FRAMES IN SPACES WITH FINITE RATE OF INNOVATION

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Abstract. Signals with finite rate of innovation are those signals having finite degrees of freedom per unit of time that specify them. In this paper, we introduce a prototypical space $V_q(\Phi, \Lambda)$ modelling signals with finite rate of innovation, such as stream of (different) pulses found in GPS applications, cellular radio and ultra wide-band communication. In particular, the space $V_q(\Phi, \Lambda)$ is generated by a family of well-localized molecules Φ of similar size located on a relatively-separated set Λ using ℓ^q coefficients, and hence is locally finitely-generated. Moreover that space $V_q(\Phi, \Lambda)$ includes finitely-generated shift-invariant spaces, spaces of non-uniform splines, and the twisted shift-invariant space in Gabor (Wilson) system as its special cases. Use the well-localization property of the generator Φ, we show that if the generator Φ is a frame for the space $V_2(\Phi, \Lambda)$ and has polynomial (subexponential) decay, then its canonical dual (tight) frame has the same polynomial (subexponential) decay. We apply the above result about the canonical dual frame to the study of the Banach frame property of the generator Φ for the space $V_q(\Phi, \Lambda)$ with $q \neq 2$, and of the polynomial (subexponential) decay property of the mask associated with a refinable function that has polynomial (subexponential) decay.

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1. Introduction

A signal that has finite degree of freedom per unit of time, the number of samples per unit of time that specify it, is called to be a signal *with finite rate of innovation* ([60]). In this paper, we introduce a prototypical space generated by a family of separated-located molecules of similar size for modelling (periodic, discrete) signals with finite rate of innovation, and study the (Banach) frame property of that generating family for that prototypical space. The non-uniform sampling and stable reconstruction problem for that prototypical space will be discussed in the subsequent paper [58].

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Our prototypical space $V_q(\Phi, \Lambda)$, for modelling time signals with finite rate of innovation, is given by

$$V_q(\Phi,\Lambda) := \Big\{ \sum_{\lambda\in\Lambda} c(\lambda)\phi_\lambda,\ (c(\lambda))_{\lambda\in\Lambda} \in \ell^q(\Lambda)\Big\}, \tag{1.1}$$

where Λ is a relatively-separated subset of $\mathbf{R}$, ℓ^q is the space of all q-summable sequences on Λ, and the generator $\Phi := \{\phi_\lambda,\ \lambda \in \Lambda\}$ is enveloped by a function h in a Wiener amalgam space $W_q(L_{p,u})$,

$$|\phi_\lambda(x)| \le h(x-\lambda) \text{ for all } \lambda \in \Lambda, \tag{1.2}$$

see Example 2.10 for details, and see also (2.13) for another convenient well-localization assumption on the generator Φ considered in the paper.

Given any function $f = \sum_{\lambda\in\Lambda} c(\lambda)\phi_\lambda \in V_q(\Phi, \Lambda)$, we see from the definition of the space $V_q(\Phi, \Lambda)$ that, on any unit interval $t + [-1/2, 1/2)$ of time, it is determined by the coefficients $c(\lambda)$ with $\lambda \in t + [-1/2, 1/2)$. Recalling that the total number of sampling locations $\lambda \in \Lambda$ on each unit interval $t+[-1/2, 1/2)$ is bounded by the upper bound $D(\Lambda)$ of the relatively-separated subset Λ (see (2.4) for the definition), we then conclude that any function in the space $V_q(\Phi, \Lambda)$ has finite rate of innovation.

The space $V_q(\Phi, \Lambda)$ is suitable for modelling (i) band-limited signals ([9, 34, 37]), (ii) signals in a shift-invariant space ([1, 3]), (iii) signals in the time-frequency plane ([28]), (iv) non-uniform splines ([53, 60]), and (v) diffusion wavelets ([20]), see Example 3.1 for details. More importantly, the space $V_q(\Phi, \Lambda)$ is very convenient for modelling most of known signals with finite rate of innovations in [22, 33, 40, 44, 46, 47, 60], for instance, (vi) stream of pulses $\sum_l a_l p(t-t_l)$, found in example in GPS applications and cellular radio, where $p(t)$ is the antenna transmit pulse shape; (vii) stream of different pulses $\sum_l a_l p_l(t - t_l)$ found in modelling ultra wide-band, where different incoming paths are subjected to different frequency-selective attentuations; (viii) bandlimited signals with additive shot noise $\sum_{k\in\mathbf{Z}} c(k)\text{sinc}(t - k) + \sum_l d(l)\delta(t - t_l)$; (ix) sum of bandlimited signals and non-uniform spline signals, convenient for modelling electrocardiogram signals.

In this paper, we consider the (Banach) frame property of the well-localized generator Φ for the space $V_q(\Phi, \Lambda)$. The principal results of this paper are Theorems 4.1 and 4.5, where it is shown that if Φ is a frame (resp. Riesz basis) for the space $V_2(\Phi, \Lambda) \subset L^2$ and Φ is enveloped by a function h in the Wiener amalgam space $W_q(L_{p,u})$ for the polynomial weight $u(x) = (1 + |x|)^\alpha$ with $\alpha > d(1 - 1/p)$ or the subexponential weight $u(x) = \exp(D|x|^\delta)$ with $D > 0$ and $\delta \in (0, 1)$, then its canonical dual frame (resp. its dual Riesz basis) $S^{-1}\Phi$ is enveloped by another function $\tilde h$ in the *same* Wiener amalgam space. Unlike in the study of frame and Riesz basis property for the shift-invariant setting, see Example 3.1 for details of that setting, the main obstacle to solve the frame (Riesz basis) problem for the space $V_2(\Phi, \Lambda)$ generated by separately-located molecules Φ comes from the non-group structure on the molecules, which makes the standard approach from Fourier analysis inapplicable. Our approach to the above frame problem is to show that the generator Φ enveloped by a function in a Wiener amalgam space is self-localized in some matrix algebra of (Schur) Sjöstrand class, see for instance [4, 24, 31, 32, 35, 56, 57] and references therein for other numerous applications of various matrix algebras in numerical analysis, wavelet theory, frame theory, and sampling theory.

The paper is organized as follow. In Section 2, we recall some basic properties for spaces of homogenous type, some basic concept related to (Banach) frames and basis, and we introduce families of separately-located molecules of size one on spaces of homogenous type. Using the families of separately-located molecules of size one in Section 2 as well-localized building blocks, we generate a prototypical space $V_q(\Phi, \Lambda)$ in Section 3 for modelling (periodic, discrete) signals with finite rate of innovation, band-limited signals, signals in shift-invariant spaces, signals in time-frequency domain, non-uniform splines, and diffusion wavelets. We show in Section 3 that the space $V_q(\Phi, \Lambda)$ is a subspace of L^q when the family Φ of building blocks are separately-located and appropriately localized, see Theorem 3.2 for details. In Section 4, we consider the canonical dual frame, canonical tight frame, dual Riesz basis, orthonormal basis for the space $V_2(\Phi, \Lambda)$, and establish the principal results of this paper that if Φ is a frame (Riesz basis) of $V_2(\Phi, \Lambda)$ and has certain polynomial (subexponential) decay, then its canonical dual frame (dual Riesz basis), canonical tight frame (orthonormal basis) have the *same* polynomial (subexponential) decay, see Theorems 4.1 and 4.5 for details. The corollaries of the above principal results for special shift-invariant setting, (periodic) finite-innovation-rate setting, and Gabor (Wilson) frame setting are mostly new, see Corollaries 4.3 and 4.6 – 4.8 for details. In Section 5, we apply Theorems 4.1 and 4.5 to show that if the generator Φ is a frame for $V_2(\Phi, \Lambda)$ and is enveloped by a function in a Wiener amalgam space then it is Banach frame for the space $V_q(\Phi, \Lambda)$ with $q \neq 2$, see Theorem 5.1. Finally in last section, we apply Theorems 4.1 and 4.5 to establish some connection between a globally-supported refinable function and its mask, particularly, we prove in Theorem 6.1 that if the generator Φ is a Riesz basis for $V_2(\Phi, \Lambda)$, has polynomial decay at infinity, and satisfies a refinement equation, then its mask has the same polynomial decay.

In this paper, the capital letter C, if unspecified, denotes an absolute constant which may be different at different occurrences.

2. Preliminary

2.1. **Spaces of homogenous type.** Let X be a set. A function $\rho : X \times X \longmapsto [0, \infty)$ is called a *quasi-metric* if (i) $\rho(x, y) \geq 0$ for every $x, y \in X$, $\rho(x, x) = 0$ for all $x \in X$, and $\#\{y \in X : \rho(x, y) = 0\} \leq D_0$ for all $x \in X$;

(ii) $\rho(x,y)=\rho(y,x)$ for all $x,y\in X$; and (iii) $\rho(x,y)\le L(\rho(x,z)+\rho(z,y))$ for all $x,y,z\in X$, where D_0 and L are positive constants, and $\#(E)$ denotes the cardinality of a set E. In the standard definition of a quasi-metric ρ, the set $\{y\in X:\rho(x,y)=0\}$ is assumed to be $\{x\}$ for all $x\in X$ instead ([21]).

The pair (X,ρ) is called a *quasi-metric space.*

A quasi-metric space (X,ρ) with Borel measure μ, to be denoted by (X,ρ,μ), is said to be a *space of homogenous type* if μ is a non-negative Borel measure that satisfies the *doubling condition*:

$$0<\mu(B(x,\tau))\le D_1\mu(B(x,\tau/2))<\infty \text{ for all } \tau>0 \text{ and } x\in X, \tag{2.1}$$

where D_1 is a positive constant and $B(x,\tau):=\{y\in X:\ \rho(x,y)<\tau\}$ is the open ball of radius τ around x ([21, 41, 42]).

Remark 2.1. We say that a Borel measure μ on the space of homogenous type satisfies *uniform boundedness conditions* if

$$D_2\le \mu(B(x,1))\le D_3 \quad \text{for all } x\in X \tag{2.2}$$

where D_2, D_3 are positive constants. For any space of homogenous type (X,ρ,μ), it is known that an equivalent quasi-metric $\tilde\rho$ to the quasi-metric ρ, in the sense that quasi-metric spaces (X,ρ) and $(X,\tilde\rho)$ have the same topology, such that the uniform boundedness condition is satisfied ([41, 42]). So in this paper, *we always assume that the Borel measure μ for a space of homogenous type (X,ρ,μ) satisfies the uniform boundedness condition.* The uniform boundedness condition will be used later to establish the connection between a space of homogenous type and its discretized subsets, such as relatively-separated subsets or lattices, see, for instance, Proposition 2.7, Remark 2.8, and Theorem 2.11.

Example 2.2. Examples of spaces of homogeneous type include, for instance, (i) Euclidean spaces of any dimension with isotropic or anisotropic metrics induced by positive-definite bilinear forms (see e.g. [8, 58]); (ii) the time-frequency plane $\mathbf{R}^d\times\mathbf{R}^d$ with the quasi-metric ρ_G defined by $\rho_G(x,y)=\min_{g_1,g_2\in G}|g_1x-g_2y|$ for $x,y\in\mathbf{R}^d\times\mathbf{R}^d$, where G is a finite subgroup of the group GL_{2d} of all nonsingular $(2d)\times(2d)$ matrices (see e.g. [29]); and (iii) compact Riemannian manifolds of bounded curvature, with respect to geodesic metric, or also with respect to metrics induced by certain classes of vector fields (see e.g. [20, 49]). Our model examples of spaces of homogenous type in our considering signals with finite rate of innovation are:

(i) Euclidean space $\mathbf{R}^d$ with standard Euclidean metric $|\cdot|$ and Lebesgue measure μ; and
(ii) the sphere $S^d\subset\mathbf{R}^{d+1}$ with dilated Euclidean metric ρ_δ and dilated Lebesgue measure μ_δ on S^d, that is, $\rho_\delta(x,y)=\delta|x-y|, x,y\in S^d$ and $\mu_\delta=\delta^{-d}m_0$ where $0<\delta\le 1$, and $|\cdot|$ and m_0 are standard Euclidean metric and Lebesgue measure on S^d. For this model example, one

may verify that the constants L and $D_i, 0 \le i \le 3$, in the definition of a quasi-metric and in the doubling and uniform boundedness conditions for a Borel measure are bounded by some constant C that is *independent* of the parameter $\delta \in (0,1]$.

2.2. **Relatively-separated subsets and lattices.** Let (X,ρ) be a quasi-metric space. A discrete subset Λ of X is said to be a *lattice* if there exists a positive constant D_4 such that

$$1 \le \sum_{\lambda \in \Lambda} \chi_{B(\lambda,1)}(x) \le D_4 \quad \text{for all } x \in X \tag{2.3}$$

([20]); and to be a *relatively-separated subset* if there exists a positive constant $D(\Lambda)$ such that

$$\sum_{\lambda \in \Lambda} \chi_{B(\lambda,1)}(x) \le D(\Lambda) \quad \text{for all } x \in X \tag{2.4}$$

([1, 3]). Relatively-separated subsets of a space of homogenous type will be used as the set of location of the molecules, see the Subsection 2.4, while the lattice is used to define separately-located molecules of size one, see Remarks 2.3 and 2.8 below.

Remark 2.3. A quasi-metric space (X,ρ) is said to be *uniformly discretizable* if there exists a lattice. Given a space of homogenous type (X,ρ,μ), if it is a separable topological space, one may easily verify that it is uniformly discretizable. So, in this paper, *we always assume that a space of homogenous type is uniformly discretizable.*

For a relatively-separated subset Λ of a space of homogenous type (X,ρ,μ), the pair (Λ,ρ) is a quasi-metric space. The natural counting measure μ_c on (Λ,ρ) satisfies uniform boundedness condition because

$$1 \le \mu_c\{\lambda \in \Lambda : \rho(\lambda,\lambda') < 1\} = \sum_{\lambda \in \Lambda} \chi_{B(\lambda,1)}(\lambda') \le D(\Lambda) \quad \text{for all } \lambda' \in \Lambda$$

by (2.4). But the counting measure μ_c does not satisfy the doubling condition in general. For example, the discrete set $\mathbf{Z}_\alpha := \{2^{n\alpha} : 0 \le n \in \mathbf{Z}\}, \alpha > 1$, is a relatively-separated subset of the real line $\mathbf{R}$ with standard norm, and the counting measure on $\mathbf{Z}_\alpha$ does not satisfy the doubling condition, since

$$\frac{\mu_c(B(2^{n\alpha}, 2^{n\alpha-1}))}{\mu_c(B(2^{n\alpha}, 2^{n\alpha}))} \to 0 \text{ as } n \to \infty.$$

Therefore for a relatively-separated subset Λ, (Λ,ρ,μ_c) is not always a space of homogenous type, while, on the other hand, it can be verified that, for a lattice Λ, the space (Λ,ρ,μ_c) is a space of homogenous type.

