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A note on the integer translates of a compactly supported distribution on $\mathbb{R}$

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1. Introduction and Result. Our object in this note is to show a relation between global and local linear independence. To this aim, we introduce some definitions. Let ϕ be a compactly supported distribution on $\mathbb{R}$. The integer translates of ϕ are called globally linearly independent if the condition $\sum_{k\in\mathbb{Z}} c(k)\,\phi(x-k)=0$ on $\mathbb{R}$ implies $c(k)=0$ for all $k\in\mathbb{Z}$. Let E^k be the shift operator defined by $E^k\,\phi(x)=\phi(x+k)$ on $\mathbb{R}$ for $k\in\mathbb{Z}$. In [1], A. Ben-Artzi and A. Ron exhibit an equivalence between global linear independence and a very weak kind of local linear independence. They hoped that the theorem below (Claim 6.1 in [1]) is true.

Theorem. *Assume that the integer translates of the compactly supported distribution ϕ are globally linearly independent. Then there exists a bounded set A such that the conditions*

$$\sum_{k\in\mathbb{Z}} c(k)\,\phi(x-k)=0 \quad \textit{on} \quad A \quad \textit{and} \quad \operatorname{supp} E^{-k}\,\phi\cap A\neq\emptyset$$

imply $c(k)=0$.

But they constructed a counterexample for higher spatial dimensions, and they also noticed that the theorem above unfortunately is true *only* for univariate splines. In this note inspired by matrix method in wavelet theory we show that the above theorem is valid for any compactly supported distribution and construct the set A explicitly.

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2. Proof of Theorem. Without loss of generality we assume $\operatorname{supp}\phi\subset[0,\infty)$ and $\operatorname{supp}\phi\cap[0,1)\neq\emptyset$, where we denote the support of ϕ by $\operatorname{supp}\phi$ and the empty set by $\emptyset$. Let N be the minimal integer n such that $\operatorname{supp}\phi\subset[0,n]$. Observe that the theorem holds true for $A=(-\frac{1}{2},\frac{1}{2})$ when $N=0$ or $\operatorname{supp}\phi=\{0\}$. Therefore we assume $N\geqq 1$ hereafter. We prove our theorem in two cases.

Case 1. $\operatorname{supp}\phi\subset\{0,1,\dots,N\}$.

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Define $A_1 = (-\frac{1}{2}, N + \frac{1}{2})$. Let $c(k)$ be a sequence such that

$$\sum_{k \in \mathbb{Z}} c(k)\, \phi(x-k) = 0, \quad \text{on} \quad A_1,$$

i.e.

(1) $$\sum_{k \in \mathbb{Z}} c(k)\, \phi(j-k) = 0, \qquad \text{for} \quad 0 \leqq j \leqq N,$$

where we denote by $\phi(j)$ the distribution with support in $\{0\}$ which fulfills $\langle \phi(j), f \rangle = \langle E^{-j} \phi, f \rangle$ for all C^∞ function f with support in $(-\frac{1}{2}, \frac{1}{2})$. In matrix notation, we can write (1) as

$$C_1 \Psi_1(0) = 0,$$

where we denote

$$C_1 = \begin{pmatrix} c_0 & c_{-1} & \cdots & c_{-N} \\ c_1 & c_0 & \cdots & c_{-N+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_N & c_{N-1} & \cdots & c_0 \end{pmatrix}$$

and

$$\Psi_1(0) = \begin{pmatrix} \phi(0) \\ \phi(1) \\ \vdots \\ \phi(N) \end{pmatrix}.$$

Recall that $\phi(0) \neq 0$ and $\phi(N) \neq 0$. Therefore if C_1 has a zero row, i.e., $c_{j-s} = 0$ for some $0 \leqq j \leqq N$ and all $0 \leqq s \leqq N$, then $c_j = 0$ for all $-N \leqq j \leqq N$ and our theorem is proved. On the other hand $\det C_1$ must be zero otherwise $\Psi_1(0) = 0$ which is a contradiction. Therefore

$$\bar{C}_{k+1} = \sum_{0 \leqq m \leqq k} a_m \bar{C}_m$$

for some $k \leqq N - 1$, where we denote $\bar{C}_k$ the k-th row of C_1. In other words

$$c_{k+1-s} = \sum_{0 \leqq m \leqq k} a_m c_{m-s}$$

for all $0 \leqq s \leqq N$. Denote by k_0 the maximal integer k' such that $a_m = 0$ for $0 \leqq m \leqq k' - 1$. Therefore $a_{k_0} \neq 0$.

