NON-UNIFORM AVERAGE SAMPLING AND RECONSTRUCTION OF SIGNALS WITH FINITE RATE OF INNOVATION

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Abstract. From an average (ideal) sampling/reconstruction process, the question arises whether and how the original signal can be recovered from its average (ideal) samples. We consider the above question under the assumption that the original signal comes from a prototypical space modelling signals with finite rate of innovation, which includes finitely-generated shift-invariant spaces, twisted shift-invariant spaces associated with Gabor frames and Wilson bases, and spaces of polynomial splines with non-uniform knots as its special cases. We show that the displayer associated with an average (ideal) sampling/reconstruction process, that has well-localized average sampler, can be found to be well-localized. We prove that the reconstruction process associated with an average (ideal) sampling process is robust, locally behaved, and finitely implementable, and thus we conclude that the original signal can be approximately recovered from its incomplete average (ideal) samples with noise in real time. Most of our results in this paper are new even for the special case that the original signal comes from a finitely-generated shift-invariant space.

Key words. average sampling, ideal sampling, signals with finite rate of innovation, shift-invariant spaces

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1. Introduction. Modern digital data processing of functions (or signal or images) uses a discretized version of the original function that is obtained by (average) sampling on a discrete set ([2]). The classical model is the Shannon sampling and reconstruction on the band-limited space B_Ω, the space of all square-integrable functions on the real line with their Fourier transform supported in $[-\Omega, \Omega]$. From the Shannon sampling theorem, sampling a function f in B_π on the uniform grid $\mathbf{Z}$ yields an ℓ^2 sequence $(f(k))_{k \in \mathbf{Z}}$, and conversely the original function f can be recovered from its sampling data $\{f(k), k \in \mathbf{Z}\}$ by the following reconstruction formula:

$$(1.1) \qquad f(x) = \sum_{k \in \mathbf{Z}} f(k)\mathrm{sinc}(x - k), \ x \in \mathbf{R},$$

where the sinc function is defined by $\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$. The above sampling and reconstruction theorem gives a framework for converting analog signals into sequences, which can be processed digitally and converted back to analog signals via the reconstruction formula (1.1). For the ideal sampling and reconstruction on the band-limited space and the finitely-generated shift-invariant spaces, there are extensive literature (see, for example, the recent review papers [2, 56] and monographs [11, 13, 42]).

In most of physical circumstance, due to the non-ideal acquisition device at the sampling location, it is not realistic to measure the sample $f(\gamma)$ of the original signal f in a space V at the location γ exactly. So a better assumption is that the sampled data are of the form $\langle f, \psi_\gamma \rangle$,

$$(1.2) \qquad A: \ V \ni f \longmapsto (\langle f, \psi_\gamma \rangle)_{\gamma \in \Gamma},$$

where ψ_γ, to be known as the *average sampling functional*, reflects the characteristic of the nonideal acquisition device at the sampling location γ. We call the above

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sampling process as an *average sampling process*, and the collection $\Psi := \{\psi_\gamma, \gamma \in \Gamma\}$ of average sampling functionals as an *average sampler*. Clearly, the average sampling process becomes an ideal sampling process if the delta function δ_γ is used as the average sampling functional ψ_γ on every sampling location γ.

An easy model for the average sampling process is the discretization of the blurring process encountered in many practical situation, such as in the process of a remote camera imaging a scene and an observer viewing the sampled image ([58]). For the average sampling and reconstruction on the band-limited space and on the finitely-generated shift-invariant spaces, the reader may refer [1, 4, 5, 6, 21, 24, 27, 52, 53, 54, 60] and references cited therein.

The question arise from the average sampling process whether and how the original function can be recovered from its average samples. Specifically, the first part of the above question, which will be discussed in Section 4, can be described as follows: Given a class of functions V on $\mathbf{R}^d$, find conditions on the average sampler $\Psi = \{\psi_\gamma : \gamma \in \Gamma\}$ under which any function f in V can be reconstructed uniquely and stably from its average samples $\{\langle f, \psi_\gamma \rangle : \gamma \in \Gamma\}$.

The second part of the above question arisen from the average sampling process is the reconstruction process from the average (ideal) samples:

$$(1.3) \qquad D : \ (c_\gamma)_{\gamma \in \Gamma} \longmapsto \sum_{\gamma \in \Gamma} c_\gamma \tilde{\psi}_\gamma \in V$$

such that

$$(1.4) \qquad DAf = f \quad \text{for all} \ \ f \in V.$$

Here for each $\gamma \in \Gamma$, the function $\tilde{\psi}_\gamma$, to be known as the *display block* at the location γ, reflects the characteristic of the display device at the sampling location γ. We call the above reconstruction process as an *average reconstruction process* and the collection $\Psi := \{\tilde{\psi}_\gamma, \ \gamma \in \Gamma\}$ of display blocks as an *displayer*.

For the efficiency and stability of the reconstruction process (1.3) and (1.4) to recover a function f in the space V from its averaging samples $\{\langle f, \psi_\gamma \rangle, \gamma \in \Gamma\}$ or from its ideal samples $\{f(\gamma), \gamma \in \Gamma\}$, we require that the corresponding displayer $\tilde{\Psi} := \{\tilde{\psi}_\gamma, \gamma \in \Gamma\}$ is *well-localized*, and that the average sampling/reconstruction process (1.3) and (1.4) is *robust, local-behaved,* and *finitely-implementable*. In this paper, we show that those natural requirements for the average (ideal) sampling/reconstruction process would be met when signals in the space V has finite rate of innovation and the average sampler Ψ is well-localized, see Section 2.3 for our reasons to consider sampling/reconstruction of signals with finite rate of innovation. Here a signal is said to have *finite rate of innovation* if it has finite degree of freedom per unit of time, see [22, 33, 39, 41, 44, 45, 57].

The paper is organized as follows. We divide Section 2 into five subsections. In the first three subsections, we make some basic assumptions on the sampling set Γ, the average sampler $\Psi = \{\psi_\gamma, \gamma \in \Gamma\}$, and the space V where the original function f for the average sampling/reconstruction process comes from. Briefly, we assume that the sampling set Γ is a relatively-separated subset of $\mathbf{R}^d$, the average sampler Ψ is well-localized in the sense that every average sampling functional ψ_γ in the average sampler Ψ is essentially located in a neighborhood of $\gamma \in \Gamma$, and the space V is the

space $V_q(\Phi, \Lambda)$, that is originally introduced in [51] for modelling signals with finite rate of innovation. In the last two subsections, we recall some basic properties of the space $V_q(\Phi, \Lambda)$ in [51], and introduce a simplified model of our average (ideal) sampling/reconstruction process for readers' convenience.

Since each display block $\tilde{\psi}_\gamma$ in the displayer $\tilde{\Psi} := \{\tilde{\psi}_\gamma, \gamma \in \Gamma\}$ reflects the characteristic of the display device at the sampling location $\gamma, \gamma \in \Gamma$, it is reasonable to *require* that for each $\gamma \in \Gamma$, the display block $\tilde{\psi}_\gamma$ is essentially supported in a neighborhood of the sampling location γ. In Section 3, derived from a general theorem for localized frames ([8, Theorem 1], [23, Theorem 3.6] and [31, Theorem 13]), it is shown that such an requirement would be met for average (ideal) sampling in the space $V_2(\Phi, \Lambda)$ if the average sampler Ψ and the generator Φ for the space $V_2(\Phi, \Lambda)$ are well-localized (Theorems 3.1 and 3.2), see Remark 3.1 for more general formulation of the well-localization of a displayer. The well-localization of displayers will play crucial roles in our study of stable average sampling in $V_r(\Phi, \Lambda)$ with $r \neq 2$ (Corollary 3.4), the robustness and local convergence of the reconstruction process from average (ideal) samples (Theorems 5.1 – 5.3 and 6.1 – 6.3), and exponential convergence of an iterative algorithm for the reconstruction process from average (ideal) samples (Theorems 7.1 and 7.2).

In Section 4, we find conditions on the average sampling sampler $\Psi = \{\psi_\gamma : \gamma \in \Gamma\}$ (respectively, on the ideal sampling set Γ) under which any function f in $V_2(\Phi, \Lambda)$ can be reconstructed uniquely and stably from its average samples $\{\langle f, \psi_\gamma \rangle : \gamma \in \Gamma\}$ (respectively, from its ideal samples $\{f(\gamma), \gamma \in \Gamma\}$), see Theorems 4.1 and 4.2.

In the average (ideal) sampling/reconstruction process, we should bring the following situations into our consideration: the average samples $\{\langle f, \psi_\gamma \rangle : \gamma \in \Gamma\}$ may involve some noises (caused, such as, by measurement, storage, or transmission), and the average sampler Ψ may not be exactly same as the one we expect (such as, because of the mathematical modelling or the measurement of the acquisition device). In Section 5, we consider the numerical stability of the reconstruction process (1.3) and (1.4). We show that if the average sampler Ψ and the generator Φ for the space $V_2(\Phi, \Lambda)$ are well-localized, then the reconstruction process (1.3) and (1.4) for $f \in V_2(\Phi, \Lambda)$ is stable under the corruption of the average (ideal) sampling data, and the perturbation of averaging samplers, ideal sampling sets and the displayers, see Theorems 5.1, 5.2, and 5.3 for details. Then we conclude that the reconstruction process (1.3) and (1.4) for $f \in V_2(\Phi, \Lambda)$ is *robust*.

By the reconstruction process (1.3) and (1.4), any function f in the space V can be recovered fully when its average (ideal) sampling data are received completely. In some situations (such as data missing in the transmission and real-time reconstruction process), we are required to recover the original function (signal) partially from incomplete (ideal) average samples. We observe from the well-localization of the average sampler Ψ and of the ideal sampling set Γ, the average sampling data $\langle f, \psi_\gamma \rangle$ and the ideal sampling data $f(\gamma)$ catches the information of the function f essentially in a neighborhood of the sampling location γ for every $\gamma \in \Gamma$, which implies that the average (ideal) sampling procedure is locally behaved. So a natural question is whether the reconstruction procedure is locally behaved, or particularly whether a function $f \in V_2(\Phi, \Lambda)$ on a certain region K can be recovered approximately (or exactly) from the average sampling data $\langle f, \psi_\gamma \rangle$ and the ideal sampling data $f(\gamma)$ for the sampling location γ in a neighborhood of that region. In Section 6, it is proved that for any bounded region K, the original function in the space $V_2(\Phi, \Lambda)$ can be approximately recovered from its average (ideal) samples in a R-neighborhood

$B(K,R) = \{y : \inf_{x \in K} |y - x| \le R\}$ of that region K via a finite algorithm (see Theorems 6.1 and 6.2 for details), and moreover that the local convergence rate of the local reconstruction procedure is almost the same as the rate of polynomial (subexponential) decay of the generator Φ and of the average sampler Ψ. Therefore we conclude that the reconstruction process (1.3) and (1.4) for $f \in V_2(\Phi, \Lambda)$ is *locally behaved* and *finitely implementable* (hence it could lead possibly to a real-time reconstruction algorithm) when the average sampler Ψ and the generator Φ are well-localized. As a byproduct of the local reconstruction theorems, we obtain a necessary condition on the location Γ of average (ideal) sampling devices, which states that, for a stable average(ideal) sampling/reconstruction procedure on the space $V_2(\Phi, \Lambda)$, there exists a positive constant R_0 such that for any domain $K \subset \mathbf{R}^d$, the number of average (ideal) sampling devices located in R_0-neighborhood $B(K, R_0)$ of that domain K should exceed the degrees of freedom of the space $V_2(\Phi, \Lambda)$ in the domain K, see Theorem 6.3 for details. The above necessary condition, which is usually known as the density property, is established in [3] for the ideal sampling on the B-spline space (see [2, 3] and references cited therein for nonuniform sampling on the band-limited space, and [8] for non-uniform Gabor system).

In the average (ideal) sampling/reconstruction process, we need efficient and fast numerical algorithm that recover any function $f \in V_r(\Phi, \Lambda)$ from its average sampling values $\langle f, \psi_\gamma \rangle, \gamma \in \Gamma$, or from its ideal sampling values $f(\gamma), \gamma \in \Gamma$. In Section 7, we modify the standard Richardson-Landweber iterative frame algorithm to implement the reconstruction process (1.3) and (1.4) for signals $f \in V_r(\Phi, \Lambda)$ when average (ideal) samples are received completely, and show that the new iterative algorithm converges exponentially for any initial data in ℓ^r and the limit agrees with the signal in the space $V_r(\Phi, \Lambda)$ whenever the initial data is obtained from average (ideal) sampling that signal (see [2, 6, 21] for similar convergence results to the standard Richardson-Landweber iterative frame algorithm in the shift-invariant setting). The Richardson-Landweber iterative algorithm is easily to be implemented but it provides slow convergence in general. Relaxation and acceleration techniques, such as the conjugate gradient acceleration, help to alleviate the convergence problem, but their consideration is beyond the scope of this paper and will be discussed in the subsequent paper.

The proofs of all results are collected in Section 8.

In this paper, the big letter C, if unspecified, denotes an absolute constant which may be different at different occurrences.

2. Preliminaries.

2.1. The sampling set Γ. Every γ in the sampling set Γ is used as the location of a (non-)ideal sampling acquisition device, which has the average sampling characteristic ψ_γ. Then reasonable assumptions on the sampling set Γ are that only finitely many such sampling acquisition devices are located in any unit interval, and that the distribution of those devices is almost location-invariant. So in this paper, we make the following basic assumption to the sampling set Γ:

(i) *The sampling set Γ is a relatively-separated subset of $\mathbf{R}^d$.*
Here, given a subset $X = \{x_j\}$ of $\mathbf{R}^d$, we say that X is *relatively-separated* if there exists a positive constant $D(X)$ such that

$$(2.1) \qquad \sum_{x_j \in X} \chi_{x_j + [0,1]^d}(x) \le D(X) \quad \text{for all } x \in \mathbf{R}^d.$$

2.2. The average sampler Ψ. We say that a positive function u on $\mathbf{R}^d$ is a *weight* if it is continuous and symmetric, and satisfies $1 = u(0) \leq u(x) < \infty$ for all $x \in \mathbf{R}^d$, and the inequality $u(x+y) \leq u(x)v(y), x, y \in \mathbf{R}^d$, holds for another continuous function v on $\mathbf{R}^d$. The model examples of weights convenient for our considering the sampling/reconstruction process are the *polynomial weights*

$$(2.2) \qquad\qquad u_\alpha(x) = (1 + |x|)^\alpha$$

with $\alpha \geq 0$, and the *subexponential weights*

$$(2.3) \qquad\qquad e_{D,\delta}(x) = \exp(D|x|^\delta)$$

with $D > 0$ and $\delta \in (0, 1)$.

Given $1 \leq p, q \leq \infty$, a weight u, a relatively-separated subset Γ of $\mathbf{R}^d$, and a family $\Psi = \{\psi_\gamma : \gamma \in \Gamma\}$ of functions on $\mathbf{R}^d$, we define $\|\Psi\|_{q,p,u}$ by

$$\|\Psi\|_{q,p,u} := \sup_{\gamma \in \Gamma} \left\| \left(\|\psi_\gamma(\cdot)u(\cdot - \gamma)\|_{L^q(k+[0,1]^d)} \right)_{k \in \mathbf{Z}^d} \right\|_{\ell^p(\mathbf{Z}^d)}$$
$$(2.4) \qquad\qquad + \sup_{k \in \mathbf{Z}^d} \left\| \left(\|\psi_\gamma(\cdot)u(\cdot - \gamma)\|_{L^q(k+[0,1]^d)} \right)_{\gamma \in \Gamma} \right\|_{\ell^p(\Gamma)},$$

where, as usual, $\| \cdot \|_{L^q(K)}$ denotes the usual L^q norm on the space $L^q(K)$ of all q-integrable functions on a measurable set K, and $\| \cdot \|_{\ell^p(X)}$ (or $\| \cdot \|_{\ell^p}$ for short) is the usual $\ell^p(X)$ norm on the space of all q-summable sequences on the index set X. For $q = p = \infty$, it is obvious that $\|\Psi\|_{q,p,u} < \infty$ if and only if $|\psi_\gamma(x)| \leq \|\Psi\|_{q,p,u}(u(x-\gamma))^{-1}$ for all $x \in \mathbf{R}^d$ and $\gamma \in \Gamma$. In general, for the family of functions $\Psi = \{\psi_\gamma : \gamma \in \Gamma\}$ with $\|\Psi\|_{q,p,u} < \infty$, each the function $\psi_\gamma, \gamma \in \Gamma$, is an L^q function "locally" and a weighted L^p function centered at γ "globally". Therefore for each $\gamma \in \Gamma$, the function ψ_γ in the collection $\Psi := \{\psi_\gamma, \gamma \in \Gamma\}$ with $\|\Psi\|_{q,p,u} < \infty$ can be thought to be essentially supported in a neighborhood of $\gamma \in \Gamma$.

For the average sampler $\Psi = \{\psi_\gamma : \gamma \in \Gamma\}$, each average sampling functional ψ_γ reflects the characteristic of the nonideal acquisition device at the location $\gamma \in \Gamma$, and hence it should be essentially supported in a neighborhood of the sampling location γ. So we make the following basic assumption on the average sampler Ψ:

(ii) *The average sampler $\Psi = \{\psi_\gamma : \gamma \in \Gamma\}$ satisfies*

$$(2.5) \qquad\qquad \|\Psi\|_{q,p,u} < \infty$$

for some $1 \leq p, q \leq \infty$ and weight u.

We interpret any average sampler, that satisfies the basic assumption (ii), to have *polynomial (subexponential) decay*, due to the interpretation of the collection Ψ of average sampling functional $\psi_\gamma, \gamma \in \Gamma$, with $\|\Psi\|_{q,p,u} < \infty$ and the model assumption on the weight u convenient for our considering sampling/reconstruction process that u is a polynomial weight u_α or a subexponential weight $e_{D,\delta}$.

