LOCAL RECONSTRUCTION FOR SAMPLING IN SHIFT-INVARIANT SPACES

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Abstract. The local reconstruction from samples is one of most desirable properties for many applications in signal processing, but it has not been given as much attention. In this paper, we will consider the local reconstruction problem for signals in a shift-invariant space. In particular, we consider finding sampling sets X such that signals in a shift-invariant space can be locally reconstructed from their samples on X. For a locally finite-dimensional shift-invariant space V we show that signals in V can be locally reconstructed from its samples on any sampling set with sufficiently large density. For a shift-invariant space $V(\phi_1, \ldots, \phi_N)$ generated by finitely many compactly supported functions $\phi_1, \ldots, \phi_N$, we characterize all periodic nonuniform sampling sets X such that signals in that shift-invariant space $V(\phi_1, \ldots, \phi_N)$ can be locally reconstructed from the samples taken from X. For a refinable shift-invariant space $V(\phi)$ generated by a compactly supported refinable function ϕ, we prove that for almost all $(x_0, x_1) \in [0, 1]^2$, any signal in $V(\phi)$ can be locally reconstructed from its samples from $\{x_0, x_1\} + \mathbb{Z}$ with oversampling rate 2. The proofs of our results on the local sampling and reconstruction in the refinable shift-invariant space $V(\phi)$ depend heavily on the linear independent shifts of a refinable function on measurable sets with positive Lebesgue measure and the almost ripplet property for a refinable function, which are new and interesting by themselves.

1. Introduction

Given a discrete sampling set $X \subset \mathbb{R}$, we say that signals (functions) in a linear space V can be *locally* determined from their samples on X if for any compact set $K \subset \mathbb{R}$ there exists another compact set $\tilde{K} \supset K$ such that the restriction of a signal (function) $f \in V$ on K is uniquely determined by (and hence can be reconstructed from) its finite number of samples $f(x_j)$ taken from $\tilde{K} \cap X$. We call such a sampling set X as a *locally determining sampling set* for the space V.

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The local reconstruction from samples is one of most desirable properties for many applications in signal processing, e.g. for implementing real-time reconstruction numerically. However, the local reconstruction problem has not been given as much attention ([33, 50]). Aldroubi and Gröchenig ([2]), and Sun and Zhou ([45]) discussed the local reconstruction of sampling for the spline space

$$\begin{aligned}\mathcal{B}_n \;&:=\; \Big\{ f\in C^{n-2}\Big| \text{ The restriction of } f \text{ on } [k,k+1) \text{ is a}\\ &\qquad\qquad \text{polynomial of degree at most } n-1 \text{ for all } k\in\mathbb{Z}\Big\}\\ &=\; \Big\{\sum_{k\in\mathbb{Z}} c(k)B_n(\cdot-k)\Big|\ (c(k))_{k\in\mathbb{Z}} \text{ is a sequence on } \mathbb{Z}\Big\},\end{aligned} \tag{1.1}$$

generated by integer shifts of the B-spline $B_n, n\ge 2$, while Gröchenig and Schwab provided several fast local reconstruction methods for sampling in [25]. Here the B-spline B_1 is the characteristic function on $[0,1)$ and the B-splines $B_n, n\ge 2$, are defined inductively by $B_n := \int_0^1 B_{n-1}(\cdot - t)dt$.

In this paper, we consider the local reconstruction of signals (functions) in a shift-invariant space from their samples on a discrete set. Particularly, we will discuss the following problem: *Given a shift-invariant space V, find sampling sets X such that any signal (function) f in V can be locally determined from its samples $f(x_j), x_j\in X$, taken from X.* Here a linear space V of functions on the real line is said to be *shift-invariant* if $f\in V$ implies $f(\cdot-k)\in V$ for all $k\in\mathbb{Z}$ ([4, 9, 10, 17]).

The paper is organized as follows. In Section 2, we prove that any linear space V of signals (functions) on the real line, in which any signal (function) could be locally determined from its samples on some weakly relatively-separated sampling set, must be locally finite-dimensional (Theorem 2.1). Here a linear space V of functions on the real line is said to be *locally finite-dimensional* if for any compact set K, the space $V|_K$ (the restriction of all functions $f\in V$ onto K) is finite-dimensional ([7]), and a discrete subset X of $\mathbb{R}$ is said to be *weakly relatively-separated* if

$$\sum_{x_j\in X} \chi_{x_j+[-1/2,1/2]}(x) < \infty \quad \text{for all } x\in\mathbb{R}, \tag{1.2}$$

where χ_E is the characteristic function on a set E. We call the set satisfying (1.2) is weakly relatively-separated since the set satisfying the following strong condition

$$\sup_{x\in\mathbb{R}} \sum_{x_j\in X} \chi_{x_j+[-1/2,1/2]}(x) < \infty \tag{1.3}$$

is called to be a *relatively-separated* set, which has been widely used in the nonuniform sampling ([3, 6, 43]). In Section 2, we also show that for any locally finite-dimensional shift-invariant space V of continuous signals (functions), any element in V can be locally reconstructed from its samples taken on a uniform sampling set with sufficiently large density (Theorem 2.2).

Given compactly supported functions $\phi_1, \ldots, \phi_N$ on the line, we let

$$V(\phi_1, \ldots, \phi_N) := \Big\{ \sum_{n=1}^{N} \sum_{k\in\mathbb{Z}} c_n(k)\phi_n(\cdot - k) :\ (c_n(k))_{k\in\mathbb{Z}} \in \ell(\mathbb{Z}), 1 \le n \le N \Big\} \tag{1.4}$$

contain all (infinite) linear combinations of integer shifts of $\phi_1, \ldots, \phi_N$. The concept of shift-invariant spaces arose in approximation theory, wavelet theory, sampling theory etc (see [4, 7, 9, 10, 12, 17, 30, 38] and extensive list of references therein). Since sampling in a shift-invariant space is a realistic model for modelling signals with smoother spectrum, and a suitable model for taking into account the real acquisition and reconstruction devices, and the numerical implementation, it was well studied in the last twenty years, see [2, 3, 5, 6, 13, 14, 15, 16, 20, 21, 25, 29, 39, 44, 43, 48, 49, 51] and extensive list of references therein.