Example 2.4. *(Example 2.2 first revisited)* The set $\Lambda_0 := \{k + \epsilon_k : k \in \mathbf{Z}^d, \epsilon_k \in [-1/2, 1/2)^d\}$ used in the study of nonharmonic analysis ([61]) is a typical example of relatively-separated subsets of the space of homogenous type $(\mathbf{R}^d, |\cdot|, m)$. Similarly, one may verify that $\Lambda_\delta := \{k/|k| : k \in$

$\delta\mathbf{Z}^{d+1}, ||k|-1| \le d\delta\}$ is a relatively-separated subset of the space of homogenous type $(S^d, \rho_\delta, \mu_\delta)$ and that a positive constant C independent of $\delta \in (0,1]$ can be found such that the constant $D(\Lambda_\delta)$ in (2.4) satisfies $D(\Lambda_\delta) \le C$ for all $\delta \in (0,1]$.

2.3. (p,r)**-admissible weights.** Let (X, ρ, μ) be a space of homogenous type. A *weight* w in this paper means a positive symmetric continuous function on $X \times X$ that satisfies

$$1 \le w(x,y) = w(y,x) < \infty \ \text{for all}\ x, y \in X; \tag{2.5}$$

$$w(x,x) \le D_5 \ \text{for all}\ x \in X;\ \text{and} \tag{2.6}$$

$$\sup_{\rho(x,\tilde{x})+\rho(y,\tilde{y})\le C_0} \frac{w(x,y)}{w(\tilde{x},\tilde{y})} \le D_6 \quad \text{for all}\ x, y, \tilde{x}, \tilde{y} \in X, \tag{2.7}$$

for any given $C_0 > 0$, where $D_5 := D_5(w)$ and $D_6 := D_6(C_0, w)$ are positive constants.

Let $1 \le p, r \le \infty$. We say that a weight w is (p,r)-*admissible* if there exist another weight v and two positive constants $D_7 := D_7(w) \in (0,\infty)$ and $\theta := \theta(w) \in (0,1)$ such that

$$w(x,y) \le D_7(w(x,z)v(z,y) + v(x,z)w(z,y)) \quad \text{for all}\ x, y, z \in X, \tag{2.8}$$

$$\sup_{x\in X} \|(vw^{-1})(x,\cdot)\|_{L^{p'}(X)} \le D_7,\ \text{and} \tag{2.9}$$

$$\inf_{\tau>0} \sup_{x\in X} \|v(x,\cdot)\|_{L^{r'}(B(x,\tau))} + t \sup_{x\in X} \|(vw^{-1})(x,\cdot)\|_{L^{p'}(X\setminus B(x,\tau))} \le D_7 t^\theta \tag{2.10}$$

for all $t \ge 1$, where $p' = p/(p-1)$ and $r' = r/(r-1)$.

The technical assumption on the weight w, (p,r)-admissibility, plays very important role in our principal results, Theorems 4.1 and 4.5. The reader may refer the following typical examples of (p,r)-admissible weights, polynomial weights w_α and subexponential weights $e_{D,\delta}$ for simplification:

Example 2.5. (i) The polynomial weight $w_\alpha(x,y) := (1+\rho(x,y))^\alpha$, where $\alpha > d(X,\rho,\mu)(1-1/p)$ and the minimal rate of polynomial growth $d(X,\rho,\mu)$ is defined by

$$d(X,\rho,\mu) := \inf\big\{d : \sup_{x\in X, \tau\ge 1} \tau^{-d}\mu(B(x,\tau)) < \infty\big\} \tag{2.11}$$

([32, 57]). We remark that

$$d(X,\rho,\mu) \le \ln_2 D_1 < \infty$$

because it follows from (2.1) and (2.2) that

$$\mu(B(x,\tau)) \le D_1^j \mu(B(x, 2^{-j}\tau)) \le D_3 D_1^j \le D_3 D_1 \tau^{\ln_2 D_1} \quad \text{for all}\ \tau \ge 1,$$

where j is the unique integer so chosen that $1/2 < 2^{-j}\tau \le 1$.

(ii) The subexponential weight $e_{D,\delta}(x,y) := \exp(D\rho(x,y)^\delta)$, where $D \in (0,\infty)$ and δ satisfies the following assumption: there exists $\theta_0 \in (0,1)$ such that

$$\rho(x,y)^\delta \le \min(\rho(x,z)^\delta + \theta_0\rho(z,y)^\delta, \theta_0\rho(x,z) + \rho(z,y)^\delta) \tag{2.12}$$

holds for all $x,y,z \in X$. In this case, one may verify that $v(x,y) = \exp(D\theta_0\rho(x,y)^\delta)$ is a weight that satisfies (2.8) – (2.10). For the case that the quasi-metric ρ is Hölder continuous, that is, there exist positive constants $\beta \in (0,1)$ and $C \in (0,\infty)$ such that

$$|\rho(x,y) - \rho(z,y)| \le C\rho(x,z)^\beta(\rho(x,y) + \rho(z,y))^{1-\beta}$$

holds for all $x,y,z \in X$ ([41, 42]), one may verify that there exists $\delta_0 \in (0,\beta)$ such that the equation (2.12) holds for all $\delta \in (0,\delta_0)$, and hence $e_{D,\delta}$ are (p,r)-admissible weights for sufficiently small positive parameter δ. For the case that ρ is a metric, one can prove that $e_{D,\delta}$ are (p,r)-admissible weights for all $\delta \in (0,1)$, particularly, the inequality (2.12) holds for $\theta_0 = 2^\delta - 1$.

Remark 2.6. *For the space of homogenous type (X,ρ,μ) having finite volume, that is, $\mu(X) < \infty$, any weight is (p,r)-admissible.* For instance, for polynomial weight $w_\alpha(x,y) := (1+\rho_\delta(x,y))^\alpha$ on the unit sphere $S^d \subset \mathbf{R}^{d+1}$ with dilated Euclidean metric ρ_δ and dilated Lebesgue measure μ_δ, see Example 2.2, the constant D_7 in (2.8)– (2.10), to be denoted by $D_7(\alpha,\delta)$, can be bounded by an absolute constant C independent of $\delta \in (0,1]$ when $\alpha > d(1-1/p)$, while $D_7(\alpha,\delta)$ tends to infinity as δ tends to zero when $\alpha \le d(1-1/p)$, see Remark 4.2 and Corollary 4.7 for the importance of a upper bound for the constant D_7 in (2.8)– (2.10) that is independent of $\delta \in (0,1]$.

In the following result, which will be used later in the proof of Theorem 4.1, we show that the restriction of a (p,r)-admissible weight on a space of homogenous type to a relatively-separated subset is a (p,r)-admissible weight on that subset.

Proposition 2.7. *Let $1 \le p,r \le \infty$, (X,ρ,μ) be a space of homogenous type, Λ be a relatively-separated subset of X, and w be a (p,r)-admissible weight on the space (X,ρ,μ). Then the restriction of the weight w on Λ, to be denoted by w_Λ, is a (p,r)-admissible weight on (Λ,ρ,μ_c). Moreover, the constants $D_5(w)$, $D_6(C_0,w)$ in (2.7) and $\theta(w)$ in (2.10) for w can be used as the corresponding constants $D_5(w_\Lambda)$, $D_6(C_0,w_\Lambda)$ and $\theta(w_\Lambda)$ for w_Λ respectively, and the constant $D_7(w_\Lambda)$ in (2.8) and (2.10) can be chosen to be dependent* **only** *on the constant $D_7(w)$ in (2.8) and (2.10) for w, the constants $D_5(w)$ in (2.6) for w, and $D_6(C_0,w)$ in (2.7) for w, the constant $D(\Lambda)$ in (2.4), and the constants $L, D_i, 0 \le i \le 3$, in the definition of the space of homogenous type (X,ρ,μ).*

We leave the proof of the above proposition to the reader.

2.4. **Separately-located molecules with size one.** Let $1 \le p, q \le \infty$, w be a weight, (X, ρ, μ) be a space of homogenous type, and Λ be a relatively-separated subset of the space X. We denote the set of all $\Phi := \{\phi_\lambda, \lambda \in \Lambda\}$ with finite $\|\Phi\|_{q,p,w}$ by $\mathcal{B}_{q,p,w}(\Lambda)$, or $\mathcal{B}_{q,p,w}$ for brevity,

$$\mathcal{B}_{q,p,w}(\Lambda) := \{\Phi : \ \|\Phi\|_{q,p,w} < \infty\}, \tag{2.13}$$

and the unit ball of $\mathcal{B}_{q,p,w}(\Lambda)$ by $\mathcal{B}^1_{q,p,w}(\Lambda)$,

$$\mathcal{B}^1_{q,p,w}(\Lambda) := \{\Phi : \ \|\Phi\|_{q,p,w} \le 1\}, \tag{2.14}$$

where

$$\begin{aligned} \|\Phi\|_{q,p,w} \ &:= \ \sup_{\lambda\in\Lambda} \big\| \|\phi_\lambda(\cdot) w(\lambda,\cdot)\|_{L^q(B(x,1))} \big\|_{L^p(X)} \\ &\quad + \sup_{x\in X} \big\| \big(\|\phi_\lambda(\cdot) w(\lambda,\cdot)\|_{L^q(B(x,1))}\big)_{\lambda\in\Lambda} \big\|_{\ell^p(\Lambda)}, \end{aligned} \tag{2.15}$$

and $\|\cdot\|_{L^q(K)}$ denotes the usual L^q norm on $L^q(K)$, the space of all q-integrable functions on a measurable set K. Elements in $\mathcal{B}_{q,p,w}(\Lambda)$ will be used as building blocks to generate a space suitable for modelling signals with finite rate of innovation, see the next section for details. In Remark 2.10 below, we provide a model family of separately-located molecules with similar size convenient for modelling signals with finite rate of innovation.

Remark 2.8. Let $1 \le p, q \le \infty$, w be a weight, (X, ρ, μ) be a uniform-discretizable space of homogenous type, X_0 and Λ be a lattice and a relatively-separated subset of the space X respectively. We define $\|\Phi\|_{q,p,w,X_0}$ for $\Phi := \{\phi_\lambda, \lambda \in \Lambda\}$, a discretized correspondence of $\|\Phi\|_{q,p,w}$, by

$$\begin{aligned} \|\Phi\|_{q,p,w,X_0} \ &:= \ \sup_{\lambda\in\Lambda} \big\| \big(\|\phi_\lambda(\cdot) w(\lambda,\cdot)\|_{L^q(B(x_0,1))}\big)_{x_0\in X_0} \big\|_{\ell^p(X_0)} \\ &\quad + \sup_{x_0\in X_0} \big\| \big(\|\phi_\lambda(\cdot) w(\lambda,\cdot)\|_{L^q(B(x_0,1))}\big)_{\lambda\in\Lambda} \big\|_{\ell^p(\Lambda)}. \end{aligned} \tag{2.16}$$

One may verify that there exist positive constants A and B such that

$$A\|\Phi\|_{q,p,w} \le \|\Phi\|_{q,p,w,X_0} \le B\|\Phi\|_{q,p,w}. \tag{2.17}$$

Due to the above equivalence between $\|\Phi\|_{q,p,w}$ and $\|\Phi\|_{q,p,w,X_0}$, we may not distinguish the norm $\|\Phi\|_{q,p,w}$ from its discretized version $\|\Phi\|_{q,p,w,X_0}$.

Remark 2.9. A function f that satisfies the size condition $\||f(\cdot)|(1+\rho(\cdot,y)/d)^\alpha\|_p \le 1$ and the vanishing moment condition $\int f(x)d\mu(x) = 0$ is called as a *molecule of the Hardy space* H^1 centered around y and with width d ([21, 48]). Since any $\phi_\lambda \in \Phi := \{\phi_\lambda : \ \lambda \in \Lambda\} \in \mathcal{B}^1_{p,p,w_\alpha}(\Lambda)$ satisfies the same size condition $\||\phi_\lambda(\cdot)|(1+\rho(\lambda,\cdot))^\alpha\|_p \le 1$ to a molecule of the Hardy space, we call any $\phi_\lambda \in \Phi \in \mathcal{B}^1_{q,p,w}(\Lambda)$ as a *molecule* centered around λ and with width one. Also we note that the center $\lambda \in \Lambda$ are essentially separated and the unit balls $B(\lambda, 1), \lambda \in \Lambda$, are finitely overlapped by the assumption that Λ is a relatively-separated subset of X. So it is reasonable to consider any Φ in $\mathcal{B}^1_{q,p,w}(\Lambda)$ as *a family of separately-located molecules* *with size one*, and also as *a family of building blocks that is well-localized and separately-located.*

Example 2.10. *(Example 2.2 third revisited)* Our model family of separately-located molecules in $\mathcal{B}_{q,p,w}(\Lambda)$ for considering signals with finite rate of innovation, see also the Introduction section, is $\Phi = \{\phi_\lambda : \lambda \in \Lambda\}$ that satisfies

$$|\phi_\lambda(x)| \le h(x-\lambda) \quad \text{for all } x \in \mathbf{R}^d \text{ and } \lambda \in \Lambda, \tag{2.18}$$

where $w(x,y) = u(x-y), x, y \in \mathbf{R}^d$ for some positive function u on $\mathbf{R}^d$, Λ is a relatively-separated subset of $\mathbf{R}^d$, and

$$h \in W_q(L_{p,u}). \tag{2.19}$$

Here we recall the definition of the Wiener amalgam space $W_q(L_{p,u})$,

$$W_q(L_{p,u}) := \Big\{f : \ \|f\|_{q,p,u} := \big\|\big(\|fu\|_{L^q(k+[0,1]^d)}\big)_{k\in\mathbf{Z}^d}\big\|_{\ell^p(\mathbf{Z}^d)} < \infty\Big\}. \tag{2.20}$$

The Wiener amalgam space $W_q(L_{p,u})$ consists of functions that are "locally" in L^q and "globally" in weighted L^p with weight u, which becomes weighted L^p space with weight u when $q = p$ ([1]).