REMARK 2.1. For adapting to different average (ideal) sampling situations, we add some flexibility to the basic assumption (ii) on the average sampler Ψ with variable exponents p and q and weights u. For instance, we may use $q = 1$ for approximating ideal sampling ($\psi_\gamma \approx \delta_\gamma$, [6]), $q = 2$ for frame sampling (for instance, $\Psi = \{\phi(\cdot - k), k \in \mathbf{Z}^d\}$ for frame sampling in the shift-invariant space $V_2(\phi)$ generated by ϕ, [7, 12, 38, 51]), $q = \infty$ in local blurring or local averaging (for instance, $\psi_\gamma = h(\cdot - \gamma)$ for some compactly supported phase function h, [5, 24, 58]), and the subexponential weight $e_{D,\delta}$ for oversampling band-limited signals ([34]).

REMARK 2.2. Given $p \in [1, \infty]$, a weight u, and two relatively-separated subsets Γ, Γ' of $\mathbf{R}^d$, we define the *matrix algebra* $\mathcal{A}_{p,u}(\Gamma, \Gamma')$ *of Schur class* by

$$(2.6) \qquad \mathcal{A}_{p,u}(\Gamma, \Gamma') := \left\{ A := (A(\gamma, \gamma'))_{\gamma \in \Gamma, \gamma' \in \Gamma'} : \ \|A\|_{\mathcal{A}_{p,u}} < \infty \right\}$$

where

$$\|A\|_{\mathcal{A}_{p,u}} := \sup_{\gamma \in \Gamma} \left\| \left(A(\gamma, \gamma') u(\gamma - \gamma') \right)_{\gamma' \in \Gamma'} \right\|_{\ell^p(\Gamma')} + \sup_{\gamma' \in \Gamma'} \left\| \left(A(\gamma, \gamma') u(\gamma - \gamma') \right)_{\gamma \in \Gamma} \right\|_{\ell^p(\Gamma)},$$

(see, e.g., [31, 36, 50]). Then the basic assumption $\|\Psi\|_{q,p,u} < \infty$ on the average sampler Ψ is characterized by:

$$(2.7) \qquad \|\Psi\|_{q,p,u} < \infty \text{ if and only if } (\|\psi_\gamma\|_{L^q(k+[0,1]^d)})_{\gamma \in \Gamma, k \in \mathbf{Z}^d} \in \mathcal{A}_{p,u}(\Gamma, \mathbf{Z}^d).$$

For the basic assumption $\|\Psi\|_{q,p,u} < \infty$ with different exponents p, q and weights u, we have the following results, which will be used frequently in the proofs:

$$(2.8) \qquad\qquad\qquad \|\Psi\|_{q_1,p,u} \leq C\|\Psi\|_{q_2,p,u}$$

if $q_1 \leq q_2$, and

$$(2.9) \qquad\qquad\qquad \|\Psi\|_{q,p_1,u} \leq C\|\Psi\|_{q,p_2,v}$$

if $p_1 \leq p_2$ and $\|uv^{-1}\|_{L^r} < \infty$ where $1/r = 1/p_1 - 1/p_2$, see [51] for details.

2.3. The space V in which functions are sampled and recovered. The band-limited space $B_\Omega, \Omega > 0$, is a prototypical space for sampling theory and for signal processing in the classical band-limited model ([2]). By Whittaker representation theorem, the band-limited space B_π is spanned by the shifted sinc function $\text{sinc}(x-k) := \frac{\sin \pi(x-k)}{\pi(x-k)}, k \in \mathbf{Z}$, using ℓ^2 coefficients, i.e., $B_\Omega = \{\sum_{k \in \mathbf{Z}} c(k)\text{sinc}(x-k) : (c(k)) \in \ell^2\}$.

Since the sinc function has infinite support and slow decay at infinity, the band-limited space is often unsuitable for numerical implementations (see e.g. [2, 34]). Hence people consider other models that retain some of the *simplicity* and *structure* of the band-limited model, but are more *amenable* to numerical implementation and are more *flexible* for approximate real data (see [2, 10, 14, 35, 55] and references cited therein).

A finitely-generated shift-invariant space is such a widely-used model other than the band-limited model, see e.g. [1, 2, 4, 5, 6, 18, 27, 40, 52, 53, 54]. Here the finitely-generated shift-invariant space $V_q(\phi_1, \ldots, \phi_N)$, that has functions $\phi_1, \ldots, \phi_N$ on $\mathbf{R}^d$ as its generators, is defined by

$$V_q(\phi_1, \ldots, \phi_N) := \Big\{ \sum_{n=1}^N \sum_{k \in \mathbf{Z}^d} c_n(k)\phi_n(\cdot - k) :$$

$$(2.10) \qquad\qquad\qquad (c_n(k))_{k \in \mathbf{Z}^d} \in \ell^q, 1 \leq n \leq N \Big\},$$

where $1 \leq q \leq \infty$ (see e.g. [7, 12, 15, 20, 38] for the applications of finitely-generated shift-invariant spaces in wavelet analysis and approximation theory). Clearly the finitely-generated shift-invariant space $V_q(\phi_1, \ldots, \phi_N)$ becomes the band-limited space B_π if we let $q = 2, N = 1$ and $\phi_1 = \text{sinc}$.

The space $V_q(\Phi, \Lambda)$,

$$(2.11) \qquad V_q(\Phi, \Lambda) := \Big\{ \sum_{\lambda \in \Lambda} c(\lambda) \phi_\lambda : \ \big\|(c(\lambda))_{\lambda \in \Lambda}\big\|_{\ell^q(\Lambda)} < \infty \Big\},$$

that was recently introduced by the author in [51], is a new model other than the above band-limited model and shift-invariant model, where $1 \le q \le \infty$, Λ is a relatively-separated subset of $\mathbf{R}^d$, and $\Phi = \{\phi_\lambda, \lambda \in \Lambda\}$ satisfies $\|\Phi\|_{q,p,u} < \infty$ for some $1 \le p, q \le \infty$ and some weight u. We call Φ as the *generator* of the space $V_q(\Phi, \Lambda)$, and Λ as the *location of the generator*.

The prototypical space $V_q(\Phi, \Lambda)$ has shift-invariant spaces, twisted shift-invariant spaces generated by (non-)uniform Gabor frame system (or Wilson basis) in the time-frequency analysis (see, e.g. [8, 16, 25, 37, 46] and references cited therein), and spaces of polynomial splines (which are widely used as approximating spaces in data fitting problems and operator-equation problems [14, 35, 47]) as its special cases. Particularly the space $V_q(\Phi, \Lambda)$ and the shift-invariant space $V_q(\phi_1, \ldots, \phi_N)$ are related as follows:

$$(2.12) \qquad V_q(\Phi, \Lambda) = V_q(\phi_1, \ldots, \phi_N)$$

and

$$(2.13) \qquad \|\Phi\|_{q,p,u} < \infty \text{ if and only if } \phi_1, \ldots, \phi_N \in W_q(L_{p,u}),$$

if we let $\Lambda := \{x_1, \ldots, x_N\} + \mathbf{Z}^d$ and $\phi_\lambda := \phi_n(\cdot - k)$ if $\lambda = x_n + k$ for some $k \in \mathbf{Z}^d$ where $\{x_1, \ldots, x_N\}$ is a discrete set in $\mathbf{R}^d/\mathbf{Z}^d$, see [51] for details. Here we recall that the *Wiener amalgam space* $W_q(L_{p,u})$, which consists of functions that are "locally" in L^q and "globally" in weighted L^p space with weight u ([2]), is defined by

$$(2.14) \ W_q(L_{p,u}) := \Big\{ f : \ \|f\|_{W_q(L_{p,u})} := \big\| \big(\|fu\|_{L^q(k+[0,1]^d)} \big)_{k \in \mathbf{Z}^d} \big\|_{\ell^p(\mathbf{Z}^d)} < \infty \Big\}.$$

The prototypical space $V_q(\Phi, \Lambda)$ is suitable for modelling signals with finite rate of innovations in [22, 33, 39, 41, 44, 45, 57], for instance, (i) stream of pulses $\sum_l a_l p(t - t_l)$ found in GPS applications and cellular radio, where $p(t)$ is the antenna transmit pulse shape; (ii) stream of different pulses $\sum_l a_l p_l(t - t_l)$ found in modelling ultra wide-band, where different incoming paths are subjected to different frequency-selective attentuations; (iii) bandlimited signals with additive shot noise $\sum_{k \in \mathbf{Z}} c(k)\mathrm{sinc}(t - k) + \sum_l d(l)\delta(t - t_l)$; (iv) sum of bandlimited signals and non-uniform spline signals, convenient for modelling electrocardiogram signals.

The prototypical space $V_q(\Phi, \Lambda)$ retains some of the simplicity and structure of a finitely-generated shift-invariant space of the form (2.10), is *amenable* to numerical implementation (see Sections 5, 6 and 7), and is more *flexible* for approximating real data than the band-limited model and the shift-invariant model (see [51] for details).

So in this paper, we make the following basic assumption on the space V in which functions are sampled and recovered:

(iii) *The space V is of the form $V_q(\Phi, \Lambda)$, where $1 \le q \le \infty$, Λ is a relatively-separated subset of $\mathbf{R}^d$, and $\Phi = \{\phi_\lambda : \lambda \in \Lambda\}$ is a family of functions on $\mathbf{R}^d$ satisfying $\|\Phi\|_{q,p,u} < \infty$ for some $1 \le p \le \infty$ and weight u.*

REMARK 2.3. Signals in the space $V := V_q(\Phi, \Lambda)$, that satisfies the above basic assumption (iii), have finite rate of innovation, because a signal $f = \sum_{\lambda \in \Lambda} c(\lambda)\phi_\lambda \in V_q(\Phi, \Lambda)$ on a unit interval $t + [-1/2, 1/2]^d$ is essentially determined by the coefficients $c(\lambda)$ with $\lambda \in t + [-1/2, 1/2]^d$ because of the well-localization property of the generator

Φ, and the total number of the locations $\lambda \in \Lambda$ on the unit interval $t + [-1/2, 1/2)^d$ is bounded by some constant C_0 independent of the center t of the unit interval due to relatively-separatedness of the location Λ of the generator Φ.

REMARK 2.4. We provide some flexibility on the assumption $\|\Phi\|_{q,p,u} < \infty$ on the generator Φ of the space $V_q(\Phi, \Lambda)$ for adapting to different modelling situations. For instance, we may use $q = 1$ and $p = \infty$ when modelling slow-varying signals with shot noises ([57]), $1 \le p, q \le \infty$ when modelling signals in a finitely-generated shift-invariant space ([2]), $q = \infty$ and $1 \le p \le \infty$ for decomposing a time signal via (non-)uniform Gabor frame system or Wilson basis ([25]).

2.4. The space $V_q(\Phi, \Lambda)$ for modelling signals with finite rate of innovations. Let $\delta_{\lambda\lambda'}$ stand for the usual Kronecker symbol. For a Hilbert space H with $E = \{e_\lambda, \lambda \in \Lambda\}$ being its Riesz basis, we say that $E^d = \{e_\lambda^d : \lambda \in \Lambda\} \subset H$ is a *dual Riesz basis* of E if E^d is a Riesz basis of H and $\langle e_\lambda, e_{\lambda'}^d \rangle = \delta_{\lambda\lambda'}$ for all $\lambda, \lambda' \in \Lambda$; and we say that $E^o = \{e_\lambda^o : \lambda \in \Lambda\}$ is an *orthonormal basis* for H if E^o is a basis of H and $\langle e_\lambda^o, e_{\lambda'}^o \rangle = \delta_{\lambda\lambda'}$ for all $\lambda, \lambda' \in \Lambda$.

Let $1 \le p \le \infty$. We say that a weight u is *p-admissible* if there exist a weight v and two positive constants $D := D(u) \in (0, \infty)$ and $\theta := \theta(u) \in (0, 1)$ such that

$$(2.15) \qquad u(x + y) \le D(u(x)v(y) + v(x)u(y)) \quad \text{for all } x, y \in \mathbf{R}^d,$$

$$(2.16) \qquad \|(vu^{-1})\|_{L^{p'}} \le D, \text{ and}$$

$$(2.17) \qquad \inf_{\tau > 0} \|v\|_{L^1(B(\tau))} + t\|vu^{-1}\|_{L^{p'}(\mathbf{R}^d \setminus B(\tau))} \le Dt^\theta \quad \text{for all } t \ge 1,$$

where $p' = p/(p - 1)$ and $B(\tau) = \{x \in \mathbf{R}^d : |x| \le \tau\}$. The p-admissibility of a weight u is a technical condition in [50] to establish the Wiener lemma for matrix algebras of Schur class and of Sjöstrand class, see also Lemmas 8.1 and 8.2. It is verified in [50] that the polynomial weight u_α with $\alpha > d(1 - 1/p)$ and the subexponential weight $e_{D,\delta}$ with $D > 0$ and $\delta \in (0, 1)$ are p-admissible weights. The reader may refer those two model examples for simplification, see also subsection 2.5.

Now we recall some properties of the space $V_q(\Phi, \Lambda)$ in [51], see e.g. [2, 38] for the similar results for our familiar shift-invariant setting.

PROPOSITION 2.1. ([51]) *Let $1 \le q \le \infty$, $u_\alpha(x) := (1 + |x|)^\alpha, \alpha \ge 0$, be the polynomial weights, Λ be a relatively-separated subset of $\mathbf{R}^d$, $\Phi = \{\phi_\lambda, \lambda \in \Lambda\}$ satisfy $\|\Phi\|_{q,1,u_0} < \infty$, and $V_q(\Phi, \Lambda)$ be defined as in (2.11). Then $V_q(\Phi, \Lambda) \subset L^q$. Moreover,*

$$(2.18) \qquad \Big\|\sum_{\lambda \in \Lambda} c(\lambda)\phi_\lambda\Big\|_{L^r} \le C\big\|(c(\lambda))_{\lambda \in \Lambda}\big\|_{\ell^r(\Lambda)}\|\Phi\|_{q,1,u_0}$$

for every sequence $(c(\lambda))_{\lambda \in \Lambda} \in \ell^r(\Lambda)$ with $1 \le r \le q$, and

$$(2.19) \qquad \big\|(\langle f, \phi_\lambda \rangle)_{\lambda \in \Lambda}\big\|_{\ell^r(\Lambda)} \le C\|f\|_{L^r}\|\Phi\|_{q,1,u_0}$$

for all $f \in L^r$ with $q/(q - 1) \le r \le \infty$.

PROPOSITION 2.2. ([51]) *Let $2 \le q \le \infty$, $1 \le p \le \infty$, u be a p-admissible weight, Λ be a relatively-separated subset of $\mathbf{R}^d$, $\Phi = \{\phi_\lambda, \lambda \in \Lambda\}$ be a family of functions on $\mathbf{R}^d$, $V_2(\Phi, \Lambda)$ be as in (2.11), and the frame operator S on $V_2(\Phi, \Lambda)$ be defined by*

$$(2.20) \qquad Sf = \sum_{\lambda \in \Lambda} \langle f, \phi_\lambda \rangle \phi_\lambda, \ f \in V_2(\Phi, \Lambda).$$

Assume that Φ is a Riesz basis for $V_2(\Phi, \Lambda)$ and satisfies $\|\Phi\|_{q,p,u} < \infty$. Then $S^{-1}\Phi :=$ $\{S^{-1}\phi_\lambda : \lambda \in \Lambda\}$ is a dual Riesz basis of Φ, $S^{-1/2}\Phi := \{S^{-1/2}\phi_\lambda : \lambda \in \Lambda\}$ is an orthonormal basis of $V_2(\Phi, \Lambda)$, and $\|S^{-1}\Phi\|_{q,p,u} + \|S^{-1/2}\Phi\|_{q,p,u} < \infty$. If we further assume that Λ is a lattice, and that the generator Φ of the space $V_2(\Phi, \Lambda)$ is enveloped by a function in the Wiener amalgam space $W_\infty(L_{p,u})$ (hence $\|\Phi\|_{q,p,u} \le \|\Phi\|_{\infty,p,u} \le C\|h\|_{W_\infty(L_{p,u})} < \infty$), i.e., $|\phi_\lambda(x)| \le h(x - \lambda)$ for some $h \in W_\infty(L_{p,u})$, then so are dual Riesz basis $S^{-1}\Phi$ and orthonormal basis $S^{-1/2}\Phi$.

REMARK 2.5. Let $1 \le p, r \le \infty$, we say that a weight u is (p, r)-*admissible*, (or we say that $w(x, y) := u(x - y)$ is a (p, r)-admissible translation-invariant weight, [50]), if there exist a weight v and two positive constants $D \in (0, \infty)$ and $\theta \in (0, 1)$ such that (2.15), (2.16), and

$$(2.21) \qquad \inf_{\tau > 0} \|v\|_{L^{r'}(B(\tau))} + t\|vu^{-1}\|_{L^{p'}(\mathbf{R}^d \backslash B(\tau))} \le Dt^\theta \quad \text{for all } t \ge 1,$$

hold, where $p' = p/(p-1)$ and $r' = r/(r-1)$. Clearly the p-admissibility of a weight agrees with its (p, ∞)-admissibility. Since for any weight v it holds that $\|v\|_{L^{r'}(B(\tau))} \le C\|v\|_{L^1(B(\tau))}$ for all $\tau \ge 1$, we then conclude that p-admissibility of a weight implies its (p, r)-admissibility for any $1 \le r \le \infty$.

