NON-UNIFORM AVERAGE SAMPLING AND RECONSTRUCTION IN MULTIPLY GENERATED SHIFT-INVARIANT SPACES

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ABSTRACT. The problem of reconstructing a function f from a set of non-uniformly distributed, weighted-average sampled values $\{\int_{\mathbb{R}^d} f(x)\psi_{x_j}(x)dx : j \in J\}$ is studied in the context of shift-invariant subspaces of $L^p(\mathbb{R}^d)$ generated by p-frames. The special but important case where the weighted-average sampled values are of the form $\{\int_{\mathbb{R}^d} f(x)\psi(\cdot - x_j)dx : j \in J\}$ is also studied. Fast approximation-projection iterative reconstruction algorithms are developed. The performance of the algorithms are analyzed when the data is corrupted by noise. Estimates are derived for the convergence rates of the algorithms in terms of the sampling density, the generators of the shift-invariant space and the sampling functionals (ψ_{x_j}).

1. INTRODUCTION

The reconstruction of a function f on $\mathbb{R}^d$ from its samples $\{f(x_j) : j \in J\}$, where J is a countable index set, is a common task in many applications in signal or image processing. The sampling set $X = \{x_j : \in J\}$ is often non-uniform and prevents the use of standard methods from Fourier analysis. For example, the loss of data packets during transmission through Internet or from satellites can be viewed as a non-uniform sampling/reconstruction problem. In geophysical exploration, the earth's magnetic field is measured by a combination of airborn, fast moving acquisition devices, as well as scattered stationary devices resulting in highly non-uniform sampling patterns, and a huge data

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set. The goal is to reconstruct the magnetic field and use it to reveal geological features.

In the sampling and reconstruction problem, the function f is usually assumed to belong to a shift-invariant space of the form

(1.1)
$$V^p(\Phi)=\left\{\sum_{i=1}^{r}\sum_{k\in\mathbb{Z}^d} c_{ik}\phi_i(\cdot-k):\ c_i=(c_{ik})\in\ell^p(\mathbb{Z}^d), i=1,\dots,r\right\},$$

where $\Phi=(\phi_1,\cdots,\phi_r)$ is called *the generator of* V. If $r=1$, $d=1$, $p=2$, and $\phi(x)=\frac{\sin(\pi x)}{\pi x}$, then $V^2(\phi)$ is the classical space of band-limited functions often used as a model in sampling theory (see for example [9, 16, 22, 29, 32] and the references therein). However, since band-limited functions are analytic, they have infinite support, thus local errors may propagate, and the reconstruction algorithms can be computationally inefficient. Moreover, many applied problems impose different a priori constraints on the type of functions. For this reason the sampling and reconstruction problems have been investigated in spline subspaces [8, 19, 27], wavelet subspaces [6, 10, 13, 12, 18, 20, 21, 30, 31, 33], and general shift-invariant spaces [2, 3, 4, 7, 28].

1.1. **Weighted average sampling in shift-invariant space.** The assumption that the sample values $\{f(x_j):\ j\in J\}$ can be measured exactly is not always valid. To take into account the characteristics of the acquisition devices, a weighted average value in the neighborhood of x_j is assumed. This means that the sampled data is of the form

$$g_{x_j}=\int_{\mathbb{R}^d} f(x)\overline{\psi_{x_j}(x)}dx, \tag{1.2}$$

where $\int_{\mathbb{R}^d}\psi_{x_j}=1$. Each function ψ_{x_j} reflects the characteristic of the sampling device used to measure the average sampling value of f in the neighborhood of x_j.

One of the goals of a sampling theory is to find conditions on the sampling set $X=\{x_j:\ j\in J\}$ such that a small change in the function f produces a small change in the sample values $\{g_{x_j}:\ j\in J\}$ and such that f can be reconstructed from $\{g_{x_j}:\ j\in J\}$ exactly and in a stable way. Equivalently, we must find conditions on X such that

$$c_p\|f\|_{L^p}\le\left(\sum_{x_j\in X}\left|g_{x_j}(f)\right|^p\right)^{1/p}\le C_p\|f\|_{L^p}, \tag{1.3}$$

where g_{x_j} are defined by (1.2), and where c_p and C_p are positive constants independent of f. Another important goal in sampling theory is to find fast algorithms for reconstructing the function f from its sample values.

When the sampling set is uniform, the weighted-average sampling and reconstruction problem has been studied in [26] for the particular case where the functionals in (1.2) are of the form $\psi_{x_j} = \psi(\cdot - x_j)$ (i.e., a single device ψ is used to obtain all the measurements), the sampling is critical (i.e., no oversampling), and in (1.1) $p = 2$, $r = 1$, and $d = 1$. The case of uniform sampling with multiple devices have been studied by Sun and Zhou [23], under the assumption that

$$\operatorname{supp} \psi_{x_j} \subset \left[x_j - \frac{\delta}{2}, x_j + \frac{\delta}{2}\right], \quad \psi_{x_j} \geq 0. \tag{1.4}$$

Define the Fourier transform $\hat{f}$ of an integrable function f by $\hat{f}(\xi) = \int f(\tau)e^{i2\pi\tau\xi}d\tau$. For non-uniform sampling, Gröchenig [17] proved that if $|x_{j+1} - x_j| \leq \delta < \sqrt{2}/2$, then any band-limited function f with $\operatorname{supp}\left(\hat{f}\right) \subset [-\frac{1}{2}, \frac{1}{2}]$ is uniquely determined from its averages $\langle f, \psi_{x_j}\rangle$, provided that (1.4) holds. He also showed that f can be reconstructed by iterative algorithms. Sun and Zhou [24] also studied average sampling under the assumption (1.4), and $\psi_{x_j}(\cdot + x_j)$ even and nondecreasing on $[0, \frac{\delta}{2}]$. They gave density conditions on X under which f satisfies (1.3) and derived frame algorithms for the reconstruction. They also gave bounds on the error of reconstruction when a non-band-limited function is reconstructed by the frame algorithms. In [25], Sun and Zhou showed that if the maximal gap between consecutive sampling points is smaller than a characteristic length, then a function in a spline subspace is uniquely determined from local averages obtained from averaging functions satisfying (1.4). For $p = 2$ and $r = 1$ in (1.1), Aldroubi gave conditions on the density of X and the diameter of the support of the sampling functionals ψ_{x_j} under which a function f can be reconstructed by iterative *approximation-projection* algorithms (A-P algorithms for short) [1]. In [1], estimates were also derived for the convergence rates of the A-P algorithms in terms of the generating function ϕ and the diameter of the support of the functionals ψ_{x_j}. It should be noted that A-P algorithms are not frame algorithms and do not require knowledge of the frames associated with $\{\psi_{x_j} : \ x_j \in X\}$. A-P algorithms are robust, their convergence is geometric, and they perform optimally even if the samples are corrupted by noise [1, 2, 4].