For the sets $\mathcal{B}_{q,p,w}(\Lambda)$ with different p, q, w, we have the following property, which will be used later in the proofs of Theorems 3.2 and 4.1.

Theorem 2.11. *Let $1 \le q, q', p, p' \le \infty$, $\alpha, \beta \in \mathbf{R}$, (X, ρ, μ) be a uniformly-discretizable space of homogenous type such that μ satisfies the uniform boundedness condition. Assume that $q \le q'$, $p \le p'$, and that w and $\tilde{w}$ are weights with the property that $\sup_{x\in X} \|(w\tilde{w}^{-1})(x,\cdot)\|_{L^r} < \infty$ where $1/r = 1/p - 1/p'$. Then*

$$\mathcal{B}_{q',p',\tilde{w}}(\Lambda) \subset \mathcal{B}_{q',p,w}(\Lambda) \subset \mathcal{B}_{q,p,w}(\Lambda).$$

Moreover, there exist positive constants A and B such that

$$\|\Phi\|_{q,p,w} \le A\|\Phi\|_{q',p,w} \le B\|\Phi\|_{q',p',\tilde{w}}. \tag{2.21}$$

Proof. Obviously it suffices to prove (2.21). The first inequality of the estimate (2.21) holds because of the equivalence between $\|\Phi\|_{q,p,w}$ and its discretized correspondence $\|\Phi\|_{q,p,w,X_0}$ given in Remark 2.8, and the estimate,

$$\|f\|_{L^q(B(x_0,1))} \le C\|f\|_{L^{q'}(B(x_0,1))}$$

for any $x_0 \in X$ and any function f, which follows from the uniform boundedness condition (2.2).

From the definition of a weight and the uniform boundedness condition for the Borel measure μ, there exists a positive constant C such that

$$\|(w\tilde{w}^{-1})(\lambda,\cdot)\|_{L^\infty(B(x,1))} \le C\|(w\tilde{w}^{-1})(\lambda,\cdot)\|_{L^r(B(x,1))}$$

and

$$\|(w\tilde{w}^{-1})(\lambda,\cdot)\|_{L^\infty(B(x,1))} \le C\|(w\tilde{w}^{-1})(\cdot,x)\|_{L^r(B(\lambda,1))}$$

for all $\lambda \in \Lambda$. This, together with the relatively-separatedness and the assumption $\sup_{x\in X} \|(w\tilde{w}^{-1})(x,\cdot)\|_{L^r} < \infty$, implies that

$$\sup_{\lambda\in\Lambda} \|\|(w\tilde{w}^{-1})(\lambda,\cdot)\|_{L^\infty(B(x,1))}\|_{L^r} + \sup_{x\in X} \|(\|(w\tilde{w}^{-1})(\lambda,\cdot)\|_{L^\infty(B(x,1))})_{\lambda\in\Lambda}\|_{\ell^r(\Lambda)} < \infty.$$

Hence the second inequality of the estimate (2.21) follows. □

2.5. **Frames and Banach frames.** Given a Hilbert space H, we say that $E = \{e_\lambda, \lambda \in \Lambda\} \subset H$ is a *frame* of the space H if there exist two positive constants $A, B > 0$ such that

$$A\|f\|_2 \le \|(\langle f, e_\lambda\rangle)_{\lambda\in\Lambda}\|_{\ell^2(\Lambda)} \le B\|f\|_2 \quad \text{for all } f \in H. \tag{2.22}$$

For a frame E of a Hilbert space H, we define the *frame operator* S by

$$Sf = \sum_{\lambda\in\Lambda} \langle f, e_\lambda\rangle \phi_\lambda. \tag{2.23}$$

It is known that if E is a frame then the frame operator S is bounded and has bounded inverse, $S^{-1}E = \{S^{-1}e_\lambda, \lambda \in \Lambda\}$ is a frame for H, and the following reconstruction formula holds for any $f \in H$:

$$f = \sum_{\lambda\in\Lambda} \langle f, S^{-1}e_\lambda\rangle e_\lambda = \sum_{\lambda\in\Lambda} \langle f, e_\lambda\rangle S^{-1}e_\lambda \tag{2.24}$$

([13, 28]). The frame $S^{-1}E$ is known as the *canonical dual frame* of E.

Let $\delta_{\lambda\lambda'}$ stand for the usual Kronecker symbol. For a Hilbert space H with $E = \{e_\lambda, \lambda \in \Lambda\}$ being its Riesz basis, we say that $E^d = \{e^d_\lambda : \lambda \in \Lambda\} \subset H$ is a *dual Riesz basis* of E if E^d is a Riesz basis of H and $\langle e_\lambda, e^d_{\lambda'}\rangle = \delta_{\lambda\lambda'}$ for all $\lambda, \lambda' \in \Lambda$. For a Hilbert space H, we say that $E^o = \{e^o_\lambda : \lambda \in \Lambda\}$ is an *orthonormal basis* for H if E^o is a basis of H and $\langle e^o_\lambda, e^o_{\lambda'}\rangle = \delta_{\lambda\lambda'}$ for all $\lambda, \lambda' \in \Lambda$.

Let $1 \le r \le \infty$, $\mathcal{B}$ be a Banach space, $E := \{e_\lambda, \lambda \in \Lambda\}$ be a sequence of elements from the dual $\mathcal{B}^*$ of the space $\mathcal{B}$. If (i) $(\langle g, e_\lambda\rangle) \in \ell^r(\Lambda)$ for all $g \in \mathcal{B}$; (ii) the norm $\|g\|_{\mathcal{B}}$ and $\|(\langle g, e_\lambda\rangle)_{\lambda\in\Lambda}\|_{\ell^r}$ are equivalent, i.e., there exist two positive constants A and B such that $A\|g\|_{\mathcal{B}} \le \|(\langle g, e_\lambda\rangle)_{\lambda\in\Lambda}\|_{\ell^r} \le B\|g\|_{\mathcal{B}}$ for all $g \in \mathcal{B}$, then we say that E is a *r-frame* for $\mathcal{B}$ ([2, 14]). If we further assume that (iii) there exists a bounded linear operator $R : \ell^r(\Lambda) \longmapsto \mathcal{B}$ such that $R(\langle g, e_\lambda\rangle) = g$ for all $g \in \mathcal{B}$, then we say that E is a *Banach frame* for $\mathcal{B}$ with respect to $\ell^r(\Lambda)$ ([11, 27]). The operator R is known as the *reconstruction operator*. If the span of Φ is dense in $\mathcal{B}$ and there exist constants $A, B > 0$ such that $A\|c\|_{\ell^r} \le \|\sum_{\lambda\in\Lambda} c(\lambda)e_\lambda\|_{\mathcal{B}} \le B\|c\|_{\ell^r}$ for all scalar sequence $c := (c(\lambda))_{\lambda\in\Lambda} \in \ell^r(\Lambda)$, then we say that E is a *r-Riesz basis* for $\mathcal{B}$ ([14]).

3. Locally finitely-generated spaces

Let (X, ρ, μ) be a space of homogenous type, Λ be a relatively-separated subset of the space X, and $\Phi := \{\phi_\lambda,\ \lambda \in \Lambda\}$ be a family of separately-located molecules of similar size that belongs to $\mathcal{B}_{q,p,w}(\Lambda)$ for some $1 \le p, q \le \infty$ and weight w. We denote the space spanned by molecules in Φ with ℓ^p coefficients by $V_q(\Phi, \Lambda)$,

$$V_q(\Phi, \Lambda) := \Big\{ \sum_{\lambda\in\Lambda} c(\lambda)\phi_\lambda :\ \big\|(c(\lambda))_{\lambda\in\Lambda}\big\|_{\ell^q(\Lambda)} < \infty \Big\},\ 1 \le q \le \infty. \tag{3.1}$$

We define the space $V_q(\Phi, \Lambda)$ on spaces of homogenous type, instead of on the real line mentioned in the Introduction section, for the unified treatment for various time signals with finite rate of innovation, such as (i) periodic signals $x(t)$ with period T, $x(t + T) = x(t)$, (ii) (periodic) discrete signals $\{x(nT), n \in \mathbf{Z}\}$ where $T > 0$, (iii) time signals on a finite interval $x(t), t \in [T_0, T_1]$, and (iv) time signals $x(t), t \in (-\infty, \infty)$ ([44, 46, 47, 60]), and also for considering diffusion wavelets on manifolds ([20]). The reader may refer the two model examples of spaces of homogenous type, the Euclidean space $\mathbf{R}^d$ and the unit sphere S^d with dilated metric ρ_δ in Example 2.2 for simplification.

We provide some flexibility on the assumption $\Phi \in \mathcal{B}_{q,p,w}(\Lambda)$ for the generator Φ in the above definition of the space $V_q(\Phi, \Lambda)$, instead of the enveloping assumption (1.2) for the generator Φ mentioned in the Introduction section, for unified treatment to different modelling situations, such as, $q = 1, p = \infty$ when modelling slow-varying signals with shot noises ([60]), $1 \le p, q \le \infty$ for modelling signals in a shift-invariant space ([1]), $q = \infty, 1 \le p \le \infty$ for decomposing a time signal via Gabor (Wilson) system ([28]), see Example 3.1 for more details about the last two setting.

As mentioned in the Introduction section, we have that any function in the space $V_q(\Phi, \Lambda)$ has finite rate of innovation, and that the space $V_q(\Phi, \Lambda)$ is a locally finitely-generated space suitable for modelling known signals with finite rate of innovations in [22, 33, 40, 44, 46, 47, 60], such as, stream of (different) pulses found in example in GPS applications, cellular radio, and ultra wide-band; bandlimited signals with additive shot noise; sum of bandlimited signals and non-uniform spline signals for modelling electrocardiogram signals. Also the space $V_q(\Phi, \Lambda)$ is suitable for modelling band-limited signals, signals in a shift-invariant space, functions in the time-frequency plane, non-uniform splines, see Example 3.1 below.

Example 3.1. (i) The Paley-Wiener space space B_Ω, since the Shannon sampling theorem states that the Paley-Wiener space B_Ω is generated by the sinc function, $B_\Omega = \{\sum_{k\in\mathbf{Z}} c(k)\mathrm{sinc}(\Omega x - k) :\ (c(k)) \in \ell^2\}$ where $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$ ([1, 9, 34, 37]).

(ii) The finitely-generated shift-invariant space $V_q(\phi_1, \ldots, \phi_N)$,

(3.2)

$$V_q(\phi_1, \ldots, \phi_N) := \Big\{ \sum_{n=1}^N \sum_{k\in \mathbf{Z}^d} c_n(k)\phi_n(\cdot - k) : \ (c_n(k))_{k\in\mathbf{Z}^d} \in \ell^q, 1 \le n \le N \Big\}$$

generated by finitely many functions $\phi_1, \ldots, \phi_N$ on $\mathbf{R}^d$. The shift-invariant space (3.2) and its various generalizations are widely used in wavelet analysis, approximation theory, sampling theory etc, see, for instance, [1, 2, 6, 7, 17, 38, 39, 52] and references therein. In the above shift-invariant setting, we have that $\Phi := \{\phi_n(\cdot - k) : \ 1 \le n \le N, k \in \mathbf{Z}^d\}$ belongs to $\mathcal{B}_{q,p,w}(\mathbf{Z}^d)$ if and only if $\phi_n, 1 \le n \le N$, belong to the Wiener amalgam space $W_q(L_{p,u})$, where $w(x,y) = u(x-y)$ for some positive function u on $\mathbf{R}^d$.

(iii) The twisted shift-invariant space

$$\Big\{ \sum_{k,l\in\mathbf{Z}^d} c(k,l)e^{-i\alpha k(\omega-\beta l)}(V\psi)(x-\alpha k, \omega-\beta l) : \sum_{k,l\in\mathbf{Z}^d} |c(k,l)|^2 < \infty \Big\}$$

in the time-frequency domain $\mathbf{R}^d \times \mathbf{R}^d$, which is generated by a Gabor system

$$\Psi := \{e^{-i\beta l x}\psi(x-\alpha k) : \ k, l \in \mathbf{Z}^d\},$$

where α, β are positive numbers, ψ is a window function on $\mathbf{R}^d$, and V is the short-time Fourier transform with the Gaussian window function defined by

$$Vf(x,\omega) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} e^{-iy\omega} f(y) e^{-|y-x|^2/2} dy, \ f \in L^2$$

(see, for instance, [4, 10, 28, 36, 51, 57] and references therein for historical remarks and recent development on Gabor system). In the above twisted-shift-invariant case, the family of building blocks

$$\Phi := \Big\{e^{-i\alpha k(\omega-\beta l)}(V\psi)(x-\alpha k, \omega-\beta l) : \ (\alpha k, \beta l) \in \alpha\mathbf{Z}^d \times \beta\mathbf{Z}^d\Big\}$$

belongs to $\mathcal{B}_{q,p,w}(\alpha\mathbf{Z}^d \times \beta\mathbf{Z}^d)$ if and only if ψ belongs to the modulation space M_u^p, where

$$M_u^p := \{f : Vf(x,\omega)u(x,\omega) \in L^p(\mathbf{R}^{2d})\},$$

where $w((x,\omega),(x',\omega')) = u(x-x', \omega-\omega')$. (The reader may refer [28] for various properties and applications of the modulation spaces). We remark that since the Gabor system with Gaussian window and $\alpha\beta < 1$ is a frame of L^2 (see e.g. [28]), the space $L^2(\mathbf{R}^d)$ is isomorphic to a twisted shift-invariant space on the time-frequency plane $\mathbf{R}^d \times \mathbf{R}^d$, and hence also to a space of the form (3.1) on the time-frequency plane.

(iv) The space S_m of all square-integrable polynomial splines of order m satisfying $m-1$ continuity conditions at each knot t_i, where m is a positive integer m and $\mathbf{T} := \{t_i\}_{i=-\infty}^{+\infty}$ is a bi-infinite increasing sequence satisfying

$$0 < T_0 := \inf_i t_{i+m} - t_i \le \sup_i t_{i+m} - t_i =: T_1 < \infty \tag{3.3}$$

(see e.g. [53]).The reason is that

$$S_m = \Big\{ \sum_{i\in\mathbf{Z}} c(i) B_i(x) : \sum_{i\in\mathbf{Z}} |c(i)|^2 < \infty \Big\} \tag{3.4}$$

by Curry-Scheonberg representation, where B_i is the normalized B-spline associated with the knots $t_i, \ldots, t_{i+m}$ ([53]). Moreover, since the normalized B-spline $B_i(x)$ has support in $[t_i, t_{i+m}]$ and satisfies $0 \le B_i(x) \le 1$ ([53]), the family of building blocks $\Phi = \{B_i, t_i \in \mathbf{T}\}$ belongs to $\mathcal{B}_{q,p,w}(\mathbf{T})$ for any $1 \le p, q \le \infty$ and any weigth w.

