbb{Z}^d} |t(\lambda,\mu)|\\
&\qquad \times|\partial_x^k K(x,\lambda+t)||\omega_\delta(\partial_y^l K)(y,\mu+t')|dtdt'\Big\|_{\tilde{\mathcal{W}}^0_{p,e_{D',1}}}\\
&+\Big\|\int_{[-\delta_0/2,\delta_0/2]^d}\int_{[-\delta_0/2,\delta_0/2]^d}\sum_{\lambda,\mu\in\delta_0\mathbb{Z}^d} |t(\lambda,\mu)|\\
&\qquad \times|\omega_\delta(\partial_x^k K)(x,\lambda+t)||\omega_\delta(\partial_y^l K)(y,\mu+t')|dtdt'\Big\|_{\tilde{\mathcal{W}}^0_{p,e_{D',1}}}\\
\le & C\Big\|\sum_{\lambda,\mu\in\delta_0\mathbb{Z}^d}|t(\lambda,\mu)|\varphi(x-\lambda)\varphi(y-\mu)\Big\|_{\tilde{\mathcal{W}}^0_{p,e_{D',1}}}\\
&\quad\times\Big(\|\omega_\delta(\partial_x^k K)(x,y)\|_{\tilde{\mathcal{W}}^0_{p,e_{D',1}}}\|(\partial_y^l K)(x,y)\|_{\tilde{\mathcal{W}}^0_{p,e_{D',1}}}\\
&\qquad+\|(\partial_x^k K)(x,y)\|_{\tilde{\mathcal{W}}^0_{p,e_{D',1}}}\|\omega_\delta(\partial_y^l K)(x,y)\|_{\tilde{\mathcal{W}}^0_{p,e_{D',1}}}\\
&\qquad\|\omega_\delta(\partial_x^k K)(x,y)\|_{\tilde{\mathcal{W}}^0_{p,e_{D',1}}}\|\omega_\delta(\partial_y^l K)(x,y)\|_{\tilde{\mathcal{W}}^0_{p,e_{D',1}}}\Big)\\
\le & C\delta^{\alpha-\alpha_0}\|A^\dagger_{T\Phi,T\Phi}\|_{\tilde{\mathcal{A}}_{p,e_{D',1}}}\|T\|^4_{\tilde{\mathcal{W}}^\alpha_{p,e_{D',1}}}\quad \forall\ \delta\in(0,1),
\end{aligned}$$

where $k,l\in\mathbb{Z}^d$ with $|k|+|l|<\alpha$, and α_0 is the largest integer strictly smaller than α. The above two estimates prove the conclusion that the pseudo-inverse of $(T^*T)^\dagger$ is an integral operator in $\tilde{\mathcal{W}}^\alpha_{p,e_{D',1}}$. This together with $T^\dagger = (T^*T)^\dagger T^*$ and (3.49) and (3.50), (3.51) completes the proof. □

4. Wiener's Lemma for Localized Integral Operators: Part II

In this section, we introduce the second class of localized integral operators with the Kurbatov class as its model, and establish the Wiener's lemma for that class of localized integral operators (Theorems 4.1 and 4.3).

Take $\alpha \ge 0, 1 \le p \le \infty$ and a weight u on $\mathbb{R}^d \times \mathbb{R}^d$, and let α_0 be the largest integer strictly smaller than α. The second class $\mathcal{C}^\alpha_{p,u}$ of localized integral operators

$$Tf(x) = \int_{\mathbb{R}^d} K_T(x,y) f(y) dy, \ \ f \in L^2 \tag{4.1}$$

to be discussed in this paper is defined as follows:

$$\mathcal{C}^\alpha_{p,u} := \Big\{ T, \ \|T\|_{\mathcal{C}^\alpha_{p,u}} := \|K_T\|_{\mathcal{C}^\alpha_{p,u}} < \infty \Big\}, \tag{4.2}$$