In Section 3, we characterize sampling sets of the form $X_0+\mathbb{Z}$ where $X_0 \subset [0, 1)$ such that any signal (function) in the finitely-generated shift-invariant space $V(\phi_1, \ldots, \phi_N)$ can be locally reconstructed from its samples taken from $X_0 + \mathbb{Z}$ (Theorem 3.1).

A sampling set of the form $X_0 + \mathbb{Z}$ with $X_0 \subset [0, 1)$ do not form a uniform sampling set but consist of nonuniformly shifted unions of a uniform sampling set $\mathbb{Z}$, which is often refered to as a *periodic nonuniform sampling set* or a *bunched sampling set* ([11, 19, 27, 37, 41, 50]). If we use the *sampling rate* to measure the sampling points per unit of time. Then the sampling rates of the periodic non-uniform sampling sets $X_0 + \mathbb{Z}$ in Theorem 3.1 and $X_0 + N\mathbb{Z}$ in Theorem 3.4 are M and M/N respectively, where $X_0 = \{x_m, 1 \le m \le M\} \subset [0, 1)$. For signals (functions) in a linear space V, Vetterli, Marziliano, Blu ([34, 47]) used the *innovative rate* to measure the degree of their freedom per unit time. We remark that the innovative rate of signals (functions) in the space $V(\phi_1, \ldots, \phi_N)$ is N. From Theorem 3.1, the sampling rate exceeds or is equal to the innovative rate N (see [43] for more general results about the sampling rate of a sampling set and the innovative rate of signals).

For most of shift-invariant spaces, the local reconstruction of signals (functions) in the shift-invariant space $V(\phi_1, \ldots, \phi_N)$ from the samples taken from $X_0 + \mathbb{Z}$ could be possible only when the sampling rate is sufficiently larger than the innovative rate (Theorem 2.2 and Remark 3.3). This inspires us to consider the periodic nonuniform sampling set X with sampling rate one that can be used to sampling signals in $V(\phi)$ (where signals have innovative rate one), particularly the sampling set of the form $X_0 + N\mathbb{Z}$ where $N \geq 1$ and $X_0 \in [0, 1)$ has cardinality N. In Theorem 3.4 we provides a sufficient condition on the generator ϕ so that signals (functions) in that single-generated shift-invariant space $V(\phi)$ can be locally reconstructed from its samples taken from $X_0+N\mathbb{Z}$ where $X_0 \subset [0, 1)$ has cardinality N. As an application of Theorem 3.4, we show that the set $X_0 + N\mathbb{Z}$ with $X_0 \subset (0, 1)$ having cardinality N is always a locally determining sampling set for the space $V(\phi)$ if ϕ is a ripplet (Corollary 3.5). Here a *ripplet* means a compactly supported continuous function ϕ such that for any $s \geq 1, x_0 < x_1 < \ldots < x_{s-1}$ and integers $n_0 < n_1 < \ldots < n_{s-1}$, $\det(\phi(x_i - n_{i'}))_{0\leq i,i'\leq s-1} \geq 0$ (see [23, 24, 36] and references therein for examples and characterizations). As a B-spline B_n is a ripplet, this recovers the local reconstruction result established in [2] for signals (functions) in a spline space.

A linear space V of functions on the real line is said to be *refinable* if $f(\cdot/2) \in V$ for any $f \in V$, and a compactly supported function ϕ is said to be *refinable* if the shift-invariant space $V(\phi)$ generated by the integer shifts of the function ϕ is refinable. The refinable space arose in wavelet theory, for instance, the zero-scaled space V_0 in a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ of L^2 is a refinable shift-invariant space ([17]), but there are only few papers specially devoted to sampling problems on a refinable shift-invariant space ([26, 27]). In Section 4, we studied the local reconstruction of signals (functions) in a refinable shift-invariant space. Precisely, we show in Theorem 4.1 that for almost all $(x_0, x_1) \in [0, 1]^2$ the periodic nonuniform sampling set $\{x_i + k, 0 \leq i \leq 1, k \in \mathbb{Z}\}$ having sampling rate 2 is a locally determining sampling set for the refinable shift-invariant space $V(\phi)$, and in Theorem 4.2 that for almost all $(x_0, \ldots, x_{N-1}) \in [0, 1]^N$ the set $\{x_i + Nk, 0 \leq i \leq N - 1, k \in \mathbb{Z}\}$ having sampling rate 1 can be used as the sampling set from which the samples taken can be used to locally reconstruct any signal (function) in the refinable shift-invariant space $V(\phi)$ if ϕ is refinable and supported on $[0, N]$. The proofs of Theorems 4.1 and 4.2 depend heavily on two new properties of refinable functions (see Theorems A.1 and A.2 in the appendix), which are interesting by themselves.

2. Sampling in locally finitely-dimensional spaces

In this section, we first provide a necessary condition on the linear space V in which any function could be locally reconstructed from its samples on a weakly relatively-separated sampling set (Theorem 2.1), and then show that continuous functions in a locally finite-dimensional shift-invariant space V can be locally reconstructed from its samples on a uniform sampling set with sufficiently large density (Theorem 2.2).

Theorem 2.1. *Let V be a linear space of functions on the line. If there exists a weakly relatively-separated subset X of $\mathbb{R}$ such that any function in V can be locally determined from its samples on X, then V is a locally finite-dimensional linear space.*

Proof. Let X be a weakly relatively-separated subset of $\mathbb{R}$ and assume that any function in V can be locally determined from its samples on X. Take a compact set K, and denote by $V|_K$ the linear space of the restriction of all functions $f \in V$ to K. By the local reconstruction assumption on functions in V, there exists a compact set $K' \supset K$ such that for any function $f \in V$, its restriction on K is uniquely determined by its finitely many samples on $K' \cap X$. Denote the dimension of the space $V|_K$ by I_1 and the cardinality of $K' \cap X$ by I_2. By the weakly-relative-separatedness of the set X, we have that $I_2 < \infty$. Therefore it suffices to prove that $I_1 \le I_2$. Suppose, on the contrary, that $I_1 \ge I_2 + 1$. Then there exist functions $g_i \in V, 1 \le i \le I_2 + 1$, whose restrictions on K are linearly independent. Since the size of the matrix $(g_i(x_j))_{1\le i\le I_2+1, x_j\in K'\cap X}$ is $(I_2+1)\times I_2$, there exists a nonzero vector $v = (v_1, \ldots, v_{I_2+1})$ such that the function $g := v_1 g_1 + \cdots + v_{I_2+1} g_{I_2+1} \in V$ satisfies that $g(x_j) = 0$ for all $x_j \in K' \cap X$. This together with the local reconstruction assumption on functions in V implies the restriction of the function g on K is identically zero, which contradicts to the linear independence of the function $g_i, 1 \le i \le I_2 + 1$, on K.