In the following result, which will be used in Sections 4 and 5 to the study of (Banach) frames, we show that the space $V_q(\Phi, \Lambda)$ is a subspace of L^q if $\Phi \in \mathcal{B}_{q,1,w_0}(\Lambda) \subset \mathcal{B}_{q,p,w}(\Lambda)$, where $1 \le p, q \le \infty$ and the weight w satisfies $\sup_{x\in X} \|w^{-1}(x,\cdot)\|_{p/(p-1)} < \infty$.

Theorem 3.2. *Let $1 \le q \le \infty$ and (X, ρ, μ) be a space of homogenous type with the property that μ satisfies the uniform boundedness condition (2.2) and X is uniformly-discretizable. Assume that Λ is a discrete subset of X, $\Phi := \{\phi_\lambda, \lambda \in \Lambda\} \in \mathcal{B}_{q,1,w_0}(\Lambda)$, and $V_q(\Phi, \Lambda)$ is defined as in (3.1). Then*

$$V_q(\Phi, \Lambda) \subset L^q. \tag{3.5}$$

Moreover,

$$\Big\| \sum_{\lambda\in\Lambda} c(\lambda)\phi_\lambda \Big\|_{L^q} \;\le\; C \big\| \big(c(\lambda)\big)_{\lambda\in\Lambda} \big\|_{\ell^q(\Lambda)} \|\Phi\|_{q,1,w_0} \tag{3.6}$$

for every sequence $(c(\lambda))_{\lambda\in\Lambda} \in \ell^q(\Lambda)$, and

$$\big\| \big(\langle f, \phi_\lambda\rangle\big)_{\lambda\in\Lambda} \big\|_{\ell^r(\Lambda)} \le C \|f\|_{L^r} \|\Phi\|_{q,1,w_0} \tag{3.7}$$

for all $f \in L^r$ with $q/(q-1) \le r \le \infty$.

Proof. Take a sequence $c = (c(\lambda))_{\lambda\in\Lambda} \in \ell^q(\Lambda)$ and let $f = \sum_{\lambda\in\Lambda} c(\lambda)\phi_\lambda$. We then have

$$\begin{aligned}\|f\|_{L^q}^q &\le \sum_{x_0\in X_0}\int_{B(x_0,1)}\Big(\sum_{\lambda\in\Lambda}|c(\lambda)|^q|\phi_\lambda(x)|\Big)\times\Big(\sum_{\mu\in\Lambda}|\phi_\mu(x)|\Big)^{q-1}dx\\ &\le \sum_{\lambda\in\Lambda}|c(\lambda)|^q\sum_{x_0\in X_0}\Big(\int_{B(x_0,1)}|\phi_\lambda(x)|^q dx\Big)^{1/q}\\ &\qquad\times\Big(\int_{B(x_0,1)}\Big(\sum_{\mu\in\Lambda}|\phi_\mu(x)|\Big)^q dx\Big)^{1-1/q}\\ &\le \|c\|_{\ell^q(\Lambda)}^q\times\Big(\sup_{\lambda\in\Lambda}\sum_{x_0\in X_0}\|\phi_\lambda\|_{L^q(B(x_0,1))}\Big)\\ &\qquad\times\Big(\sup_{x_0\in X_0}\sum_{\lambda\in\Lambda}\|\phi_\lambda\|_{L^q(B(x_0,1))}\Big)^{q-1}\\ &\le \|c\|_{\ell^q(\Lambda)}^q\|\Phi\|_{q,1,w_0}^q\end{aligned}$$

for $1 \le q < \infty$, and similarly

$$\|f\|_{L^\infty} \le \|c\|_{\ell^\infty(\Lambda)}\Big(\sup_{x_0\in X_0}\sum_{\lambda\in\Lambda}\|\phi_\lambda\|_{L^\infty(B(x_0,1))}\Big) \le \|c\|_{\ell^\infty(\Lambda)}\|\Phi\|_{\infty,1,w_0}$$

for $q = \infty$. Therefore the estimate (3.6), and hence (3.5), follows.

Let $f \in L^r$. Using Hölder inequality, we obtain

$$\begin{aligned}\|(\langle f,\phi_\lambda\rangle)_{\lambda\in\Lambda}\|_{\ell^r(\Lambda)} &\le \Big\|\Big(\sum_{x_0\in X_0}\|f\|_{L^r(B(x_0,1))}\|\phi_\lambda\|_{L^{r/(r-1)}(B(x_0,1))}\Big)_{\lambda\in\Lambda}\Big\|_{\ell^r(\Lambda)}\\ &\le \|\Phi\|_{r/(r-1),1,w_0}\|f\|_r \le \|\Phi\|_{q,1,w_0}\|f\|_r,\end{aligned}$$

where the last inequality follows form (2.21). This proves (3.7). □

Remark 3.3. For the special case that the space $V_q(\Phi,\Lambda)$ is a shift-invariant space $V_q(\phi_1,\dots,\phi_N)$ generated by finitely many functions $\phi_1,\dots,\phi_N$, similar estimates to the ones in (3.6) and (3.7) are established by Jia and Micchelli under weak assumption $\phi_1,\dots,\phi_N \in \mathcal{L}^q$, where

$$\mathcal{L}^q := \Big\{f : \Big\|\sum_{k\in\mathbf{Z}^d}|f(\cdot+k)|\Big\|_{L^q([0,1]^d)} < \infty\Big\}$$

([39, Theorems 2.1 and 3.1]).

4. Canonical Dual Frames and Dual Riesz Basis

In this section, we consider the frame property of the generator Φ for the space $V_2(\Phi,\Lambda) \subset L^2$. The main results of this section are Theorems 4.1 and 4.5.

Theorem 4.1. *Let $2 \le q \le \infty$, $1 \le p \le \infty$, (X,ρ,μ) be a space of homogenous type such that the space X is uniformly discretizable and that the Borel measure μ satisfies the uniform boundedness condition (2.2), Λ be a* *relatively-separated subset of* X, $\Phi := \{\phi_\lambda, \lambda \in \Lambda\}$, $V_2(\Phi, \Lambda)$ *be as in (3.1), and define the frame operator* S *by*

$$Sf = \sum_{\lambda\in\Lambda} \langle f, \phi_\lambda\rangle \phi_\lambda. \tag{4.1}$$

Assume that w *is a* $(p, 2)$*-admissible weight and that* $\Phi \in \mathcal{B}_{q,p,w}(\Lambda)$*. Then the following statements are true.*

(i) *If* Φ *is a frame of* $V_2(\Phi, \Lambda)$*, then the canonical dual frame* $S^{-1}\Phi := \{S^{-1}\phi_\lambda : \lambda \in \Lambda\} \in \mathcal{B}_{q,p,w}(\Lambda)$ *with* $S^{-1}\phi_\lambda \in V_1(\Phi, \Lambda)$ *for all* $\lambda \in \Lambda$*.*

(ii) *If* Φ *is a frame of* $V_2(\Phi, \Lambda)$*, then the canonical tight frame* $S^{-1/2}\Phi := \{S^{-1/2}\phi_\lambda : \lambda \in \Lambda\} \in \mathcal{B}_{q,p,w}(\Lambda)$ *with* $S^{-1/2}\phi_\lambda \in V_1(\Phi, \Lambda)$ *for all* $\lambda \in \Lambda$*.*

(iii) *If* Φ *is a Riesz basis of* $V_2(\Phi, \Lambda)$*, then the dual Riesz basis* $S^{-1}\Phi$ *of* Φ *belongs to* $\mathcal{B}_{q,p,w}(\Lambda)$*.*

(iv) *If* Φ *is a Riesz basis of* $V_2(\Phi, \Lambda)$*, then* $S^{-1/2}\Phi$ *is an orthonormal basis of* $V_2(\Phi, \Lambda)$ *and belongs to* $\mathcal{B}_{q,p,w}(\Lambda)$*.*

Remark 4.2. Keeping track on the constants in the proof of Theorem 4.1, we conclude that $S^{-1}\Phi$ and $S^{-1/2}\Phi$ in the first and second statements (resp. the third and forth statements) of Theorem 4.1 satisfies

$$\|S^{-1}\Phi\|_{q,p,w} + \|S^{-1/2}\Phi\|_{q,p,w} \le C \tag{4.2}$$

for all $\Phi \in \mathcal{B}_{q,p,w}(\Lambda)$ with $\|\Phi\|_{q,p,w} \le 1$, where the constant C depends only on the parameters p, q, the constants D_0, L in the definition of the quasi-metric ρ, the constants D_1, D_2, D_3 in the double condition and uniform boundedness condition for the Borel measure μ, the constant D_4 in the lattice X_0, the constant $D(\Lambda)$ for the relatively-separated subset Λ, the constants D_5, D_6, D_7 and θ in (2.5)– (2.10) for the weight w, and the upper and lower frame bounds for the frame Φ (resp. the upper and lower Riesz bounds for the Riesz basis Φ). The uniform boundedness result (4.2) for $S^{-1}\Phi$ and $S^{-1/2}\Phi$ can be used to reach the *uniform stability* for the reconstruction procedure for functions in scaling and wavelet spaces at different scales, such as scaling and wavelet spaces on a bounded interval (see e.g. [16, 19]), periodic scaling and wavelet spaces (see e.g. [12, 50]), nonstationary multiresolution analysis (see e.g. [18]), and diffusion scaling and wavelet spaces (see e.g. [20]). The following Corollary 4.7 for periodic signals with finite rate of innovation is such an example.

As an easy application of Theorem 4.1, we have the following result for the shift-invariant setting, see Example 3.1.

Corollary 4.3. *Let* $1 \le p \le \infty, 2 \le q \le \infty$, $u(x) = (1 + |x|)^\alpha$ *for some* $\alpha > d(1 - 1/p)$ *or* $u(x) = \exp(D|x|^\delta)$ *for some* $D > 0$ *and* $\delta \in (0, 1)$*, and let* $\phi_1, \ldots, \phi_N$ *belong to the Wiener amalgam space* $W_q(L_{p,u})$*. Then*

(i) *If* $\Phi := \{\phi_n(\cdot - k) : 1 \le n \le N, k \in \mathbf{Z}^d\}$ *is a frame of the shift-invariant space* $V_2(\phi_1, \ldots, \phi_N)$*, then* $\{(S^{-1}\phi_n)(\cdot - k) : 1 \le n \le$

$N, k \in \mathbf{Z}^d\}$ is the canonical dual frame of Φ, $\{(S^{-1/2}\phi_n)(\cdot - k) : 1 \le n \le N, k \in \mathbf{Z}^d\}$ is the canonical tight frame for $V_2(\phi_1, \dots, \phi_N)$, and $S^{-1}\phi_n, S^{-1/2}\phi_n \in W_q(L_{p,u}), 1 \le n \le N$.

(ii) *If $\Phi := \{\phi_n(\cdot - k) : 1 \le n \le N, k \in \mathbf{Z}^d\}$ is a Riesz basis of the shift-invariant space $V_2(\phi_1, \dots, \phi_N)$, then $\{(S^{-1}\phi_n)(\cdot - k) : 1 \le n \le N, k \in \mathbf{Z}^d\}$ is a dual Riesz basis of Φ, $\{(S^{-1/2}\phi_n)(\cdot - k) : 1 \le n \le N, k \in \mathbf{Z}^d\}$ is an orthonormal basis for $V_2(\phi_1, \dots, \phi_N)$, and $S^{-1}\phi_n, S^{-1/2}\phi_n \in W_q(L_{p,u}), 1 \le n \le N$.*

Remark 4.4. The results in Corollary 4.3 about canonical dual frame, canonical tight frame, and orthnormal basis are new, while similar results about dual Riesz basis can be found in [1, 39]. In particular, the dual Riesz basis property for $q = \infty$ and $p = 1$ was established by Aldroubi and Gröchenig ([1, Theorem 2.3]), while the result in [39, Theorem 4.1] states that $S^{-1}\phi_1, \dots, S^{-1}\phi_N \in \mathcal{L}^q$ if $\phi_1, \dots, \phi_N \in \mathcal{L}^q$. Since $\mathcal{L}^q \subset W_q(L_{p,u})$, comparing with Jia and Micchelli's result in [39], we make strong assumption on the functions $\phi_1, \dots, \phi_N$ which leads to better results for their dual Riesz basis.

A weight w on $\mathbf{R}^d$ is said to be *translation-invariant* if $w(x, y) = u(x-y)$ for some positive function u on $\mathbf{R}^d$. Similarly for $1 \le p, r \le \infty$, a weight w on $\mathbf{R}^d$ is said to be *(p, r)-admissible translation-invariant* if it is translation invariant and if there exists another translation-invariant weight v with the property that (2.8) – (2.10) hold.

For the case that the space (X, ρ, μ) of homogenous type is the Euclidean space $\mathbf{R}^d$ with standard Euclidean metric and Lebesgue measure, we have a result similar to Theorem 4.1 but with different but very convenient localization assumption for the generator Φ.

Theorem 4.5. *Let $1 \le p \le \infty$ and Λ be a relatively-separated subset of $\mathbf{R}^d$. Assume that w is a $(p, \min(2, p))$-admissible translation-invariant weight with $w(x, y) := u(x-y)$ for some positive function u on $\mathbf{R}^d$, and that $\Phi := \{\phi_\lambda, \lambda \in \Lambda\}$ is a frame for $V_2(\Phi, \Lambda)$ and satisfies*

$$|\phi_\lambda(x)| \le h(x - \lambda) \text{ for all } x \in \mathbf{R}^d \text{ and } \lambda \in \Lambda \tag{4.3}$$

for some function h in the Wiener amalgam space $W_\infty(L_{p,u})$. Then there exists another function $\tilde{h}$ in the **same** *Wiener amalgam space $W_\infty(L_{p,u})$ such that the canonical dual frame $S^{-1}\Phi$ and the canonical tight frame $S^{-1/2}\Phi$ satisfy*

$$|S^{-1}\phi_\lambda(x)| + |S^{-1/2}\phi_\lambda(x)| \le \tilde{h}(x - \lambda) \text{ for all } x \in \mathbf{R}^d \text{ and } \lambda \in \mathbf{R}^d. \tag{4.4}$$

Taking our familiar polynomial weights and subexponential weights as the weight w in Theorem 4.5, we have the following corollary.