where for a kernel function K on $\mathbb{R}^d \times \mathbb{R}^d$

$$\mathcal{G}(K)(x) = \sup_{y \in \mathbb{R}^d} |K(y, x+y)| \tag{4.3}$$

and

$$\|K\|_{\mathcal{C}^\alpha_{p,u}} = \begin{cases} \|\mathcal{G}(Ku)\|_p & \text{if } \alpha = 0, \\ \displaystyle\sum_{k,l \in \mathbb{Z}^d, |k|+|l|<\alpha} \|\partial_x^k \partial_y^l K(x,y)\|_{\mathcal{C}^0_{p,u}} \\ \quad + \displaystyle\sum_{k,l \in \mathbb{Z}^d, |k|+|l|<\alpha} \sup_{0<\delta\le 1} \delta^{-\alpha+\alpha_0} \|\omega_\delta(\partial_x^k \partial_y^l K(x,y))\|_{\mathcal{C}^0_{p,u}} \\ & \text{if } \alpha > 0. \end{cases}$$

For $\alpha = 0$ and the polynomial weight $u(x,y) = (1+|x-y|)^\gamma$ with $\gamma \ge 0$, we see that $T \in \mathcal{C}^\alpha_{p,u}$ if its kernel $K_T(x,y)$ is enveloped by the kernel $K_0(x-y)$ of a localized convolution operator,

$$|K_T(x,y)| \le K_0(x-y), \quad x, y \in \mathbb{R}^d$$

for some function K_0 with $\|K_0(\cdot)(1+|\cdot|)^\gamma\|_p < \infty$ (see [20, 37, 38] for $p=1$ and $\gamma = 0$). Clearly for any $1 \le p \le \infty$, regularity exponent $\alpha \ge 0$ and weight u, the class $\mathcal{C}^\alpha_{p,u}$ of integral operators just defined is a subset of the class $\mathcal{W}^\alpha_{p,u}$ of integral operator introduced in Section 2,

$$\mathcal{C}^\alpha_{p,u} \subset \mathcal{W}^\alpha_{p,u}, \tag{4.4}$$

but the converse is not true. For instance, the integral operator with kernel $K(x+y)$ belongs to $\mathcal{W}^\alpha_{1,u_0}$ but not to $\mathcal{C}^\alpha_{1,u_0}$ where K is a nonzero compactly supported C^∞ function and $u_0 \equiv 1$ is the trivial weight.

To state our results for localized integral operators in $\mathcal{C}^\alpha_{p,u}$, we recall a technical condition on the weight u ([46]). We say that a weight u is *strongly p-admissible* if there exists another weight v, two positive constants $C \in (0,\infty)$ and $\theta \in (0,1)$ such that (2.9) holds and

$$\inf_{\tau \ge 1} \big\|\mathcal{G}(v)\big\|_{L^1(B(0,\tau))} + t\big\|\mathcal{G}(vu^{-1})\big\|_{L^{p'}(\mathbb{R}^d \setminus B(0,\tau))} \le Ct^\theta \tag{4.5}$$

for all $t \ge 1$ ([46]). The model examples of strongly p-admissible weights are polynomial weights $u_\gamma(x,y) := (1+|x-y|)^\gamma$ with $\gamma > d(p-1)/p$, and the subexponential weights $e_{D,\delta}(x,y) = \exp(D|x-y|^\delta)$ with $D > 0$ and $0 < \delta < 1$. Clearly a strongly p-admissible weight is p-admissible. I believe that those two concepts for weights are different, but I do not have any particular example of p-admissible weight that is not strongly p-admissible.