$\square$

Theorem 2.2. *Let V be a locally finite-dimensional shift-invariant space of continuous functions on the line. Then there exists $M \in \mathbb{N}$ such that any function $f \in V$ can be locally determined from the samples $f(k/M), k \in \mathbb{Z}$.*

Proof. By the assumption on the space V, there exist continuous functions $g_n, 1 \le n \le N$, such that their restriction on $[0, 1]$ form a basis of $V|_{[0,1]}$, the set of the restriction of all functions in V on $[0,1]$. By the linear independence of the continuous functions $g_1,\dots,g_N$ on $[0,1]$, there exist positive constants A and B such that

$$A\Big(\sum_{n=1}^N |c(n)|^2\Big)^{1/2} \le \Big(\int_0^1 \Big|\sum_{n=1}^N c(n)g_n(x)\Big|^2 dx\Big)^{1/2} \le B\Big(\sum_{n=1}^N |c(n)|^2\Big)^{1/2}$$

for any vector $c=(c(1),\dots,c(N))$. On the other hand, by the continuity of the functions $g_1,\dots,g_N$, there exists $M\in\mathbb{N}$ such that

$$\Big(\int_0^1 \Big|\sum_{n=1}^N c(n)(g_n-\tilde g_n^M)(x)\Big|^2 dx\Big)^{1/2} \le \frac{A}{2}\Big(\sum_{n=1}^N |c(n)|^2\Big)^{1/2}$$

for any vector $c=(c(1),\dots,c(N))$, where

$$\tilde g_n^M(x)=\sum_{m=0}^{M-1} g_n(m/M)\chi_{[m/M,(m+1)/M)}(x).$$

Combining the above two inequalities yields

$$\begin{aligned}\Big(\sum_{m=0}^{M-1}\frac{1}{M}\Big|\sum_{n=1}^N c(n)g_n(m/M)\Big|^2\Big)^{1/2} &= \Big(\int_0^1\Big|\sum_{n=1}^N c(n)\tilde g_n^M(x)\Big|^2 dx\Big)^{1/2}\\ &\ge \frac{A}{2}\Big(\sum_{n=1}^N |c(n)|^2\Big)^{1/2} \qquad (2.1)\end{aligned}$$

for any vector $c=(c(1),\dots,c(N))$. Thus there exists an $M\times N$ matrix $B=(b_{mn})_{0\le m\le M-1,1\le n\le N}$ by (2.1) such that

$$\sum_{m=0}^{M-1} g_n(m/M)b_{mn'}=\delta_{nn'},\ \ 1\le n,n'\le N. \qquad (2.2)$$

Now we start to establish the local reconstruction formula for functions in V from their samples on $\mathbb{Z}/M$. Take any $f\in V$. Then there exist sequences $(c_n(k))_{k\in\mathbb{Z}}\in\ell(\mathbb{Z}), 1\le n\le N$, by the shift-invariance of the space V and by the construction of the functions $g_n, 1\le n\le N$, such that

$$f(x)=\sum_{k\in\mathbb{Z}} f(x)\chi_{[0,1)}(x-k)=\sum_{n=1}^N\sum_{k\in\mathbb{Z}} c_n(k)\tilde g_n(x-k) \qquad (2.3)$$

where $\tilde{g}_n = g_n \chi_{[0,1)}, 1 \le n \le N$. Thus by (2.2) and (2.3), we have the following local reconstruction formula:

$$\begin{aligned} f &= \sum_{n=1}^{N} \sum_{k\in\mathbb{Z}} \Big(\sum_{n'=1}^{N} \sum_{m=0}^{M-1} c_{n'}(k) \tilde{g}_{n'}(m/M) b_{mn} \Big) \tilde{g}_n(x-k) \\ &= \sum_{n=1}^{N} \sum_{k\in\mathbb{Z}} \Big(\sum_{m=0}^{M-1} f\Big(k + \frac{m}{M}\Big) b_{mn} \Big) \tilde{g}_n(x-k) \end{aligned} \tag{2.4}$$

for any $f \in V$.

□

From the proof of Theorem 2.2, we have

Corollary 2.3. *Let V be a locally finite-dimensional shift-invariant space of continuous functions on the line. If $X_0 := \{x_m \in [0,1), 1 \le m \le M\}$ is so chosen that any function g in $V|_{[0,1]}$ is uniquely determined by its samples $g(x_m), 1 \le m \le M$, then the periodic nonuniform sampling set $X_0 + \mathbb{Z} := \{x_m + k, x_m \in X_0, k \in \mathbb{Z}\}$ is a sampling set such that any function in V can be locally determined from its samples on X.*

Remark 2.4. For a continuous function f on the line, define its *modulus of continuity* $\omega_\delta(f)(x)$ by

$$\omega_\delta(f)(x) = \sup_{|y|\le\delta} |f(x+y) - f(x)|.$$

The modulus of continuity is a delicate tool in mathematical analysis ([18, 46]), and it has also been used to estimate the density of a stable sampling set (see [1, 3, 6] and references therein). The modulus of continuity can also be used to estimate the density of a local determining sampling set. Given a locally finite-dimensional shift-invariant space V of continuous function on the real line if there exists $a \in [0,1]$ and $\delta_0 \in (0,\infty)$ such that

$$\|\omega_{\delta_0}(f)\|_{L^2([a,a+1])} \le \beta \|f\|_{L^2([a,a+1])} \text{ for all } f \in V \tag{2.5}$$

where $\beta \in (0,1)$, then using the argument in the proof of Theorem 2.2, any weakly relatively-separated set $X = \{\ldots < x_{i-1} < x_i < x_{i+1} < \ldots\}$ satisfying the following density property $\sup_{i\in\mathbb{Z}} |x_{i+1} - x_i| \le \delta_0$ (such as the uniform sampling set $b + \delta\mathbb{Z}$ with $b \in \mathbb{R}$ and $\delta \in [0, \delta_0]$) is a locally determining sampling set for the space V.