Corollary 4.6. *Let $1 \le p \le \infty$, $u(x) = (1 + |x|)^\alpha$ for some $\alpha > d(1 - 1/p)$ or $u(x) = \exp(D|x|^\delta)$ for some $D > 0$ and $\delta \in (0, 1)$, and Λ be a relatively-separated subset of $\mathbf{R}^d$. Assume that $\Phi := \{\phi_\lambda : \lambda \in \Lambda\}$ satisfies*

(4.3) for some function h in the Wiener amalgam space $W_\infty(L_{p,u})$, and that Φ is a frame (Riesz basis) of $V_2(\Phi,\Lambda)$. Then the canonical dual frame (dual Reisz basis) $S^{-1}\Phi$ and the canonical tight frame (orthonormal basis) $S^{-1/2}\Phi$ satisfy (4.4) for some function $\tilde h$ in the same Wiener amalgam space $W_\infty(L_{p,u})$.

For $1\le q,p\le\infty, T\ge 1$ and u be a weight on $\mathbf{R}^d$, we define the periodic Wiener amalgam space $W_q^T(L_{p,u})$ be the space of all periodic functions f with period T such that

$$\|f\|_{q,p,u,T}=\|\|fu\|_{L^q(k+[-1/2,1/2]^d)}\|_{\ell^p([-T/2,T/2)^d\cap\mathbf{Z}^d)}.$$

As an application of Theorem 4.5 with standard modification, we have the following results for the model setting for (periodic) signals with finite rate of innovation (see Example 2.10), and for Gabor (Wilson) frame setting (see Example 3.1).

Corollary 4.7. *Let $1\le p\le\infty$, $T\ge 2$, $E=\{0,1\}^d$, $u(x)=(1+|x|)^\alpha$ for some $\alpha>d(1-1/p)$ or $u(x)=\exp(D|x|^\delta)$ for some $D>0$ and $\delta\in(0,1)$, and Λ be a relatively-separated subset of $T[-1/2,1/2)^d$. Assume that $\Phi:=\{\phi_\lambda:\ \lambda\in\Lambda\}$ satisfies the following properties: (i) $\phi_\lambda,\lambda\in\Lambda$, are periodic function with period T, i.e. $\phi_\lambda(\cdot-kT)=\phi_\lambda$ for $k\in\mathbf{Z}^d$; (ii) $|\phi_\lambda(x)|\le h(x-\lambda)$ for some periodic function h with period T in the periodic Wiener amalgam space $W_\infty^T(L_{p,u})$, and (iii) there exist positive constants A and B with the property that*

$$A^2\sum_{\lambda\in\Lambda}|c(\lambda)|^2\le\Big\|\sum_{\lambda\in\Lambda}c(\lambda)\phi_\lambda\Big\|^2_{L^2([-\frac{T}{2},\frac{T}{2})^d)}\le B^2\sum_{\lambda\in\Lambda}|c(\lambda)|^2 \tag{4.5}$$

holds for all sequences $(c(\lambda))_{\lambda\in\Lambda}\in\ell^2(\Lambda)$. Define the frame operator S by

$$Sf=\sum_{\lambda\in\Lambda}\langle f,\phi_\lambda\rangle_{L^2([-T/2,T/2])}\phi_\lambda,$$

Then the dual Riesz basis $S^{-1}\Phi$ and the orthonormal basis $S^{-1/2}\Phi$ satisfy

$$|S^{-1}\phi_\lambda(x)|+|S^{-1/2}\phi_\lambda(x)|\le\tilde h(x-\lambda),\ x\in\mathbf{R}^d$$

for some function $\tilde h$ in the same periodic Wiener amalgam space $W_\infty(L_{p,u})$. Moreover, the periodic Wiener amalgam norm $\|\tilde h\|_{q,p,u,T}$ of $\tilde h$ is bounded by some constant C **independent** *of the period T (particularly depending only on $p,d,u,D(\Lambda)$, the upper and lower Riesz bounds in (4.5), and the periodic Wiener amalgam norm $\|h\|_{q,p,u,T}$ of h).*

Corollary 4.8. *Let $1\le p\le\infty$, V be the short-time Fourier transform with Gaussian window, $u(x)=(1+|x|)^\alpha$ for some $\alpha>2d(1-1/p)$ or $u(x)=\exp(D|x|^\delta)$ for some $D>0$ and $\delta\in(0,1)$, G be a finite group of $(2d)\times(2d)$ nonsingular matrices, and Λ be a relatively-separated subset of the time-frequency plane $\mathbf{R}^d\times\mathbf{R}^d$. Assume that $\Phi:=\{\phi_\lambda:\ \lambda\in\Lambda\}$ satisfies*

$$|V\phi_\lambda(x,\omega)|\le\inf_{g\in G}h((x,\omega)g-\lambda),\ (x,\omega)\in\mathbf{R}^d\times\mathbf{R}^d$$

for some function h in the Wiener amalgam space $W_\infty(L_{p,u})$, and that Φ is a frame (Riesz basis) of $V_2(\Phi,\Lambda)$. Then the canonical dual frame (dual Riesz basis) $S^{-1}\Phi$ and the canonical tight frame (orthonormal basis) $S^{-1/2}\Phi$ satisfy

$$|VS^{-1}\phi_\lambda(x,\omega)|+|VS^{-1/2}\phi_\lambda(x,\omega)|\le \inf_{g\in G}\tilde h((x,\omega)g-\lambda),\ (x,\omega)\in \mathbf{R}^d\times\mathbf{R}^d$$

for some function $\tilde h$ in the same Wiener amalgam space $W_\infty(L_{p,u})$.

Remark 4.9. The results in Corollary 4.8 for the Gabor system, i.e. the finite group G contains only the unit matrix, was established in [31, 32] for $p=1$ and in [4] for $p=1$ and $u(x)\equiv 1$. The results in Corollary 4.8 for the Gabor system are new for $1<p<\infty$, and the ones for the Wilson system, i.e. the finite group G contains more than the unit matrix, are completely new. The author believe that the results in Corollary 4.8 will be useful to establish the Littlewood-Paley decomposition of an L^p function via a Gabor (Wilson) system or a local (co)sine system, c.f. [26].

4.1. **Proof of Theorem 4.1.** To prove Theorem 4.1, we recall a matrix algebra in [57]. Let $1\le p\le\infty$, (X,ρ,μ) be a space of homogenous type, Λ and Λ' be two relatively-separated subsets of the space X, and w be a weight on X. We define the matrix algebra $\mathcal{A}_{p,w}(\Lambda,\Lambda')$ of Schur class by

$$\mathcal{A}_{p,w}(\Lambda,\Lambda'):=\Big\{A:=(A(\lambda,\lambda'))_{\lambda\in\Lambda,\lambda'\in\Lambda'}:\ \|A\|_{\mathcal{A}_{p,w}}<\infty\Big\},\tag{4.6}$$

where

$$\begin{aligned}\|A\|_{\mathcal{A}_{p,w}} &:= \sup_{\lambda\in\Lambda}\big\|\big(A(\lambda,\lambda')w(\lambda,\lambda')\big)_{\lambda'\in\Lambda'}\big\|_{\ell^p(\Lambda')}\\ &\quad+\sup_{\lambda'\in\Lambda'}\big\|\big(A(\lambda,\lambda')w(\lambda,\lambda')\big)_{\lambda\in\Lambda}\big\|_{\ell^p(\Lambda)}\end{aligned}$$

(see, e.g., [32, 35, 57]). By Remark 2.8, we have that, for $\Phi:=\{\phi_\lambda,\lambda\in\Lambda\}$,

$$\Phi\in\mathcal{B}_{q,p,w}(\Lambda)\ \text{if and only if}\ (\|\phi_\lambda\|_{L^q(B(x_0,1))})_{x_0\in X_0,\lambda\in\Lambda}\in\mathcal{A}_{p,w}(X_0,\Lambda).\tag{4.7}$$

For a matrix A in $\mathcal{A}_{p,w}(\Lambda,\Lambda')$, we define its transpose A^* by

$$A^*:=(\overline{A(\lambda,\lambda')})_{\lambda'\in\Lambda',\lambda\in\Lambda}\in\mathcal{A}_{p,w}(\Lambda',\Lambda).$$

To prove Theorem 4.1, we need some properties of the matrices in the Schur class $\mathcal{A}_{p,w}(\Lambda,\Lambda')$. The third property in Lemma 4.10 below is usually known as the Wiener's lemma, see, for instance, [4, 24, 31, 32, 35, 56, 57] and references therein for its recent development and various applications.

Lemma 4.10. *Let $1\le p\le q\le\infty$, set $r=pq/(q-p)$, X be a space of homogenous type such that X is uniformly discretizable and that μ satisfies the uniform boundedness condition, w and $\tilde w$ are weights on X, and $\Lambda,\Lambda',\Lambda''$ are relatively-separated subsets of X. Then the following statements are true.*

(i) *If* $\sup_{x\in X} \|(w\tilde{w}^{-1})(x,\cdot)\|_{L^r} < \infty$, *then there exists a positive constant* C *such that*

$$\|A\|_{\mathcal{A}_{p,w}} \le C\|A\|_{\mathcal{A}_{q,\tilde{w}}} \text{ for all } A \in \mathcal{A}_{q,\tilde{w}}(\Lambda,\Lambda'). \tag{4.8}$$

(ii) *If there exists another weight* v *on* X *such that (2.8) and (2.9) holds, then there exists a positive constant* C *such that*

$$\|AB\|_{\mathcal{A}_{p,w}} \le C\|A\|_{\mathcal{A}_{p,w}}\|B\|_{\mathcal{A}_{p,w}} \tag{4.9}$$

for all $A \in \mathcal{A}_{p,w}(\Lambda,\Lambda')$ *and* $B \in \mathcal{A}_{p,w}(\Lambda',\Lambda'')$, *where*

$$AB := \Big(\sum_{\lambda'\in\Lambda'} A(\lambda,\lambda')B(\lambda',\lambda'')\Big)_{\lambda\in\Lambda,\lambda''\in\Lambda''}.$$

(iii) *If* w *is a* $(p,2)$*-admissible weight on* X *and if* A *is a matrix in* $\mathcal{A}_{p,w}(\Lambda,\Lambda)$ *with the property that* $A^* = A$,

$$\langle Ac,c\rangle \ge D\langle c,c\rangle \text{ for all } c \in H, \tag{4.10}$$

and

$$PA = AP = A, \tag{4.11}$$

where D *is a positive constant,* H *is a Hilbert subspace of* $\ell^2(\Lambda)$, *and* P *is the projection operator from* $\ell^2(\Lambda)$ *to* H, *then for any* $\lambda > 0$, *both* A^γ *and its Moore-Penrose pseudo-inverse* $(A^\gamma)^\dagger$ *belong to* $\mathcal{A}_{p,w}(\Lambda,\Lambda)$. *Here the Moore-Penrose pseudo-inverse of* A^γ *is the unique matrix that satisfies* $P(A^\gamma)^\dagger = (A^\gamma)^\dagger P = (A^\gamma)^\dagger$ *and* $A^\gamma(A^\gamma)^\dagger = (A^\gamma)^\dagger A^\gamma = P$.

Proof. Lemma 4.10 is essentially established in [57], except we further use Proposition 2.7 to restrict weights on the space (X,ρ,μ) of homogenous type to its relatively-separated subset (Λ,ρ,μ_c). We omit the details of the proof here. □

To prove Theorem 4.1, we also need a result about the Gram matrix, which implies that Φ in Theorem 4.1 is self-localized in the matrix algebra $\mathcal{A}_{p,w}(\Lambda,\Lambda)$.

Lemma 4.11. *Let* $1 \le p,q \le \infty$, (X,ρ,μ) *be a space of homogenous type with the property that* X *is uniformly discretizable and that* μ *satisfies the uniform bounded condition,* Λ *and* Γ *be relatively-separated subsets of the space* (X,ρ,μ), w *be a weight on* X *with the property that (2.8) and (2.9) hold for some weight* v *on* X, *and* $\Phi = \{\phi_\lambda : \lambda \in \Lambda\}$ *and* $\Psi = \{\psi_\gamma : \gamma \in \Gamma\}$ *satisfy*

$$\|\Phi\|_{q,p,w} + \|\Psi\|_{q/(q-1),p,w} < \infty. \tag{4.12}$$

Define the Gram matrix $A_{\Psi,\Phi} = (A_{\Psi,\Phi}(\gamma,\lambda))_{\gamma\in\Gamma,\lambda\in\Lambda}$ *by*

$$A_{\Psi,\Phi}(\gamma,\lambda) := \int_X \psi_\gamma(x)\overline{\phi_\lambda(x)}d\mu(x). \tag{4.13}$$

Then

$$A_{\Psi,\Phi} \in \mathcal{A}_{p,w}(\Gamma, \Lambda). \tag{4.14}$$

Moreover, there exists a positive constant C such that

$$\|A_{\Psi,\Phi}\|_{\mathcal{A}_{p,w}} \le C\|\Psi\|_{q/(q-1),p,w}\|\Phi\|_{q,p,w}. \tag{4.15}$$

Proof. Obviously it suffices to prove (4.15). Let X_0 be a uniformly discretizable set of X. Define

$$Q_{\Psi,q/(q-1)} = \big(\|\psi_\gamma\|_{L^{q/(q-1)}(B(x_0,1))}\big)_{\gamma\in\Gamma, x_0\in X_0}$$

and

$$Q_{\Phi,q} = \big(\|\phi_\lambda\|_{L^q(B(x_0,1))}\big)_{\lambda\in\Lambda, x_0\in X_0}.$$

By (4.7), (4.12), Lemma 4.10, and the following estimate for any $\gamma \in \Gamma$ and $\lambda \in \Lambda$:

$$|A_{\Psi,\Phi}(\gamma, \lambda)| \le \sum_{x_0\in X_0} \|\psi_\gamma\|_{L^{q/(q-1)}(B(x_0,1))}\|\phi_\lambda\|_{L^q(B(x_0,1))},$$

we obtain

$$\begin{aligned}\|A_{\Psi,\Phi}\|_{\mathcal{A}_{p,w}} &\le C\|Q_{\Psi,q/(q-1)}\|_{\mathcal{A}_{p,w}}\|Q^*_{\Phi,q}\|_{\mathcal{A}_{p,w}} \\ &\quad + C\|Q_{\Phi,q}\|_{\mathcal{A}_{p,w}}\|Q^*_{\Psi,q/(q-1)}\|_{\mathcal{A}_{p,w}} \\ &\le C\|\Psi\|_{q/(q-1),p,w}\|\Phi\|_{q,p,w} < \infty. \end{aligned} \tag{4.16}$$

Therefore the estimate (4.15) follows. □

Now we are ready to start the proof of Theorem 4.1.