Observe that if we construct a new sequence $\{\tilde{c}_k\}_{k \in \mathbb{Z}}$ such that

(2) $$\tilde{c}_j = c_j$$

for $k_0 - N \leqq j \leqq k + 1$ and

(3) $$\sum_{j \in \mathbb{Z}} \tilde{c}_j \phi(x-j) = 0 \quad \text{on} \quad \mathbb{R},$$

then $\tilde{c}_k = 0$ for all $k \in \mathbb{Z}$ and C_1 has a zero row, which implies our theorem in Case 1, by the assumption that the integer translates of ϕ are globally linearly independent.

Therefore the problem reduces to the construction of the sequence $\{\tilde{c}_k\}$ satisfying (2) and (3). We inductively define

(4) $$\tilde{c}_j = \sum_{k_0 \leqq m \leqq k} a_m \tilde{c}_{m-k-1+j}$$

for $j \geqq k+2$ and

(5) $$\tilde{c}_j = -\frac{1}{a_{k_0}} \sum_{k_0+1 \leqq m \leqq k} a_m \tilde{c}_{m-k_0+j} + \frac{1}{a_{k_0}} \tilde{c}_{k+1-k_0+j}$$

for $j \leqq k - N - 1$. From the construction above we have

(6) $$\tilde{c}_j = \sum_{k_0 \leqq m \leqq k} a_m \tilde{c}_{j+m-k-1}$$

and

(7) $$\tilde{c}_j = - \sum_{k_0+1 \leqq m \leqq k} \frac{a_m}{a_{k_0}} \tilde{c}_{j+m-k_0} + \frac{1}{a_{k_0}} \tilde{c}_{j+k-k_0+1}$$

for all $j \in \mathbb{Z}$. Therefore by (4) and (6) we have

$$\begin{aligned} &\sum_{j\in\mathbb{Z}} \tilde{c}_j \phi(m+1-j) \\ &\quad = \sum_{j\in\mathbb{Z}} \tilde{c}_{j+1} \phi(m-j) \\ &\quad = \sum_{k_0 \leqq s \leqq k} a_s \sum \tilde{c}_{j+s-k} \phi(m-j) \\ &\quad = \sum_{k_0 \leqq s \leqq k} a_s E^{m+s-k} \left(\sum_{j\in\mathbb{Z}} \tilde{c}_j \phi(\cdot - j) \right)(0). \end{aligned}$$

Recall that $\tilde{c}_j = c_j$ for $k_0 - N \leqq j \leqq k+1$ and (1). Therefore we have

$$\sum_{j\in\mathbb{Z}} \tilde{c}_j \phi(k+1-j) = \sum_{k_0 \leqq s \leqq k} a_s E^{m+s-k} \left(\sum_{j\in\mathbb{Z}} c_j \phi(\cdot - j) \right)(0) = 0.$$

Inductively we have

$$\sum_{j\in\mathbb{Z}} \tilde{c}_j \phi(n-j) = 0$$

for $n \geqq k+1$. Similarly by (5) and (7) we have

$$\sum_{j\in\mathbb{Z}} \tilde{c}_j \phi(n-j) = 0$$

for $n \leqq k_0 - 1$. Therefore

$$\sum_{j\in\mathbb{Z}} \tilde{c}_j \phi(n-j) = 0 \quad \text{for all} \quad n \in \mathbb{Z}.$$

Recall that $\operatorname{supp} \phi \subset \{0, 1, \ldots, N\}$. Therefore

$$\sum_{j\in\mathbb{Z}} \tilde{c}_j \phi(x-j) = 0 \quad \text{for} \quad x \in \mathbb{R}$$

and the construction of a sequence $\{\tilde{c}_j\}$ satisfying (2) and (3) is finished. This proves our theorem for $A_1 = (-\frac{1}{2}, N + \frac{1}{2})$ in Case 1.

Case 2. $\operatorname{supp}\phi \not\subseteq \{0,1,\ldots,N\}$.

Define $A_2 = (0,N)$. Let $c(k)$ be a sequence such that

$$\sum_{k\in\mathbb{Z}} c(k)\,\phi(x-k) = 0 \quad \text{on} \quad A_2. \tag{8}$$

Therefore we have

$$\sum_{k\in\mathbb{Z}} c(k)\,\phi(x-k+j) = 0 \quad \text{on} \quad (0,1) \tag{9}$$

for $0 \leqq j \leqq N-1$. In matrix notation, we wrote (9) as

$$C_2\,\Psi_2(x) = 0$$

for $x\in(0,1)$, where we denote

$$C_2 = \begin{pmatrix} c_0 & c_{-1} & \cdots & c_{-N+1} \\ c_1 & c_0 & \cdots & c_{-N+2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{N-1} & c_{N-2} & \cdots & c_0 \end{pmatrix}$$

and

$$\Psi_2(x) = \begin{pmatrix} \phi(x) \\ \phi(x+1) \\ \vdots \\ \phi(x+N-1) \end{pmatrix}.$$