In this paper, we will consider the sampling problem in $V^p(\Phi)$, where $\{\phi_i(\cdot - j) : \ j \in \mathbb{Z}^d, i = 1, \ldots, r\}$ is a p-frame for $V^p(\Phi)$, i.e., there exists a positive constant A (depending on Φ and p) such that

(1.5)

$$A^{-1}\|f\|_{L^p} \le \sum_{i=1}^{r} \Big\| \Big(\int_{\mathbb{R}^d} f(x)\overline{\phi_i(x-j)}dx \Big)_{j\in\mathbb{Z}^d} \Big\|_{\ell^p} \le A\|f\|_{L^p}, \quad f \in V^p(\Phi).$$

We also assume throughout this paper (see the notation in Section 2) that:

$$\Phi = (\phi_1, \dots, \phi_r) \in W(L^1)^{(r)}, \text{ i.e., } \phi_i \in W(L^1), i = 1, \dots, r. \tag{1.6}$$

Under these conditions, the space $V^p(\Phi)$ in (1.1) is well defined, and it is a closed linear subspace of $L^p(\mathbb{R}^d)$ (see Theorem 1 in [5]). For this case, the well-posedness sampling condition (1.3) can then be written as

$$c\,\|f\|_{L^p} \le \left(\sum_{x_j\in X} \left|\langle f, \psi_{x_j}\rangle\right|^p \right)^{1/p} \le C\,\|f\|_{L^p} \quad f \in V^p(\Phi), \tag{1.7}$$

which is similar to a frame condition. However, the set $\{\psi_{x_j} : x_j \in X\}$ does not necessarily form a frame for $V^p(\Phi)$ since the functions ψ_{x_i}, $x_i \in X$, are not necessarily in $V^p(\Phi)$. The sampling theory in such spaces is new, since all previous results consider spaces in which $r = 1$ (single generator), and assume $\{\phi(\cdot - j) : \ j \in \mathbb{Z}^d\}$ to be a Riesz basis, instead of a (possibly redundant) frame. Moreover for average sampling in shift-invariant spaces, only the case $p = 2$ has been considered so far [1].

In this work, we prove that a function $f \in V^p(\Phi)$ can be reconstructed from its average samples by an iterative A-P algorithm, provided that the sampling set X satisfies a density condition that depends on Φ and the set $\{\psi_{x_j} : \ x_j \in X\}$ (Section 3.1). Our results treat the case of averaging functions in which the only requirement is that $\operatorname{supp}\psi_{x_j}$ is compact for each $x_j \in X$ (Theorem 3.2). But we also treat the important case where $\psi_{x_j} = \psi(\cdot - x_j)$ for each $x_j \in X$ (Theorem 3.1). However, for this case, we do not assume that ψ has compact support. In Section 3.2, we prove that the A-P algorithms converge even if the samples are corrupted by noise, and that the reconstruction result is optimal in some sense (Theorem 3.3). In Section 3.3, we present estimates for the rate of convergence of the A-P algorithms of Theorems 3.1 and 3.2 in term of the generator Φ and the sampling functions $\{\psi_{x_j} : \ x_j \in X\}$. The proofs of the results are collected in Section 4.

2. Notations and preliminaries

For the sampling problem we need to impose regularity requirements on the space $V^p(\Phi)$. Wiener amalgam spaces are useful in this context and they are defined as follows: A measurable function f belongs to $W(L^p)$, $1 \le p < \infty$, if it satisfies

$$\|f\|_{W(L^p)} = \left(\sum_{k \in \mathbb{Z}^d} \text{ess sup}\{|f(x+k)|^p : x \in [0,1]^d\} \right)^{1/p} < \infty. \tag{2.1}$$

If $p = \infty$, a measurable function f belongs to $W(L^\infty)$ if it satisfies

$$\|f\|_{W(L^\infty)} = \sup_{k \in \mathbb{Z}^d} \{\text{ess sup}\{|f(x+k)| : x \in [0,1]^d\}\} < \infty. \tag{2.2}$$

In this case $W(L^\infty)$ coincides with $L^\infty(\mathbb{R}^d)$.

Endowed with this norm, $W(L^p)$ becomes a Banach space [14, 15]. The subspace of continuous functions $W_0(L^p) = W(C, L^p) \subset W(L^p)$ is a closed subspace of $W(L^p)$ and thus also a Banach space [14, 15]. We have the following inclusions between the various spaces:

$$W_0(L^p) \subset W_0(L^q) \subset W(L^q) \subset L^q(\mathbb{R}^d), \quad 1 \le p \le q \le \infty. \tag{2.3}$$

The following convolution relations hold for $1 \le p \le \infty$ [4]:

(i) If $f \in L^p(\mathbb{R}^d)$ and $g \in W(L^1)$, then $f * g \in W(L^p)$ and

$$\|f * g\|_{W(L^p)} \le C \|f\|_{L^p} \|g\|_{W(L^1)}. \tag{2.4}$$

(ii) If $c = (c_k) \in \ell^p(\mathbb{Z}^d)$ and $\varphi \in W(L^1)$, then $\sum_{k \in \mathbb{Z}^d} c_k \varphi(\cdot - k) \in W(L^p)$ and

$$\Big\| \sum_{k \in \mathbb{Z}^d} c_k \varphi(\cdot - k) \Big\|_{W(L^p)} \le \|c\|_{\ell^p} \|\varphi\|_{W(L^1)}. \tag{2.5}$$

(iii) If $f \in L^p(\mathbb{R}^d)$ and $g \in W(L^1)$, then the sequence $d = (d_k)$ defined by $d_k = \int_{\mathbb{R}^d} f(x)\overline{g(x-k)}dx$, $k \in \mathbb{Z}^d$, belongs to $\ell^p(\mathbb{Z}^d)$ and

$$\|d\|_{\ell^p} \le \|f\|_{L^p} \|g\|_{W(L^1)}. \tag{2.6}$$

2.1. **Shift invariant spaces.**

In addition to the requirement that the generator Φ of $V^p(\Phi)$ satisfies (1.5) and (1.6), we also require ϕ_i, $i = 1, \cdots, r$, to be continuous. Thus, together, the requirements are that Φ satisfies (1.5) and belongs to $W_0(L^1)^{(r)}$ (here $W_0(L^1)^{(r)}$ denotes the Cartesian product $W_0(L^1) \times \cdots \times W_0(L^1)$ of r copies of $W_0(L^1)$). With these requirements, it is well known that the space $V^p(\Phi)$ is a space of continuous L^p-functions, and we have the following properties [4, 5]:

(i) The space $V^p(\Phi)$ is a closed linear subspace of $L^p(\mathbb{R}^d)$, and there exists a positive constant B (depending on Φ and p) such that

$$B^{-1}\|f\|_{L^p} \le \inf_{f=\sum_{i=1}^r \phi_i *' c_i} \sum_{i=1}^r \|c_i\|_{\ell^p} \le B\|f\|_{L^p} \quad \forall f \in V^p(\Phi), \tag{2.7}$$

where $\phi_i *' c_i = \sum_{k\in\mathbb{Z}^d} c_{ik}\phi_i(\cdot - k)$ and $c_i = (c_{ik}) \in \ell^p(\mathbb{Z}^d)$.

(ii) The space $V^p(\Phi)$ is a closed linear subspace of $W_0(L^p)$, and we have the norm equivalence $\|f\|_{L^p} \approx \|f\|_{W(L^p)}$.

(iii) There exists $\tilde\phi_1, \dots, \tilde\phi_r \in W_0(L^1) \cap V^p(\Phi)$ such that for every $f \in V^p(\Phi)$,

$$f = \sum_{i=1}^r \sum_{j\in\mathbb{Z}^d} \langle f, \tilde\phi_i(\cdot - j)\rangle \phi_i(\cdot - j) = \sum_{i=1}^r \sum_{j\in\mathbb{Z}^d} \langle f, \phi_i(\cdot - j)\rangle \tilde\phi_i(\cdot - j). \tag{2.8}$$

Hence the operator P defined by

$$Pf = \sum_{i=1}^r \sum_{j\in\mathbb{Z}^d} \langle f, \tilde\phi_i(\cdot - j)\rangle \phi_i(\cdot - j), \quad f \in L^p(\mathbb{R}^d), \tag{2.9}$$

is a bounded projection from $L^p(\mathbb{R}^d)$ onto $V^p(\Phi)$.