Proof of Theorem 4.1. First we prove the first statement of Theorem 4.1. Let $A_{\Phi,\Phi} = (A_{\Phi,\Phi}(\gamma, \lambda))_{\gamma,\lambda\in\Lambda}$ be as in (4.13). Then

$$A_{\Phi,\Phi} \in \mathcal{A}_{p,w}(\Lambda, \Lambda) \tag{4.17}$$

by Theorem 2.11, Lemma 4.11, and the assumption $\Phi \in \mathcal{B}_{q,p,w}(\Lambda)$.

By the frame property of Φ in $V_2(\Phi, \Lambda)$, there exists a positive constant C_0 such that

$$\sum_{\lambda,\lambda'\in\Lambda} \overline{\langle\phi_\lambda, f\rangle} A_{\Phi,\Phi}(\lambda, \lambda')\langle\phi_{\lambda'}, f\rangle = \Big\|\sum_{\lambda\in\Lambda}\langle f, \phi_\lambda\rangle\phi_\lambda\Big\|_2^2 \ge C_0 \sum_{\lambda\in\Lambda} |\langle\phi_\lambda, f\rangle|^2 \tag{4.18}$$

for any $f \in V_2(\Phi, \Lambda)$. Let

$$H = \big\{(\langle\phi_\lambda, f\rangle)_{\lambda\in\Lambda} : f \in V_2(\Phi, \Lambda)\big\}. \tag{4.19}$$

Then H is a Hilbert subspace of $\ell^2(\Lambda)$ by the frame property of Φ. For any sequence $d = (d(\lambda))_{\lambda\in\Lambda} \in H^\perp$, we have

$$\sum_{\lambda'\in\Lambda} A_{\Phi,\Phi}(\lambda, \lambda')d(\lambda') = \sum_{\lambda'\in\Lambda} \overline{\langle\phi_{\lambda'}, \phi_\lambda\rangle} d(\lambda') = 0. \tag{4.20}$$

Using (4.17), (4.18) and (4.20), and applying Lemma 4.10 with H being as in (4.19), we obtain that

$$(A_{\Phi,\Phi})^\dagger \in \mathcal{A}_{p,w}(\Lambda,\Lambda). \tag{4.21}$$

We write $(A_{\Phi,\Phi})^\dagger := (A^d_\Phi(\lambda,\lambda'))_{\lambda,\lambda'\in\Lambda}$, and define $\Phi^d = \{\phi^d_\lambda : \lambda\in\Lambda\}$ by

$$\phi^d_\lambda := \sum_{\lambda'\in\Lambda} A^d_\Phi(\lambda,\lambda')\phi_{\lambda'}. \tag{4.22}$$

Then

$$\Phi^d \subset V_1(\Phi,\Lambda) \tag{4.23}$$

by (4.21) and the fact that $\mathcal{A}_{p,w}(\Lambda,\Lambda)\subset \mathcal{A}_{1,w_0}(\Lambda,\Lambda)$ by Lemma 4.10 and the assumption on the weight w;

$$\begin{aligned}\|\Phi^d\|_{q,p,w} &\le \Big\|\Big(\sum_{\lambda'\in\Lambda}|A^d_\Phi(\lambda,\lambda')|\|\phi_{\lambda'}\|_{L^q(B(x_0,1))}\Big)_{\lambda\in\Lambda,x_0\in X_0}\Big\|_{\mathcal{A}_{p,w}}\\ &\le C\|(A_{\Phi,\Phi})^\dagger\|_{\mathcal{A}_{p,w}}\|\Phi\|_{q,p,w}<\infty \qquad (4.24)\end{aligned}$$

by (4.9), (4.21) and (4.22); and

$$\Phi^d = S^{-1}\Phi \tag{4.25}$$

because

$$\begin{aligned}\big(\langle S\phi^d_\lambda, f\rangle\big)_{\lambda\in\Lambda} &= \Big(\sum_{\lambda_1,\lambda_2\in\Lambda} A^d_\Phi(\lambda,\lambda_2)\langle\phi_{\lambda_2},\phi_{\lambda_1}\rangle\langle\phi_{\lambda_1},f\rangle\Big)_{\lambda\in\Lambda}\\ &= A^\dagger_\Phi A_\Phi\big(\langle\phi_{\lambda_1},f\rangle\big)_{\lambda_1\in\Lambda} = \big(\langle\phi_\lambda,f\rangle\big)_{\lambda\in\Lambda} \text{ for all } f\in V_2(\Phi,\Lambda).\end{aligned}$$

Hence the first statement follows from (4.23), (4.24), and (4.25).

Now we prove the second statement of Theorem 4.1. Using (4.17), (4.18), and (4.20), and applying Lemma 4.10 with H being as in (4.19), we obtain that

$$((A_{\Phi,\Phi})^{1/2})^\dagger \in \mathcal{A}_{p,w}(\Lambda,\Lambda). \tag{4.26}$$

We write $((A_{\Phi,\Phi})^{1/2})^\dagger = (A^o_\Phi(\lambda,\lambda'))_{\lambda,\lambda'\in\Lambda}$ and define $\Phi^o = \{\phi^o_\lambda : \lambda\in\Lambda\}$ by

$$\phi^o_\lambda := \sum_{\lambda'\in\Lambda} A^o_\Phi(\lambda,\lambda')\phi_{\lambda'}. \tag{4.27}$$

Similar to the above argument for Φ^d, we have

$$\Phi^o \subset V_1(\Phi,\Lambda) \tag{4.28}$$

and

$$\|\Phi^o\|_{q,p,w} \le C\|((A_{\Phi,\Phi})^{1/2})^\dagger\|_{\mathcal{A}_{p,w}}\|\Phi\|_{q,p,w}<\infty. \tag{4.29}$$

Define the operator T on $V_2(\Phi,\Lambda)$ by $Tf=\sum_{\lambda\in\Lambda}\langle f, S^{-1}\phi_\lambda\rangle\phi^o_\lambda$. From (4.21) and (4.25) it follows that $\langle T\phi_\lambda, f\rangle = \langle \phi^o_\lambda, f\rangle$ for all $\lambda\in\Lambda$ and $f\in V_2(\Phi,\Lambda)$, which implies that

$$T\phi_\lambda = \phi^o_\lambda \quad \text{for all } \lambda\in\Lambda. \tag{4.30}$$

For any $f, h \in V_2(\Phi, \Lambda)$,

$$\begin{aligned}\langle ST^2 f, h\rangle &= \sum_{\lambda_i\in\Lambda, 1\le i\le 5} \langle f, S^{-1}\phi_{\lambda_1}\rangle A^o_{\Phi,\Phi}(\lambda_1,\lambda_2)\langle \phi_{\lambda_2}, S^{-1}\phi_{\lambda_3}\rangle \\ &\qquad A^o_{\Phi,\Phi}(\lambda_3,\lambda_4)\langle \phi_{\lambda_4},\phi_{\lambda_5}\rangle\langle\phi_{\lambda_5}, h\rangle \\ &= \Big\langle ((A_{\Phi,\Phi})^{1/2})^\dagger Q((A_{\Phi,\Phi})^{1/2})^\dagger A_{\Phi,\Phi} c, d\Big\rangle \\ &= \langle c, d\rangle = \langle f, h\rangle,\end{aligned}$$

where $c = (\langle \phi_\lambda, h\rangle)_{\lambda\in\Lambda}$, $d = (\langle \phi_\lambda, S^{-1} f\rangle)_{\lambda\in\Lambda}$, and Q is the projection operator from $\ell^2(\Lambda)$ to the sequence space H in (4.19). Thus

$$T = S^{-1/2}. \tag{4.31}$$

Therefore the second statement in Theorem 4.1 follow from (4.28), (4.29), (4.30), and (4.31).

Then the third statement of Theorem 4.1. By the first statement, we have

$$\phi_\lambda = \sum_{\lambda'\in\Lambda} \langle \phi_\lambda, S^{-1}\phi_{\lambda'}\rangle \phi_{\lambda'} \quad \text{for all } \lambda\in\Lambda. \tag{4.32}$$

This together with the Riesz basis assumption proves the third statement.

Finally the last statement of Theorem 4.1 follows from

$$S^{-1/2}\phi_\lambda = \sum_{\lambda'\in\Lambda} \langle S^{-1/2}\phi_\lambda, S^{-1/2}\phi_{\lambda'}\rangle S^{-1/2}\phi_{\lambda'} \quad \text{for all } \lambda\in\Lambda.$$

and the Riesz basis assumption on Φ. □

4.2. **Proof of Theorem 4.5.** To prove Theorem 4.5, we apply similar argument to the one used in the proof of Theorem 4.1, essentially except using the matrix algebra $\mathcal{C}_{p,w}(\Lambda)$ of Sjöstrand class instead of the matrix algebra $\mathcal{A}_{p,w}(\Lambda,\Lambda)$ of Schur class. Here for $1\le p\le\infty$, a translation-invariant weight w on $\mathbf{R}^d$ of the form $w(x,y) = u(x-y)$ for some positive function u, and a relatively-separated subset Λ of $\mathbf{R}^d$, the Sjöstrand class $\mathcal{C}_{p,w}(\Lambda)$ is defined by

$$\begin{aligned}\mathcal{C}_{p,\alpha}(\Lambda) \;:=\; \Big\{ & A := (A(\lambda,\lambda'))_{\lambda,\lambda'\in\Lambda}, \\ & \|A\|_{\mathcal{C}_{p,\alpha}} := \|(A_*(k)u(k))_{k\in\mathbf{Z}^d}\|_{\ell^p} < \infty\Big\},\end{aligned} \tag{4.33}$$

where

$$A_*(k) = \sup_{\lambda\in m+[-1/2,1/2)^d, \lambda'\in m+k+[-1/2,1/2)^d, m\in\mathbf{Z}^d\times\mathbf{Z}^d} |A(\lambda,\lambda')|,\ k\in\mathbf{Z}^d.$$

The Sjöstrand class $\mathcal{C}_{1,w_0}(\mathbf{Z}^d)$ was introduced by Sjöstrand for the study of pseudo-differential operators, see [4, 5, 30, 54, 57] and references therein for the recent developments and various applications.

Similar to Lemma 4.10, we have the following properties for the matrix algebra $\mathcal{C}_{p,w}(\Lambda)$ of Sjöstrand class.

Lemma 4.12. ([57]) *Let $1 \le p \le q \le \infty$, set $r = pq/(q-p)$, w and $\tilde{w}$ are translation-invariant weights on $\mathbf{R}^d$ of the form $w(x,y) = u(x,y)$ and $\tilde{w}(x,y) = \tilde{u}(x-y)$ where $u, \tilde{u}$ are some positive functions on $\mathbf{R}^d$, and Λ is a relatively-separated subset of $\mathbf{R}^d$. Then the following statements are true.*

(i) *If $\|u\tilde{u}^{-1}\|_{L^r} < \infty$, then there exists a positive constant C such that*

$$\|A\|_{\mathcal{C}_{p,w}} \le C\|A\|_{\mathcal{C}_{q,\tilde{w}}} \text{ for all } A \in \mathcal{C}_{q,\tilde{w}}(\Lambda, \Lambda'). \tag{4.34}$$

(ii) *If there exists another translation-invariant weight v such that (2.8) and (2.9) holds, then there exists a positive constant C such that*

$$\|AB\|_{\mathcal{C}_{p,w}} \le C\|A\|_{\mathcal{C}_{p,w}}\|B\|_{\mathcal{C}_{p,w}} \tag{4.35}$$

for all $A, B \in \mathcal{C}_{p,w}(\Lambda)$.

(iii) *If w is a $(p, \min(2,p))$-admissible translation-invariant weight on $\mathbf{R}^d$ and if A is a matrix in $\mathcal{C}_{p,w}(\Lambda)$ that satisfies (4.10), (4.11) and $A^* = A$, then for any $\lambda > 0$, both A^γ and its Moore-Penrose pseudo-inverse $(A^\gamma)^\dagger$ belong to $\mathcal{C}_{p,w}(\Lambda)$.*

Now we start the proof of Theorem 4.5.

Proof of Theorem 4.5. By (4.3), it follows that for any $\lambda \in k + [-1/2, 1/2)^d$ and $\lambda' \in k' + [-1/2, 1/2)^d$ with $k, k' \in \mathbf{Z}^d$,

$$\begin{aligned} & |\langle \phi_\lambda, \phi_{\lambda'} \rangle| \\ \le & \int_{\mathbf{R}^d} |h(x-\lambda)||h(x-\lambda')|dx \\ \le & \sum_{\epsilon \in \{-1,0,1\}^d, l \in \mathbf{Z}^d} \|h\|_{L^\infty(l+[-1/2,1/2]^d)} \|h\|_{L^\infty(k-k'+\epsilon-l+[-1/2,1/2]^d)}. \end{aligned}$$

This, together with the Wiener amalgam space assumption on h and admissibility assumption on the translation-invariant weight w, implies that the Gram matrix $A_{\Phi,\Phi}$ in (4.13) belongs to $\mathcal{C}_{p,w}(\Lambda)$,

$$A_{\Phi,\Phi} \in \mathcal{C}_{p,w}(\Lambda). \tag{4.36}$$

Using the similar argument used in the proof of Theorem 4.1 and applying (4.36) and the Wiener's lemma for infinite matrices in the Sjöstrand class $\mathcal{C}_{p,w}(\Lambda)$ (the third statement in Lemma 4.12), we have that

$$(A_{\Phi,\Phi})^\dagger, ((A_{\Phi,\Phi})^{1/2})^\dagger \in \mathcal{C}_{p,\alpha}(\Lambda), \tag{4.37}$$

which implies that there exists a sequence $G = (g(k))$ with

$$(g(k)u(k)) \in \ell^p \tag{4.38}$$

such that

$$|(A_{\Phi,\Phi})^\dagger(\lambda, \lambda')| + |((A_{\Phi,\Phi})^{1/2})^\dagger(\lambda, \lambda')| \le g(k-k')$$

for all $\lambda \in k + [-1/2, , 1/2)^d, \lambda' \in k' + [-1/2, 1/2)^d$ with $k, k' \in \mathbf{Z}^d$. Therefore for any $\lambda \in k + [-1/2, 1/2)^d$ and $x \in k' + [-1/2, 1/2)^d$ with $k, k' \in \mathbf{Z}^d$, we have that

$$|S^{-1}\phi_\lambda(x)| + |S^{-1/2}\phi_\lambda(x)|$$
$$\le \sum_{k''\in\mathbf{Z}^d} \sum_{\lambda''\in\Lambda\cap k''+[-1/2,1/2)^d} g(k-k'')H(x-\lambda'')$$
$$\le C_0 \sum_{k''\in\mathbf{Z}^d} g(k-k'') \sup_{t\in[-1,1)^d} |h(k'-k''-t)| \le \tilde h(x-\lambda), \tag{4.39}$$

where

$$\tilde h(x) = C_0 \sup_{l\in(x+[-1,1]^d)\cap\mathbf{Z}^d} \sum_{k''\in\mathbf{Z}^d} g(k'') \sup_{t\in[-1,1)^d} |h(l-k''-t)|. \tag{4.40}$$

From (4.38), (4.40) and the assumption $h \in W_\infty(L_{p,u})$, it follows that $\tilde h \in W_\infty(L_{p,u})$. This together with (4.39) proves (4.4), and hence completes the proof. □

5. Banach Frames

In this section, we discuss the Banach frame property of the generator Φ for the space $V_r(\Phi,\Lambda)$ with $r \ne 2$.