Remark 2.5. A *locally finitely-generated space* generated by $\Phi = \{\phi_\lambda\}_{\lambda\in\Lambda}$ is given by

$$V(\Phi) = \Big\{ \sum_{\lambda\in\Lambda} c(\lambda)\phi_\lambda,\ (c(\lambda)) \in \ell(\Lambda)\Big\} \tag{2.6}$$

where Λ is a weakly relatively-separated subset of $\mathbb{R}$, Φ is a family of continuous functions $\phi_\lambda, \lambda \in \Lambda$, that are supported in a fixed neighborhood $\lambda + K_0$ of the center λ, $\ell(\Lambda)$ is the space of all sequences on Λ, and K_0 is a bounded set ([43, 44]). The locally finitely-generated space $V(\Phi)$ in (2.6) was introduced in [44] to model signals with finite rate of innovations (see [8, 34, 43, 47] and extensive references therein for the study of sampling for signals with finite rate of innovations). A model space of locally finitely-generated spaces is the shift-invariant space $V(\phi_1, \ldots, \phi_N)$ in (1.4) generated by the shifts of compactly supported functions $\phi_1, \ldots, \phi_N$. One may easily verify that functions in the locally finitely-generated space $V(\Phi)$ in (2.6) is locally finite-dimensional if ϕ_λ is supported in $\lambda + K_0$ for some fixed compact set K_0, where $\Phi = \{\phi_\lambda, \lambda \in \Lambda\}$. Applying the argument used in the proof of Theorem 2.1 we conclude that $X = \cup_{k\in\mathbb{Z}} X_k$ is a locally determining sampling set for the locally finitely-generated space $V(\Phi)$ if for any $k \in \mathbb{Z}$, $X_k \subset [k, k+1)$ is a finite set such that any functions in $V(\Phi)|_{[k,k+1)}$ is determined from its samples taken from X_k.

3. Sampling in finitely-generated shift-invariant spaces

The shift-invariant space $V(\phi_1, \ldots, \phi_N)$ generated by compactly supported continuous functions $\phi_1, \ldots, \phi_N$ is locally finite-dimensional. Let $X_0 := \{x_m \in [0, 1), 1 \le m \le M\}$ be so chosen that any function f in the space spanned by $\{\phi_n(\cdot - k)\chi_{[0,1)}, 1 \le n \le N, k \in \mathbb{Z}\}$ can be reconstructed from its samples $f(x_m), 1 \le m \le M$. Then it is noticed by Corollary 2.3 that the periodic nonuniform sampling set $X_0 + \mathbb{Z} := \{x_m + k, 1 \le m \le M, k \in \mathbb{Z}\}$ is a locally determining sampling set for the space $V(\phi_1, \ldots, \phi_N)$. In this section, we first characterize all locally determining sampling sets of the form $X_0 + \mathbb{Z}$ for a finitely-generated shift-invariant space where $X_0 \subset [0, 1)$, and then we give a sufficient condition on the set $X_0 \subset [0, 1)$ such that the periodic nonuniform sampling set $X_0 + N\mathbb{Z}$ with sampling rate one is a locally determining sampling set for a single-generated shift-invariant space where $N \ge 1$.

To state our results, we recall the concept of linear independent shifts. The nonzero compactly supported functions $\phi_1, \ldots, \phi_N$ are said to have *linear independent shifts* if the semi-convolution defined by

$$(\ell(\mathbb{Z}))^N \ni \big((c_1(k), \ldots, c_N(k))^T\big)_{k\in\mathbb{Z}} \longmapsto \sum_{n=1}^N \sum_{k\in\mathbb{Z}} c_n(k)\phi_n(\cdot - k)$$

is one-to-one, where $(\ell(\mathbb{Z}))^N$ is the N-copies of $\ell(\mathbb{Z})$ ([30, 38]). We also recall the *Zak transform* $Z\phi(x,\xi)$ of a function ϕ on the line by

$$Z\phi(x,\xi) = \sum_{k\in\mathbb{Z}} \phi(x+k)e^{-ik\xi}$$

([29]).

Theorem 3.1. *Let $\phi_1, \ldots, \phi_N$ be compactly supported functions that have linear independent shifts, and $X_0 := \{x_m \in [0,1), 1 \le m \le M\}$. Then the periodic nonuniform sampling set $X_0 + \mathbb{Z} := \{x_m + k, 1 \le m \le M, k \in \mathbb{Z}\}$ is a locally determining sampling set for the space $V(\phi_1, \ldots, \phi_N)$ if and only if the matrix*

$$Z\Phi(X_0,\xi) := \big(Z\phi_n(x_m,\xi)\big)_{1\le n\le N, 1\le m\le M} \tag{3.1}$$

has rank N for any complex number ξ, where $\Phi = (\phi_1, \ldots, \phi_N)^T$.

Proof. ($\Longrightarrow$) Suppose, on the contrary, that the matrix $Z\Phi(X_0,\xi)$ in (3.1) has rank less than N for some complex number ξ_0. Then there exists a nonzero vector $v = (v_1, \ldots, v_N)^T$ such that the function $\phi := \sum_{n=1}^N v_n\phi_n$ satisfies

$$\sum_{k\in\mathbb{Z}} \phi(x_m+k)e^{-ik\xi_0} =: Z(\phi)(x_m,\xi_0) = 0, \ 1 \le m \le M.$$

Therefore the function $f := \sum_{k\in\mathbb{Z}} e^{-ik\xi_0}\phi(\cdot + k) \in V(\phi_1, \ldots, \phi_N)$ satisfies

$$f(x_m + k') = e^{ik'\xi_0} Z(\phi)(x_m,\xi_0) = 0 \quad \forall\, k' \in \mathbb{Z},$$

and hence is identically zero by the local reconstruction assumption. This contradicts to the assumption that $\phi_1, \ldots, \phi_N$ have linear independent shifts.