(iv) If $X = \{x_j : j \in J\}$ is *separated*, i.e., $\inf_{j\neq l} |x_j - x_l| > 0$, then

$$\left(\sum_{x_j\in X} |f(x_j)|^p\right)^{1/p} \le C\,\|f\|_{L^p} \qquad \text{for all } f \in V^p(\Phi). \tag{2.10}$$

3. Main Results

We will assume throughout that the sampling set X is separated, and that the sampling functionals ψ_{x_j} satisfy the following properties:

(i) $\sup_j \|\psi_{x_j}\|_{W(L^1)} < \infty$;
(ii) $\int_{\mathbb{R}^d} \psi_{x_j} = 1$.

3.1. Fast approximation-projection iterative reconstruction algorithms.

Fast approximation-projection (A-P) iterative algorithms for the reconstruction of functions from their samples have been introduced by Feichtinger and Gröchenig for the case of band-limited functions [16]. These schemes have been extended by Aldroubi and Feichtinger to general shift-invariant spaces [2]. In this paper, we will develop the theory of fast A-P iterative reconstruction schemes for the case of average sampling. First, we need to introduce the notion of γ-density useful in this regard.

Definition 3.1. A set $X = \{x_j : j \in J\}$ is γ_0-dense in $I\!R^d$ if

$$\mathbb{R}^d = \cup_j B_\gamma(x_j) \qquad \text{for every } \gamma > \gamma_0\,, \tag{3.1}$$

where $B_\gamma(x_j)$ are balls centered at x_j, and with radius γ.

This definition implies that the distance of any sampling point to its next neighbor is at most $2\gamma_0$. Thus strictly speaking, γ_0 is the inverse of a density, i.e., if γ_0 increases, the number of points per unit cube decreases.

A special but important case for average sampling is when the sampling functions ψ_{x_j} are obtained by translation of a single function ψ. Thus, $\psi_{x_j} = \psi(\cdot - x_j)$ and the weighted samples are of the form $g_{x_j} = \langle f, \psi(\cdot - x_j)\rangle$. For this case, the iterative algorithm that we develop uses a quasi-reconstruction operator $\mathrm{A}_{\mathrm{X},a}$ in the iteration scheme. To define this operator, we start from a partition of unity $\{\beta_j\}_{j\in J}$ defined as follows:

Definition 3.2. A bounded uniform partition of unity (BUPU) associated with $\{B_\gamma(x_j)\}_{j\in J}$ is a set of functions $\{\beta_j\}_{j\in J}$ that satisfy

(1) $0 \le \beta_j \le 1$, $\forall j \in J$;
(2) supp $\beta_j \subset B_\gamma(x_j)$; and
(3) $\sum_{j\in J} \beta_j = 1$.

The operator $\mathrm{A}_{\mathrm{X},a}$ is then defined by

$$\mathrm{A}_{\mathrm{X},a}\, f = \sum\nolimits_{j\in J} \langle f, \psi_a(\cdot - x_j)\rangle \beta_j = \sum\nolimits_{j\in J} (f * \psi_a^*)(x_j)\beta_j, \tag{3.2}$$

where $\psi_a(\cdot) = \frac{1}{a^d}\psi(\frac{\cdot}{a})$, and where $\psi_a^*(x) = \overline{\psi_a(-x)}$. Obviously the quasi-reconstruction operator $\mathrm{A}_{\mathrm{X},a}\, f$ does not belong to the space $V^p(\Phi)$. However, we can use this operator in an A-P iterative scheme to reconstruct the exact function $f \in V^p(\Phi)$ as follows:

Theorem 3.1. *Let Φ be in $W_0(L^1)^{(r)}$, let ψ be a function in $W(L^1)$ such that $\int_{\mathbb{R}^d} \psi = 1$, and let P be a bounded projection from L^p onto $V^p(\Phi)$. Then there exists a density $\gamma = \gamma(\Phi, \psi) > 0$ and $a_0 > 0$ such that any $f \in V^p(\Phi)$ can be recovered from its weighted average samples $\{\langle f, \psi_a(\cdot - x_j)\rangle : j \in J\}$ on any γ-dense set $X = \{x_j : j \in J\}$ and for any $0 < a < a_0$, by the following A-P iterative algorithm:*

$$\begin{cases} f_1 = \mathrm{P}\,\mathrm{A}_{\mathrm{X},a}\, f \\ f_{n+1} = \mathrm{P}\,\mathrm{A}_{\mathrm{X},a}(f - f_n) + f_n\ . \end{cases} \tag{3.3}$$

In this case, the iterate f_n converges to f uniformly, and also in the $W(L^p)$-norm and the L^p-norm. Moreover, the convergence is geometric, that is,

$$\|f - f_n\|_{L^p} \le \|f - f_n\|_{W(L^p)} \le C_1 \alpha^n \|f - f_1\|_{W(L^p)}$$

for some $\alpha = \alpha(\gamma, a, \Phi, \psi) < 1$ *and* $C_1 < \infty$.

Theorem 3.1 treats the case of a single averaging function ψ_a shifted to the points $\{x_j\}$ for obtaining the measurements $\langle f, \psi_a(\cdot - x_j)\rangle$. In practice, this is the situation when a single measuring device is used to obtain the discrete data. For this case, ψ_a is what is called the *impulse response* of the measuring device. More generally, we can allow the averaging function ψ_{x_j} to depend on the point x_j. Thus, the averaging functions can be described by the infinite vector $\Psi_X = (\psi_{x_j})_{j\in J}$. For this case, and under some uniformity on the size of the averaging functions ψ_{x_j}, we can recover the function f exactly by using the quasi reconstruction operator

$$\mathrm{A_X}\, f = \sum\nolimits_{j\in J} \langle f, \psi_{x_j}\rangle \beta_j \tag{3.4}$$

in the following A-P iterative algorithm:

Theorem 3.2. *Let* Φ *be in* $W_0(L^1)^{(r)}$, P *be a bounded projection from* L^p *onto* $V^p(\Phi)$, *and let the averaging sampling functionals* $\psi_{x_j} \in W(L^1)$ *satisfy* $\int_{\mathbb{R}^d} \psi_{x_j} = 1$ *and* $\int_{\mathbb{R}^d} \left|\psi_{x_j}\right| \leq M$, *where* $M > 0$ *is independent of* x_j. *Then there exists a density* $\gamma = \gamma(\Phi, M) > 0$ *and* $a_0 = a_0(\Phi, M) > 0$ *such that if* $X = \{x_j : j \in J\}$ *is separated and* γ*-dense in* $\mathbb{R}^d$, *and if the average sampling functionals* ψ_{x_j} *satisfy* $\operatorname{supp}\psi_{x_j} \subset x_j + [-a, a]^d$ *for some* $0 < a < a_0$, *then any* $f \in V^p(\Phi)$ *can be recovered from its weighted average samples* $\{\langle f, \psi_{x_j}\rangle : j \in J\}$ *by the following iterative algorithm:*

$$\begin{cases} f_1 = \mathrm{P}\,\mathrm{A_X}\, f \\ f_{n+1} = \mathrm{P}\,\mathrm{A_X}(f - f_n) + f_n\ . \end{cases} \tag{3.5}$$

In this case, the iterate f_n *converges to* f *uniformly, and also in the* $W(L^p)$*-norm and the* L^p*-norm. Moreover, the convergence is geometric, that is,*

$$\|f - f_n\|_{L^p} \leq \|f - f_n\|_{W(L^p)} \leq C_1\alpha^n \|f - f_1\|_{W(L^p)}$$

for some $\alpha = \alpha(\gamma, a, \Phi, M) < 1$ *and* $C_1 < \infty$.