2.5. Model. The reader may consider the following model for simplification:
(i) The generator $\Phi := \{\phi_\lambda, \lambda \in \Lambda\}$ of the space $V_r(\Phi, \Lambda)$ is enveloped by some function g in the Wiener amalgam space $W_q(L_{p,u})$ with $2 \le q \le \infty$, i.e.,

$$(2.22) \qquad |\phi_\lambda(x)| \le g(x - \lambda) \quad \text{for all } x \in \mathbf{R}^d \text{ and } \lambda \in \Lambda.$$

(The above envelopment assumption (2.22) for the generator Φ implies that the basic assumption (iii) $\|\Phi\|_{q,p,u} < \infty$ for the generator Φ is satisfied. The converse is true in the shift-invariant setting, $V_r(\Phi, \Lambda) = V_r(\phi_1, \ldots, \phi_N)$, particularly, the above envelopment property for the generator Φ of the space $V_r(\Phi, \Lambda)$ is equivalent to the basic assumption (iii) $\|\Phi\|_{q,p,u} < \infty$, and also equivalent to the property that the generators $\phi_1, \ldots, \phi_N$ of the space $V_2(\phi_1, \ldots, \phi_N)$ belong to the Wiener amalgam space $W_q(L_{p,u})$. The envelopment assumption (2.22) for the generator Φ is not satisfied when the space $V_q(\Phi, \Lambda)$ is used for modelling slow-varying signals with shot noises, [41].)
(ii) The average sampler $\Psi = \{\psi_\gamma : \gamma \in \Gamma\}$ is enveloped by some function h in the Wiener amalgam space $W_{q^*}(L_{p,u})$ with $q/(q-1) \le q^* \le \infty$, i.e.,

$$(2.23) \qquad |\psi_\gamma(x)| \le h(x - \gamma) \quad \text{for all } x \in \mathbf{R}^d \text{ and } \gamma \in \Gamma.$$

(The above envelopment assumption (2.23) for the average sampler Ψ implies that the basic assumption (ii) $\|\Psi\|_{q^*,p,u} < \infty$ for the average sampler Ψ is satisfied. The above envelopment assumption (2.23) for the average sampler Ψ is not satisfied when it is a family of approximating delta functionals with variable width, [27].)
(iii) The weight u in the above envelopment assumptions on the generator Φ and the average sampler Ψ is the polynomial weight $u_\alpha(x) := (1 + |x|)^\alpha$ with $\alpha > d(1 - 1/p)$ or the exponential weight $e_{D,\delta}(x) := \exp(D|x|^\delta)$ with $D > 0$ and $\delta \in (0, 1)$.

REMARK 2.6. Given an exponent $p \in [1, \infty]$, a weight u, and two relatively-separated subsets Γ, Γ' of $\mathbf{R}^d$, we define the *matrix algebra $\mathcal{C}_{p,u}(\Gamma, \Gamma')$ of Sjöstrand class* by

$$(2.24) \qquad \mathcal{C}_{p,u}(\Gamma, \Gamma') := \left\{ A := (A(\gamma, \gamma'))_{\gamma, \gamma' \in \Gamma}, \ \|A\|_{\mathcal{C}_{p,u}} < \infty \right\}$$

where

$$\|A\|_{\mathcal{C}_{p,u}} := \|((Au)_*(k))_{k\in\mathbf{Z}^d}\|_{\ell^p}$$

and

$$(Au)_*(k) = \sup_{\gamma\in m+[-1/2,1/2)^d,\gamma'\in m+k+[-1/2,1/2)^d,m\in\mathbf{Z}^d} |A(\gamma,\gamma')u(\gamma-\gamma')|, \ k \in \mathbf{Z}^d$$

(see, e.g. [8, 9, 26, 48, 50]). For the case that $q = \infty, 1 \leq p \leq \infty$, and Γ is a lattice, the envelopment property (2.23) for the average sampler $\Psi := \{\psi_\gamma, \ \gamma \in \Gamma\}$ is characterized by $(\|\psi_\gamma\|_{L^q(\mu+[0,1]^d)})_{\gamma,\mu\in\Gamma} \in \mathcal{C}_{p,u}(\Gamma,\Gamma)$. Here a subset X of $\mathbf{R}^d$ is said to be a *lattice* if $1 \leq \sum_{x_j\in X} \chi_{x_j+[0,1]^d}(x) \leq D(X)$ for all $x \in \mathbf{R}^d$ and some $D(X) < \infty$.

3. Well-localized displayer. Let $1 \leq q \leq \infty$ and V be a subspace of L^q. We say that Γ, a subset of $\mathbf{R}^d$, is a *stable ideal sampling set* for the space V if there exist two positive constants A, B such that

$$(3.1) \qquad A\|f\|_{L^q} \leq \|(f(\gamma))_{\gamma\in\Gamma}\|_{\ell^q(\Gamma)} \leq B\|f\|_{L^q} \quad \text{for all } f \in V,$$

and that $\Psi = \{\psi_\gamma : \gamma \in \Gamma\}$, a family of average sampling functionals, is a *stable averaging sampler* for the space V if there exist two positive constants A', B' such that

$$(3.2) \qquad A'\|f\|_{L^q} \leq \|(\langle f,\psi_\gamma\rangle)_{\gamma\in\Gamma}\|_{\ell^q(\Gamma)} \leq B'\|f\|_{L^q} \quad \text{for all } f \in V$$

([5]). From the above definitions of a stable ideal sampling set Γ and a stable average sampler Ψ, we have that any function $f \in V$ can be reconstructed uniquely and stably from its samples $\{f(\gamma) : \gamma \in \Gamma\}$ if Γ is a stable ideal sampling set for V, and similarly that any function $f \in V$ can be reconstructed uniquely and stably from its average samples $\{\langle f,\psi_\gamma\rangle : \ \gamma \in \Gamma\}$ if Ψ is a stable averaging sampler for V.

For average (ideal) sampling on the space $V_2(\Phi,\Lambda)$, derived from a general theorem for localized frames ([8, Theorem 1], [23, Theorem 3.6] and [31, Theorem 13]), we have the following well-localization results for its displayer (Theorems 3.1 and 3.2), see Remark 3.1 for more general formulation of the well-localization of a displayer. From those results, it concludes that the displayer $\tilde{\Psi}$ associated with a stable (ideal) averaging sampler Ψ has the same polynomial (subexponential) decay when both the average sampler Ψ and the generator Φ for the space $V_2(\Phi,\Lambda)$ have polynomial (subexponential) decay.

THEOREM 3.1. *Let* $2 \leq q \leq \infty, q/(q-1) \leq q^* \leq \infty, 1 \leq r \leq q, 1 \leq p \leq \infty, u$ *is a p-admissible weight, the subsets* Λ, Γ *of* $\mathbf{R}^d$ *be relatively-separated, the generator* $\Phi = \{\phi_\lambda : \lambda \in \Lambda\}$ *satisfy*

$$(3.3) \qquad \|\Phi\|_{q,p,u} < \infty,$$

the average sampler $\Psi = \{\psi_\gamma : \gamma \in \Gamma\}$ *satisfy*

$$(3.4) \qquad \|\Psi\|_{q^*,p,u} < \infty,$$

and the space $V_r(\Phi,\Lambda)$ *be as in (2.11). Assume that* Φ *is a Riesz basis of* $V_2(\Phi,\Lambda)$ *and that* Ψ *is a stable averaging sampler for* $V_2(\Phi,\Lambda) \subset L^2$. *Then there exists a displayer* $\tilde{\Psi} = \{\tilde{\psi}_\gamma : \gamma \in \Gamma\} \subset V_1(\Phi,\Lambda)$ *such that*

$$(3.5) \qquad \|\tilde{\Psi}\|_{q,p,u} < \infty,$$

and

$$(3.6) \qquad f = \sum_{\gamma \in \Gamma} \langle f, \psi_\gamma \rangle \tilde{\psi}_\gamma \quad \text{for all } f \in V_r(\Phi, \Lambda).$$

If we further assume that Λ is a lattice and that the generator Φ and the average sampler Ψ are enveloped by some functions in the Wiener amalgam space $W_\infty(L_{p,u})$, then so is the displayer $\tilde{\Psi}$.

THEOREM 3.2. *Let $1 \le p, r \le \infty$, u be a p-admissible weight, the subsets Λ, Γ of $\mathbf{R}^d$ be relatively-separated, $\Phi = \{\phi_\lambda : \lambda \in \Lambda\}$ be a family of continuous functions on $\mathbf{R}^d$ that satisfies*

$$(3.7) \qquad \|\Phi\|_{\infty, p, u} < \infty,$$

and the space $V_r(\Phi, \Lambda)$ be as in (2.11). Assume that Φ is a Riesz basis of $V_2(\Phi, \Lambda)$, and that Γ is a stable ideal sampling set for $V_2(\Phi, \Lambda) \subset L^2$. Then there exists a displayer $\tilde{\Psi} = \{\tilde{\psi}_\gamma : \gamma \in \Gamma\}$ such that $\tilde{\Psi} \subset V_1(\Phi, \Lambda)$, $\|\tilde{\Psi}\|_{\infty, p, u} < \infty$, and

$$(3.8) \qquad f = \sum_{\gamma \in \Gamma} f(\gamma) \tilde{\psi}_\gamma \quad \text{for all } f \in V_r(\Phi, \Lambda).$$

If we further assume that Γ is a lattice and that Φ is enveloped by a function in the Wiener amalgam space $W_\infty(L_{p,u})$, then so is the displayer $\tilde{\Psi}$.

REMARK 3.1. For the well-localization of the displayer $\tilde{\Psi}$, the following general principle can be derived from theorems for localized frames ([8, Theorem 1], [23, Theorem 3.6] and [31, Theorem 13]): *Let Λ and Γ be two index sets, and let the families $\mathcal{A}(\Lambda) = \{(a(\lambda, \lambda'))_{\lambda, \lambda' \in \Lambda}\}$ and $\mathcal{A}(\Gamma, \Lambda) = \{(a(\gamma, \lambda))_{\gamma \in \Gamma, \lambda \in \Lambda}\}$ of infinite matrices have the following algebraic properties that (i) $\mathcal{A}(\Lambda)$ is an inverse-closed matrix algebra in $\mathcal{B}(\ell^2(\Lambda))$ (the space of all bounded operators on $\ell^2(\Lambda)$), (ii) $A^T A \in \mathcal{A}(\Lambda)$ for any $A \in \mathcal{A}(\Gamma, \Lambda)$, and (iii) $AB \in \mathcal{A}(\Gamma, \Lambda)$ for any $A \in \mathcal{A}(\Gamma, \Lambda)$ and $B \in \mathcal{A}(\Lambda)$. Then on the space $V_2(\Phi, \Lambda)$ whose generator $\Phi := (\phi_\lambda)_{\lambda \in \Lambda}$ is a frame, there exists a $\mathcal{A}(\Gamma, \Lambda)$-localized displayer $\tilde{\Psi} := (\tilde{\psi}_\gamma)_{\gamma \in \Gamma}$ associated with a stable average sampling processing*

$$(3.9) \qquad V_2(\Phi, \Lambda) \ni f \longmapsto \langle f, \Psi \rangle \in \ell^2(\Gamma)$$

whose average sampler $\Psi := (\psi_\gamma)_{\gamma \in \Gamma}$ is $\mathcal{A}(\Gamma, \Lambda)$-localized. Here we say that $\Psi := (\psi_\gamma)_{\gamma \in \Gamma}$ is $\mathcal{A}(\Gamma, \Lambda)$-localized on $V_2(\Phi, \Lambda)$ if $A_{\Psi, \Phi} := (\langle \psi_\gamma, \phi_\lambda \rangle)_{\gamma \in \Gamma, \lambda \in \Lambda} \in \mathcal{A}(\Gamma, \Lambda)$ and $A_{\Psi, \tilde{\Phi}} := (\langle \psi_\gamma, \tilde{\phi}_\lambda \rangle)_{\gamma \in \Gamma, \lambda \in \Lambda} \in \mathcal{A}(\Gamma, \Lambda)$ where $\tilde{\Phi} = (\tilde{\phi}_\lambda)_{\lambda \in \Lambda}$ is the dual frame generator $\tilde{\Phi}$ associated with the frame Φ. Thus the well-localization for the displayer in Theorems 3.1 and 3.2 becomes essentially a concrete example of the above general principle, (particularly in Theorems 3.1 and 3.2, the Schur class $\mathcal{A}_{p,u}(\Gamma, \Lambda)$ and the Sjöstrand class $\mathcal{C}_{p,u}(\Gamma, \Lambda)$ are used as $\mathcal{A}(\Gamma, \Lambda)$ in the above principle, and $\mathcal{A}_{p,u}(\Lambda, \Lambda)$ and $\mathcal{C}_{p,u}(\Lambda, \Lambda)$ as $\mathcal{A}(\Lambda)$.) The well-localization of the displayer in the above principle and the localization of the dual frame in the theory of frames are equivalent since for the case that $\Psi \in V_2(\Phi, \Lambda)$, (otherwise replacing Ψ by the projection $P\Psi$ of Ψ on $V_2(\Phi, \Lambda)$, see Remark 3.2 below,) the stability of the average sampling process (3.9) is equivalent to the frame property for Ψ, and the displayer $A_{\Psi, \Phi}(A_{\Psi, \Phi}^T A_{\Psi, \Phi})^\dagger \Phi$ associated with the average sampling processing (3.9) is the canonical dual frame associated with the frame Ψ, where $(A_{\Psi, \Phi}^T A_{\Psi, \Phi})^\dagger$ is the pseudo-inverse of the matrix $A_{\Psi, \Phi}^T A_{\Psi, \Phi}$. Therefore the above principle for the well-localization of a displayer is

established: (i) in [8, Theorem 1] for the case that $\Psi = \Phi$, $\Gamma = \Lambda$, $\mathcal{A}(\Lambda, \Lambda)$ and $\mathcal{A}(\Lambda)$ are the Sjöstrand class $\mathcal{C}_{1,0}(\Lambda, \Lambda)$ defined by

$$\mathcal{C}_{1,0}(\Lambda, \Lambda) = \Big\{ (c(\lambda, \lambda'))_{\lambda, \lambda' \in \Lambda} : \ \big(\sup_{a(\lambda) - a(\lambda') = g \in G} |c(\lambda, \lambda')| \big)_{g \in G} \in \ell^1(G) \Big\},$$

where G is a additive discrete group of the form $B_0 \mathbf{Z}^d \times (\mathbf{Z}^{d'}/B_1 \mathbf{Z}^{d'})$ for some non-singular diagonal matrices B_0 and B_1, and $a : \Lambda \to G$ is a map with $\sup_{g \in G} \#\{\lambda \in \Lambda : a(\lambda) = g\} < \infty$; (ii) in [23, Theorem 3.6] for the case that $\Psi = \Phi$, $\Gamma = \Lambda$, $\mathcal{A}(\Lambda, \Lambda)$ and $\mathcal{A}(\Lambda)$ are a solid, inverse-closed, involutive Banach algebra; and (iii) in [31, Theorem 13] for the case that $\Lambda = \mathbf{Z}^d$, Γ is a relatively-separated subset of $\mathbf{R}^d$, $\mathcal{A}(\Lambda) = \mathcal{A}$, and $\mathcal{A}(\Gamma, \Lambda) = \{(a(\gamma, k))_{\gamma \in \Gamma, k \in \mathbf{Z}^d} : (\tilde{a}(m, n))_{m,n \in \mathbf{Z}^d} \in \mathcal{A}\}$, where $\mathcal{A}$ is a solid, inverse-closed, involutive Banach algebra, and

$$\tilde{a}(m, n) = \begin{cases} \sup_{\gamma \in (m + [0,1)^d) \cap \Gamma} |a(\gamma, n)| & \text{if } (m + [0,1)^d) \cap \Gamma \neq \emptyset, \\ 0 & \text{if } (m + [0,1)^d) \cap \Gamma = \emptyset. \end{cases}$$

The above principle for the well-localization of the displayer can be derived from [31, Theorem 13] with identical proof. The author thanks the anonymous referee for the suggestion that leads to the above general principle for the well-localization of the displayer, and for pointing out that the results in Theorems 3.1 and 3.2 can be derived from theorems for localized frames in [8, 23, 31].

REMARK 3.2. Let $\Phi = \{\phi_\lambda : \lambda \in \Lambda\}$ and $\Psi = \{\psi_\gamma : \gamma \in \Gamma\}$ be as in either Theorem 3.1 or Theorem 3.2, and $S^{-1}\Phi = \{S^{-1}\phi_\lambda : \lambda \in \Lambda\}$ be the dual Riesz basis for the space $V_2(\Phi, \Lambda)$ in Proposition 2.2. By Propositions 2.1 and 2.2, the operator P defined by $Pf := \sum_{\lambda \in \Lambda} \langle f, S^{-1}\phi_\lambda \rangle \phi_\lambda$ is a projection operator from L^2 to $V_2(\Phi, \Lambda)$. We can extend the domain of the projection operator P so that $P\psi_\gamma$ is well defined for every sampling functional ψ_γ in the average sampling case and for the delta functional δ_γ in the ideal sampling case. Moreover, $\langle f, \psi_\gamma \rangle = \langle f, P\psi_\gamma \rangle$ for all $\gamma \in \Gamma$ and $f \in V_2(\Phi, \Lambda)$. We then have that if Ψ is a stable average sampler then $P\Psi = \{P\psi_\gamma : \gamma \in \Gamma\}$ is a frame of $V_2(\Phi, \Lambda)$, that is, there exist two positive constants $A, B > 0$ such that

$$(3.10) \qquad A\|f\|_2 \leq \|(\langle f, P\psi_\gamma \rangle)_{\gamma \in \Gamma}\|_{\ell^2(\Gamma)} \leq B\|f\|_2 \quad \text{for all } f \in V_2(\Phi, \Lambda).$$

So by the general frame theory, a displayer $\tilde{\Psi} \subset V_2(\Phi, \Lambda)$, which *may or may not* have polynomial (subexponential) decay, can be constructed, while we show in Theorems 3.1 and 3.2 that a displayer $\tilde{\Psi}$ can be constructed to has polynomial (subexponential) decay whenever the generator Φ and the average sampler Ψ have. The reader may refer [2] and references cited therein for the connection among average (ideal) sampling, reproducing kernel Hilbert space, and frame theory in the shift-invariant setting.