($\Longleftarrow$) By the assumption on the matrix $Z\Phi(X_0,\xi)$, similar to the Euclidean algorithm for co-prime polynomials we can extend that $N \times M$ matrix $Z\Phi(X_0,\xi)$ to an $M \times M$ square matrix $A(\xi)$ whose entries are trigonometrical polynomials and whose determinant is a nonzero monomial ([31]). Then the inverse matrix $A(\xi)^{-1}$ of the $M \times M$ square matrix $A(\xi)$ has trigonometrical polynomial entries, and the $M \times N$

matrix $B(\xi) = \big(\sum_{k\in\mathbb{Z}} b_{mn}(k)e^{-ik\xi}\big)_{1\le m\le M, 1\le n\le N}$ obtained from placing the first N columns of the inverse matrix $A(\xi)^{-1}$ side-by-side satisfies

$$Z\Phi(X_0,\xi)B(\xi) = I_N \quad \text{for all complex number } \xi,$$

or equivalently

(3.2)
$$\sum_{m=1}^{M}\sum_{k'\in\mathbb{Z}} \phi_n(x_m+k')b_{mn'}(k-k') = \delta_{nn'}\delta_{k0}, \ \forall\ 1\le n,n'\le N, k\in\mathbb{Z}.$$

Thus for any $f = \sum_{n=1}^{N}\sum_{k\in\mathbb{Z}} c_n(k)\phi_n(\cdot - k) \in V(\phi_1,\ldots,\phi_N)$, we have

$$\begin{aligned} &\sum_{m=1}^{M}\sum_{k'\in\mathbb{Z}} f(x_m+k')b_{mn}(k-k') \\ = &\sum_{m=1}^{M}\sum_{n'=1}^{N}\sum_{k',k''\in\mathbb{Z}} c_{n'}(k'')\phi_{n'}(x_m+k'-k'')b_{mn}(k-k') \\ = &\sum_{n'=1}^{N}\sum_{k''\in\mathbb{Z}} c_{n'}(k'')\delta_{nn'}\delta_{kk''} = c_n(k), \ \forall\ 1\le n\le N, k\in\mathbb{Z}. \end{aligned}$$

This yields the following local reconstruction formula:

$$f(x) = \sum_{n=1}^{N}\sum_{k\in\mathbb{Z}}\Big(\sum_{m=1}^{M}\sum_{k'\in\mathbb{Z}} f(x_m+k')b_{mn}(k-k')\Big)\phi_n(x-k)$$

for any $f\in V(\phi_1,\ldots,\phi_N)$.

□

For a single-generated shift-invariant space $V(\phi)$, we obtain the following result from Theorem 3.1, which was established in [19].

Corollary 3.2. *Let ϕ be a compactly supported function that has linear independent shifts, and $X_0 := \{x_m \in \mathbb{R}, 1\le m\le M\}$. Then any function in the shift-invariant space $V(\phi)$ can be locally reconstructed from its sample taken from $X_0+\mathbb{Z} := \{x_m+k:\ 1\le m\le M, k\in\mathbb{Z}\}$ if and only if the trigonometric polynomials $Z\phi(x_1,\xi),\ldots,Z\phi(x_M,\xi)$ do not have common zero.*

Remark 3.3. Recall that the innovative rate of functions in the space $V(\phi_1,\ldots,\phi_N)$ are equal to N, while the sampling rate of $\{x_1,\ldots,x_M\}+\mathbb{Z}$ is equal to M. From Theorem 3.1, the sampling rate is always larger than or equal to the rate of innovation of functions when the reconstruction procedure is local. If the number M of sampling points taken in $[0,1)$ is the same as the number N of generators in the shift-invariant space $V(\phi_1,\ldots,\phi_N)$, we can reformulate the assumption that the matrix $Z\Phi(X_0,\xi)$ has rank N for any complex number ξ in Theorem 3.1 as

$$\det Z\Phi(X_0,\xi) = c_0 e^{-ik_0\xi} \tag{3.3}$$

for some $0 \neq c_0 \in \mathbb{R}$ and $k_0 \in \mathbb{Z}$. For most of finitely-generated shift-invariant spaces $V(\phi_1,\ldots,\phi_N)$, a local reconstruction procedure from a periodic nonuniform sampling set of the form $X_0 + \mathbb{Z}$ exists only for oversampling (i.e., $M > N$). For instance, noting that for $n \geq 3$ and $x_0 \in [0,1)$, the Zak transform $ZB_n(x_0,\xi) = \sum_{k=0}^{n-1} B_n(x_0 + k)e^{-ik\xi}$ of the B-spline is not a monomial, we conclude from (3.3) that oversampling is necessary for locally reconstructing signals in the spline spaces $\mathcal{B}_n, n \geq 3$, in (1.1) from their samples in a periodic nonuniform sampling set $\{x_0 + k|\ k \in \mathbb{Z}\}$. For a refinable function ϕ supported on $[0,N]$ that has linear independent shifts, its was proved in [26] that $\{\frac{l}{2^{N-2}}|\ 0 \leq l \leq 2^{N-2} - 1\} + \mathbb{Z}$ with sampling rate 2^{N-2} is a locally determining sampling set for the refinable shift-invariant space $V(\phi)$.

The above remark about the sampling rate of a sampling set and the innovative rate of functions inspires us to consider the local reconstruction of non-uniform sampling on a sampling set with its sampling rate being the same as the rate of innovation of functions in the shift-invariant space. We discuss such sampling and reconstruction procedure in the following result, while the corresponding result for the spline space case has been established in [2].