Remark 3.1. Theorems 3.1 and 3.2 require bounded projections from L^p onto $V^p(\Phi)$. Bounded projections on closed subspaces of Banach spaces do not always exist. However, in the context of the shift-invariant space $V^p(\Phi)$ described in Theorems 3.1 and 3.2, Remark 2 in [5, p. 7] assures us that we can always find a bounded projector P from L^p onto $V^p(\Phi)$ for any p with $1 \leq p \leq \infty$ (see (2.9)). Since the same operator P in (2.9) works simultaneously for all L^p, $1 \leq p \leq \infty$, we call it a *universal projector*. In addition, this universal projector can be implemented using *filtering algorithms*, i.e., simple convolutions that are easily implementable with fast algorithms.

3.2. **Reconstruction in presence of noise.** In practice, the sampled data is often corrupted by noise. Moreover, the assumption that the function f belongs to some specific space $V^p(\Phi)$ is often an idealization. Thus, it is important to know whether the A-P algorithms (3.3) and (3.5) still converge under non-ideal circumstances. To investigate these situations, we only assume that the data $f' = \{f'_j : \ j \in J\}$ belongs to ℓ^p, but we do not assume that $f' = \{f'_j : \ j \in J\}$ are local averages of a function $f \in V^p(\Phi)$. For this case we use the initialization

$$f_1 = \mathrm{P}\,\mathrm{Q}_\mathrm{X}\{f'_j\} := \mathrm{P}\left(\sum_{j\in J} f'_j \beta_j\right) \in V^p(\Phi), \tag{3.6}$$

where $\{\beta_j : \ j \in J\}$ is the BUPU in Definition (3.2). Algorithm (3.3) becomes

$$f_{n+1} = f_1 + (\mathrm{I} - \mathrm{P}\,\mathrm{A}_{\mathrm{X},a})f_n, \tag{3.7}$$

and algorithm (3.5) becomes

$$f_{n+1} = f_1 + (\mathrm{I} - \mathrm{P}\,\mathrm{A}_\mathrm{X})f_n. \tag{3.8}$$

We have:

Theorem 3.3. *Under the same assumptions as in Theorem 3.1, the algorithm (3.7), with the initialization (3.6), converges to a function* $f_\infty \in V^p(\Phi)$ *which satisfies* $\mathrm{P}(\mathrm{A}_{\mathrm{X},a}\, f_\infty - \mathrm{Q}_\mathrm{X}\{f'_j\}) = 0$. *Correspondingly, under the assumptions of Theorem 3.2, algorithm (3.8) converges to a function* $f_\infty \in V^p(\Phi)$ *which satisfies* $\mathrm{P}(\mathrm{A}_\mathrm{X}\, f_\infty - \mathrm{Q}_\mathrm{X}\{f'_j\}) = 0$.

3.3. **Convergence rate.** When both Φ and ψ satisfy additional regularity conditions, an estimate of the convergence rate α in Theorem 3.1 in terms of the γ-density, a, Φ, and ψ is given by:

Theorem 3.4. *Assume that* Φ *and* ψ *satisfy the conditions of Theorem 3.1, and that* $|\nabla\phi_i| \in W(L^1)$ *for every* $i = 1, \dots, r$, *and* $\|\psi\|_{1,\eta} := \int_{\mathbb{R}^d} |\psi(t)||t|^\eta dt < \infty$ *for some* $0 < \eta \le 1$. *Then the convergence rate* α *in Theorem 3.1 satisfies*

$$\begin{aligned}\alpha \ \le \ B\,\|\mathrm{P}\|_{op} \Bigg(& 3^d\gamma \max_{1\le i\le r} |||\nabla\phi_i|||_{W(L^1)} + (6^d+1)a^\eta \|\psi\|_{1,\eta} \\ & \times\Big((1+2^d)\max_{1\le i\le r}\|\phi_i\|_{W(L^1)} + 3^d \max_{1\le i\le r}|||\nabla\phi_i|||_{W(L^1)}\Big)\Bigg),\end{aligned}$$

where B *is the upper bound constant in (2.7).*

We have a corresponding result for the situation in Theorem 3.2.

Theorem 3.5. *Assume that Φ and $\Psi_X = (\psi_{x_j})_j \in J$ satisfy the conditions of Theorem 3.2, and that $|\nabla\phi_i| \in W(L^1)$ for every $i = 1, \dots, r$. Then the convergence rate α in Theorem 3.2 satisfies*

$$\alpha \le 3^d B \, \|\mathrm{P}\|_{op} \left(\gamma + M(6^d+1)a\right) \max_{1\le i\le r} \||\nabla\phi_i|\|_{W(L^1)}\,, \tag{3.9}$$

where B is the upper bound constant in (2.7) and M is the upper bound in Theorem 3.2.

Remark 3.2. The above estimates allow us to find sufficient density conditions and size conditions on the averaging functions for the A-P iterative algorithms to be convergent, as well as the convergence rates.

4. Proofs

4.1. **Proof of Theorem 3.1.** To prove Theorem 3.1, we need to introduce the quasi-interpolant $\mathrm{Q_X}$ of the sampled values $g|_X$ of a function $g \in W_0(L^p)$. Given a bounded uniform partition of unity $\{\beta_j : j \in J\}$ associated with a separated sampling set X as in Definition (3.2), we define a quasi-interpolant $\mathrm{Q_X}\, c$ on sequences by

$$\mathrm{Q_X}\, c = \sum_{j\in J} c_j \beta_j\,.$$

If $f \in W_0(L^p)$, we write

$$\mathrm{Q_X}\, f = \sum_{j\in J} f(x_j)\beta_j$$

for the quasi-interpolant of the sequence $c_j = f(x_j)$. We will need the following property of the quasi-interpolant $\mathrm{Q_X}$:

Lemma 4.1. *Let X be any sampling set with γ-density $\gamma(X)$, $\{\beta_j : j \in J\}$ be a BUPU associated with X (see Definition 3.2), and let $\varphi \in W_0(L^1)$. Then there exists a constant $C = C(\gamma, d)$ such that for any $f = \sum_k c_k \varphi(\cdot - k)$, we have*

$$\|\mathrm{Q_X}\, f\|_{L^p} \le \|\mathrm{Q_X}\, f\|_{W(L^p)} \le C(\gamma, d)\, \|c\|_{\ell^p}\, \|\varphi\|_{W(L^1)} \quad \forall c = (c_k) \in \ell^p(\mathbb{Z}^d),$$

where the constant $C(\gamma, d) \le (\lceil 2\gamma + 4\rceil^d + 1)$ does not depend explicitly on the sampling set X, or the partition of unity in Definition (3.2). Here $\lceil t\rceil$ denotes the smallest integer greater than or equal to t.

To prove the above lemma, we need the following lemma from [2, 4].