Theorem 5.1. *Let $X, \rho, \mu, q, p, w, \Lambda, \Phi, S$ be as in Theorem 4.1, $q/(q-1) \le r \le q$, and $V_r(\Phi,\Lambda)$ as in (3.1). Then Φ is a Banach frame for the space $V_r(\Phi,\Lambda)$ with respect to $\ell^r(\Lambda)$ (resp. r-Riesz basis for the space $V_r(\Phi,\Lambda)$) if Φ is a frame (resp. Riesz basis) of $V_2(\Phi,\Lambda)$.*

As an application of Theorem 5.1, we have the following result for the shift-invariant setting.

Corollary 5.2. *Let $1 \le p \le \infty, 2 \le q \le \infty, q/(q-1) \le r \le q$, $u(x) = (1+|x|)^\alpha$ for some $\alpha > d(1-1/p)$ or $u(x) = \exp(D|x|^\delta)$ for some $D > 0$ and $\delta \in (0,1)$, and $\phi_1,\dots,\phi_N$ belong to the Wiener amalgam space $W_q(L_{p,u})$. Then $\Phi := \{\phi_n(\cdot - k) : 1 \le n \le N, k \in \mathbf{Z}^d\}$ is a Banach frame (resp. Riesz basis) of the shift-invariant space $V_r(\phi_1,\dots,\phi_N)$ if Φ is a frame (resp. Riesz basis) of $V_2(\phi_1,\dots,\phi_N)$.*

Remark 5.3. For the case that $q = \infty$, the frame result in Corollary 5.2 was proved by Aldroubi, Tang, and the author under the weak assumption $\phi_1,\dots,\phi_N \in W_\infty(L_{1,0}) \subset W_\infty(L_{p,u})$, while the result for $1 \le q < \infty$ is new. The Riesz basis result in Corollary 5.2 is true under the weak assumption that $\phi_1,\dots,\phi_N \in \mathcal{L}^q \subset W_q(L_{p,u})$, see [39, Theorems 4.1 and 4.2]. For the shift-invariant setting, using a characterization of the frame property in the Fourier technique ([2, 6, 39]), it is shown that the converse in Corollary 5.2 is true. The author believes that the converse in Theorem 5.1 is true for the non-shift-invariant setting, that is, Φ is a frame (resp. Riesz basis) for $V_2(\Phi,\Lambda)$ if Φ is a Banach frame (resp. Riesz basis) for the space $V_r(\Phi,\Lambda)$ with $r \ne 2$.

By Theorems 3.2 and 4.1, the proof of Theorem 5.1 depends, in turn, on the following result about the extension of an operator P on $V_2(\Phi, \Lambda)$.

Theorem 5.4. *Let $X, \rho, \mu, q, p, w, \Lambda, \Phi, S$ be as in Theorem 4.1. Let $q/(q-1) \le r \le q$ and define $V_r(\Phi, \Lambda)$ as in (3.1). If Φ is a frame for $V_2(\Phi, \Lambda)$, then the operator P, to be defined by*

$$P: \ f \longmapsto \sum_{\lambda\in\Lambda} \langle f, \phi_\lambda\rangle S^{-1}\phi_\lambda, \tag{5.1}$$

is a bounded operator from L^r to $V_r(\Phi, \Lambda)$ and satisfies

$$Pf = f \text{ for all } f \in V_r(\Phi, \Lambda). \tag{5.2}$$

Proof. By Theorem 3.2, the operator P in (5.1) defines a bounded operator from L^r to $V_r(\Phi, \Lambda)$ for any r between $q/(q-1)$ and q. The reconstruction formula (5.2) for $1 \le r \le q$ and $r < \infty$ follows the frame reconstruction formula

$$f = \sum_{\lambda\in\Lambda} \langle f, \phi_\lambda\rangle S^{-1}\phi_\lambda \quad \text{for all } f \in V_2(\Phi, \Lambda) \tag{5.3}$$

([13, 28]), the density of $\ell^2 \cap \ell^r$ in ℓ^r, and the following estimate

$$\begin{aligned}
&\Big\| \sum_{\lambda_1,\lambda_2,\lambda_3\in\Lambda} |c(\lambda_1)| \times |\langle \phi_{\lambda_1}, \phi_{\lambda_2}\rangle| \times |A_\Phi^d(\lambda_2, \lambda_3)| \times |\phi_{\lambda_3}| \Big\|_r \\
\le\ & C\|c\|_{\ell^r(\Lambda)} \|A_{\Phi,\Phi}\|_{\mathcal{A}_{1,w_0}} \|(A_{\Phi,\Phi})^\dagger\|_{\mathcal{A}_{1,w_0}} \|\Phi\|_{r,1,w_0} \\
\le\ & C\|c\|_{\ell^r(\Lambda)} \|A_{\Phi,\Phi}\|_{\mathcal{A}_{p,w}} \|(A_{\Phi,\Phi})^\dagger\|_{\mathcal{A}_{p,w}} \|\Phi\|_{r,p,w} \\
\le\ & C\|c\|_{\ell^r(\Lambda)} \|A_{\Phi,\Phi}\|_{\mathcal{A}_{p,w}} \|(A_{\Phi,\Phi})^\dagger\|_{\mathcal{A}_{p,w}} \|\Phi\|_{q,p,w} < \infty
\end{aligned}$$

for any $c := (c(\lambda))_{\lambda\in\Lambda} \in \ell^r(\Lambda)$ by (4.21), Theorem 3.2, and Lemmas 4.10 and 4.11.

For $r = \infty$, we have $q = \infty$. Take any $x_0 \in X$ and $f = \sum_{\lambda\in\Lambda} c(\lambda)\phi_\lambda \in V_\infty(\Phi, \Lambda)$ with $c := (c(\lambda))_{\lambda\in\Lambda} \in \ell^\infty(\Lambda)$. We define $f_{x_0,N} = \sum_{\rho(\lambda,x_0)\le N} c(\lambda)\phi_\lambda$ for $N \ge 1$. Then by (2.2), there exists a positive constant C for any $\delta > 0$ (independent of $x_0 \in X$ and $N \ge 1$) such that

$$\begin{aligned}
&\sup_{x\in B(x_0,1)} |f(x) - f_{x_0,N}(x)| \\
\le\ & \sum_{\rho(\lambda,x_0)>N} |c(\lambda)| \sup_{x\in B(x_0,1)} |\phi_\lambda(x)| \\
\le\ & C\|c\|_{\ell^\infty(\Lambda)} \|\Phi\|_{\infty,p,w} \Big(\sum_{\rho(\lambda,x_0)>N} w(\lambda, x_0)^{-p/(p-1)} \Big)^{(p-1)/p} \\
\le\ & C\|c\|_{\ell^\infty(\Lambda)} \|\Phi\|_{\infty,p,w} \|w(\cdot, x_0)\chi_{X\setminus B(x_0,N/L-1)}(\cdot)\|_{L^{p/(p-1)}}, \qquad (5.4)
\end{aligned}$$

and

$$\begin{aligned}
& \sup_{x\in B(x_0,1)} \Big|\sum_{\lambda\in\Lambda} \langle f-f_{x_0,N}, \phi_\lambda\rangle S^{-1}\phi_\lambda(x)\Big| \\
\le\ & \|c\|_{\ell^\infty} \sum_{\lambda\in\Lambda}\sum_{\rho(\lambda',k)>N,\lambda'\in\Lambda} |\langle \phi_{\lambda'},\phi_\lambda\rangle| \sup_{x\in B(x_0,1)} |S^{-1}\phi_\lambda(x)| \\
\le\ & \|c\|_{\ell^\infty}\Big(\sum_{\rho(\lambda,x_0)\ge N/(2L),\lambda\in\Lambda}\sum_{\lambda'\in\Lambda} + \sum_{\lambda\in\Lambda}\sum_{\rho(\lambda,\lambda')\ge N/(2L),\lambda'\in\Lambda}\Big) \\
& \qquad |\langle \phi_{\lambda'}, S^{-1}\phi_\lambda\rangle| \sup_{x\in B(x_0,1)} |S^{-1}\phi_\lambda(x)| \\
\le\ & \|c\|_{\ell^\infty(\Lambda)}\|A_{\Phi,\Phi}\|_{\mathcal{A}_{1,w_0}} \sum_{\rho(\lambda,x_0)\ge N/(2L)} \sup_{x\in B(x_0,1)} |S^{-1}\phi_\lambda(x)| \\
& + \|c\|_{\ell^\infty}\|S^{-1}\Phi\|_{\infty,1,w_0} \sup_{\lambda\in\Lambda} \sum_{\rho(\lambda',\lambda)\ge N/(2L),\lambda'\in\Lambda} |\langle\phi_{\lambda'},\phi_\lambda\rangle| \\
\le\ & C\|c\|_{\ell^\infty(\Lambda)}\|A_{\Phi,\Phi}\|_{\mathcal{A}_{p,w}} \\
& \times \|S^{-1}\Phi\|_{\infty,p,w} \sup_{x\in X} \|(w(x,\cdot))^{-1}\chi_{X\setminus B(x,N/(2L^2)-1)}(\cdot)\|_{L^{p/(p-1)}}, \qquad (5.5)
\end{aligned}$$

where L is the constant in the definition of the quasi-metric ρ. By (2.9), (5.3), (5.4), (5.5), Theorem 4.1, Lemma 4.11, and the assumption $\Phi \in \mathcal{B}_{\infty,p,\alpha}(\Lambda)$, we obtain

$$\begin{aligned}
& \sup_{x\in B(x_0,1)} |f(x)-Pf(x)| \\
=\ & \sup_{x\in B(x_0,1)} \Big|f(x)-\sum_{\lambda\in\Lambda}\langle f,\phi_\lambda\rangle S^{-1}\phi_\lambda(x)\Big| \\
=\ & \sup_{x\in B(x_0,1)} \Big|(f-f_{x_0,N})(x)-\sum_{\lambda\in\Lambda}\langle f-f_{x_0,N},\phi_\lambda\rangle S^{-1}\phi_\lambda(x)\Big| \\
\le\ & C\|c\|_{\ell^\infty(\Lambda)} \sup_{x\in X} \|(w(x,\cdot))^{-1}\chi_{X\setminus B(x,N/(2L^2)-1)}(\cdot)\|_{L^{p/(p-1)}} \to 0
\end{aligned}$$

as $N\to\infty$. This proves (5.2) for $r=\infty$, and hence completes the proof. □

6. Refinable Functions With Global Support

In this section, we consider an application of Theorem 4.1 to the study of refinable functions with global support.

Fix a matrix M whose eigenvalues have norm strictly larger than one. We say that functions $\phi_1,\ldots,\phi_N$ are *refinable* if there exists a matrix-valued ℓ^2-sequence $(a(k))_{k\in\mathbf{Z}^d}$ such that

$$\phi(M^{-1}\cdot) = \sum_{k\in\mathbf{Z}^d} a(k)\phi(\cdot-k), \qquad (6.1)$$

where $\phi=(\phi_1,\ldots,\phi_N)^T$. The sequence $(a(k))_{k\in\mathbf{Z}^d}$ is known as the *mask* of the refinable function ϕ. The reader may refer [15, 23, 43, 48, 59] and references therein for its various properties and applications to the theory of multiresolution analysis and the construction of wavelets. In the following result, we show that if a refinable function has polynomial decay then the mask has the same polynomial decay.