Theorem 3.4. *Let ϕ be a function supported on $[0,N]$ where $N \in \mathbb{N}$. Assume that $x_0,\ldots,x_{N-1} \in [0,1)$ and that $\big(\phi(x_n + n')\big)_{0\leq n,n'\leq N-1}$ is nonsingular. Then the set $\{x_n + Nk, 0 \leq n \leq N-1, k \in \mathbb{Z}\}$ having sampling rate one is a locally determining sampling set for the shift-invariant space $V(\phi)$.*

Proof. Let $f := \sum_{k\in\mathbb{Z}} c(k)\phi(\cdot - k) \in V(\phi)$. Then for any $k \in \mathbb{Z}$,

$$\begin{aligned} f(x_n + Nk) &= \sum_{l\in\mathbb{Z}}\sum_{n'=0}^{N-1} c(lN - n')\phi(x_n + Nk - Nl + n') \\ &= \sum_{n'=0}^{N-1} c(kN - n')\phi(x_n + n') \end{aligned} \tag{3.4}$$

by the support property of ϕ. Let $B = (b_{nn'})_{1\le n,n'\le N}$ be the inverse of the matrix $(\phi(x_n + n'))_{0\le n,n'\le N-1}$. Then it follows from (3.4) that

$$c(kN - n') = \sum_{n=0}^{N-1} b_{nn'} f(x_n + Nk) \quad \text{for all } k \in \mathbb{Z}. \tag{3.5}$$

This yields the following local reconstruction formula:

$$f(x) = \sum_{k\in\mathbb{Z}} \sum_{n'=0}^{N-1} \Big(\sum_{n=0}^{N-1} b_{nn'} f(x_n + Nk) \Big) \phi(x - Nk + n') \tag{3.6}$$

for any $f \in V(\phi)$, and hence completes the proof.

□

In [23], it is proved that for any ripplet ϕ, $\det(\phi(x_i - n_{i'}))_{0\le i,i'\le s-1} > 0$ if $\phi(x_i - n_i) > 0$ for all $0 \le i \le s-1$. Therefore by Theorem 3.4, we have the following result about local reconstruction of sampling in a shift-invariant space generated by a ripplet.

Corollary 3.5. *Let ϕ be a continuous ripplet such that $\phi(x) > 0$ for all $x \in (0, N)$ and $\phi(x) = 0$ otherwise, and let $x_0, \ldots, x_{N-1} \in (0, 1)$. Then $\{x_n + Nk, 0 \le n \le N-1, k \in \mathbb{Z}\}$ is a locally determining sampling set for the shift-invariant space $V(\phi)$.*

Remark 3.6. It is known that $B_N, N \ge 2$, are ripplets ([23]), and hence any function in the spline space (1.1) can be locally reconstructed from its samples on $\{x_n + Nk, 1 \le n \le N, k \in \mathbb{Z}\}$ if $0 < x_1 < \cdots < x_{N-1} < 1$, which was established in [2], see [45] for a complete characterization of locally determining sampling sets for the spline space.

4. Sampling in refinable shift-invariant spaces

In this section, we consider the local reconstruction of functions in a refinable shift-invariant space. Based on the results obtained in this section, we believe that the periodic non-uniform sampling has better performance than the uniform sampling from the viewpoint of local reconstruction.

Theorem 4.1. *Let ϕ be compactly supported refinable function that has linear independent shifts. Then for almost all $(x_0, x_1) \in [0, 1]^2$, the periodic nonuniform sampling set $\{x_i + k,\ 0 \le i \le 1, k \in \mathbb{Z}\}$ with sampling rate 2 is a locally determining sampling set for the refinable shift-invariant space $V(\phi)$.*

Proof. For a refinable function ϕ, it follows from Theorem A.1 in the Appendix that the vector $(\phi(x+k))_{k\in\mathbb{Z}}$ is a nonzero vector for almost all $x \in (0,1)$, and that for any nonzero complex number ξ', $\sum_{k\in\mathbb{Z}} \phi(x+k)e^{-ik\xi'} \neq 0$ for almost all $x \in [0,1)$. Thus for any $x_0 \in [0,1]$ and any root ξ' of the trigonometric polynomial $\sum_{k\in\mathbb{Z}} \phi(x_0+k)e^{-ik\xi}$, $\sum_{k\in\mathbb{Z}} \phi(x_1+k)e^{-ik\xi'} \neq 0$ for almost all $x_1 \in [0,1]$. Hence the trigonometric polynomials $\sum_{k\in\mathbb{Z}} \phi(x_0+k)e^{-ik\xi}$ and $\sum_{k\in\mathbb{Z}} \phi(x_1+k)e^{-ik\xi}$ do not have common (complex) roots for almost all $(x_0,x_1) \in [0,1]^2$. This together with (3.1) and Theorem 3.1 proves the desired conclusion.

□

Unlike B-spline $B_N, N \geq 2$, most of refinable functions ϕ such as the Daubechies' scaling functions ([17]) are not ripplets ([23, 24, 36]). Thus not all sets of the form $\{x_n + Nk, 0 \leq n \leq N-1, k \in \mathbb{Z}\}$ with $\{x_n, 0 \leq n \leq N-1\} \subset (0,1)$ can be used as a locally determining sampling set. In Theorem A.2, we show that any refinable function is "almost" a ripplet. Therefore by Theorems 3.4 and A.2, we have the following result about periodic nonuniform sampling in a refinable shift-invariant space with the sampling rate being the same as the innovative rate of signals in the refinable shift-invariant space.

Theorem 4.2. *Let ϕ be a refinable function that has linear independent shifts and is supported on $[0,N]$, where $N \geq 1$. Then for almost all $(x_0,\ldots,x_{N-1}) \in [0,1]^N$, the periodic nonuniform sampling set $\{x_n + Nk, 0 \leq n \leq N-1, k \in \mathbb{Z}\}$ with sampling rate 1 is a locally determining sampling set for the refinable shift-invariant space $V(\phi)$.*

Appendix A. Two properties of refinable functions

In the appendix, we show that the shifts of a refinable function on the line are linearly independent on every measurable set with positive Lebesgue measure (Theorem A.1), and that any refinable function on the line is "almost" a ripplet (Theorem A.2). Those two properties of refinable functions, a strong version of locally linearly independent shifts and a weak version of ripplets, are interesting by themselves and have been used in the study of sampling in a refinable shift-invariant space, particularly in the proofs of Theorems 4.1 and 4.2. The affine self-similarity of refinable functions plays the essential role in the proof of two above new properties ([12, 28]).