Lemma 4.2. *Let $\varphi \in W_0(L^1)$ and let $f = \sum_k c_k \varphi(\cdot - k)$, where $c = (c_k) \in \ell^p(\mathbb{Z}^d)$. Then*

(i) *the oscillation (or modulus of continuity)*

$$\operatorname{osc}_\gamma(f)(x) = \sup_{|y|\le\gamma} |f(x+y) - f(x)|$$

belongs to $W(L^p)$;

(ii) *the oscillation* $\operatorname{osc}_\gamma(\varphi)$ *satisfies*

$$\|\operatorname{osc}_\gamma(\varphi)\|_{W(L^1)} \le C'(\gamma, d)\, \|\varphi\|_{W(L^1)}, \tag{4.1}$$

where $C'(\gamma,d) \le \lceil 2\gamma + 4\rceil^d$, *and*

$$\|\operatorname{osc}_\gamma(\varphi)\|_{W(L^1)} \to 0 \quad \text{as} \quad \gamma \to 0;$$

(iii) *the oscillation* $\operatorname{osc}_\gamma(f)$ *satisfies*

$$\|\operatorname{osc}_\gamma(f)\|_{W(L^p)} \le \|c\|_{\ell^p}\, \|\operatorname{osc}_\gamma(\varphi)\|_{W(L^1)} \qquad \textit{for all} \;\; c \in \ell^p. \tag{4.2}$$

In particular, $\|\operatorname{osc}_\gamma(f)\|_{W(L^p)} \to 0$ *as* $\gamma \to 0$.

Proof of Lemma 4.1. Let $f = \sum_k c_k \varphi(\cdot - k)$, where $c = (c_k) \in \ell^p(\mathbb{Z}^d)$. From (2.5), we have $f \in W(L^p)$ and

$$\begin{aligned}
|f(x) - (\mathrm{Q_X}\, f)(x)| &= \Big| f(x) - \sum_{j\in J} f(x_j)\beta_j(x) \Big| \\
&= \Big| f(x) \sum_{j\in J} \beta_j(x) - \sum_{j\in J} f(x_j)\beta_j(x) \Big| \\
&\le \sum_{j\in J} |f(x) - f(x_j)|\, \beta_j(x) \\
&\le \sum_{j\in J} \operatorname{osc}_\gamma(f)(x)\beta_j(x) \\
&\le \operatorname{osc}_\gamma(f)(x) \sum_{j\in J} \beta_j(x) = \operatorname{osc}_\gamma(f)(x).
\end{aligned}$$

From this pointwise estimate and Lemma 4.2, we get that

$$\|f - \mathrm{Q_X}\, f\|_{W(L^p)} \le \|\operatorname{osc}_\gamma(f)\|_{W(L^p)} \le \|c\|_{\ell^p}\, \|\operatorname{osc}_\gamma(\varphi)\|_{W(L^1)}. \tag{4.3}$$

Thus using (2.5), (4.1) and (4.3), we obtain

$$\begin{aligned}
\|\mathrm{Q_X}\, f\|_{W(L^p)} &\le \|f - \mathrm{Q_X}\, f\|_{W(L^p)} + \|f\|_{W(L^p)} \\
&\le \left(\lceil 2\gamma + 4\rceil^d + 1\right) \|c\|_{\ell^p}\, \|\varphi\|_{W(L^1)}.
\end{aligned} \tag{4.4}$$

□

For the proof of Theorem 3.1, we will also need the following lemma.

Lemma 4.3. *Let $\psi \in L^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \psi(x)dx = 1$, and define $\psi_a(\cdot) = a^{-d}\psi(\cdot/a)$, where $a > 0$ is any positive real number. Then for every $\phi \in W(L^1)$,*

$$\|\phi - \phi * \psi_a^*\|_{W(L^1)} \to 0 \quad as \quad a \to 0^+.$$

Proof. We will estimate the $W(L^1)$-norm of $\varphi^a = \phi - \phi * \psi_a^*$. Since $\int_{\mathbb{R}^d} \psi(x)dx = 1$ and $\psi_a(x) = a^{-d}\psi(x/a)$, we have

$$\varphi^a(x) = \phi(x) - \phi * \psi_a^*(x) = \int_{\mathbb{R}^d} (\phi(x) - \phi(x+t))\overline{\psi_a(t)}\, dt.$$

Therefore

$$\begin{aligned} |\varphi^a(x)| &\le \int_{\mathbb{R}^d} |\phi(x) - \phi(x+t)||\psi_a(t)|dt \\ &= \Big(\int_{|t|\le 1} + \int_{|t|\ge 1}\Big)|\phi(x) - \phi(x+t)||\psi_a(t)|dt \\ &= I_1(x) + I_2(x). \end{aligned} \tag{4.5}$$

By direct computations, we have

$$\begin{aligned} \|I_2\|_{W(L^1)} &= \sum_{k\in\mathbb{Z}^d} \sup_{x\in k+[0,1)^d} \int_{|t|\ge 1} |\phi(x) - \phi(x+t)||\psi_a(t)|dt \\ &\le \sum_{k\in\mathbb{Z}^d} \int_{|t|\ge 1} \Big(\sup_{x\in k+[0,1)^d} |\phi(x)| + \sup_{x\in k+[0,1)^d} |\phi(x+t)|\Big)|\psi_a(t)|dt \\ &\le \int_{|t|\ge 1} \Big(\|\phi\|_{W(L^1)} + \sum_{k\in\mathbb{Z}^d} \sup_{x\in k+[0,1)^d} |\phi(x+t)|\Big)|\psi_a(t)|dt \\ &\le (1+2^d)\|\phi\|_{W(L^1)} \int_{|t|\ge 1} |\psi_a(t)|dt \\ &= (1+2^d)\|\phi\|_{W(L^1)} \int_{|t|\ge a^{-1}} |\psi(t)|dt, \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \|I_1\|_{W(L^1)} &\le \int_{|t|\le 1} \sum_{k\in\mathbb{Z}^d} \sup_{x\in k+[0,1)^d} |\phi(x) - \phi(x+t)||\psi_a(t)|dt \\ &\le \int_{|t|\le 1} \sum_{k\in\mathbb{Z}^d} \sup_{x\in k+[0,1)^d} \operatorname{osc}_{|t|}(\phi)(x)|\psi_a(t)|dt \\ &= \int_{|t|\le 1} \|\operatorname{osc}_{|t|}(\phi)\|_{W(L^1)}|\psi_a(t)|dt =: I. \end{aligned} \tag{4.7}$$

By Lemma 4.2, for any $\epsilon > 0$, there exists $\delta_0 > 0$ such that

$$\|\mathrm{osc}_\delta(\phi)\|_{W(L^1)} < \epsilon \quad \forall\ \delta < \delta_0.$$

Write

$$I = \Big(\int_{|t|\le\delta_0} + \int_{\delta_0\le|t|\le 1}\Big)\|\mathrm{osc}_{|t|}(\phi)\|_{W(L^1)}|\psi_a(t)|dt =: I_3 + I_4. \tag{4.8}$$

Then

$$I_3 \le \epsilon \int_{|t|\le\delta_0} |\psi_a(t)|dt \le \epsilon\|\psi\|_1,$$

and

$$\begin{aligned} I_4 &\le \int_{1\ge|t|\ge\delta_0} \|\,\mathrm{osc}_1(\phi)\|_{W(L^1)}|\psi_a(t)|dt \\ &\le \|\,\mathrm{osc}_1(\phi)\|_{W(L^1)} \int_{|s|\ge\delta_0/a} |\psi(s)|ds \\ &\to 0 \quad \text{as } a \to 0^+. \end{aligned}$$

By (4.8), $I \to 0$ as $a \to 0^+$. Combining (4.5), (4.6) and (4.7), we have

$$\|\varphi^a\|_{W(L^1)} \le \|I_1\|_{W(L^1)} + \|I_2\|_{W(L^1)} \to 0 \quad \text{as} \quad a \to 0^+.$$

□

Lemma 4.4. *Let* P *be a bounded projection from* $L^p(I\!R^d)$ *onto* $V^p(\Phi)$. *Then there exist* $\gamma_0 > 0$ *and* $a_0 > 0$ *such that for every separated* γ*-dense set* X *with* $\gamma \le \gamma_0$ *and for every positive* $a \le a_0$*, the operator* $\mathrm{I} - \mathrm{P}\, A_{X,a}$ *is a contraction on* $V^p(\Phi)$.