Theorem 4.1. *Let $\alpha > 0, 1 \le p \le \infty$, u be a strongly p-admissible weight on $\mathbb{R}^d \times \mathbb{R}^d$, and $\mathcal{C}_{p,u}^\alpha$ be as in (4.2). Assume that $T \in \mathcal{C}_{p,u}^\alpha$, $0 \ne \lambda \in \mathbb{C}$ and $\lambda I + T$ has bounded inverse on $\mathcal{B}^2$. Then $(\lambda I + T)^{-1} = \lambda^{-1} I + \tilde{T}$ for some $\tilde{T} \in \mathcal{C}_{p,u}^\alpha$, or equivalently $(\lambda I + T)^{-1} T \in \mathcal{C}_{p,u}^\alpha$.*

Using the standard holomorphic calculus ([40]), we have the Wiener's lemma for the class $\mathcal{C}_{p,u}^\alpha$ of localized integral operators.

Corollary 4.2. *Let $\alpha > 0, 1 \le p \le \infty$, u be a strongly p-admissible weight on $\mathbb{R}^d \times \mathbb{R}^d$, and $T \in \mathcal{C}_{p,u}^\alpha$. Then the following statements hold.*

(i) *If T has bounded pseudo-inverse $T^\dagger$, then $T^\dagger \in \mathcal{C}_{p,u}^\alpha$.*
(ii) *If g is analytic on a neighborhood of the spectrum of the operator T with $g(0) = 0$, then $g(T) \in \mathcal{C}_{p,u}^\alpha$.*

The exponential weight $u(x, y) = \exp(D|x - y|)$ are not strong p-admissible weights. For localized integral operators $T \in \mathcal{C}_{p,u}^\alpha$ with exponential weights, we have

Theorem 4.3. *Let $\alpha, D > 0$, $e_{D,1}(x, y) = \exp(D|x - y|)$, and $\mathcal{C}_{p,e_{D,1}}^\alpha$ be as in (4.2). Assume that $T \in \mathcal{C}_{p,e_{D,1}}^\alpha$ and T has bounded pseudo-inverse $T^\dagger$. Then $T^\dagger \in \mathcal{C}_{p,e_{D',1}}^\alpha$ for some $D' \in (0, D)$.*

4.1. **Proof of Theorem 4.1.** We can prove Theorem 4.1 by following the proof of Theorem 3.1 line by line, except using Proposition 4.4 below instead of Theorem 2.2 and Lemma 3.10. We omit the details of the proof here.

Proposition 4.4. *Let $\alpha, \beta \ge 0$, $1 \le p, q \le \infty$ and u be a weight on $\mathbb{R}^d \times \mathbb{R}^d$. Then the following statements are true.*

(i) $(\mathcal{C}_{p,u}^\alpha)^* = \mathcal{C}_{p,u}^\alpha$.
(ii) *If $\beta \ge \alpha$ then $\mathcal{C}_{p,u}^\beta \subset \mathcal{C}_{p,u}^\alpha$.*
(iii) *If $\left\|\mathcal{G}(vu^{-1})\right\|_{pq/(p-q)} < \infty$ and $1 \le q \le p \le \infty$ then $\mathcal{C}_{p,u}^\alpha \subset \mathcal{C}_{q,v}^\alpha$.*
(iv) *If $T \in \mathcal{C}_{p,u}^\alpha$ and $\|\mathcal{G}(u^{-1})\|_{p/(p-1)} < \infty$, then T is a bounded operator on $L^q, 1 \le q \le \infty$, but T does not have bounded inverse on $L^q, 1 \le q < \infty$.*
(v) *If $T_1, T_2 \in \mathcal{C}_{p,u}^\alpha$ then $T_1 T_2 \in \mathcal{C}_{p,u}^\alpha$.*
(vi) *If $T_1, T_3 \in \mathcal{C}_{p,u}^\alpha$ and $T_2 \in \mathcal{C}_{p,u}^0$ then $T_1 T_2 T_3 \in \mathcal{C}_{p,u}^\alpha$.*
(vii) *If $\alpha > 0$ then there exists $C \in (0, \infty)$ and $\tilde{\theta} \in (0, 1)$ such that*

$$\|T^2\|_{\mathcal{C}_{p,u}^0} \le C \|T\|_{\mathcal{C}_{p,u}^\alpha}^{1+\tilde{\theta}} \|T\|_{\mathcal{B}^2}^{1-\tilde{\theta}} \quad \text{for all } T \in \mathcal{C}_{p,u}^\alpha. \tag{4.6}$$

Proof. We will use the similar argument to the one in the proofs of Theorem 2.2 and Lemma 3.10, except when proving the paracompact property (vii). We include a complete proof for the convenience.