By Theorem 3.1 and 3.2, we have the following corollary for the uniform sampling in the familiar shift-invariant space:

COROLLARY 3.3. *Let* $1 \leq p \leq \infty, 2 \leq q \leq \infty, 1 \leq r \leq q, q/(q-1) \leq q^* \leq \infty$, u *be the polynomial weight* u_α *with* $\alpha > d(1 - 1/p)$ *or the subexponential weight* $e_{D,\delta}$ *with* $D > 0$ *and* $\delta \in (0, 1)$, *and* $\phi_1, \ldots, \phi_N \in W_q(L_{p,u})$, *and* $\Phi = \{\phi_n(\cdot - k) : 1 \leq n \leq N, k \in \mathbf{Z}^d\}$ *be a Riesz basis of* $V_2(\phi_1, \ldots, \phi_N)$. *Then we have*
   (i) If $\psi_1, \ldots, \psi_L \in W_{q^*}(L_{p,u})$, *and* $\Psi := \{\psi_l(\cdot - k) : 1 \leq l \leq L, k \in \mathbf{Z}^d\}$ *is a stable average sampler for* $V_2(\phi_1, \ldots, \phi_N)$, *then there exist functions* $\tilde{\psi}_1, \ldots, \tilde{\psi}_L \in W_q(L_{p,u})$ *such that* $f = \sum_{l=1}^{L} \sum_{k \in \mathbf{Z}^d} \langle f, \psi_l(\cdot - k) \rangle \tilde{\psi}_l(\cdot - k)$ *holds for all* $f \in V_r(\phi_1, \ldots, \phi_N)$.

(ii) *If $\phi_1, \ldots, \phi_N$ are continuous functions in $W_\infty(L_{p,u})$, and $X_0 + \mathbf{Z}^d$ is a stable ideal sampling set for $V_2(\phi_1, \ldots, \phi_N)$ where $X_0 = \{x_1, \ldots, x_L\} \subset [0,1)^d$, then there exist continuous functions $\tilde{\psi}_1, \ldots, \tilde{\psi}_L \in W_\infty(L_{p,u})$ such that $f = \sum_{l=1}^L \sum_{k \in \mathbf{Z}^d} f(x_l + k)\tilde{\psi}_l(\cdot - k)$ holds for all $f \in V_r(\phi_1, \ldots, \phi_N)$.*

REMARK 3.3. For the shift-invariant setting $V_2(\Phi, \Lambda) = V_2(\phi_1, \ldots, \phi_N)$ for some functions $\phi_1, \ldots, \phi_N$, it is shown in [27] that if the generator $\phi_1, \ldots, \phi_N$ satisfy $|\phi_n(x)| \leq C_0(1 + |x|)^{-\alpha}$ for all $1 \leq n \leq N$ and $x \in \mathbf{R}^d$, (i.e., $\|\Phi\|_{\infty,\infty,u_\alpha} < \infty$ where $\Phi := \{\phi_n(\cdot - k), \ 1 \leq n \leq N, k \in \mathbf{Z}^d\}$), and if the average sampling functional ψ_γ is supported in $\gamma + [-a, a]$ and $\|\psi_\gamma\|_{L^1} \leq C_0$ for some positive constants C_0 and a, (which implies that $\|\Psi\|_{1,\infty,u} < \infty$ for any weight u where $\Psi := \{\psi_\gamma, \ \gamma \in \Gamma\}$), then the corresponding displayer $\tilde{\Psi} := \{\tilde{\psi}_\gamma : \gamma \in \Gamma\}$ can be chosen to satisfy $|\tilde{\psi}_\gamma(x)| \leq C_1(1 + |x - \gamma|)^{-\alpha}$ for some positive constant C_1 (i.e., $\|\tilde{\Psi}\|_{\infty,\infty,u_\alpha} < \infty$). A similar result for the ideal sampling process is also established in [27]. The above results for average (ideal) sampling process follow from Theorems 3.1 and 3.2 with $p = q = \infty, q^* = 1, u = u_\alpha$ and $V_2(\Phi, \Lambda)$ being a finitely-generated shift-invariant space. Other than those exponents p, q, q^* and weight u mentioned above, the results in Theorems 3.1 and 3.2 are new even for our familiar shift-invariant setting.

By Theorems 3.1 and 3.2, we have the following result for the stability of the average sampler Ψ in the space $V_r(\Phi, \Lambda) \subset L^r$ with $r \neq 2$, which can also derived from [23, Theorem 2.7] and [27, Theorem 10].

COROLLARY 3.4. *Let $2 \leq q \leq \infty, 1 \leq p \leq \infty$, u be a p-admissible weight, the subsets Λ, Γ of $\mathbf{R}^d$ be relatively-separated. Then we have*

(i) *If Φ is a Riesz basis of $V_2(\Phi, \Lambda)$ and satisfies (3.3), and if Ψ is a stable averaging sampler for $V_2(\Phi, \Lambda) \subset L^2$ and satisfies (3.4) for some $q/(q-1) \leq q^* \leq \infty$, then the stable average sampler Ψ for $V_2(\Phi, \Lambda)$ is also a stable averaging sampler for $V_r(\Phi, \Lambda) \subset L^r$ for all $q^*/(q^* - 1) \leq r \leq q$.*

(ii) *If the family $\Phi = \{\phi_\lambda : \lambda \in \Lambda\}$ of continuous functions on $\mathbf{R}^d$ is a Riesz basis of $V_2(\Phi, \Lambda)$ and satisfies (3.7), and if Γ is a stable ideal sampling set for $V_2(\Phi, \Lambda)$, then the ideal sampling set Γ for $V_2(\Phi, \Lambda)$ is also a stable ideal sampling set for $V_r(\Phi, \Lambda) \subset L^r$ for all $1 \leq r \leq \infty$.*

REMARK 3.4. By Corollary 3.4, the stability of average (ideal) sampler for $V_2(\Phi, \Lambda) \subset L^2$ implies the stability of average (ideal) sampler for $V_r(\Phi, \Lambda) \subset L^r$ with $r \neq 2$. Such an implication can also be derived from [23, Theorem 2.7] and [27, Theorem 10] where the formulation of localized frames are used. (The author thank the anonymous referee to point out that derivation.) The above implication is observed in [21, 27] for the shift-invariant setting under a bit stronger assumptions on the average sampler Ψ, the generator Φ, and the non-uniform sampling set Γ than the ones in Corollary 3.4. As for the case that the average sampler and the generator are identical and that the grid Λ and the sampling set Γ are $\mathbf{Z}^d$, the above implication has long been known (and even the converse is also true), see, for instance, [7, 38].

4. Stability of the average (ideal) sampling/reconstruction process. For a matrix $A := (A(\lambda, \lambda'))_{\lambda \in \Lambda, \lambda \in \Lambda'}$, we define its transpose A^* by $A^* := (\overline{A(\lambda, \lambda')})_{\lambda' \in \Lambda', \lambda \in \Lambda}$. For a space $V_2(\Phi, \Lambda)$ generated by $\Phi := \{\phi_\lambda, \lambda \in \Lambda\}$, an average sampling on the space $V_2(\Phi, \Lambda)$ with the average sampler $\Psi := \{\psi_\gamma, \gamma \in \Gamma\}$, and an ideal sampling on the space $V_2(\Phi, \Lambda)$ with the sampling set Γ, we define the subspace H of ℓ^2 by

$$(4.1) \qquad H = \{(\langle f, \phi_\lambda \rangle)_{\lambda \in \Lambda} : \ f \in V_2(\Phi, \Lambda)\},$$

and two Gram matrices $A_{\Psi,\Phi}$ and $A_{\delta_\Gamma,\Phi}$ by

$$(4.2) \qquad\qquad A_{\Psi,\Phi} = \big(\langle \psi_\gamma, \phi_\lambda\rangle\big)_{\gamma\in\Gamma, \lambda\in\Lambda},$$

and

$$(4.3) \qquad\qquad A_{\delta_\Gamma,\Phi} = \big(\phi_\lambda(\gamma)\big)_{\gamma\in\Gamma, \lambda\in\Lambda}.$$

THEOREM 4.1. *Let $2 \le q \le \infty, q/(q-1) \le q^* \le \infty, q^*/(q^*-1) \le r \le q, 1 \le p \le \infty$, u be a p-admissible weight, the subsets Λ, Γ of $\mathbf{R}^d$ be relatively-separated, the generator $\Phi = \{\phi_\lambda : \lambda \in \Lambda\}$ satisfy $\|\Phi\|_{q,p,u} < \infty$, the average sampler $\Psi = \{\psi_\gamma : \gamma \in \Gamma\}$ satisfy $\|\Psi\|_{q^*,p,u} < \infty$, and the space $V_r(\Phi, \Lambda)$, the subspace H of ℓ^2 and the matrix $A_{\Psi,\Phi}$ be as in (2.11), (4.1) and (4.2) respectively. Assume that Φ is a frame of $V_2(\Phi, \Lambda)$. Then Ψ is a stable averaging sampler for $V_2(\Phi, \Lambda) \subset L^2$ if and only if there exists a positive constant C such that*

$$(4.4) \qquad C^{-1}\|c\|_{\ell^2(\Lambda)}^2 \le \langle (A_{\Psi,\Phi})^* A_{\Psi,\Phi} c, c\rangle \le C\|c\|_{\ell^2(\Lambda)}^2 \text{ for all } c \in H.$$

THEOREM 4.2. *Let $1 \le p, r \le \infty$, u be a p-admissible weight, the subsets Λ, Γ of $\mathbf{R}^d$ be relatively-separated, $\Phi = \{\phi_\lambda : \lambda \in \Lambda\}$ be a family of continuous functions on $\mathbf{R}^d$ that satisfies (3.7), and the space $V_r(\Phi, \Lambda)$, the space H and the matrix $A_{\delta_\Gamma,\Phi}$ be as in (2.11), (4.1) and (4.3) respectively. Assume that Φ is a frame of $V_2(\Phi, \Lambda)$. Then Γ is a stable ideal sampling set for $V_2(\Phi, \Lambda)$ if and only if there exists a positive constant C such that*

$$(4.5) \qquad C^{-1}\|c\|_{\ell^2(\Lambda)}^2 \le \langle (A_{\delta_\Gamma,\Phi})^* A_{\delta_\Gamma,\Phi} c, c\rangle \le C\|c\|_{\ell^2(\Lambda)}^2 \text{ for all } c \in H.$$

REMARK 4.1. For the uniform average sampling on finitely-generated shift-invariant spaces, that is, $V_2(\Phi, \Lambda) = V_2(\phi_1, \ldots, \phi_N)$ and $\Psi = \{\psi_l(\cdot - k), 1 \le l \le L, k \in \mathbf{Z}^d\}$ for some functions $\phi_n, 1 \le n \le N$, and $\psi_l, 1 \le l \le L$, the matrix $(A_{\Psi,\Phi})^* A_{\Psi,\Phi}$ in (4.4) can be written as

$$(A_{\Psi,\Phi})^* A_{\Psi,\Phi} = \Big(\sum_{l=1}^N \sum_{j\in\mathbf{Z}^d} \langle \phi_n(\cdot - k), \psi_l(\cdot - j)\rangle\langle \psi_l(\cdot - j), \phi_{n'}(\cdot - k')\rangle\Big)_{(n,k),(n',k')\in\{1,\ldots,N\}\times\mathbf{Z}^d}.$$

Due to the shift-invariant structure of the matrix $(A_{\Psi,\Phi})^* A_{\Psi,\Phi}$, we may use the Fourier technique to interpret (4.4) in Theorem 4.1 as

$$(4.6) \qquad\qquad C^{-1}G(\xi) \le A_{as}(\xi) \le CG(\xi) \text{ a.e. } \xi \in \mathbf{R}^d,$$

where the Fourier transform $\hat{f}$ of an integrable function f is defined by $\hat{f}(\xi) = \int_{\mathbf{R}^d} f(x)e^{-ix\xi}dx$, and the $N \times N$ matrices $G(\xi)$ and $A_{as}(\xi)$ are defined by

$$G(\xi) = \Big(\sum_{k\in\mathbf{Z}^d} \widehat{\phi}_n(\xi + 2k\pi)\overline{\widehat{\phi}_{n'}(\xi + 2k\pi)}\Big)_{1\le n, n'\le N}$$

and

$$A_{as}(\xi) = \sum_{l=1}^L \sum_{k,k'\in\mathbf{Z}^d} \Big(\widehat{\phi}_n(\xi + 2k\pi)\overline{\widehat{\psi}_l(\xi + 2k\pi)}\widehat{\psi}_l(\xi + 2k'\pi)\overline{\widehat{\phi}_{n'}(\xi + 2k'\pi)}\Big)_{1\le n, n'\le N}.$$

The above characterization (4.6) of the stable averaging sampler Ψ is given in [5] under weaker assumptions on the generator $\phi_1, \ldots, \phi_N$ and the average sampler $\psi_1, \ldots, \psi_L$ than the ones for the generator Φ and the average sampler Ψ in Theorem 4.1.

REMARK 4.2. For the uniform ideal sampling on a single generated shift-invariant space, that is, $V_2(\Phi, \Lambda) = V_2(\phi)$ and $\Gamma = \mathbf{Z}^d$, the matrix $(A_{\delta_\Gamma, \Phi})^* A_{\delta_\Gamma, \Phi}$ in (4.5) can be written as $(A_{\delta_\Gamma, \Phi})^* A_{\delta_\Gamma, \Phi} = (\sum_{j \in \mathbf{Z}^d} \phi(j-k)\phi(j-k'))_{k,k' \in \mathbf{Z}^d}$. Similar to the uniform average sampling case, we may use the Fourier technique to interpret (4.5) in Theorem 4.2 as follows: $C^{-1} \leq |\sum_{k \in \mathbf{Z}^d} \widehat{\phi}(\xi + 2k\pi)| \leq C$ for almost all $\xi \in \mathbf{R}^d$, which was given in [59].

REMARK 4.3. For the characterization of stable average sampler and stable ideal sampling set for various spaces, there are extensive literature (see, for instance, the recent review papers [2, 56] and monographs [11, 13, 42] for ideal sampling, and [1, 2, 4, 5, 6, 27, 52, 53, 54] for average sampling.)

5. Robustness of the reconstruction process. For the numerical stability reconstruction formula (3.6) and (3.8) when the average (ideal) sampling data and the displayer are corrupted, such as, by the noise in the measurement, we have the following result.

THEOREM 5.1. *Let* $p, q, q^*, r, u, \Lambda, \Gamma, \Phi, \Psi, \tilde{\Psi}$ *be as either in Theorem 3.1 or in Theorem 3.2. Assume that the original function* f *belongs to* $V_r(\Phi, \Lambda)$*, and that* $G = \{g_\gamma : \gamma \in \Gamma\}$ *and* $\tilde{\Psi}' = \{\tilde{\psi}'_\gamma : \gamma \in \Gamma\}$ *are the corrupted average sampling data and displayer respectively. Then there exists a positive constant* C *(independent of* f, G *and* $\tilde{\Psi}'$*) such that*

$$\Big\| f - \sum_{\gamma \in \Gamma} g_\gamma \tilde{\psi}'_\gamma \Big\|_r \leq C \|(g_\gamma - \langle f, \psi_\gamma \rangle)_{\gamma \in \Gamma}\|_{\ell^r(\Gamma)} \|\tilde{\Psi}\|_{q,p,u}$$

(5.1)
$$+ C \|\tilde{\Psi} - \tilde{\Psi}'\|_{q,p,u} \|(g_\gamma)_{\gamma \in \Gamma}\|_{\ell^r(\Gamma)}.$$

For the numerical stability of the reconstruction process (3.6) and (3.8) when there is certain perturbation for the average sampler and for the ideal sampling set, we have the following results.

THEOREM 5.2. *Let* $p, q, q^*, u, \Lambda, \Gamma, \Phi, \Psi, \tilde{\Psi}, V_q(\Phi, \Lambda)$ *be as in Theorem 3.1. Then there exist a sufficiently small positive number* δ_0 *and a positive constant* C *such that any average sampler* $\Psi' = \{\psi'_\gamma : \gamma \in \Gamma\}$ *with the property that*

(5.2)
$$\|\Psi' - \Psi\|_{q^*,p,u} \leq \delta_0$$

is a stable average sampler for the space $V_2(\Phi, \Lambda)$*, and the corresponding displayer* $\tilde{\Psi}'$ *satisfies*

(5.3)
$$\|\tilde{\Psi}' - \tilde{\Psi}\|_{q^*,p,u} \leq C \|\Psi' - \Psi\|_{q^*,p,u}.$$

THEOREM 5.3. *Let* $p, u, \Lambda, \Gamma, \Phi, \tilde{\Psi}, V_q(\Phi, \Lambda)$ *be as in Theorem 3.2. Assume that* $\Phi_\delta = \{\phi_{\gamma,\delta} : \gamma \in \Gamma\}$ *satisfies*

(5.4)
$$\|\Phi_\delta\|_{\infty,p,u} \to 0 \text{ as } \delta \to 0,$$

where $\phi_{\gamma,\delta}(x) = \sup_{|t| \leq \delta} |\phi_\gamma(x+t) - \phi_\gamma(x)|$*. Then there exist a sufficiently small positive number* δ_0 *and a positive constant* C *such that any sampling set* $\Gamma' := \{\gamma + \delta_\gamma :$

$\gamma \in \Gamma\}$ with $\sup_{\gamma \in \Gamma} |\delta_\gamma| \leq \delta_0$ is a stable ideal sampling set for the space $V_2(\Phi, \Lambda)$, and the corresponding displayer $\tilde{\Psi}'$ satisfies $\|\tilde{\Psi}' - \tilde{\Psi}\|_{\infty,p,u} \leq C\|\Phi_{\delta_0}\|_{\infty,p,u}$.