Proof. Let $f = \sum_{i=1}^r \sum_k c_{ik}\phi_i(\cdot - k) \in V^p(\Phi)$. We have

$$\begin{aligned} \|f - \mathrm{P}\, A_{X,a}f\|_{L^p} &= \|f - \mathrm{P}\,\mathrm{Q_X}\, f + \mathrm{P}\,\mathrm{Q_X}\, f - \mathrm{P}\, A_{X,a}f\|_{L^p} \\ &\le \|\mathrm{P}\, f - \mathrm{P}\,\mathrm{Q_X}\, f\|_{L^p} + \|\mathrm{P}\,\mathrm{Q_X}\, f - \mathrm{P}\, A_{X,a}f\|_{L^p} \\ &\le \|\mathrm{P}\|_{\mathrm{op}}\,(\|\, f - \mathrm{Q_X}\, f\|_{L^p} + \|\mathrm{Q_X}\, f - \mathrm{Q_X}(f * \psi_a^*)\|_{L^p}). \end{aligned} \tag{4.9}$$

Using (4.3) and the upper bound inequality of (2.7), the first term of the last inequality in (4.9) can be estimated as follows:

$$\|\, f - \mathrm{Q_X}\, f\|_{L^p} \le \|\, f - \mathrm{Q_X}\, f\|_{W(L^p)} \le B \max_{1\le i\le r} \|\mathrm{osc}_\gamma(\phi_i)\|_{W(L^1)}\, \|f\|_{L^p}\, . \tag{4.10}$$

The second term $\|\mathrm{Q_X}\, f - \mathrm{Q_X}(f * \psi_a^*)\|_{L^p}$ can be estimated as follows. Write $\varphi_i^a = \phi_i - \phi_i * \psi_a^*$ for $i = 1, \dots, r$. Since each $\phi_i \in W_0(L^1)$ and

$\psi \in L^1$, (2.4) implies that $\varphi_i^a \in W_0(L^1)$. Noting that $\mathrm{Q_X}\, f - \mathrm{Q_X}(f * \psi_a^*) = \mathrm{Q_X}(\sum_{i=1}^r \sum_k c_{ik}\varphi_i^a(\cdot - k))$, and using Lemma 4.1, we obtain

$$\begin{aligned} \|\mathrm{Q_X}\, f - \mathrm{Q_X}(f * \psi_a^*)\|_{L^p} &\leq \Big\| \mathrm{Q_X}(\sum_{i=1}^r \sum_k c_{ik}\varphi_i^a(\cdot - k))\Big\|_{W(L^p)} \\ &\leq C(\gamma, d) \sum_{i=1}^r \|c_i\|_{\ell^p} \|\varphi_i^a\|_{W(L^1)} . \end{aligned}$$

Hence by (2.7),

$$\|\mathrm{Q_X}\, f - \mathrm{Q_X}(f * \psi_a^*)\|_{L^p} \leq C(\gamma, d) B \|f\|_{L^p} \max_{1\leq i\leq r} \|\varphi_i^a\|_{W(L^1)} . \tag{4.11}$$

By combining (4.9), (4.10) and (4.11), we get

$$\begin{aligned} \| f - \mathrm{P}\, A_{X,a} f\|_{L^p} &\leq \|\mathrm{P}\|_{\mathrm{op}} \big(\max_{1\leq i\leq r} \|\mathrm{osc}_\gamma(\phi_i)\|_{W(L^1)} \\ &\quad + (\lceil 2\gamma + 4\rceil^d + 1) \max_{1\leq i\leq r} \|\varphi_i^a\|_{W(L^1)} \big) B \|f\|_{L^p} . \end{aligned} \tag{4.12}$$

Let $\epsilon > 0$ be any positive real number. Using Lemma 4.2 (ii), we may choose γ_0 so small so that $\max_{1\leq i\leq r} \|\mathrm{osc}_\gamma(\phi_i)\|_{W(L^1)} \leq \epsilon/2$ for all $\gamma \leq \gamma_0$. Then by Lemma 4.3, we may choose a_0 so small that $(\lceil 2\gamma_0 + 4\rceil^d + 1) \max_{1\leq i\leq r} \|\varphi_i^a\|_{W(L^1)} \leq \epsilon/2$ for all $a \leq a_0$. Therefore, we can choose γ_0 and a_0 so that for any $\gamma \leq \gamma_0$ and $a \leq a_0$, we have

$$\| f - \mathrm{P}\, A_{X,a} f\|_{L^p} \leq B\epsilon \|\mathrm{P}\|_{\mathrm{op}} \|f\|_{L^p} \quad \text{for all } f \in V^p(\Phi). \tag{4.13}$$

To get a contraction, we choose $B\epsilon \|\mathrm{P}\|_{\mathrm{op}} < 1$. □

Proof of Theorem 3.1. Let $e_n = f - f_n$ be the error after n iterations of Algorithm (3.3). Then the sequence e_n satisfies the recursion

$$\begin{aligned} e_{n+1} &= f - f_{n+1} \\ &= f - f_n - \mathrm{P}\, A_{X,a}(f - f_n) \\ &= (I - \mathrm{P}\, A_{X,a}) e_n. \end{aligned} \tag{4.14}$$

Using Lemma 4.4, we may choose γ_0 and a_0 so small that $\|\, \mathrm{I} - \mathrm{P}\, A_{X,a}\|_{\mathrm{op}} = \alpha < 1$. Therefore by (4.14) we obtain

$$\|e_{n+1}\|_{L^p} \leq \alpha \|e_n\|_{L^p} \tag{4.15}$$

and

$$\|e_n\|_{L^p} \leq \alpha^{n-1} \|e_1\|_{L^p} .$$

Thus $\|e_n\|_{L^p} \to 0$ as $n \to \infty$. Since for $V^p(\Phi)$, the $W(L^p)$-norm and the L^p-norm are equivalent, the inequality above also holds in the $W(L^p)$-norm and the proof is completed. □

4.2. **Proof of Theorem 3.2.** To prove Theorem 3.2, we only need to modify the proof of Lemma 4.4 and prove that $\mathrm{I} - \mathrm{P}\,\mathrm{A_X}$ is a contraction on $V^p(\Phi)$.

Proof. Let $f = \sum_{i=1}^r \sum_k c_{ik}\phi_i(\cdot - k) \in V^p(\Phi)$. We have

$$\begin{aligned}
\|f - \mathrm{P}\,\mathrm{A_X} f\|_{L^p} &= \|f - \mathrm{P}\,\mathrm{Q_X} f + \mathrm{P}\,\mathrm{Q_X} f - \mathrm{P}\,\mathrm{A_X} f\|_{L^p} \\
&\le \|\mathrm{P} f - \mathrm{P}\,\mathrm{Q_X} f\|_{L^p} + \|\mathrm{P}\,\mathrm{Q_X} f - \mathrm{P}\,\mathrm{A_X} f\|_{L^p} \\
&\le \|\mathrm{P}\|_{\mathrm{op}} (\| f - \mathrm{Q_X} f\|_{L^p} + \|\mathrm{Q_X} f - \mathrm{A_X} f\|_{L^p}). \qquad (4.16)
\end{aligned}$$