Observe that $\det C_2 \neq 0$ implies $\Psi_2(x) = 0$ on $(0,1)$ and $\operatorname{supp}\phi \subset \{0,1,\ldots,N\}$, which contradicts the assumption of Case 2. Therefore we must have $\det C_2 = 0$. As in Case 1, we have

$$c_{k+1-s} = \sum_{k_0 \leqq m \leqq k} a_m\, c_{m-s}$$

for some $k \leqq N-2$ and all $0 \leqq s \leqq N-1$, where we assume $a_{k_0} \neq 0$. Also we can construct a sequence $\{\tilde{c}_j\}$ satisfying (2), (4) and (5) with $j \leqq k-N-1$ in (5) replaced by $j \leqq k-N$ and $k_0 - N \leqq j \leqq k+1$ in (2) replaced by $k_0 - N + 1 \leqq j \leqq k+1$. Therefore by the same procedure as in Case 1 we can prove

$$\sum_{j\in\mathbb{Z}} \tilde{c}_j\,\phi(x-j) = 0 \quad \text{on} \quad \mathbb{R}\setminus\mathbb{Z}. \tag{10}$$

Denote

$$\tilde{\phi}(x) = \sum_{j\in\mathbb{Z}} \tilde{c}_j\,\phi(x-j).$$

From the construction of $\{\tilde{c}_j\}$ we have the formula

$$E'\tilde{\phi} = \sum_{k_0-k \leqq m \leqq 0} a_{m+k_0}\, E^m \tilde{\phi}. \tag{11}$$

Recall from (8) that $\tilde{\phi}(x) = \sum c(j)\,\phi(x-j)$ on $(k_0, k+2)$, and $\operatorname{supp}\tilde{\phi} \subset \mathbb{Z}$. By (11) and the same procedure as in Case 1 we have $\tilde{\phi} = 0$. Hence $\tilde{c}_j = 0$ for all $j\in\mathbb{Z}$ and

$c_j = 0$ for $k_0 - N + 1 \leqq j \leqq k + 1$ by the assumption that the integer translates of ϕ are globally linearly independent.

From the proof above we know that C_2 has a zero row, i.e., $c_{j_0 - s} = 0$ for all $0 \leqq s \leqq N - 1$ and some $0 \leqq j_0 \leqq N - 1$. Recall from (8) that

$$\sum_{j_0 - N \leqq j \leqq j_0} c_j \phi(x - j) = 0 \quad \text{on} \quad (j_0 - 1, j_0 + 1),$$

when $j_0 \geqq 1$, i.e.,

$$c_{j_0 - N} \phi(x - j_0 + N) = 0 \quad \text{on} \quad (j_0 - 1, j_0 + 1),$$

or

$$c_{j_0 - N} \phi(y) = 0 \quad \text{on} \quad (N - 1, N + 1).$$

Recall that $\operatorname{supp} \phi \cap (N - 1, N] \neq \emptyset$. Hence $c_{j_0 - N} = 0$. Inductively we have $c_j = 0$ for $-N + 1 \leqq j \leqq j_0 - N$. Further by $\operatorname{supp} \phi \cap [0, 1) \neq \emptyset$ we have $c_j = 0$ for $j \geqq j_0$. Thus we prove our theorem for $A_2 = (0, N)$ in Case 2. The proof of the theorem is finished.

In conclusion, we construct a counterexample such that the above mentioned set A can not be chosen as a small neighborhood of $\operatorname{supp} \phi$.

Example. Define $\delta^i (i = 0, 1)$ be a distribution defined by $\langle \delta^0, f \rangle = f(0)$ and $\langle \delta^1, f \rangle = f'(0)$ for any smooth function f. Let $\phi(x) = \delta^0(x) + 2\delta^0(x - 3) + \delta^0(x - 5) + \delta^1(x) + \delta^1(x - 3)$. Therefore $\operatorname{supp} \phi = \{0, 3, 5\}$. It is easy to check that the integer translates of ϕ are globally linearly independent since $\hat{\phi}(\xi) = (1 + 2e^{3i\xi} + e^{5i\xi}) + (i\xi)(1 + e^{3i\xi})$. On the other hand,

$$\begin{aligned} \phi(x - 5) - \phi(x - 3) - \phi(x - 2) + \phi(x) - \phi(x + 2) - \phi(x + 3) \\ + \phi(x + 5) = 0 \end{aligned}$$

on $(-\frac{1}{2}, \frac{1}{2}) \cup (\frac{5}{2}, \frac{7}{2}) \cup (\frac{9}{2}, \frac{11}{2})$, which is a neighborhood of $\{0, 3, 5\}$.

References

[1] A. Ben-Artzi and A. Ron, On the integer translates of a compactly supported function: dual bases and linear projectors. SIAM J. Math. Anal. **21**, 1550–1562 (1990).

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