By Theorems 5.2 and 5.3, we have the following results about perturbation for non-uniform average (ideal) sampling on a finitely-generated shift-invariant space.

COROLLARY 5.4. *Let* $1 \leq p \leq \infty, 2 \leq q \leq \infty, q/(q-1) \leq q^* \leq \infty$, *u be the polynomial weight* u_α *with* $\alpha > d(1 - 1/p)$ *or the subexponential weight* $e_{D,\delta}$ *with* $D > 0$ *and* $\delta \in (0,1)$, $\phi_1, \ldots, \phi_N \in W_q(L_{p,u})$, Γ *be a relatively-separated subset of* $\mathbf{R}^d$, *and* $\Phi := \{\phi_n(\cdot - k), \ 1 \leq n \leq N, k \in \mathbf{Z}^d\}$ *be a Riesz basis of the shift-invariant space* $V_2(\phi_1, \ldots, \phi_N)$. *Then*

    (i) If $\psi_1, \ldots, \psi_L \in W_{q^*}(L_{p,u})$, *and* $\Psi := \{\psi_l(\cdot - \gamma), \ 1 \leq l \leq L, \gamma \in \Gamma\}$ *is a stable average sampler for* $V_2(\phi_1, \ldots, \phi_N)$, *then there exists a positive constant* δ_0 *such that for any functions* $\theta_1, \ldots, \theta_L$ *with* $\sum_{l=1}^L \|\psi_l - \theta_l\|_{W_{q^*}(L_{p,u})} \leq \delta_0$, *their generating average sampler* $\Theta := \{\theta_l(\cdot - \gamma), \ 1 \leq l \leq L, \gamma \in \Gamma\}$ *is a stable average sampler for* $V_2(\phi_1, \ldots, \phi_N)$.

    (ii) If $\phi_1, \ldots, \phi_N$ *are continuous functions in* $W_\infty(L_{p,u})$, *and* Γ *is a stable ideal sampling set for* $V_2(\phi_1, \ldots, \phi_N)$, *then there exists a positive constant* δ_0 *such that any relatively-separated set* $\tilde{\Gamma} = \{\gamma + \delta_\gamma : \gamma \in \Gamma\}$ *with* $\sup_{\gamma \in \Gamma} |\delta_\gamma| \leq \delta_0$ *is a stable ideal sampling set for* $V_2(\phi_1, \ldots, \phi_N)$.

REMARK 5.1. From Corollary 5.4, we see that for the shift-invariant setting, the stability of the non-uniform average (ideal) sampling is preserved under small perturbation. Such a phenomenon is observed in [18, 40, 56] for the ideal sampling process in the band-limited spaces and the finite-generated shift-invariant spaces, and in [4] for the average sampling process in the finitely-generated shift-invariant spaces. We use different approach to consider the perturbation problem than the ones in [4, 18, 40, 56] and then the stability under small perturbation is shown to be preserved under weak assumptions on the generator Φ and the average sampler Ψ, see for instance, such a preservation is established in [4] only for the case $q = \infty, q^* = 1, p = 1$ and $u = u_0$.

REMARK 5.2. Unlike in the shift-invariant setting, the stability of the ideal sampling set is not preserved under small perturbation in our general setting, or in some way the assumption (5.4) in Theorem 5.3 cannot be eliminated if we expect that the stability of the ideal sampling set is preserved under small perturbation. For instance, let $\Lambda = \mathbf{Z}$ and $\Phi := \{\phi_k(x) := h(2(x - k)) \cos^2(4k\pi(x - k)) : k \in \mathbf{Z}\}$, where $h(x) = \max(1 - |x|, 0)$ is the hat function. For the space $V_2(\Phi, \mathbf{Z})$ generated by that family of functions Φ, we see that $f(k) = c(k), k \in \mathbf{Z}$, for any $f = \sum_{k \in \mathbf{Z}} c(k)\phi_k \in V_2(\Phi, \mathbf{Z})$. Thus $\|f\|_2 \leq \|(f(k))_{k \in \mathbf{Z}}\|_{\ell^2(\mathbf{Z})} \leq 4\|f\|_2$ for all $f \in V_2(\Phi, \mathbf{Z})$, where we have also used the facts that $\phi_k, k \in \mathbf{Z}$, are supported in $k + [-1/2, 1/2]$ and satisfy $\frac{1}{4} \leq \|\phi_k\|_2 \leq 1$. This shows that $\mathbf{Z}$ is a stable sampling set for $V_2(\Phi, \mathbf{Z})$. Noting that $f(k + 1/(8k)) = 0, k \in \mathbf{Z}$, for any $f = \sum_{k \in \mathbf{Z}} c(k)\phi_k \in V_2(\Phi, \mathbf{Z})$, we then conclude that for *any* $0 < \delta < 1/2$, the small perturbation $\mathbf{Z}_\delta = \{k + (-1)^k \min(\delta, 1/(8|k|)), \ 0 \neq k \in \mathbf{Z}\}$ of the stable sampling set $\mathbf{Z}$ is *not* a stable sampling set for $V_2(\Phi, \Lambda)$. Moreover, the assumption (5.4) does not hold for that family of functions Φ, since for any $\delta > 0$,

$$\|\Phi_\delta\|_{q,p,\alpha} \geq \|\phi_k - \phi_k(\cdot + 1/(8k))\|_{L^q(k+[0,1])} \geq \frac{1}{2} \int_0^{1/4} |\cos 8k\pi x| dx \geq \frac{1}{16},$$

where the integer k is chosen so that $8k\delta \geq 1$.

6. Locally finite reconstruction process. For a bounded set K and a positive number R, we let $B(K, R) := \{y \in \mathbf{R}^d : \inf_{x \in K} |y - x| \leq R\}$ be the R-neighborhood

of the set K. For an average sampling process on the space $V_r(\Phi, \Lambda)$ with the average sampler $\Psi := \{\psi_\gamma, \gamma \in \Gamma\}$, we define the *locally finite reconstruction approximation* of a function $f \in V_r(\Phi, \Lambda)$ on a bounded set K using average sampling data on the R-neighborhood $B(K, R)$ of the set K by

$$(6.1) \qquad \tilde{f}^a_{K,R} = \sum_{\gamma \in \Gamma \cap B(K,R)} \langle f, \psi_\gamma \rangle \tilde{\psi}^a_{\gamma,K,R}$$

where

$$(6.2) \qquad \tilde{\psi}^a_{\gamma,K,R} = \sum_{\lambda_1, \lambda_2 \in B(K,2R)} \langle \psi_\gamma, \phi_{\lambda_1} \rangle (\tilde{A}_{\Psi,\Phi,K,R})^{-1}(\lambda_1, \lambda_2) \phi_{\lambda_2}$$

and

$$(6.3) \qquad \tilde{A}_{\Psi,\Phi,K,R} = \Big(\sum_{\gamma \in \Gamma \cap B(K,4R)} \langle \phi_\lambda, \psi_\gamma \rangle \langle \psi_\gamma, \phi_{\lambda'} \rangle \Big)_{\lambda,\lambda' \in B(K,3R)}.$$

Similarly for the ideal sampling process on the space $V_r(\Phi, \Lambda)$ with the sampling set Γ, we define the locally finite reconstruction approximation of a function $f \in V_r(\Phi, \Lambda)$ on a bounded set K using the ideal sampling data on the R-neighborhood $B(K, R)$ of the set K by

$$(6.4) \qquad \tilde{f}^i_{K,R} = \sum_{\gamma \in \Gamma \cap B(K,R)} f(\gamma) \tilde{\psi}^i_{\gamma,K,R}$$

where

$$(6.5) \qquad \tilde{\psi}^i_{\gamma,K,R} = \sum_{\lambda_1, \lambda_2 \in B(K,2R)} \phi_{\lambda_1}(\gamma) (\tilde{A}_{\delta_\Gamma,\Phi,K,R})^{-1}(\lambda_1, \lambda_2) \phi_{\lambda_2}$$

and

$$(6.6) \qquad \tilde{A}_{\Psi,\Phi,K,R} = \Big(\sum_{\gamma \in \Gamma \cap B(K,4R)} \phi_\lambda(\gamma) \phi_{\lambda'}(\gamma) \Big)_{\lambda,\lambda' \in B(K,3R)}.$$

For any bounded set K, we observe that the locally finite reconstruction approximation $\tilde{f}^a_{K,R}$ for the average sampling/reconstruction process and $\tilde{f}^i_{K,R}$ for the ideal sampling/reconstruction process are obtained by using the samples in a finite neighborhood of that set K with finitely many steps. Then we conclude from Theorems 6.1 and 6.2 that the locally finite reconstruction approximation could be possibly used in the real-time reconstruction by selecting the parameter R properly.

Using the similar idea to the finite section method in frame theory (see e.g. [17, 19, 34]), we have the following locally finite reconstruction approximation for the average (ideal) sampling/reconstruction process:

THEOREM 6.1. *Let $p, q, q^*, r, u, \Lambda, \Gamma, \Phi, \Psi, \tilde{\Psi}$ be as in Theorem 3.1, and $\tilde{f}^a_{K,R}$ be defined as in (6.1) for any bounded set K, positive number $R \geq 1$ and function $f \in V_r(\Phi, \Lambda)$. Then there exists a positive constant C (independent of the bounded set K, the positive number $R \geq 1$, and the function $f \in V_r(\Phi, \Lambda)$) such that*

$$\|\tilde{f}^a_{K,R} - f\|_{L^r(K)} \leq C\|u^{-1}\|^2_{L^{p'}(\mathbf{R}^d \setminus B(R))} \|(\langle f, \psi_\gamma \rangle)_{\gamma \in \Gamma}\|_{\ell^r(\Gamma)}$$

$$(6.7) \qquad\qquad + C\|u^{-1}\|_{L^{p'}(\mathbf{R}^d \setminus B(R))} \|(\langle f, \psi_\gamma \rangle)_{\gamma \in \Gamma \setminus B(K,R)}\|_{\ell^r(\Gamma \setminus B(K,R))}$$

holds for any bounded set K, any positive number $R \geq 1$, and any $f \in V_r(\Phi, \Lambda)$.

THEOREM 6.2. *Let* $p, r, u, \Lambda, \Gamma, \Phi, \tilde{\Psi}$ *be as in Theorem 3.2, and* $\tilde{f}^i_{K,R}$ *be defined as in (6.4) for any bounded set* K*, positive number* $R \geq 1$ *and function* $f \in V_r(\Phi, \Lambda)$*. Then there exists a positive constant* C *(independent of the bounded set* K*, the positive number* $R \geq 1$ *and the function* $f \in V_r(\Phi, \Lambda)$*) such that*

$$\|\tilde{f}^i_{K,R} - f\|_{L^r(K)} \leq C\|u^{-1}\|^2_{L^{p'}(\mathbf{R}^d \setminus B(R))}\|(f(\gamma))_{\gamma \in \Gamma}\|_{\ell^r(\Gamma)}$$
$$(6.8) \qquad\qquad + C\|u^{-1}\|_{L^{p'}(\mathbf{R}^d \setminus B(R))}\|(f(\gamma))_{\gamma \in \Gamma \setminus B(K,R)}\|_{\ell^r(\Gamma \setminus B(K,R))}$$

holds for any bounded set K*, any positive number* $R \geq 1$*, and any* $f \in V_r(\Phi, \Lambda)$*.*

As a byproduct of Theorems 6.1 and 6.2, we have the following result about the location of (non-)ideal sampling devices.

THEOREM 6.3. *Let* $p, q, q^*, r, u, \Lambda, \Gamma, \Phi, \Psi, \tilde{\Psi}$ *be as in either Theorem 3.1 or Theorem 3.2. Then there exists a positive constant* R_0 *such that for any bounded set* K*, the number of (non-)ideal sampling devices located in* $B(K, R_0)$ *exceeds the degrees of freedom of the space* $V_2(\Phi, \Lambda)$ *in the domain* K*, that is,*

$$(6.9) \qquad\qquad \#(\Gamma \cap B(K, R_0)) \geq \#(\Lambda \cap K)$$

where $\#(E)$ *denotes the cardinality of a set* E*.*

REMARK 6.1. If $V_2(\Phi, \Lambda)$ is a shift-invariant space generated by a compactly supported continuous function ϕ, then one may verify that the matrix $\left(\sum_{\gamma \in \Gamma} \phi_\lambda(\gamma) \phi_{\lambda'}(\gamma)\right)$ associated with the ideal sampling on Γ is a band-limited matrix. Furthermore for the case that finite truncation of that matrix is invertible, (which is true if ϕ is a B-spline and Γ is a sampling set with sampling density strictly less than the optimal density, see [3, 4]), Gröchenig and Schwab ([32]) proposed an efficient local reconstruction algorithm to recover the original function in a domain *exactly*, instead of *approximately* in Theorems 6.1 and 6.2, from its samples in a neighborhood of that domain. Comparing with the locally perfect recovery in [32], we see that the locally finite approximation in Theorems 6.1 and 6.2 works for well-localized average sampling process as well as ideal sampling process, and for most signals with finite rate of innovation instead of signals of B-spline type.

REMARK 6.2. The density property (6.9) in Corollary 6.3 is established in [3] for the ideal sampling of signals in a B-spline space, that is, $V_2(\Phi, \Lambda)$ is the shift-invariant space generated by the integer shifts of a B-spline. The readers may refer [2] for similar results to average (ideal) sampling in the band-limited space, and [8] to the non-uniform Gabor system.

7. The Richardson-Landweber iterative reconstruction process. Let Φ be the generator of the space $V_2(\Phi, \Lambda)$, Ψ be an average sampler, and Γ be an ideal sampling set. We define an iterative reconstruction algorithm from average sampling data $(a_\gamma)_{\gamma \in \Gamma} \in \ell^r$ by:

$$(7.1) \qquad \begin{cases} f_0 = A^{-2} \sum_{\gamma \in \Gamma, \lambda \in \Lambda} a_\gamma \langle \psi_\gamma, \phi_\lambda \rangle \phi_\lambda, \\ f_n = f_0 + f_{n-1} - A^{-2} T_{as} f_{n-1} \text{ if } n \geq 1, \end{cases}$$

where A is a positive parameter and the operator T_{as} is defined by

$$(7.2) \qquad\qquad T_{as}f = \sum_{\gamma \in \Gamma, \lambda \in \Lambda} \langle f, \psi_\gamma \rangle \langle \psi_\gamma, \phi_\lambda \rangle \phi_\lambda.$$

Similarly we define an iterative reconstruction algorithm from ideal sampling data $(a_\gamma)_{\gamma \in \Gamma} \in \ell^r$ by:

$$(7.3) \qquad \begin{cases} f_0 = A^{-2} \sum_{\gamma \in \Gamma, \lambda \in \Lambda} a_\gamma \phi_\lambda(\gamma) \phi_\lambda \\ f_n = f_0 + f_{n-1} - A^{-2} T_{is} f_{n-1}, \ n \geq 1, \end{cases}$$

where A is a positive parameter and the operator T_{is} is defined by

$$(7.4) \qquad T_{is} f = \sum_{\gamma \in \Gamma, \lambda \in \Lambda} f(\gamma) \phi_\lambda(\gamma) \phi_\lambda.$$

Since the operators A_{as} in (7.2) and A_{is} in (7.4) can be written as $T_{as} = T_\Phi A_{\Psi,\Phi}^* A_{\Psi,\Phi} (T_\Phi)^{-1}$, and $T_{is} = T_\Phi A_{\delta_\Gamma,\Phi}^* A_{\delta_\Gamma,\Phi} (T_\Phi)^{-1}$, where $T_\Phi : (c(\lambda)) \longmapsto \sum_{\lambda \in \Lambda} c(\lambda) \phi_\lambda$, we then have that the iterative reconstruction algorithms from average sampling data and from ideal sampling data are equivalent to the familiar Richardson-Landweber iterative algorithms for the positive operators $A_{\Psi,\Phi}^* A_{\Psi,\Phi}$ and $A_{\delta_\Gamma,\Phi}^* A_{\delta_\Gamma,\Phi}$ on $\ell^2(\Lambda)$ respectively. For the iterative reconstruction algorithms from average sampling data and from ideal sampling data, we have the following exponential convergence for signals in $V_r(\Phi, \Lambda)$.

THEOREM 7.1. *Let $p, q, q^*, r, u, \Lambda, \Gamma, \Phi, \Psi, V_r(\Phi, \Lambda)$ be as in Theorem 3.1. We assume that the parameter A in (7.1) is a positive constant larger than the operator norm of the matric $A_{\Psi,\Phi}$ from $\ell^2(\Lambda)$ to $\ell^2(\Gamma)$. Then the sequence $\{f_n, n \geq 1\}$ in (7.1) converges to a function f_∞ in $V_r(\Phi, \Lambda)$ in L^r norm exponentially for any initial data $(a_\gamma)_{\gamma \in \Gamma} \in \ell^r$, that is, there exist two positive constants $C \in (0, \infty)$ and $s \in (0, 1)$ such that*

$$(7.5) \qquad \|f_n - f_\infty\|_r \leq Cs^n \|(a_\gamma)_{\gamma \in \Gamma}\|_{\ell^r(\Gamma)} \text{ for all } n \geq 1.$$

Furthermore if the initial data $(a_\gamma)_{\gamma \in \Gamma}$ are obtained from average sampling a function f in $V_r(\Phi, \Lambda)$, that is, $a_\gamma = \langle f, \psi_\gamma \rangle$ for all $\gamma \in \Gamma$, then the limit function f_∞ of the sequence $\{f_n, n \geq 1\}$ agrees with the original function f.