(i): The first conclusion holds because of $\|T^*\|_{\mathcal{C}_{p,u}^\alpha} = \|T\|_{\mathcal{C}_{p,u}^\alpha}$ for all $T \in \mathcal{C}_{p,u}^\alpha$.

(ii): The second conclusion is true since $\|T\|_{\mathcal{C}_{p,u}^\alpha} \le \|T\|_{\mathcal{C}_{p,u}^\beta}$ for all $T \in \mathcal{C}_{p,u}^\beta$ with $\beta \ge \alpha$.

(iii): The third conclusions follows from the fact that

$$\|T\|_{\mathcal{C}^\alpha_{q,v}} \le \big\|\mathcal{G}(vu^{-1})\big\|_{pq/(p-q)}\|T\|_{\mathcal{C}^\alpha_{p,u}} \quad \text{for all } T\in \mathcal{C}^\alpha_{p,u}. \tag{4.7}$$

(iv): The fourth conclusion can be easily deducted from (4.4) and the fourth conclusion of Theorem 2.2.

(v): Take two integral operators $T_1, T_2\in \mathcal{C}^\alpha_{p,u}$. From (2.1) – (2.3), (2.9), (2.19) and (2.20) it follows that $\|T_1T_2\|_{\mathcal{C}^\alpha_{p,u}} \le C\|T_1\|_{\mathcal{C}^\alpha_{p,u}}\|T_2\|_{\mathcal{C}^\alpha_{1,v}} + C\|T_1\|_{\mathcal{C}^\alpha_{1,v}}\|T_2\|_{\mathcal{C}^\alpha_{p,u}}$ where v is the weight in (2.19). This together with (4.7) implies that

$$\|T_1T_2\|_{\mathcal{C}^\alpha_{p,u}} \le C\|\mathcal{G}(vu^{-1})\|_{p/(p-1)}\|T_1\|_{\mathcal{C}^\alpha_{p,u}}\|T_2\|_{\mathcal{C}^\alpha_{p,u}}, \tag{4.8}$$

and hence completes the proof of the fifth conclusion.

(vi): Take $T_1, T_3\in \mathcal{C}^\alpha_{p,u}$ and $T_2\in\mathcal{C}^0_{p,u}$. By (3.9), (3.10) and (4.8), we have

$$\|T_1T_2T_3\|_{\mathcal{C}^\alpha_{p,u}} \le C\|T_1\|_{\mathcal{C}^\alpha_{p,u}}\|T_2\|_{\mathcal{C}^0_{p,u}}\|T_3\|_{\mathcal{C}^\alpha_{p,u}}. \tag{4.9}$$

The conclusion (vi) is then proved.

(vii): Take an integral operator $T\in \mathcal{C}^\alpha_{p,u}$ and let K be its kernel. Then by (2.1) – (2.3), (2.9), (3.14) and (4.5), we obtain