Theorem 6.1. *Let $2 \le p \le \infty$, $\alpha > d(1-1/p)$, and $\phi_1, \dots, \phi_N$ be functions on $\mathbf{R}^d$. Assume that $\phi_n(x)(1+|x|)^\alpha \in L^p, 1 \le n \le N$, $\{\phi_n(\cdot - k) : 1 \le n \le N, k \in \mathbf{Z}^d\}$ is a Riesz basis for the shift-invariant space $V_2(\phi_1, \dots, \phi_N)$, and $\phi = (\phi_1, \dots, \phi_N)^T$ satisfies the refinement equation (6.1). Then the mask $(a(k))_{k \in \mathbf{Z}^d}$ belongs to $\ell^p_\alpha(\mathbf{Z}^d)$, that is, $(a(k)(1+|k|)^\alpha)_{k \in \mathbf{Z}^d} \in \ell^p$.*

Proof. By Corollary 4.3, there exist functions $\tilde\phi_1, \dots, \tilde\phi_N$ such that $\tilde\phi_n(x)(1+|x|)^\alpha \in L^p, 1 \le n \le N$, and

$$\int_{\mathbf{R}^d} \phi_n(\cdot - k)\overline{\tilde\phi_{n'}(x-k')}dx = \delta_{nn'}\delta_{kk'} \tag{6.2}$$

for all $1 \le n, n' \le N$ and $k, k' \in \mathbf{Z}^d$. From (6.1) and (6.2), it follows that

$$\begin{aligned} |a(k)| &\le \sum_{n,n'=1}^N \int_{\mathbf{R}^d} |\phi_n(M^{-1}x)||\tilde\phi_{n'}(x-k)|dx \\ &\le |\det M| \sum_{n,n'=1}^N \sum_{l \in \mathbf{Z}^d} \|\phi_n\|_{L^p(M^{-1}(l+[0,1]^d))} \|\tilde\phi_{n'}\|_{L^p(l-k+[0,1]^d)}, \ k \in \mathbf{Z}^d. \end{aligned}$$

Therefore

$$\begin{aligned} & \big\|\big(a(k)(1+|k|)^\alpha\big)_{k\in\mathbf{Z}^d}\big\|_{\ell^p} \\ \le\ & C \sum_{n,n'=1}^N \big\|\big(\|\phi_n\|_{L^p(M^{-1}(l+[0,1]^d))}\big)_{l\in\mathbf{Z}^d}\big\|_{\ell^1} \big\|\big(\|\tilde\phi_n\|_{L^p(l+[0,1]^d)}(1+|l|)^\alpha\big)_{l\in\mathbf{Z}^d}\big\|_{\ell^p} \\ & + C \sum_{n,n'=1}^N \big\|\big(\|\phi_n\|_{L^p(M^{-1}(l+[0,1]^d))}(1+|M^{-1}l|)^\alpha\big)_{l\in\mathbf{Z}^d}\big\|_{\ell^p} \\ & \qquad \times \big\|\big(\|\tilde\phi_n\|_{L^p(l+[0,1]^d)}\big)_{l\in\mathbf{Z}^d}\big\|_{\ell^1} \\ \le\ & C \sum_{n,n'=1}^N \|\phi_n(x)(1+|x|)^\alpha\|_p \|\tilde\phi_{n'}(x)(1+|x|)^\alpha\|_p < \infty. \end{aligned}$$

This completes the proof. □

Using the similar argument with some modification, we have the following result about the mask of a refinable function that has (sub)exponential decay.

Theorem 6.2. *Let $2 \le p \le \infty$, $\delta \in (0,1]$, $D > 0$, and $\phi_1, \dots, \phi_N$ be functions on $\mathbf{R}^d$. Assume that $\phi_n(x)\exp(D|x|^\delta) \in L^p, 1 \le n \le N$, $\{\phi_n(\cdot - k) : 1 \le n \le N, k \in \mathbf{Z}^d\}$ is a Riesz basis for the shift-invariant space $V_2(\phi_1, \dots, \phi_N)$, and $\phi = (\phi_1, \dots, \phi_N)^T$ satisfies the refinement equation*

(6.1). Then there exists a positive constant D', which is usually strictly smaller than D, such that the mask $(a(k))_{k\in\mathbf{Z}^d}$ satisfies $(a(k)\exp(D'(1+|k|)^\delta))_{k\in\mathbf{Z}^d}\in\ell^p$.

Remark 6.3. Similar results to the ones in Theorems 6.1 and 6.2 about the interaction between the decay at infinity of a refinable function and its mask was discussed in [39, 55]. In particular, Jia and Micchelli proved that if $\phi\in\mathcal{L}^2$ then the mask $(a(k))_{k\in\mathbf{Z}^d}\in\ell^1$ ([39]). We remark that for $\alpha>d(1-1/p)$ with $p\ge 2$, $\phi(x)(1+|x|)^\alpha\in L^p$ implies that $\phi\in\mathcal{L}^2$ and $(a(k))_{k\in\mathbf{Z}^d}\in\ell^p_\alpha$ implies that $(a(k))_{k\in\mathbf{Z}^d}\in\ell^1$. Thus comparing with Jia and Micchelli's result in [39] and the result in Theorem 6.1, we make stronger assumption on the refinable function which leads to better decay for its mask.

Remark 6.4. The author believes that the conclusion in Theorem 6.1 is true for $1\le p<2$ with the Riesz basis assumption for the space $V_2(\phi_1,\ldots,\phi_N)$ replaced by the corresponding assumption for the space $V_p(\phi_1,\ldots,\phi_N)$. The author further conjectures that the following statement is true: *Let $1\le p\le\infty, \alpha>d(1-1/p)$, $(a(k))_{k\in\mathbf{Z}^d}$ be a summable sequence of $N\times N$ matrices such that $H(\xi)=|\det M|^{-1}\sum_{k\in\mathbf{Z}^d}a(k)e^{-ik\xi}$ is Hölder continuous at the origin, that one is an eigenvalue of $H(0)$ with one-dimensional eigenspace, and that all other eigenvalues of the matrix $H(0)$ lie inside the open unit circle. Define $\phi=(\phi_1,\ldots,\phi_N)^T$ by $\widehat{\phi}(\xi)=\prod_{j=1}^\infty H((M^T)^{-j}\xi)v_0$, where v_0 is a nonzero eigenvector of the matrix $H(0)$ associated with the eigenvalue one. Assume that $(\widehat{\phi}_n(\xi+2k\pi))_{1\le n\le N,k\in\mathbf{Z}^d}$ has full rank N for all $\xi\in\mathbf{R}^d$. Then $(a(k)(1+|k|)^\alpha)_{k\in\mathbf{Z}^d}\in\ell^p$ if and only if $\langle\phi(\cdot-x),h\rangle(1+|x|)^\alpha\in L^p, 1\le n\le N$, for all C^∞ function h supported in $[-1,1]^d$.* The reader may refer [55] for a partial result to the above conjecture.

References

[1] A. Aldroubi and K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant space, *SIAM Review*, **43**(2001), 585–620.

[2] A. Aldroubi, Q. Sun, and W.-S. Tang, p-frames and shift invariant subspaces of L^p, *J. Fourier Anal. Appl.*, **7**(2001), 1–21.

[3] A. Aldroubi, Q. Sun, and W.-S. Tang, Convolution, average sampling and a Calderon resolution of the identity for shift-invariant spaces, *J. Fourier Anal. Appl.*, **11**(2005), 215–244.

[4] R. Balan, P. G. Casazza, C. Heil, and Z. Landau, Density, overcompleteness and localization of frames I: theorey; II: Gabor system, Preprint, 2004.

[5] A. G. Baskakov, Wiener's theorem and asymptotic estimates for elements in inverse matrices, *Funktsional Anal. o. Prilozhen.*, **24**(1990), 64–65.

[6] J. J. Benedetto and S. Li, The theory of multiresolution analysis frames and applications to filter banks, *Appl. Comput. Harmon. Anal.*, **5**(1998), 389–427.

[7] M. Bownik, The structure of shift-invariant subspaces of $L^2(R^n)$, *J. Funct. Anal.*, **177**(2000), 282–309.

[8] M. Bownik, Anisotropic Hardy spaces and wavelets, *Memoir Amer. Math. Soc.*, 2003.

[9] P. L. Butzer, A survey of the Whittaker-Shannon sampling theorem and some of its extensions, *J. Math. Res. Exposition*, **3**(1983), 185–212.

[10] P. Casazza, The art of frame theory, *Taiwanese J. Math.*, **4**(2000), 129–201.

[11] P. Casazza, D. Han, and D. R. Larson, Frames for Banach spaces, In *The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999)* , *Contemp. Math.*, **247** (1999), 149–182.

[12] H.-L. Chen and S.-L. Peng, Localization of dual periodic scaling and wavelet functions, *Adv. Comput. Math.*, **19**(2003), 195–210.

[13] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, 2003.

[14] O. Christensen and D. T. Stoeva, p-frames in separable Banach spaces, *Adv. Comput. Math.*, **18**(2003), 117–126.

[15] C. K. Chui, *An Introduction to Wavelets*, Academic Press, 1992.

[16] C. K. Chui, W. He, and J.Stöckler, Nonstationary tight wavelet frames: I. Bounded intervals; II. Unbounded intervals, *Appl. Comput. Harmon. Anal.*, **17**(2004), 141–197; **18**(2005), 25–66.

[17] C. K. Chui and Q. Sun, Affine frame decompositions and shift-invariant spaces, **20**(2006), 74–107.

[18] A. Cohen and N. Dyn, Nonstationary subdivision schemes and multiresolution analysis, *SIAM J. Math. Anal.*, **27**(1996), 1745–1769.

[19] A. Cohen, I. Daubechies, and P. Vial, Wavelets on the interval and fast wavelet transforms, *Appl. Comput. Harmon. Anal.* **1**(1993), 54–81.

[20] R. Coifman and M. Maggioni, Diffusion wavelets, *Appl. Comput. Harmon. Anal.*, To appear.

[21] R. Coifman and G. Weiss, *Analyses Harmoniques Noncommutative sur Certains Espaces Homogenes*, Springer, 1971.

[22] R. J.-M. Cramer, R. A. Scholtz, and M. Z. Win, Evaluation of an ultra wide-band propagation channel, *IEEE Trans. Antennas and Propagation*, **50**(2002), 561–569.

[23] I. Daubechies, *Ten Lectures on Wavelets*, CBMF Conference Series in Applied Mathematics, 61, SIAM, Philadelphia, 1992.

[24] M. Fornasier and K. Gröchenig, Intrinsic localization of frames, *Constr. Approx.*, **22**(2005), 395–415.

[25] M. Fornasier and H. Rauhut, Continuous frames, function spaces, and the discretization problem, *J. Fourier Anal. Appl.*, **11**(2005), 245–287.

[26] L. Grafakos and C. Lennerd, Characterization of $L_p(R_n)$ using Gabor frames, *J. Fourier Anal. Appl.*, **7**(2001), 101–126.

[27] K. Gröchenig, Describing functions: atomic decompositions versus frames, *Monatsh. Math.*, **112**(1991), 1–42.

[28] K. Gröchenig, *Foundation of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.

[29] K. Gröchenig, Localization of frames, Banach frames, and the invertibility of the frame operator, *J. Fourier Anal. Appl.*, **10**(2004), 105–132.

[30] K. Gröchenig, Time-frequency analysis of Sjöstrand class, *Rev. Mat. Iberoam.*, to appear.

[31] K. Gröchenig and M. Leinert, Wiener's lemma for twisted convolution and Gabor frames, *J. Amer. Math. Soc.*, **17**(2003), 1–18.

[32] K. Gröchenig and M. Leinert, Symmetry of matrix algebras and symbolic calculus for infinite matrices, *Trans, Amer. Math. Soc.*, to appear.

[33] Y. Hao, P. Marziliano, M. Vetterli, and T. Blu, Sampling and reconstruction of ECG as a signal with a finite rate of innovation, submitted to IEEE Trans.Biomedical Engineering.

[34] J. R. Higgins, Five short stories about the cardinal series, *Bull. Amer. Math. Soc.*, **12**(1985), 45–89.

[35] S. Jaffard, Properiétés des matrices bien localisées prés de leur diagonale et quelques applications, *Ann. Inst. Henri Poincaré*, **7**(1990), 461–476.

[36] A. J. E. M. Janssen, Duality and biorthogonality for Weyl-Heisenberg frames, *J. Fourier Anal. Appl.*, **1**(1995), 403–436.
[37] A. Jerri, The Shannon sampling theorem — its various extensions and applications: A tutorial review, *Proc. IEEE*, **65**(1977), 1565–1596.
[38] R.-Q. Jia, Shift-invariant spaces and linear operator equations, *Israel J. Math.*, **103**(1998), 259–288.
[39] R.-Q. Jia and C. A. Micchelli, Using the refinement equations for the construction of pre-wavelets. II. Powers of two, In *Curves and Surfaces (Chamonix-Mont-Blanc, 1990)*, Academic Press, Boston, MA, 1991, pp. 209–246.
[40] J. Kusuma, I. Maravic and M. Vetterli, Sampling with finite rate of innovation: channel and timing estimatation for UWB and GPS, *IEEE Conference on Communication 2003*, Achorage, AK.
[41] R. Macias and C. Segovia, Lipschitz functions on spaces of homogenous type, *Adv. Math.*, **33**(1979), 257–270.
[42] R. Macias and C. Segovia, A decomposition into atoms of distribution on spaces of homogenous type, *Adv. Math.*, **33**(1979), 271–309.
[43] S. G. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, 1999.
[44] I. Maravic and M. Vetterli, Sampling and reconstruction of signals with finite rate of innovation in the presence of noise, *IEEE Trans. Signal Proc.*, **53**(2005), 2788–2805.
[45] P. Marziliano and M. Vetterli, Reconstruction of Irregular sampled discrete-time bandlimited signals with unknown sampling locations, *IEEE Trans. Signal Proc.*, **48**(2000), 3462–3471.
[46] P. Marziliano, M. Vetterli, and T. Blu, Sampling and exact reconstruction of bandlimited signals with shot noise, submitted.
[47] P. Marziliano, Sampling Innovations, PhD thesis Swiss Federal Institute of Technology, April 2001.
[48] Y. Meyer, *Ondelettes et Operateurs*, Herman, Paris, 1990.
[49] A. Nagel, E. Stein, and S. Waigner, Ball and metrics defined by vector fields I: basic properties, *Acta Math.*, **155**(1985), 103–147.
[50] G. Plonka, Periodic spline interpolation with shifted nodes, *J. Approx. Theory*, **76**(1994), 1–20.
[51] A. Ron and Z. Shen, Weyl-Heisenberg frames and Riesz bases in $L_2(R^d)$, *Duke Math. J.*, **89**(1997), 237–282.
[52] A. Ron and Z. Shen, Generalized shift-invariant systems, *Constr. Approx.*, **22**(2005), 1–45.
[53] L. L. Schumaker, *Spline Functions: Basic Theory*, John Wiley & Sons, New York, 1981.
[54] J. Sjöstrand, An algebra of pseudodifferential operators, *Math. Res. Lett.*, **1**(1994), 185–192.
[55] Q. Sun, Stability of the shifts of global supported distributions, *J. Math. Anal. Appl.*, **261**(2001), 113–125.
[56] Q. Sun, Wiener's lemma for infinite matrices with polynomial off-diagonal decay, *C. R. Acad. Sci. Paris, Ser I*, **340**(2005), 567–570.
[57] Q. Sun, Wiener's lemma for infinite matrices, *Trans. Amer. Math. Soc.*, Accepted.
[58] Q. Sun, Non-uniform average sampling and reconstruction of signals with finite rate of innovation, *SIAM J. Math. Anal.*, submitted.
[59] Q. Sun, N. Bi, and D. Huang, *An Introduction to Multiband Wavelets*, Zhejiang University Press, Hangzhou 2000.
[60] M. Vetterli, P. Marziliano, and T. Blu, Sampling signals with finite rate of innovation, *IEEE Trans. Signal Proc.*, **50**(2002), 1417–1428.
[61] R. M. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, New York (1980).

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