THEOREM 7.2. *Let $p, r, u, \Lambda, \Gamma, \Phi, V_q(\Phi, \Lambda)$ be as in Theorem 3.2. Assume that the parameter A in (7.3) is a positive constant larger than the operator norm of the matrix $A_{\delta_\Gamma,\Phi} := (\phi_\lambda(\gamma))_{\gamma \in \Gamma, \lambda \in \Lambda}$ from $\ell^2(\Lambda)$ to $\ell^2(\Gamma)$. Then the sequence $\{f_n, n \geq 1\}$ in (7.3) converges to a function $f_\infty \in V_r(\Phi, \Lambda)$ in L^r norm exponentially. Moreover if $a_\gamma = f(\gamma), \gamma \in \Gamma$, holds for some $f \in V_r(\Phi, \Lambda)$, then $f_\infty = f$.*

REMARK 7.1. For $\Phi = \{\phi_\lambda : \lambda \in \Lambda\}$, we let P be the projection operator from L^2 to $V_2(\Phi, \Lambda)$, see Remark 3.2. Then for the average sampling/reconstuction process with Ψ as its average sampler (resp. the ideal sampling/reconstruction process with Γ as its sampling set), $P\Psi := \{P\psi_\gamma, \ \psi_\gamma \in \Psi\}$ (resp. $P\delta_\Gamma := \{P\delta_\gamma, \ \gamma \in \Gamma\}$) is a frame for $V_2(\Phi, \Lambda)$, and hence the corresponding frame algorithm is the familiar Richardson-Landweber iterative algorithm for the positive operator $(A_{\Phi,\Phi})^{-1/2} A_{\Psi,\Phi}^* A_{\Psi,\Phi} (A_{\Phi,\Phi})^{-1/2}$ (resp. $(A_{\Phi,\Phi})^{-1/2} A_{\delta_\Gamma,\Phi}^* A_{\delta_\Gamma,\Phi} (A_{\Phi,\Phi})^{-1/2}$) on the space $\ell^2(\Lambda)$, see [2]. Clearly the iterative frame algorithm becomes the iterative algorithm proposed in the paper when Φ is an orthonormal basis of $V_2(\Phi, \Lambda)$. In general, we need more computation for each iterative step of the iterative frame algorithm than the one of the iterative algorithm in the paper. The consideration which iterative algorithm converges faster and other implementation of the reconstruction process associated with average (ideal) sampling/reconstruction process are beyond the scope of this paper and will be discussed in the subsequent paper. By the general frame theory, the iterative frame algorithm associated with the average (ideal) sampling/reconstruction process on the

space $V_2(\Phi, \Lambda) \subset L^2$ converges exponentially. Similar to the proofs of Theorems 7.1 and 7.2, we have the exponential convergence of the iterative frame algorithm on the space $V_r(\Phi, \Lambda)$ with $r \neq 2$, under the assumption that $p, q, q^*, r, u, \Lambda, \Lambda, \Phi, \Psi$ are as in Theorems 3.1 or 3.2. The above exponential convergence theorem for iterative frame algorithm is established in [2, 21] for the shift-invariant setting with some minor additional assumptions on the exponents p, q, q^* and weight u.

8. Proofs. In this section, we collect the proofs of all theorems and corollaries stated in Sections 3, 4, 5, 6, and 7.

8.1. Proof of Theorem 3.1. Unlike in the study of average (ideal) sampling signals in a shift-invariant space, the main obstacle to consider well-localization of the displayer comes from the non-group structure on the generator Φ and the average sampler Ψ, which makes the standard approach from Fourier analysis inapplicable. In the proof of Theorem 3.1, we will use the procedure used in the study of localized frames (see [8, 23, 30, 31, 36, 49, 50, 51] and references cited therein) with some nonessential modification (see also Remark 3.1). For the completeness of this paper, we include a complete proof.

For a matrix $A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in \Lambda}$, we denote by $\|A\|_{\mathcal{B}^2}$ its operator norm on $\ell^2(\Lambda)$. To prove Theorem 3.1, we recall some properties of the matrix algebras of the Schur class $\mathcal{A}_{p,u}(\Lambda, \Lambda')$ and of the Sjöstrand class $\mathcal{C}_{p,u}(\Lambda, \Lambda)$ in [50]. The third property in Lemmas 8.1 and 8.2 below is usually known as the Wiener's lemma, see, for instance, [8, 9, 23, 26, 30, 31, 36, 48, 49, 50] and references cited therein for its recent development and various applications.

LEMMA 8.1. ([50]) *Let* $1 \leq p \leq \tilde{p} \leq \infty$, u *and* $\tilde{u}$ *are weights, and* $\Lambda, \Lambda', \Lambda''$ *are relatively-separated subsets of* $\mathbf{R}^d$. *Then the following statements are true.*

(i) If $\|u\tilde{u}^{-1}\|_{L^r} < \infty$ *where* $r = p\tilde{p}/(\tilde{p} - p)$, *then*

$$(8.1) \qquad \|A\|_{\mathcal{A}_{p,u}} \leq C\|A\|_{\mathcal{A}_{\tilde{p},\tilde{u}}} \text{ for all } A \in \mathcal{A}_{\tilde{p},\tilde{u}}(\Lambda, \Lambda').$$

(ii) If there exists another weight v *such that (2.15) and (2.16) hold, then*

$$(8.2) \qquad \|AB\|_{\mathcal{A}_{p,u}} \leq C\|A\|_{\mathcal{A}_{p,u}}\|B\|_{\mathcal{A}_{p,u}}$$

for all $A \in \mathcal{A}_{p,u}(\Lambda, \Lambda')$ *and* $B \in \mathcal{A}_{p,w}(\Lambda', \Lambda'')$, *where*

$$AB := \Big(\sum_{\lambda' \in \Lambda'} A(\lambda, \lambda')B(\lambda', \lambda'') \Big)_{\lambda \in \Lambda, \lambda'' \in \Lambda''}.$$

(iii) If u *is a* p-*admissible weight and* A *is a matrix in* $\mathcal{A}_{p,u}(\Lambda, \Lambda)$ *satisfying*

$$(8.3) \qquad \|Ac\|_{\ell^2} \geq D_0\|c\|_{\ell^2} \text{ for all } c \in \ell^2$$

for some positive constant D_0, *then the inverse* A^{-1} *of the matrix* A *belongs to* $\mathcal{A}_{p,u}(\Lambda, \Lambda)$.

(iv) If u *is a* p-*admissible weight, then there exist positive constants* $C_1, C_2 \in (0, \infty)$ *and* $\theta \in (0, 1)$ *such that the following estimate hold for all* $A \in \mathcal{A}_{p,u}(\Lambda, \Lambda)$:

$$(8.4) \qquad \|A^n\|_{\mathcal{A}_{p,u}} \leq \left(C_1 \frac{\|A\|_{\mathcal{A}_{p,u}}}{\|A\|_{\mathcal{B}^2}} \right)^{C_2 n^\theta} \|A\|_{\mathcal{B}^2}^n, \ n \geq 1.$$

LEMMA 8.2. ([50]) *Let* $1 \leq p \leq \tilde{p} \leq \infty$, *$u$ and $\tilde{u}$ are weights on $\mathbf{R}^d$, and Λ is a lattice of $\mathbf{R}^d$. Then the following statements are true.*
 (i) *If* $\|u\tilde{u}^{-1}\|_{L^r} < \infty$ *where* $r = p\tilde{p}/(\tilde{p} - p)$, *then*

$$(8.5) \qquad \|A\|_{\mathcal{C}_{p,u}} \leq C\|A\|_{\mathcal{C}_{\tilde{p},\tilde{u}}} \text{ for all } A \in \mathcal{C}_{\tilde{p},\tilde{u}}(\Lambda, \Lambda).$$

 (ii) *If there exists another weight v such that (2.15) and (2.16) hold, then*

$$(8.6) \qquad \|AB\|_{\mathcal{C}_{p,u}} \leq C\|A\|_{\mathcal{C}_{p,u}}\|B\|_{\mathcal{C}_{p,u}} \quad \text{for all } A, B \in \mathcal{C}_{p,u}(\Lambda, \Lambda).$$

 (iii) *If u is a p-admissible weight on $\mathbf{R}^d$ and if A is a matrix in $\mathcal{C}_{p,u}(\Lambda, \Lambda)$ that satisfies (8.3), then the inverse A^{-1} of the matrix A belong to $\mathcal{C}_{p,u}(\Lambda, \Lambda)$.*

Now we are ready to start the proof of Theorem 3.1.

Proof of Theorem 3.1. By Proposition 2.2, without loss of generality, we may assume that Φ is an orthonormal basis of $V_2(\Phi, \Lambda)$ for otherwise replacing Φ by $S^{-1/2}\Phi$. Therefore

$$(8.7) \qquad \|c\|_{\ell^2(\Lambda)} = \Big\|\sum_{\lambda \in \Lambda} c(\lambda)\phi_\lambda\Big\|_2 \text{ for all } c := \big(c(\lambda)\big)_{\lambda \in \Lambda} \in \ell^2(\Lambda),$$

and

$$(8.8) \qquad f = \sum_{\lambda \in \Lambda}\langle f, \phi_\lambda\rangle\phi_\lambda \text{ for all } f \in V_2(\Phi, \Lambda).$$

From (2.8), (3.3) and (3.4), it follows that $\|\Psi\|_{q/(q-1),p,u} + \|\Phi\|_{q,p,u} < \infty$. This, together with (2.7), Lemma 8.1, and the following trivial estimate,

$$|A_{\Psi,\Phi}(\gamma, \lambda)| \leq \sum_{k \in \mathbf{Z}^d} \|\psi_\gamma\|_{L^{q/(q-1)}(k+[0,1]^d)}\|\phi_\lambda\|_{L^q(k+[0,1)^d)},$$

proves that the matrix $A_{\Psi,\Phi} = (A_{\Psi,\Phi}(\gamma, \lambda))_{\gamma \in \Gamma, \lambda \in \Lambda}$ in (4.2) belongs to $\mathcal{A}_{p,u}(\Gamma, \Lambda)$,

$$(8.9) \qquad A_{\Psi,\Phi} \in \mathcal{A}_{p,u}(\Gamma, \Lambda).$$

Furthermore, there exists a positive constant C such that

$$(8.10) \qquad \|A_{\Psi,\Phi}\|_{\mathcal{A}_{p,u}} \leq C\|\Psi\|_{q/(q-1),p,u}\|\Phi\|_{q,p,u}.$$

Clearly the transpose A^* of a matrix $A \in \mathcal{A}_{p,u}$ has the same $\mathcal{A}_{p,u}$ norm as the one of the matrix A,

$$(8.11) \qquad \|A^*\|_{\mathcal{A}_{p,u}} = \|A\|_{\mathcal{A}_{p,u}}.$$

Combining (8.2), (8.9) and (8.11) then yields

$$(8.12) \qquad A_{\Psi,\Phi}^* A_{\Psi,\Phi} \in \mathcal{A}_{p,u}(\Lambda, \Lambda).$$

For the matrix $A_{\Psi,\Phi}$, we obtain from (8.7) and the stable assumption on the averaging sampler Ψ that

$$(8.13) \qquad C^{-1}\|c\|_2 \leq \Big\|\Big(\sum_{\lambda \in \Lambda} A_{\Psi,\Phi}(\gamma, \lambda)c(\lambda)\Big)_{\gamma \in \Gamma}\Big\|_{\ell^2(\Gamma)} \leq C\|c\|_2$$

for any $\ell^2(\Lambda)$-sequence $c = (c(\lambda))_{\lambda \in \Lambda}$.

Combining (8.2), (8.12) and (8.13), and applying Lemma 8.1 to the matrix $A_{\Psi,\Phi}^* A_{\Psi,\Phi}$, we conclude that

$$(8.14) \qquad\qquad (A_{\Psi,\Phi}^* A_{\Psi,\Phi})^{-1} \in \mathcal{A}_{p,u}(\Lambda, \Lambda).$$

Let

$$(8.15) \qquad\qquad R := A_{\Psi,\Phi}(A_{\Psi,\Phi}^* A_{\Psi,\Phi})^{-1},$$

write $R = (R(\gamma, \lambda))_{\gamma \in \Gamma, \lambda \in \Lambda}$, and define $\tilde{\Psi} := \{\tilde{\psi}_\gamma : \gamma \in \Gamma\}$ by

$$(8.16) \qquad\qquad \tilde{\psi}_\gamma = \sum_{\lambda \in \Lambda} R(\gamma, \lambda)\phi_\lambda, \ \gamma \in \Gamma.$$

Now we prove that $\tilde{\Psi}$ satisfies all requirements of the displayer associated with the average sampler Ψ. By (8.2), (8.9) and (8.14), we have

$$(8.17) \qquad\qquad R \in \mathcal{A}_{p,u}(\Gamma, \Lambda).$$

This implies that the sequence $(R(\gamma, \lambda))_{\lambda \in \Lambda} \in \ell^1(\Lambda)$ for any $\gamma \in \Gamma$, and hence $\tilde{\Psi} \subset V_1(\Phi, \Lambda)$.

From (2.7), (8.2), (8.17), and the trivial estimate for $\tilde{\Psi}$: $\|\tilde{\psi}_\gamma\|_{L^q(k+[0,1]^d)} \le \sum_{\lambda \in \Lambda} |R(\gamma, \lambda)| \|\psi_\lambda\|_{L^q(k+[0,1]^d)}$, we have

$$(8.18) \qquad\qquad \|\tilde{\Psi}\|_{q,p,u} \le C\|R\|_{\mathcal{A}_{p,u}} \|\Phi\|_{q,p,u} < \infty.$$

For any $f \in V_2(\Phi, \lambda)$, it follows from (8.8), (8.15) and (8.17) that

$$\left(\left\langle \sum_{\gamma \in \Gamma} \langle f, \psi_\gamma \rangle \tilde{\psi}_\gamma, \phi_\lambda \right\rangle \right)_{\lambda \in \Lambda}$$

$$= \left(\sum_{\lambda_1 \in \Lambda} \langle f, \phi_{\lambda_1} \rangle \times \sum_{\lambda_2 \in \Lambda} \langle \phi_{\lambda_2}, \phi_\lambda \rangle \left(\sum_{\gamma \in \Gamma} A_{\Psi,\Phi}^*(\lambda_1, \gamma) R(\gamma, \lambda_2) \right) \right)_{\lambda \in \Lambda}$$

$$(8.19) \qquad = \left(\sum_{\lambda_1 \in \Lambda} \langle f, \phi_{\lambda_1} \rangle \langle \phi_{\lambda_1}, \phi_\lambda \rangle \right)_{\lambda \in \Lambda} = \left(\langle f, \phi_\lambda \rangle \right)_{\lambda \in \Lambda}.$$

This proves the reconstruction formula (3.6) for $r = 2$,

$$(8.20) \qquad\qquad f = \sum_{\gamma \in \Gamma} \langle f, \psi_\gamma \rangle \tilde{\psi}_\gamma \quad \text{for any } f \in V_2(\Phi, \Lambda).$$

For $1 \le r < \infty$, we obtain from (2.8), (2.9), (8.2), (8.9), (8.17), and Proposition 2.1 that

$$\left\| \sum_{\gamma \in \Gamma} \sum_{\lambda, \lambda' \in \Lambda} |c(\lambda)| \times |\langle \phi_\lambda, \psi_\gamma \rangle| \times |R(\gamma, \lambda')| \times |\phi_{\lambda'}| \right\|_{L^r}$$

$$\le C\|c\|_{\ell^r(\Lambda)} \|A_{\Psi,\Phi}\|_{\mathcal{A}_{p,u}} \|R\|_{\mathcal{A}_{p,u}} \|\Phi\|_{r,p,u}$$

$$(8.21) \qquad \le C\|c\|_{\ell^r(\Lambda)} \|A_{\Psi,\Phi}\|_{\mathcal{A}_{p,u}} \|R\|_{\mathcal{A}_{p,u}} \|\Phi\|_{q,p,u} < \infty$$

for any sequence $c := (c(\lambda))_{\lambda \in \Lambda} \in \ell^r(\Lambda)$. Then for $1 \le r < \infty$, the reconstruction formula (3.6) follows easily from (8.20), (8.21), and the density of $\ell^2 \cap \ell^r$ in ℓ^r.