The second term $\|\mathrm{Q_X} f - \mathrm{A_X} f\|_{L^p}$ of the last inequality can be estimated as follows: Write $f_i = \sum_k c_{ik}\phi_i(\cdot - k)$ for $i = 1, \cdots, r$. Clearly, $f_i \in V^p(\Phi)$ for $i = 1, \cdots, r$, and $f = \sum_{i=1}^r f_i$. For each f_i, we have the following pointwise estimate:

$$\begin{aligned}
|(\mathrm{Q_X} f_i - \mathrm{A_X} f_i)(x)| &= \left| \sum_j (f_i(x_j) - \langle f_i, \psi_{x_j} \rangle)\beta_j(x) \right| \\
&= \left| \sum_j \left(\int_{\mathbb{R}^d} (f_i(x_j) - f_i(\xi))\overline{\psi_{x_j}(\xi)} d\xi \right) \beta_j(x) \right| \\
&\le \sum_j \int_{\mathbb{R}^d} |f_i(x_j) - f_i(\xi)| \left|\psi_{x_j}(\xi)\right| d\xi \beta_j(x) \\
&\le \sum_j \mathrm{osc}_a(f_i)(x_j) \int_{\mathbb{R}^d} \left|\psi_{x_j}(\xi)\right| d\xi \beta_j(x) \\
&\le M \sum_j \mathrm{osc}_a(f_i)(x_j)\beta_j(x) \\
&\le M \sum_j \big(\sum_k |c_{ik}| \,\mathrm{osc}_a(\phi_i)(x_j - k)\big)\beta_j(x). \qquad (4.17)
\end{aligned}$$

From this pointwise estimate and Lemma 4.1, it follows that

$$\|\mathrm{Q_X} f_i - \mathrm{A_X} f_i\|_{L^p} \le MC(\gamma, d) \|c_i\|_{\ell^p} \|\mathrm{osc}_a(\phi_i)\|_{W(L^1)} .$$

Thus we conclude that

$$\|\mathrm{Q_X} f - \mathrm{A_X} f\|_{L^p} \le MC(\gamma, d) \sum_{i=1}^r \|c_i\|_{\ell^p} \|\mathrm{osc}_a(\phi_i)\|_{W(L^1)} .$$

Hence by (2.7),

$$\|\mathrm{Q_X} f - \mathrm{A_X} f\|_{L^p} \le MC(\gamma, d)B \|f\|_{L^p} \max_{1\le i\le r} \|\mathrm{osc}_a(\phi_i)\|_{W(L^1)} . \qquad (4.18)$$

By combining (4.10), (4.16), and (4.18), we get

$$\begin{aligned} \| f - \mathrm{P}\,\mathrm{A_X} f\|_{L^p} &\le \|\mathrm{P}\|_{\mathrm{op}} \Big(\max_{1\le i\le r} \|\mathrm{osc}_\gamma(\phi_i)\|_{W(L^1)} \\ &\quad + M(\lceil 2\gamma+4\rceil^d+1) \max_{1\le i\le r} \|\mathrm{osc}_a(\phi_i)\|_{W(L^1)} \Big) B\, \|f\|_{L^p}. \end{aligned} \tag{4.19}$$

The rest of the proof is similar to the last part of the proof of Lemma 4.4. Let $\epsilon > 0$ be any positive real number. Using Lemma 4.2 (ii), we may choose γ_0 so small so that $\max_{1\le i\le r} \|\mathrm{osc}_\gamma(\phi_i)\|_{W(L^1)} \le \epsilon/2$ for all $\gamma \le \gamma_0$. Then we may choose a_0 so small that $M(\lceil 2\gamma_0 + 4\rceil^d + 1) \max_{1\le i\le r} \|\mathrm{osc}_a(\phi_i)\|_{W(L^1)} \le \epsilon/2$ for all $a \le a_0$. Therefore, we can choose γ_0 and a_0 so that for any $\gamma \le \gamma_0$ and $a \le a_0$, we have

$$\| f - \mathrm{P}\,\mathrm{A_X} f\|_{L^p} \le B\epsilon\, \|\mathrm{P}\|_{\mathrm{op}}\, \|f\|_{L^p} \quad \text{for all } f \in V^p(\Phi). \tag{4.20}$$

To get a contraction, we choose $B\epsilon\, \|\mathrm{P}\|_{\mathrm{op}} < 1$. □

4.3. **Proof of Theorem 3.3.**

Proof. By Lemma 4.4, the operator $\mathrm{I} - \mathrm{P}\,\mathrm{A}_{\mathrm{X},a}$ is a contraction on $V^p(\Phi)$. It follows that the sequence of functions f_n in (3.7) is convergent to a function f_∞ in $V^p(\Phi)$. By taking the limits of both sides of (3.7), and using (3.6), we get $\mathrm{P}(\mathrm{A}_{\mathrm{X},a} f_\infty - \mathrm{Q_X}\{f_j'\}) = 0$. The proof of the second part of Theorem 3.3 is almost identical, except using the contractive property of the operator $I - \mathrm{P}\,\mathrm{A_X}$ on $V^p(\Phi)$. □

4.4. **Proof of Theorem 3.4.**

Proof. Consider ϕ and φ^a as in Lemma 4.3. Assume further that $|\nabla\phi| \in W(L^1)$. Let us first estimate $\|\mathrm{osc}_\delta(\phi)\|_{W(L^1)}$ for $0 < \delta \le 1$. Note that

$$\phi(x+y) - \phi(x) = \int_0^1 y \cdot \nabla\phi(x+sy)ds.$$

Therefore

$$|\phi(x+y) - \phi(x)| \le \int_0^1 |y||\nabla\phi(x+sy)|ds \le |y| \sup_{|t|\le|y|} |\nabla\phi(x+t)|,$$

which leads to the following estimate to $\mathrm{osc}_\delta(\phi)$:

$$\begin{aligned} \mathrm{osc}_\delta(\phi)(x) &= \sup_{|y|\le\delta} |\phi(x+y) - \phi(x)| \\ &\le \sup_{|y|\le\delta} |y| \sup_{|t|\le|y|} |\nabla\phi(x+t)| \le \delta \sup_{|t|\le\delta} |\nabla\phi(x+t)|. \end{aligned}$$

Thus for every $k \in \mathbb{Z}^d$,

$$\sup_{x\in k+[0,1)^d} \mathrm{osc}_\delta(\phi)(x) \le \sup_{x\in k+[0,1)^d} \delta \sup_{|t|\le\delta} |\nabla\phi(x+t)| \le \delta \sup_{y\in k+[-1,2)^d} |\nabla\phi(y)|.$$

Hence,

$$\begin{aligned} \|\mathrm{osc}_\delta(\phi)\|_{W(L^1)} &= \sum_{k\in\mathbb{Z}^d}\sup_{x\in k+[0,1)^d}\mathrm{osc}_\delta(\phi)(x) \le \delta\sum_{k\in\mathbb{Z}^d}\sup_{y\in k+[-1,2)^d}|\nabla\phi(y)| \\ &\le 3^d\delta\sum_{k\in\mathbb{Z}^d}\sup_{y\in k+[0,1)^d}|\nabla\phi(y)| \;=\; 3^d\delta\,\||\nabla\phi|\|_{W(L^1)}. \end{aligned} \tag{4.21}$$

Next we estimate the $W(L^1)$-norm of $\varphi^a = \phi - \phi * \psi_a^*$. We follow the proof of Lemma 4.3. By (4.7) and (4.21),

$$\begin{aligned} \|I_1\|_{W(L^1)} &\le \int_{|t|\le 1} 3^d|t|\,\||\nabla\phi|\|_{W(L^1)}|\psi_a(t)|dt \\ &= 3^d\,\||\nabla\phi|\|_{W(L^1)}\int_{|t|\le 1}|t||\psi_a(t)|dt \\ &= 3^d\,\||\nabla\phi|\|_{W(L^1)}a\int_{|t|\le a^{-1}}|t||\psi(t)|dt. \end{aligned} \tag{4.22}$$