$$\begin{aligned}
\|T^2\|_{\mathcal{C}^0_{p,u}} &= \Big\|\sup_{x\in\mathbb{R}^d}\Big|\int_{\mathbb{R}^d} K(x,x+z)K(x+z,x+\cdot)dz\Big|u(x,x+\cdot)\Big\|_p\\
&\le C\inf_{\tau\ge 1}\Big\{\Big\|\sup_{x\in\mathbb{R}^d}\int_{|z-\cdot|\le\tau}\mathcal{G}(Ku)(z)\big\{|(Kv)(x+z,x+\cdot)|\\
&\qquad +|(Kv)(x,x+\cdot-z)|\big\}dz\Big\|_p\\
&\qquad +C\Big\|\int_{|z-\cdot|\ge\tau}\mathcal{G}(Ku)(z)\mathcal{G}(Kv)(\cdot-z)dz\Big\|_p\Big\}\\
&\le C\inf_{\tau\ge1}\inf_{0<\delta_0\le1}\Big\{\Big\|\int_{|z-\cdot|\le\tau}\mathcal{G}(Ku)(z)\mathcal{G}(\omega_{\delta_0}(K)v)(\cdot-z)dz\Big\|_p\\
&\qquad +\delta_0^{-d}\|T\|_{\mathcal{B}^2}\Big\|\int_{|z-\cdot|\le\tau}\mathcal{G}(Ku)(z)\mathcal{G}(v)(\cdot-z)dz\Big\|_p\\
&\qquad +\|\mathcal{G}(Ku)\|_p\|\mathcal{G}(Kv)\|_{L^1(\mathbb{R}^d\backslash B(0,\tau))}\Big\}\\
&\le C\inf_{\tau\ge1}\inf_{0<\delta_0\le1}\Big\{\delta_0^{\alpha-\alpha_0}\|T\|_{\mathcal{C}^0_{p,u}}\|T\|_{\mathcal{C}^\alpha_{p,u}}\|\mathcal{G}(vu^{-1})\|_{L^{p/(p-1)}(B(0,\tau))}\\
&\qquad +\delta_0^{-d}\|T\|_{\mathcal{B}^2}\|T\|_{\mathcal{C}^0_{p,u}}\|\mathcal{G}(v)\|_{L^1(B(0,\tau))}\\
&\qquad +\|T\|^2_{\mathcal{C}^0_{p,u}}\|\mathcal{G}(vu^{-1})\|_{L^{p/(p-1)}(\mathbb{R}^d\backslash B(0,\tau))}\Big\}\\
&\le C\inf_{\tau\ge1}\Big\{\|T\|_{\mathcal{C}^0_{p,u}}\|\mathcal{G}(v)\|_{L^1(B(0,\tau))}\inf_{0<\delta_0\le1}\Big(\delta_0^{\alpha-\alpha_0}\|T\|_{\mathcal{C}^\alpha_{p,u}}+\delta_0^{-d}\|T\|_{\mathcal{B}^2}\Big)\\
&\qquad +\|T\|^2_{\mathcal{C}^0_{p,u}}\|\mathcal{G}(vu^{-1})\|_{L^{p/(p-1)}(\mathbb{R}^d\backslash B(0,\tau))}\Big\}\\
&\le C\|T\|_{\mathcal{C}^\alpha_{p,u}}^{\frac{\alpha-\alpha_0+2d}{\alpha-\alpha_0+d}}\inf_{\tau\ge1}\Big\{\|T\|_{\mathcal{B}^2}^{\frac{\alpha-\alpha_0}{\alpha-\alpha_0+d}}\|\mathcal{G}(v)\|_{L^1(B(0,\tau))}\\
&\qquad +\|T\|_{\mathcal{C}^\alpha_{p,u}}^{\frac{\alpha-\alpha_0}{\alpha-\alpha_0+d}}\|\mathcal{G}(vu^{-1})\|_{L^{p/(p-1)}(\mathbb{R}^d\backslash B(0,\tau))}\Big\}\\
&\le C\|T\|^{1+\tilde\theta}_{\mathcal{C}^\alpha_{p,u}}\|T\|^{1-\tilde\theta}_{\mathcal{B}^2}, \qquad (4.10)
\end{aligned}$$

where v and θ are the weight and the constant associated with the strong p-admissible weight u, and $\tilde\theta = \frac{\theta(\alpha-\alpha_0)+d}{\alpha-\alpha_0+d}$. Therefore the conclusion (vii) follows. $\square$