A.1. **Linearly independent shifts on a measurable set with positive Lebesgue measure.** We say that a nonzero compactly supported distribution ϕ on the real line has *locally linearly independent* *shifts* if for any open set A, $\sum_{k\in\mathbb{Z}} c(k)\phi(\cdot-k)=0$ on A if and only if $c(k)=0$ for all integers k such that $\phi(\cdot-k)\not\equiv 0$ on A ([22, 32, 35, 42]). Clearly, a compactly supported distribution that has locally linearly independent shifts must have linearly independent shifts, while the converse is not true in general. For instance the function $\phi(x)=\chi_{[0,3/2]}$ has linearly independent shifts, but it does not have locally linearly independent shifts. But for a compactly supported refinable distribution ϕ, it was proved that ϕ has linearly independent shifts if and only if ϕ has locally linearly independent shifts, see [32, 35] for an integrable refinable function and [42] for a refinable distribution. The locally linearly independent shifts of a function ϕ can be interpreted as the linear independence of the shifts of the function ϕ on any *open* set. In the following, we establish a result for refinable functions which can roughly be thought as the linear independence of the shifts of the function ϕ on any *measurable* set with positive Lebesgue measure.

Theorem A.1. *Let ϕ be compactly supported, integrable and refinable, and have linear independent shifts. Then for any measurable set E with positive Lebesgue measure, $\sum_{k\in\mathbb{Z}} c(k)\phi(\cdot-k)=0$ on E if and only if $c(k)=0$ for all integers k with $\phi(\cdot-k)\not\equiv 0$ on E.*

Proof. ($\Longleftarrow$) For any measurable set E, we have that $\sum_{k\in\mathbb{Z}} c(k)\phi(\cdot-k)=0$ on E if $c(k)=0$ for all integers k with $\phi(\cdot-k)\not\equiv 0$ on E.

($\Longrightarrow$) By the refinability of the compactly supported function ϕ and by the assumption that ϕ has linear independent shifts, $\phi(\cdot/2)$ is a finite linear combination of $\phi(\cdot-k), k\in\mathbb{Z}$ ([30, 42]). Therefore without loss of generality, we may assume that ϕ satisfies the following refinement equation

$$\phi(x)=\sum_{k=0}^{N} c_0(k)\phi(2x-k) \tag{A.1}$$

where $N\ge 1$, $\sum_{k=0}^{N} c_0(k)=2$ and $c_0(0)c_0(N)\neq 0$. Define

$$B_0=(c_0(2i-j))_{0\le i,j\le N-1} \text{ and } B_1=(c_0(2i-j+1))_{0\le i,j\le N-1},$$

where we set $c_0(i)=0$ if $i<0$ or $i>N$. Then

$$B_0 \text{ and } B_1 \text{ are nonsingular matrices} \tag{A.2}$$

by the linear independent shifts of ϕ ([32, 35, 42]), and the vector $\Phi(x):=(\phi(x),\ldots,\phi(x+N-1))^T$ satisfies

$$\Phi(x/2)=B_0\Phi(x) \text{ and } \Phi((x+1)/2)=B_1\Phi(x),\ x\in[0,1] \tag{A.3}$$

by the refinement equation (A.1), see [12].

Let $\mathcal{E}_n, 1 \le n \le N$, be the family of measurable sets $E \subset [0,1]$ with positive Lebesgue measure $|E|$ such that

$$\sum_{k=0}^{N-1} c_s^E(k)\phi(\cdot + k) = 0 \text{ on } E,\ 1 \le s \le n$$

for some vectors $c_s^E := (c_s^E(0), \dots, c_s^E(N-1))^T, 1 \le s \le n$, such that $\{c_s^E, 1 \le s \le n\}$ spans a n-dimensional subspace of $\mathbb{R}^N$. Clearly our conclusion on linear independent shifts of the refinable function ϕ on measurable sets follows from $\mathcal{E}_1 = \emptyset$. In the following we will prove that $\mathcal{E}_n$ are empty sets for all $1 \le n \le N$ by induction.

Claim: $\mathcal{E}_n, 1 \le n \le N$, *are empty sets.*

First we prove that $\mathcal{E}_N = \emptyset$. Suppose on the contrary that $\mathcal{E}_N \ne \emptyset$. Notice that $\mathcal{E}_N$ contains all measurable sets $E \subset [0,1]$ with positive Lebesgue measure such that $\phi(\cdot + n) = 0$ on E for all $0 \le n \le N-1$. Then $E_0 = \cup_{E \in \mathcal{E}_N} E \in \mathcal{E}_N$ and

$$|E_0| = \sup_{E \in \mathcal{E}_N} |E| > 0. \tag{A.4}$$

For any $l \ge 1$ and $0 \le k := \sum_{l'=0}^{l-1} \epsilon_{l'} 2^{l'} \le 2^l - 1$ where $\epsilon_{l'} \in \{0,1\}$, applying (A.3) iteratively yields

$$B_{\epsilon_{l-1}} \cdots B_{\epsilon_0} \Phi(x) = \Phi(2^{-l}x + 2^{-l}k). \tag{A.5}$$

This together with (A.2) and (A.4) implies that

$$|(2^l E_0 - k) \cap [0,1]| \le |E_0| \tag{A.6}$$

for all $0 \le k \le 2^l - 1$. On the other hand,

$$\sum_{k=0}^{2^l-1} |(2^l E_0 - k) \cap [0,1]| = \sum_{k=0}^{2^l-1} |(2^l E_0) \cap [k, k+1]| = 2^l |E_0|. \tag{A.7}$$

Thus

$$|E_0 \cap [2^{-l}k, 2^{-l}(k+1))| = 2^{-l}|E_0|,\ 0 \le k \le 2^l - 1, l \in \mathbb{N} \tag{A.8}$$

by (A.6) and (A.7). For a measurable function f on $[0,1]$, we define the average function

$$f_l(x) = 2^{-l} \int_{2^{-l}k}^{2^{-l}(k+1)} f(y)dy$$

if $x \in [2^{-l}k, 2^{-l}(k+1))$ for some $0 \le k \le 2^l - 1$. By Lebesgue theorem ([40]),

$$\lim_{l \to +\infty} f_l(x) = f(x) \text{ for almost all } x \in [0,1]. \tag{A.9}$$

For the characteristic function χ_{E_0}, the corresponding average functions $f_l, l \geq 1$, are identically $|E_0|$ by (A.8), which together with (A.4) and (A.9) implies that $E_0 \subset [0,1]$ has measure 1, or equivalently $E_0 = [0,1]$, which in turn implies that $\phi \equiv 0$, a contradiction to the nonzero assumption on ϕ.