Combining (4.5), (4.6) and (4.22), we obtain the following estimate for the $W(L^1)$-norm of φ^a:

$$\begin{aligned} \|\varphi^a\|_{W(L^1)} &\le (1+2^d)\|\phi\|_{W(L^1)}\int_{|t|\ge a^{-1}}|\psi(t)|dt \\ &\quad +3^d\,\||\nabla\phi|\|_{W(L^1)}a\int_{|t|\le a^{-1}}|t||\psi(t)|dt. \end{aligned} \tag{4.23}$$

If $\|\psi\|_{1,\eta} := \int_{\mathbb{R}^d}|\psi(t)||t|^\eta dt < \infty$ for some $0<\eta\le 1$, then by (4.23)

$$\begin{aligned} \|\varphi^a\|_{W(L^1)} &\le (1+2^d)\|\phi\|_{W(L^1)}a^\eta\|\psi\|_{1,\eta} + 3^d\,\||\nabla\phi|\|_{W(L^1)}a^\eta\|\psi\|_{1,\eta} \\ &= a^\eta((1+2^d)\|\phi\|_{W(L^1)} + 3^d\,\||\nabla\phi|\|_{W(L^1)})\|\psi\|_{1,\eta}. \end{aligned} \tag{4.24}$$

The desired result in Theorem 3.4 then follows from (4.12), (4.21) and (4.24). □

4.5. **Proof of Theorem 3.5.**

Proof. The proof of Theorem 3.5 is a direct consequence of (4.19) and (4.21). □

References

[1] A. Aldroubi. Non-uniform weighted average sampling and exact reconstruction in shift invariant and wavelet spaces. *Appl. Comput. Harmon. Anal.*,13: 151–161, 2002.

[2] A. Aldroubi and H. Feichtinger. Exact iterative reconstruction algorithm for multivariate irregularly sampled functions in spline-like spaces: the L_p theory. *Proc. Amer. Math. Soc.*, 126(9):2677–2686, 1998.

[3] A. Aldroubi and K. Gröchenig. Beurling-Landau-type theorems for non-uniform sampling in shift invariant spaces. *J. Fourier Anal. Appl.*, 6(1):91–101, 2000.

[4] A. Aldroubi and K. Gröchenig. Non-uniform sampling and reconstruction in shift-invariant spaces. *SIAM Rev.*, 43(4):585-562, 2001.

[5] A. Aldroubi, Q. Sun, and W.-S. Tang. p-frames and shift invariant spaces of L^p. *J. Fourier Anal. Appl.*, 7(1):1–21, 2001.

[6] A. Aldroubi and M. Unser. Families of wavelet transforms in connection with Shannon's sampling theory and the Gabor transform. In [11], pages 509–528. Academic Press, 1992.

[7] A. Aldroubi and M. Unser. Sampling procedure in function spaces and asymptotic equivalence with Shannon's sampling theory. *Numer. Funct. Anal. and Optimiz.*, 15(1):1–21, 1994.

[8] A. Aldroubi, M. Unser and M. Eden. Cardinal spline filters: stability and convergence to the ideal sinc interpolator. *Signal Processing*, 28:127–138, 1992.

[9] J. J. Benedetto. Irregular sampling and frames. In [11], pages 445–507. 1992.

[10] W. Chen, S. Itoh, and J. Shiki. Irregular sampling theorems for wavelet subspaces. *IEEE Trans. Inform. Theory*, 44(3):1131–1142, 1998.

[11] C. K. Chui, editor. *Wavelets: A Tutorial in Theory and Applications.* Academic Press, San Diego, CA, 1992.

[12] Z. Cvetkovic and M. Vetterli. Oversampled filter banks. *IEEE Transactions on Signal Processing*, 46(5):1245–1255, 1998.

[13] I. Djokovic and P. P. Vaidyanathan. Generalized sampling theorems in multiresolution subspaces. *IEEE Trans. Signal Process.*, 45(3):583–599, 1997.

[14] H. G. Feichtinger. Generalized amalgams, with applications to Fourier transform. *Can. J. Math.*, 42(3):395–409, 1990.

[15] H. G. Feichtinger. Wiener amalgams over Euclidean spaces and some of their applications. In *Proc. Conf. Function spaces*, K. Jarosz, editor, Vol. 136 of Lect. Notes in Math., pages 123–137, Edwardsville, IL, 1991.

[16] H. G. Feichtinger and K. Gröchenig. Iterative reconstruction of multivariate band-limited functions from irregular sampling values. *SIAM J. Math. Anal.*, 231:244–261, 1992.

[17] K. Gröchenig. Reconstruction algorithms in irregular sampling. *Mathematics of Computation*, 59:181–194, 1992.

[18] A. J. E. M. Janssen. The Zak transform and sampling theorems for wavelet subspaces. *IEEE Trans. Signal Processing*, 41:3360–3364, 1993.

[19] Y. Liu. Irregular sampling for spline wavelet subspaces. *IEEE Trans. Inform. Theory*, 42:623–627, 1996.

[20] Y. Liu and G. G. Walter. Irregular sampling in wavelet subspaces. *J. Fourier Anal. Appl.*, 2(2):181–189, 1996.

[21] I. W. Selesnick. Interpolating multiwavelet and the sampling theorem. *IEEE Trans. Signal Process.*, 46(11):2898–2908, 1999.

[22] T. Strohmer. Numerical analysis of the non-uniform sampling problem. *J. Comp. Appl. Math.*, Special issue on Numerical Analysis in the 20th Century, 122:297-316, 2000.

[23] W. Sun and X. Zhou. Average sampling theorems for shift invariant subspaces. *Science in China (Series E)*, 4(5):524–530, 2000.

[24] W. Sun and X. Zhou. Reconstruction of bandlimited functions from local averages. *Constr. Approx.*, 8:205-222,2002.

[25] W. Sun and X. Zhou. Average sampling in spline subspaces. *Applied Mathematics Letters*, 15: 233-237, 2002.

[26] M. Unser and A. Aldroubi. A general sampling theory for non-ideal acquisition devices. *IEEE Trans. on Signal Processing*, 42(11):2915–2925, 1994.

[27] M. Unser, A. Aldroubi, and M. Eden. Polynomial spline signal approximations: filter design and asymptotic equivalence with Shannon's sampling theorem. *IEEETIP*, 38:95–103, 1992.

[28] P. P. Vaidyanathan and B. Vrcelj, Biorthogonal partners and applications. *IEEE Trans. Signal Processing* 49:1013-1027, 2001.

[29] D. Walnut. Nonperiodic sampling of bandlimited functions on union of rectangular. *J. Fourier Anal. Appl.*, 2(5):436–451, 1996.

[30] G. G. Walter. A sampling theorem for wavelet subspaces. *IEEE Trans. Inform. Theory*, 38(2):881–884, 1992.

[31] X. G. Xia and Z. Z. Zhang. On sampling theorem, wavelets, and wavelet transforms. *IEEE Trans. Signal Processing*, 41(12):3524–3535, 1993.

[32] A. I. Zayed. On Kramer's sampling theorem associated with general Sturm-Liouville problems and Lagrange interpolation. *SIAM J. Appl. Math.*, 51:575-604, 1991.

[33] X. Zhou and W. Sun. On the sampling theorem for wavelet subspaces. *J. Fourier Anal. Appl.*, 5(4):347–354, 1999.

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