For $r = \infty$, we have that $p = \infty$. Take $c = (c(\lambda))_{\lambda \in \Lambda} \in \ell^\infty(\Lambda)$. We let $f = \sum_{\lambda \in \Lambda} c(\lambda) \phi_\lambda$ and $f_{k,\tau} = \sum_{|\lambda - k| \le \tau} c(\lambda) \phi_\lambda$ for $\tau \ge 1$. Then there exists a positive constant C (independent of $k \in \mathbf{Z}^d$ and $\tau \ge 1$) such that

$$\sup_{x \in k + [0,1]^d} |f(x) - f_{k,\tau}(x)| \le \sum_{|\gamma - k| > \tau} |c(\lambda)| \sup_{x \in k + [0,1]^d} |\phi_\lambda(x)|$$

$$\le C \|c\|_{\ell^\infty(\Lambda)} \|\Phi\|_{\infty, p, u} \|u^{-1}\|_{L^{p'}(\mathbf{R}^d \setminus B(\tau))},$$

and

$$\sup_{x \in k + [0,1]^d} \left| \sum_{\gamma \in \Gamma} \langle f - f_{k,\tau}, \psi_\gamma \rangle \tilde{\psi}_\gamma(x) \right|$$

$$\le \|c\|_{\ell^\infty} \left(\sum_{|\gamma - k| \ge \tau/2, \gamma \in \Gamma} \sum_{\lambda \in \Lambda} + \sum_{\gamma \in \Gamma} \sum_{|\lambda - \gamma| \ge \tau/2, \lambda \in \Lambda} \right) |\langle \phi_\lambda, \psi_\gamma \rangle| \sup_{x \in k + [0,1]^d} |\tilde{\psi}_\gamma(x)|$$

$$\le C \|c\|_{\ell^\infty(\Lambda)} \|A_{\Psi,\Phi}\|_{\mathcal{A}_{p,u}} \|\tilde{\Psi}\|_{\infty, p, u} \|u^{-1}\|_{L^{p'}(\mathbf{R}^d \setminus B(\tau/2))}.$$

The above two estimates, together with (8.20), lead to

$$\sup_{x \in k + [0,1]^d} |f(x) - \sum_{\gamma \in \Gamma} \langle f, \psi_\gamma \rangle \tilde{\psi}_\gamma(x)|$$

$$= \sup_{x \in k + [0,1]^d} |(f - f_{k,\tau})(x) - \sum_{\gamma \in \Gamma} \langle f - f_{k,\tau}, \psi_\gamma \rangle \tilde{\psi}_\gamma(x)|$$

$$\le C \|c\|_{\ell^\infty(\Lambda)} \|u^{-1}\|_{L^{p'}(\mathbf{R}^d \setminus B(\tau/2))} \to 0 \text{ as } \tau \to \infty,$$

for all $k \in \mathbf{Z}^d$, where we have used the assumption (2.17) to obtain the last limit. This proves the reconstruction formula (3.6) for $r = \infty$, and hence completes the verification that the collection $\tilde{\Psi}$ in (8.16) is the desired displayer associated with the average sampler Ψ.

Now we prove that the displayer $\tilde{\Psi}$ in (8.16) is enveloped by a function in $W_\infty(L_{p,u})$ when Λ is a lattice, and the average sampler Ψ and the generator Φ for the space $V_2(\Phi, \Lambda)$ are enveloped by some functions $g, h \in W_\infty(L_{p,u})$ respectively. Let $A_{\Psi,\Phi} = (A_{\Psi,\Phi}(\gamma, \lambda))_{\gamma \in \Gamma, \lambda \in \Lambda}$ be as in (4.2). Then for any $\lambda \in m + [-1/2, 1/2)^d$ and $\lambda' \in m' + [-1/2, 1/2)^d$ with $m, m' \in \mathbf{Z}^d$,

$$|(A_{\Psi,\Phi}^* A_{\Psi,\Phi})(\lambda, \lambda')| \le \sum_{k,l \in \mathbf{Z}^d} \sum_{\mu \in \Gamma} \|h(\cdot + \lambda - k)\|_{L^\infty([0,1]^d)} \|g(\cdot + \mu - k)\|_{L^\infty([0,1]^d)}$$

$$\times \|g(\cdot + \mu - l)\|_{L^\infty([0,1]^d)} \|h(\cdot + \lambda' - l)\|_{L^\infty([0,1]^d)}$$

$$\le C \sum_{k,l,n \in \mathbf{Z}^d} \|h(\cdot + m - k)\|_{L^\infty([-1,2]^d)} \|g(\cdot + n - k)\|_{L^\infty([-1,2]^d)}$$

$$\times \|g(\cdot + n - l)\|_{L^\infty([-1,2]^d)} \|h(\cdot + m' - l)\|_{L^\infty([-1,2]^d)}$$

$$\le d(m - m')$$

for some sequence $(d(m))_{m \in \mathbf{Z}^d}$ with $(d(m)u(m))_{m \in \mathbf{Z}^d} \in \ell^p$. This implies that

$$(8.22) \qquad A_{\Psi,\Phi}^* A_{\Psi,\Phi} \in \mathcal{C}_{p,u}(\Lambda, \Lambda).$$

Therefore using (8.13) and (8.22), and applying Lemma 8.2 to the matrix $A_{\Psi,\Phi}^* A_{\Psi,\Phi}$, we conclude that

$$(8.23) \qquad (A_{\Psi,\Phi}^* A_{\Psi,\Phi})^{-1} \in \mathcal{C}_{p,u}(\Lambda, \Lambda).$$

Let R be the matrix in (8.15) and $\tilde{\Psi} := \{\tilde{\psi}_\gamma, \gamma \in \Gamma\}$ be the displayer defined in (8.16). Then for $\gamma \in k + [0,1)^d$ and $x \in l + [0,1)^d$,

$$
\begin{aligned}
|\tilde{\psi}_\gamma(x)| &\leq \sum_{\lambda,\lambda' \in \Lambda} \sum_{m \in \mathbf{Z}^d} \|g(\cdot + \gamma - m)\|_{L^\infty([0,1]^d)} \|h(\cdot + \lambda' - m)\|_{L^\infty([0,1]^d)} \\
&\qquad |(A_{\Psi,\Phi}^* A_{\Psi,\Phi})^{-1}(\lambda',\lambda)| h(x - \lambda) \\
&\leq C \sum_{n,n',m \in \mathbf{Z}^d} \|g\|_{L^\infty(m-k+[-1,2]^d)} \|h\|_{L^\infty(m-n'+[-1,2]^d)} \\
&\qquad b(n - n') \sup_{y \in l-n+[-1,2]^d} |h(y)| \leq c(k - l),
\end{aligned}
$$

where $(b(k))_{k \in \mathbf{Z}^d}$ and $(c(k))_{k \in \mathbf{Z}^d}$ are sequences with $(b(k)u(k))_{k \in \mathbf{Z}^d}$ and $(c(k)u(k))_{k \in \mathbf{Z}^d}$ belonging to ℓ^p. This proves that $\tilde{\Psi} := \{\tilde{\psi}_\gamma, \gamma \in \Gamma\}$ is enveloped by some function in $W_\infty(L_{p,u})$, and hence completes the proof. $\square$

8.2. Proof of Theorem 3.2. By (3.7) and relatively-separatedness of the sets Γ and Λ, the matrix $A_{\delta_\Gamma,\Phi} = (\phi_\lambda(\gamma))_{\gamma \in \Gamma, \lambda \in \Lambda}$ in (4.3) belongs to $\mathcal{A}_{p,u}(\Gamma, \Lambda)$. Then we may reach the conclusion of Theorem 3.2 using the same argument as the one in the proof of Theorem 3.1 except replacing the average sampler Ψ by the ideal sampler δ_Γ. We omit the details of the proof here. $\square$

8.3. Proof of Theorem 4.1. Let S be the frame operator (2.20) on the space $V_2(\Phi, \Lambda)$. By the frame assumption on the generator Φ, we have

(8.24) $C^{-1}\|f\|_2 \leq \|Sf\|_2 \leq C\|f\|_2$ for all $f \in V_2(\Phi, \Lambda)$

and

(8.25) $C^{-1}\|f\|_2 \leq \left\|\big(\langle f, \phi_\lambda \rangle\big)_{\lambda \in \Lambda}\right\|_{\ell^2(\Lambda)} \leq C\|f\|_2$ for all $f \in V_2(\Phi, \Lambda)$.

Then the conclusion follows from (8.24), (8.25), and

$$
\langle A_{\Psi,\Phi}^* A_{\Psi,\Phi} c, c \rangle = \left\|\big(\langle Sf, \psi_\gamma \rangle\big)_{\gamma \in \Gamma}\right\|_{\ell^2(\Lambda)}^2
$$

for any $c = (\langle f, \phi_\lambda \rangle)_{\lambda \in \Lambda}$, where $f \in V_2(\Phi, \Lambda)$. $\square$

8.4. Proof of Theorem 4.2. Note that $\langle A_{\delta_\Gamma,\Phi}^* A_{\delta_\Gamma,\Phi} c, c \rangle = \|(Sf(\gamma))_{\gamma \in \Gamma}\|_{\ell^2(\Lambda)}^2$ for $c = (\langle f, \phi_\lambda \rangle)_{\lambda \in \Lambda}$, where $f \in V_2(\Phi, \Lambda)$. This, together with (8.24) and (8.25), proves the conclusion. $\square$

8.5. Proof of Theorem 5.1. The estimate (5.1) follows from (3.6), (3.8), Proposition 2.1 and the following inequality:

$$
\left| f - \sum_{\gamma \in \Gamma} g_\gamma \tilde{\psi}_\gamma' \right| \leq \sum_{\gamma \in \Gamma} |\langle f, \psi_\gamma \rangle - g_\gamma| |\tilde{\psi}_\gamma| + \sum_{\gamma \in \Gamma} |g_\gamma| |\tilde{\psi}_\gamma - \tilde{\psi}_\gamma'|. \qquad \square
$$

8.6. Proof of Theorem 5.2. Let $A_{\Psi,\Phi}$ and $A_{\Psi',\Phi}$ be the Gram matrices as in (4.2). Then there exists a positive constant C independent of the average samplers Ψ' and Ψ by (2.8) and (8.10) such that

$$
\begin{aligned}
\|A_{\Psi,\Phi} - A_{\Psi',\Phi}\|_{\mathcal{A}_{p,u}} &= \|A_{\Psi-\Psi',\Phi}\|_{\mathcal{A}_{p,u}} \\
&\leq C\|\Psi - \Psi'\|_{q/(q-1),p,\alpha} \|\Phi\|_{q,p,u} \leq C\|\Psi - \Psi'\|_{q^*,p,u}\|\Psi\|_{q,p,u}.
\end{aligned}
$$
(8.26)

This, together with (8.2), yields the following estimate:

$$(8.27) \qquad \|(A_{\Psi,\Phi})^* A_{\Psi,\Phi} - (A_{\Psi',\Phi})^* A_{\Psi',\Phi}\|_{\mathcal{A}_{p,u}} \le C \|\Psi - \Psi'\|_{q^*,p,u},$$

which then implies that

$$\|((A_{\Psi',\Phi})^* A_{\Psi',\Phi})^{-1} - ((A_{\Psi,\Phi})^* A_{\Psi,\Phi})^{-1}\|_{\mathcal{A}_{p,u}}$$
$$\le C \|((A_{\Psi,\Phi})^* A_{\Psi,\Phi})^{-1}\|_{\mathcal{A}_{p,u}}$$
$$\times \sum_{k=1}^{\infty} \left\| \left(((A_{\Psi,\Phi})^* A_{\Psi,\Phi})^{-1} ((A_{\Psi,\Phi})^* A_{\Psi,\Phi} - (A_{\Psi',\Phi})^* A_{\Psi',\Phi}) \right)^k \right\|_{\mathcal{A}_{p,u}}$$
$$\le C \|((A_{\Psi,\Phi})^* A_{\Psi,\Phi})^{-1}\|_{\mathcal{A}_{p,u}}$$
$$\times \sum_{k=1}^{\infty} \left(C \|((A_{\Psi,\Phi})^* A_{\Psi,\Phi})^{-1}\|_{\mathcal{A}_{p,u}} \|(A_{\Psi,\Phi})^* A_{\Psi,\Phi} - (A_{\Psi',\Phi})^* A_{\Psi',\Phi}\|_{\mathcal{A}_{p,u}} \right)^k$$
$$\le C \sum_{k=1}^{\infty} \left(C \|\Psi - \Psi'\|_{q^*,p,u} \right)^k \le \frac{C^2 \|\Psi - \Psi'\|_{q^*,p,u}}{1 - C \|\Psi - \Psi'\|_{q^*,p,u}} \le \frac{C^2 \delta_0}{1 - C \delta_0} < \infty$$

when δ_0 in (5.2) is sufficiently small, where we have used the estimates in (5.2), (8.2), (8.14) and (8.27). This proves that $((A_{\Psi',\Phi})^* A_{\Psi',\Phi})^{-1} \in \mathcal{A}_{p,u}(\Lambda,\Lambda)$ as δ_0 in (5.2) is sufficiently small. Moreover using the argument in the proof of Theorem 3.1, we conclude that Ψ' is a stable average sampler for the space $V_2(\Phi,\Lambda)$, and

$$(8.28) \qquad \|R' - R\|_{p,\alpha} \le C \|\Psi - \Psi'\|_{q^*,p,\alpha}$$

for some positive constant C, where $R' = A_{\Psi',\Phi}((A_{\Psi',\Phi})^* A_{\Psi',\Phi})^{-1} = (R'(\gamma,\lambda))_{\gamma \in \Gamma, \lambda \in \Lambda}$ and $R = A_{\Psi,\Phi}((A_{\Psi,\Phi})^* A_{\Psi,\Phi})^{-1} = (R(\gamma,\lambda))_{\gamma \in \Gamma, \lambda \in \Lambda}$. Therefore the displayers $\tilde{\Psi}' = \{\tilde{\psi}'_\gamma : \gamma \in \Gamma\}$ and $\tilde{\Psi} = \{\tilde{\psi}_\gamma : \gamma \in \Gamma\}$ associated with the stable average samplers Ψ' and Ψ respectively, which are defined by $\psi'_\gamma = \sum_{\lambda \in \Lambda} R'(\gamma,\lambda) \phi_\lambda$ and $\psi_\gamma = \sum_{\lambda \in \Lambda} R(\gamma,\lambda) \phi_\lambda$, $\gamma \in \Gamma$, satisfy

$$(8.29) \qquad \|\tilde{\Psi}' - \tilde{\Psi}\|_{q^*,p,u} \le C \|R - R'\|_{\mathcal{A}_{p,u}} \|\Phi\|_{q^*,p,u} \le C \|\Psi - \Psi'\|_{q^*,p,u}.$$

Hence (5.3) follows. □

8.7. Proof of Theorem 5.3. We can use the same technique as the one in the proof of Theorem 5.2 except the matrices $A_{\Psi,\Phi}$ and $A_{\Psi',\Phi}$ and the estimate (8.26) being replaced by the matrices $A_{\delta_\Gamma,\Phi} = (\phi_\lambda(\gamma))_{\gamma \in \Gamma, \lambda \in \Lambda}$ and $A_{\delta_{\Gamma'},\Phi} = (\phi_\lambda(\gamma + \delta_\gamma))_{\gamma \in \Gamma, \lambda \in \Lambda}$, and the estimate $\|A_{\delta_\Gamma,\Phi} - A_{\delta_{\Gamma'},\Phi}\|_{\mathcal{A}_{p,u}} \le \|\Phi_{\delta_0}\|_{\infty,p,u}$ for sequences $\{\delta_\gamma\}$ with $\sup_{\gamma \in \Gamma} |\delta_\gamma| \le \delta_0$ respectively. We omit the details of the proof here. □

8.8. Proof of Corollary 5.4. The first conclusion follows from Theorem 5.2 and the equivalence between $\sum_{l=1}^{L} \|\theta_n\|_{W_{q^*}(L_{p,u})}$ and $\|\Theta\|_{q^*,p,\alpha}$ where $\Theta = \{\theta_l(\cdot - \gamma) : 1 \le l \le L, \gamma \in \Gamma\}$.

For any continuous function ϕ in $W_\infty(L_{p,u})$, there exist continuous functions $\phi_n, n \ge 1$, with compact support such that $\lim_{n \to \infty} \|\phi_n - \phi\|_{W_\infty(L_{p,u})} = 0$. Therefore the modulus of continuity $\omega(\phi,\delta)(x) := \sup_{|t| \le \delta} |\phi(x + t) - \phi(x)|$ of the continuous function ϕ in $W_\infty(L_{p,u})$ has the property that $\|\omega(\phi,\delta)\|_{W_\infty(L_{p,u})} \to 0$ as $\delta \to 0$ ([2]). This together with (2.13) and Theorem 5.3 proves the second conclusion. □

8.9. Proof of Theorem 6.1. For average sampling on the space $V_r(\Phi, \Lambda)$ with the average sampler $\Psi := \{\psi_\gamma, \gamma \in \Gamma\}$, we introduce two local reconstruction approximation of a function $f \in V_r(\Phi, \Lambda)$ on a bounded set K using average sampling data on the R-neighborhood $B(K, R)$ of the set K by

$$(8.30) \qquad f_{K,R} = \sum_{\gamma \in \Gamma \cap B(K,R)} \langle f, \psi_\gamma \rangle \tilde{\psi}_\gamma,$$

and

$$(8.31) \qquad f_{K,R}^1 = \sum_{\gamma \in \Gamma \cap B(K,R)} \langle f, \psi_\gamma \rangle \tilde{\psi}_{\gamma,K,R}^1$$

where

$$\tilde{\psi}_\gamma = \sum_{\lambda_1, \lambda_2 \in \Lambda} \langle \psi_\gamma, \phi_{\lambda_1} \rangle (A_{\Psi,\Phi}^* A_{\Psi,\Phi})^{-1}(\lambda_1, \lambda_2) \phi_{\lambda_2}$$

and

$$\tilde{\psi}_{\gamma,K,R}^1 = \sum_{\lambda_1, \lambda_2 \in B(K,2R) \cap \Lambda} \langle \psi_\gamma, \phi_{\lambda_1} \rangle (A_{\Psi,\Phi}^* A_{\Psi,\Phi})^{-1}(\lambda_1, \lambda_2) \phi_{\lambda_2}.$$

For a bounded domain K and a positive number $R > 0$, define the projection matrices $P_{K,R}$ and $Q_{K,R}$ by

$$(P_{K,R}c)(\lambda) = \begin{cases} c(\lambda) & \text{if } \lambda \in \Lambda \cap B(K,R) \\ 0 & \text{if } \lambda \notin \Lambda \cap B(K,R) \end{cases}$$

for any $c := (c(\lambda))_{\lambda \in \Lambda}$, and

$$(Q_{K,R}d)(\gamma) = \begin{cases} d(\gamma) & \text{if } \gamma \in \Gamma \cap B(K,R) \\ 0 & \text{if } \gamma \notin \Gamma \cap B(K,R) \end{cases}$$

for any $d := (d(\gamma))_{\gamma \in \Gamma}$.