Inductively we assume that $\mathcal{E}_n = \emptyset$ for all $n_0 + 1 \leq n \leq N$, where $0 \leq n_0$. If $n_0 = 0$, the inductive proof is done. So we assume that $n_0 \geq 1$ hereafter. Suppose on the contrary that $\mathcal{E}_{n_0} \neq \emptyset$. We observe that for any sets $E_1, E_2 \in \mathcal{E}_{n_0}$, either $E_1 \cup E_2$ belongs to $\mathcal{E}_{n_0}$ or $E_1 \cap E_2$ has zero Lebesgue measure by the inductive hypothesis, since the space generated by the family of vectors $c_s^{E_1}$ and $c_s^{E_2}, 1 \leq s \leq n_0$ corresponding to those two sets E_1 and E_2 has its dimension being either equal to or strictly larger than n_0. Therefore the maximal sets in $\mathcal{E}_{n_0}$ are disjoint each other by the above observation. Here a set $E \in \mathcal{E}_{n_0}$ is said to be a maximal set if there does not exist $E' \in \mathcal{E}_{n_0}$ such that $E \subset E'$ and $|E| < |E'|$. This implies that for any given $\epsilon > 0$, the family $\mathcal{E}_{n_0}(\epsilon)$ of all maximal sets $E \in \mathcal{E}_{n_0}$ with $|E| > \epsilon$ has finite cardinality since they are disjoint. Hence there exists $\tilde{E}_0 \in \mathcal{E}_{n_0}$ such that

$$|\tilde{E}_0| = \sup_{E \in \mathcal{E}_{n_0}} |E| > 0. \tag{A.10}$$

Similar to the argument used in the proof of the equation (A.8), we have

$$|\tilde{E}_0 \cap [2^{-l}k, 2^{-l}(k+1))| = 2^{-l}|\tilde{E}_0|, \ 0 \leq k \leq 2^l - 1, l \in \mathbb{N}. \tag{A.11}$$

Therefore $\tilde{E}_0 = [0,1]$ by (A.11) and the Lebesgue theorem (A.10) for the characteristic function $\chi_{\tilde{E}_0}$. This implies that there exists a nonzero vector $c = (c(0), \ldots, c(N-1))^T$ such that $\sum_{k=0}^{N-1} c(k)\phi(\cdot + k) = 0$ on $[0,1]$, which contradicts to the locally linear independent shifts of the refinable function ϕ ([32, 35, 42]). This completes the inductive proof of the Claim and also the theorem.

□

A.2. **Ripplet property of a refinable function.** In this subsection, we show that any refinable function on the line is "almost" a ripplet.

Theorem A.2. *Let ϕ be compactly supported and integrable, have linear independent shifts, and satisfy a refinement equation*

$$\phi(x) = \sum_{k=0}^{N} c_0(k)\phi(2x - k) \tag{A.12}$$

where $N \ge 1$ *and the sequence* $\{c_0(k)\}_{k=0}^N$ *satisfies* $\sum_{k=0}^N c_0(k) = 2$ *and* $c_0(0)c_0(N) \ne 0$. *Then* $(\phi(x_n + n'))_{0\le n,n'\le N-1}$ *is nonsingular for almost all* $(x_0, \ldots, x_{N-1}) \in [0,1]^N$.

Proof. Set $\Phi(x) = (\phi(x), \ldots, \phi(x+N-1))^T$. Let

$$E_0 = \{x \in [0,1],\ \Phi(x) = 0\}.$$

Inductively we define

$$E(x_0) = \{x \in [0,1],\ \Phi(x) \text{ is in the space spanned by } \Phi(x_0)\}$$

for $x_0 \in [0,1]\backslash E_0$,

$$E(x_0, \ldots, x_n) = \{x \in [0,1],\ \Phi(x) \text{ is in the space spanned by } \Phi(x_0), \ldots, \Phi(x_n)\}$$

for $x_n \in [0,1]\backslash E(x_0, \ldots, x_{n-1})$ where $1 \le n \le N-2$. By Theorem A.1, the sets E_0, $E(x_0)$ with $x_0 \notin E_0$, $E(x_0, \ldots, x_n)$ with $x_n \notin E(x_0, \ldots, x_{n-1})$ where $n \ge 1$ have zero Lebesgue measures.

Let E be the set of all $(x_0, \ldots, x_{N-1})^T \in [0,1]^N$ such that $\big(\phi(x_n + n')\big)_{0\le n,n'\le N-1}$ is singular. Then

$$\begin{aligned}
|E| \ \le\ & \int_{x_0\in E_0}\Big(\int_{x_1,\ldots,x_{N-1}\in[0,1]^{N-1}} 1\ dx_1\ldots dx_{N-1}\Big)dx_0 \\
& + \int_{x_0\in[0,1]\backslash E_0}\int_{x_1\in E(x_0)}\Big(\int_{x_2,\ldots,x_{N-1}\in[0,1]^{N-2}} 1\ dx_2\ldots dx_{N-1}\Big)dx_1 dx_0 \\
& + \cdots \\
& + \int_{x_0\in[0,1]\backslash E_0}\int_{x_1\in[0,1]\backslash E(x_0)}\cdots\int_{x_{N-3}\in[0,1]\backslash E(x_0,\ldots,x_{N-4})} \\
& \quad \times \int_{x_{N-2}\in E(x_0,\ldots,x_{N-3})}\Big(\int_{[0,1]} 1\ dx_{N-1}\Big)dx_{N-2}\cdots dx_1 dx_0 \\
& + \int_{x_0\in[0,1]\backslash E_0}\int_{x_1\in[0,1]\backslash E(x_0)}\cdots\int_{x_{N-2}\in[0,1]\backslash E(x_0,\ldots,x_{N-3})} \\
& \quad \times \int_{x_{N-1}\in E(x_0,\ldots,x_{N-2})} 1\ dx_{N-1}dx_{N-2}\cdots dx_1 dx_0 \\
=\ & 0,
\end{aligned}$$

and the conclusion follows. □

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