4.2. **Proof of Theorem 4.3.** For $1 \le p \le \infty$ and a weight u, we recall the Sjöstrand class $\mathcal{C}_{p,u}(\delta_0\mathbb{Z}^d)$ of infinite matrices

$$\mathcal{C}_{p,u}(\delta_0\mathbb{Z}^d) := \Big\{A = (A(\lambda,\mu))_{\lambda,\mu\in\delta_0\mathbb{Z}^d},\ \ \|A\|_{\mathcal{C}_{p,u}(\delta_0\mathbb{Z}^d)} < \infty\Big\}, \tag{4.11}$$

where $\delta_0 > 0$ and

$$\|A\|_{\mathcal{C}_{p,u}(\delta_0\mathbb{Z}^d)} := \Big\|\Big(\sup_{\lambda\in\delta_0\mathbb{Z}^d} |A(\lambda,\lambda+\mu)|u(\lambda,\lambda+\mu)\Big)_{\mu\in\delta_0\mathbb{Z}^d}\Big\|_{\ell^p} \tag{4.12}$$

(see [41] for $p=1$, [34] for $p=\infty$, and [46] for $1 \le p \le \infty$). Define

$$\tilde{\mathcal{C}}_{p,u}(\delta_0\mathbb{Z}^d) = \mathcal{C}_{p,u}(\delta_0\mathbb{Z}^d) \cap \mathcal{C}_{1,u}(\delta_0\mathbb{Z}^d) \tag{4.13}$$

and

$$\|A\|_{\tilde{\mathcal{C}}_{p,u}(\delta_0\mathbb{Z}^d)} = \max\big(\|A\|_{\mathcal{C}_{p,u}(\delta_0\mathbb{Z}^d)}, \|A\|_{\mathcal{C}_{1,u}(\delta_0\mathbb{Z}^d)}\big) \text{ for } A \in \tilde{\mathcal{C}}_{p,u}(\delta_0\mathbb{Z}^d). \tag{4.14}$$

For the Sjöstrand class $\mathcal{C}_{p,u}(\delta_0\mathbb{Z}^d)$ with exponential weight, we have the following result.

Lemma 4.5. *Let $1 \le p \le \infty$ and $D, \delta_0 > 0$. Then the following statements are true.*

(i) *$A \in \tilde{\mathcal{C}}_{p,e_{D,1}}(\delta_0\mathbb{Z}^d)$ if and only if $A^* \in \tilde{\mathcal{C}}_{p,e_{D,1}}(\delta_0\mathbb{Z}^d)$.*

(iii) *If $A, B \in \tilde{\mathcal{C}}_{p,e_{D,1}}(\delta_0\mathbb{Z}^d)$, then $AB \in \tilde{\mathcal{C}}_{p,e_{D,1}}(\delta_0\mathbb{Z}^d)$.*

(iv) *If $A \in \tilde{\mathcal{C}}_{p,e_{D,1}}(\delta_0\mathbb{Z}^d)$ and the pseudo-inverse $A^\dagger$ of the operator A is a bounded operator on ℓ^2, then there exists a positive constant $D' \in (0, D)$ such that $A^\dagger \in \tilde{\mathcal{C}}_{p,e_{D',1}}(\delta_0\mathbb{Z}^d)$.*

Proof. The conclusions can be proved by following the proof of Lemma 3.11 step by step, except replacing $\tilde{\mathcal{A}}_{p,e_{D,1}}(\delta_0\mathbb{Z}^d)$ by $\mathcal{C}_{p,e_{D,1}}(\delta_0\mathbb{Z}^d)$. We omit the details of the proof here. $\square$

Now we start to prove Theorem 4.3.

Proof of Theorem 4.3. The conclusion can be proved by copying the proof of Theorem 3.5 line by line, except replacing $\tilde{\mathcal{A}}_{p,e_{D,1}}(\delta_0\mathbb{Z}^d)$ by $\tilde{\mathcal{C}}_{p,e_{D,1}}(\delta_0\mathbb{Z}^d)$, $\mathcal{C}^\alpha_{p,u}$ by $\mathcal{W}^\alpha_{p,u}$, and using Lemma 4.5 instead of Lemma 3.11. $\square$

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