To prove Theorem 6.1, we need the following estimates for $f - f_{K,R}$ and $f_{K,R} - f_{K,R}^1$:

LEMMA 8.3. *Let* $p, q, q^*, r, u, \Lambda, \Gamma, \Phi, \Psi, \tilde{\Psi}$ *be as in Theorem 3.1, and set* $p' = p/(p-1)$. *Then there exists a positive constant C (independent of the bounded set K, the positive number $R \geq 1$ and the function $f \in V_r(\Phi, \Lambda)$) such that*

$$(8.32) \|f_{K,R} - f\|_{L^r(K)} \leq C \|u^{-1}\|_{L^{p'}(\mathbf{R}^d \setminus B(R))} \|(\langle f, \psi_\gamma \rangle)_{\gamma \in \Gamma \setminus B(K,R)}\|_{\ell^r(\Gamma \setminus B(K,R))}$$

and

$$(8.33) \|f_{K,R} - f_{K,R}^1\|_{L^r(K)} \leq C \|u^{-1}\|_{L^{p'}(\mathbf{R}^d \setminus B(R))}^2 \|(\langle f, \psi_\gamma \rangle)_{\gamma \in B(K,R)}\|_{\ell^r(B(K,R))}$$

for any compact set K, any positive number $R \geq 1$, and any $f \in V_r(\Phi, \Lambda)$.

Proof. Let $\tilde{K} \subset \mathbf{Z}^d$ be the minimal subset of $\mathbf{Z}^d$ such that $K \subset \tilde{K} + [0,1)^d$. For $1 \leq r \leq q$ and $r < \infty$,

$$\|f - f_{K,R}\|_{L^r(K)}^r = \Big\| \sum_{\gamma \notin \Gamma_{K,R}} \langle f, \psi_\gamma \rangle \tilde{\psi}_\gamma \Big\|_{L^r(K)}^r$$

$$\leq \int_{\tilde{K}+[0,1]^d} \Big(\sum_{\gamma \notin \Gamma_{K,R}} |\langle f, \psi_\gamma \rangle|^r |\tilde{\psi}_\gamma(x)| \Big) \times \Big(\sum_{\gamma \notin \Gamma_{K,R}} |\tilde{\psi}_\gamma(x)| \Big)^{r-1} dx$$

$$\leq \sum_{\gamma \notin \Gamma_{K,R}} |\langle f, \psi_\gamma \rangle|^r \sum_{k \in \tilde{K}} \|\tilde{\psi}_\gamma\|_{L^r(k+[0,1]^d)} \times \Big(\sum_{\gamma \notin \Gamma_{K,R}} \|\tilde{\psi}_\gamma\|_{L^r(k+[0,1]^d)} \Big)^{r-1}$$

$$\leq \big\| (\langle f, \psi_\gamma \rangle)_{\gamma \in \Gamma \setminus \Gamma_{K,R}} \big\|_{\ell^r(\Gamma \setminus \Gamma_{K,R})}^r \times \Big(\sup_{\gamma \notin \Gamma_{K,R}} \sum_{k \in \tilde{K}} \|\tilde{\phi}_\gamma\|_{L^r(k+[0,1]^d)} \Big)$$

$$(8.34) \qquad \times \Big(\sup_{k \in \tilde{K}} \sum_{\gamma \notin \Gamma_{K,R}} \|\tilde{\psi}_\gamma\|_{L^r(k+[0,1]^d)} \Big)^{r-1}$$

where we set $\Gamma_{K,R} = \Gamma \cap B(K,R)$. Then for all $r \in [1, \infty)$ with $1 \leq r \leq q$, we have

$$\|f - f_{K,R}\|_{L^r(K)} \leq C \big\| (\langle f, \psi_\gamma \rangle)_{\gamma \in \Gamma \setminus \Gamma_{K,R}} \big\|_{\ell^r(\Gamma \setminus \Gamma_{K,R})}$$

$$(8.35) \qquad\qquad\qquad \times \|u\|_{L^{p'}(\mathbf{R}^d \setminus B(R))} \|\tilde{\Psi}\|_{r,p,u}.$$

For $r = \infty$, it follows from $r \leq q$ that $q = \infty$. Then using standard modification to the estimate (8.35), we obtain

$$\|f - f_{K,R}\|_{L^\infty(K)} \leq C \big\| (\langle f, \psi_\gamma \rangle)_{\gamma \in \Gamma \setminus \Gamma_{K,R}} \big\|_{\ell^\infty(\Gamma \setminus \Gamma_{K,R})}$$

$$(8.36) \qquad\qquad\qquad \times \|u\|_{L^{p'}(\mathbf{R}^d \setminus B(R))} \|\tilde{\Psi}\|_{\infty,p,u}.$$

Then the local estimate (8.32) follows from (8.18), (8.35) and (8.36).

Set $R_{\Phi,r} = (\|\phi_\lambda\|_{L^r(k+[0,1]^d)})_{\lambda \in \Lambda, k \in \mathbf{Z}^d}$ and $|A| = (|a_{\lambda,\lambda'}|)$ for a matrix $A = (a_{\lambda,\lambda'})$. Then it follows from (2.7), (2.9) and Lemma 8.1 that

$$\sup_{k \in \tilde{K}} \sum_{\gamma \in B(K,R) \cap \Gamma} \|\tilde{\psi}_\gamma - \tilde{\psi}_{\gamma,K,R}^1\|_{L^r(k+[0,1]^d)}$$

$$+ \sup_{\gamma \in B(K,R) \cap \Gamma} \sum_{k \in \tilde{K}} \|\tilde{\psi}_\gamma - \tilde{\psi}_{\gamma,K,R}^1\|_{L^r(k+[0,1]^d)}$$

$$\leq C \Big(\sup_{k \in \tilde{K}} \sum_{\gamma \in B(K,R) \cap \Gamma} + \sup_{\gamma \in B(K,R) \cap \Gamma} \sum_{k \in \tilde{K}} \Big)$$

$$\Big(\sum_{\lambda \in B(K,2R) \cap \Lambda} \sum_{\lambda' \in \Lambda \setminus B(K,2R)} + \sum_{\lambda \in \Lambda \setminus B(K,2R)} \sum_{\lambda' \in \Lambda} \Big)$$

$$|\langle \psi_\gamma, \phi_\lambda \rangle| \, |(A_{\Psi,\Phi}^* A_{\Psi,\Phi})^{-1}(\lambda, \lambda')| \, \|\phi_{\lambda'}\|_{L^r(k+[0,1]^d)}$$

$$\leq C \|Q_{K,R} \, |A_{\Psi,\Phi}| \, P_{K,2R} \, |(A_{\Psi,\Phi}^* A_{\Psi,\Phi})^{-1}| \, (I - P_{K,2R}) R_{\Phi,r} P_{K,1}\|_{\mathcal{A}_{1,u_0}}$$

$$+ C \|Q_{K,R} \, |A_{\Psi,\Phi}| \, (I - P_{K,2R}) \, |(A_{\Psi,\Phi}^* A_{\Psi,\Phi})^{-1}| \, R_{\Phi,r} P_{K,1}\|_{\mathcal{A}_{1,u_0}}$$

$$\leq C \|u\|_{L^{p'}(\mathbf{R}^d \setminus B(R))}^2 \| \, |A_{\Psi,\Phi}| \, P_{K,2R} \, |(A_{\Psi,\Phi}^* A_{\Psi,\Phi})^{-1}| \, \|_{\mathcal{A}_{p,u}} \|R_{\Phi,r}\|_{\mathcal{A}_{p,u}}$$

$$+ C \|u\|_{L^{p'}(\mathbf{R}^d \setminus B(R))}^2 \|A_{\Psi,\Phi}\|_{\mathcal{A}_{p,u}} \| \, |(A_{\Psi,\Phi}^* A_{\Psi,\Phi})^{-1}| \, R_{\Phi,r}\|_{\mathcal{A}_{p,u}}$$

$$(8.37) \quad \leq C \|u\|_{L^{p'}(\mathbf{R}^d \setminus B(R))}^2 \|A_{\Psi,\Phi}\|_{\mathcal{A}_{p,u}} \|(A_{\Psi,\Phi}^* A_{\Psi,\Phi})^{-1}\|_{\mathcal{A}_{p,u}} \|\Phi\|_{r,p,u}.$$

This together with (8.10) and (8.14) proves (8.33). □

To prove Theorem 6.1, we need another lemma.

LEMMA 8.4. *Let $p, q, q^*, r, u, \Lambda, \Gamma, \Phi, \Psi, \tilde{\Psi}$ be as in Theorem 3.1, and $A_{\Psi,\Phi}$ be as in (4.2). Then there exist positive constants C and δ_0 (independent of the bounded*

set K and the positive number $R \geq 1$) such that

$$\|P_{K,2R}(A_{\Psi,\Phi}^* A_{\Psi,\Phi})^{-1} P_{K,2R} - P_{K,2R} R_{\Psi,\Phi,K,3R} P_{K,2R}\|_{\mathcal{A}_{1,u_0}}$$

(8.38) $\leq C\|u\|_{L^{p'}(\mathbf{R}^d \setminus B(R))}^2$

holds for all bounded sets K and all positive numbers R with $\|u\|_{L^{p'}(\mathbf{R}^d \setminus B(R))} \leq \delta_0$, where $R_{\Psi,\Phi,K,3R}$ is the generalized inverse of the matrix $P_{K,3R}\tilde{A}_{\Psi,\Phi,K,R}P_{K,3R}$, that is, $R_{\Psi,\Phi,K,3R}P_{K,3R}\tilde{A}_{\Psi,\Phi,K,R}P_{K,3R} = P_{K,3R}$ and $P_{K,3R}R_{\Psi,\Phi,K,3R}P_{K,3R} = R_{\Psi,\Phi,K,3R}$.

Proof. By (8.13), for any $A \geq (\|A_{\Psi,\Phi}^* A_{\Psi,\Phi}\|_{\mathcal{B}^2})^{1/2}$ there exists a matrix $B \in \mathcal{A}_{p,u}(\Lambda, \Lambda)$ such that

(8.39) $A_{\Psi,\Phi}^* A_{\Psi,\Phi} = A^2(I - B),$

and

(8.40) $\|B\|_{\mathcal{B}^2} < 1,$

where I is the usual unit matrix. By (8.4), (8.40), and the estimates $\|P_{K,3R}BP_{K,3R}\|_{\mathcal{B}^2} \leq \|B\|_{\mathcal{B}^2}$ and $\|P_{K,3R}BP_{K,3R}\|_{\mathcal{A}_{p,u}} \leq \|B\|_{\mathcal{A}_{p,u}}$, we have

(8.41) $\|B^k\|_{\mathcal{A}_{p,u}} + \|(P_{K,3R}BP_{K,3R})^k\|_{\mathcal{A}_{p,u}} \leq C\left(\frac{\|B\|_{\mathcal{B}^2}+1}{2}\right)^k, \; k \geq 1.$

Therefore for $k \geq 2$,

$$\left\|P_{K,2R}\sum_{k=1}^{\infty} B^k P_{K,2R} - P_{K,2R}\sum_{k=1}^{\infty}(P_{K,3R}BP_{K,3R})^k P_{K,2R}\right\|_{\mathcal{A}_{1,u_0}}$$

$$\leq \sum_{k=2}^{\infty}\sum_{l=0}^{k-2}\|P_{K,2R}(P_{K,3R}BP_{K,3R})^l B(I - P_{K,3R})B^{k-1-l}P_{K,2R}\|_{\mathcal{A}_{1,u_0}}$$

$$\leq C\sum_{k=2}^{\infty}\sum_{l=0}^{k-2}\|P_{K,2R}(P_{K,3R}BP_{K,3R})^l B(I - P_{K,3R})\|_{\mathcal{A}_{1,u_0}}$$

$$\times\|(I - P_{K,3R})B^{k-1-l}P_{K,2R}\|_{\mathcal{A}_{1,u_0}}$$

$$\leq C\|u\|_{L^{p'}(\mathbf{R}^d \setminus B(R))}^2 \sum_{k=2}^{\infty}\sum_{l=0}^{k-2}\|(P_{K,3R}BP_{K,3R})^l B\|_{\mathcal{A}_{p,u}}\|B^{k-1-l}\|_{\mathcal{A}_{p,u}}$$

$$\leq C\|u\|_{L^{p'}(\mathbf{R}^d \setminus B(R))}^2 \|B\|_{\mathcal{A}_{p,u}}\sum_{k=2}^{\infty}(k-1)\left(\frac{\|B\|_{\mathcal{B}^2}+1}{2}\right)^{k-1}$$

(8.42) $\leq C\|u\|_{L^{p'}(\mathbf{R}^d \setminus B(R))}^2,$

where we have used (8.40) and (8.41) to obtain the last inequality.

Write

(8.43) $P_{K,3R}\tilde{A}_{\Psi,\Phi,K,R}P_{K,3R} = P_{K,3R}A_{\Psi,\Phi}^* Q_{K,4R}A_{\Psi,\Phi}P_{K,3R} = A^2(P_{K,3R} - B'),$

where A is the positive constant in (8.39). Since

$$\|P_{K,3R}A_{\Psi,\Phi}^* Q_{K,4R}A_{\Psi,\Phi}P_{K,3R} - P_{K,3R}A_{\Psi,\Phi}^* A_{\Psi,\Phi}P_{K,3R}\|_{\mathcal{B}^2}$$

$$\leq \|P_{K,3R}A_{\Psi,\Phi}^* Q_{K,4R}A_{\Psi,\Phi}P_{K,3R} - P_{K,3R}A_{\Psi,\Phi}^* A_{\Psi,\Phi}P_{K,3R}\|_{\mathcal{A}_{1,u_0}}$$

(8.44) $\leq C\|u\|_{L^{p'}(\mathbf{R}^d \setminus B(R))}^2 \|A_{\Psi,\Phi}\|_{\mathcal{A}_{p,u}}^2,$

we have that

$$(8.45) \quad \|P_{K,3R}BP_{K,3R} - B'\|_{\mathcal{B}^2} \leq D_0 \|u\|^2_{L^{p'}(\mathbf{R}^d \setminus B(R))} \|A_{\Psi,\Phi}\|^2_{\mathcal{A}_{p,u}} \leq \frac{1 - \|B\|_{\mathcal{B}^2}}{3},$$

when $\|u\|^2_{L^{p'}(\mathbf{R}^d \setminus B(R))} \leq \delta_0$ for some sufficiently small positive number δ_0, where D_0 is a positive constant. Therefore

$$(8.46) \quad \|(B')^k\|_{\mathcal{A}_{p,u}} \leq C \Big(\frac{\|B\|_{\mathcal{B}^2} + 1}{2} \Big)^k, \ k \geq 1$$

by (8.4), (8.40) and (8.45). Similar to the argument in the proof of the estimate (8.42), we have

$$\Big\| P_{K,2R} \sum_{k=1}^{\infty} (B')^k P_{K,2R} - P_{K,2R} \sum_{k=1}^{\infty} (P_{K,3R}BP_{K,3R})^k P_{K,2R} \Big\|_{\mathcal{A}_{1,u_0}}$$

$$\leq \sum_{k=1}^{\infty} \sum_{l=0}^{k-1} \|(B')^l (B' - P_{K,3R}BP_{K,3R})(P_{K,3R}BP_{K,3R})^{k-1-l}\|_{\mathcal{A}_{1,u_0}}$$

$$(8.47) \leq C\|u\|^2_{L^{p'}(\mathbf{R}^d \setminus B(R))} \|A_{\Psi,\Phi}\|_{\mathcal{A}_{p,u}} \sum_{k=1}^{\infty} k \Big(\frac{\|B\|_{\mathcal{B}^2} + 1}{2} \Big)^k \leq C\|u\|^2_{L^{p'}(\mathbf{R}^d \setminus B(R))},$$

where the second inequality follows from (8.41), (8.44), (8.45) and (8.46). Combining (8.39), (8.42), (8.43) and (8.47) proves the desired estimate (8.38). □

Now we start to prove Theorem 6.1.

Proof of Theorem 6.1 By Lemma 8.4,

$$\sup_{\lambda \in B(K,2R)} \sum_{\lambda' \in B(K,2R)} |(A^*_{\Psi,\Phi}A_{\Psi,\Phi})^{-1}(\lambda, \lambda') - (\tilde{A}_{\Psi,\Phi,K,R})^{-1}(\lambda, \lambda')| \leq C\|u\|^2_{L^{p'}(\mathbf{R}^d \setminus B(R))},$$

which, together with (8.9) and Lemma 8.1, implies that

$$\Big(\sup_{k \in \tilde{K}} \sum_{\gamma \in B(K,R) \cap \Gamma} + \sup_{\gamma \in B(K,R) \cap \Gamma} \sum_{k \in \tilde{K}} \Big) \|\tilde{\psi}^a_{\gamma,K,R} - \tilde{\psi}^1_{\gamma,K,R}\|_{L^r(k+[0,1]^d)} \leq C\|u\|^2_{L^{p'}(\mathbf{R}^d \setminus B(R))}.$$

(8.48)

Therefore the estimate (6.7) follows from (8.48), Proposition 2.1 and Lemma 8.3. □

8.10. Proof of Theorem 6.2. Theorem 6.2 can be proved using the similar argument to the one in the proof of Theorem 6.1 except the average sampler Ψ being replaced by the ideal sampler δ_Γ. We omit the details of the proof here. □

8.11. Proof of Theorem 6.3. For any bounded set K, we let

$$V_2(\Phi, \Lambda \cap K) = \Big\{ \sum_{\lambda \in \Lambda \cap K} c(\lambda)\phi_\lambda, \ \sum_{\lambda \in \Lambda \cap K} |c(\lambda)|^2 < \infty \Big\}.$$

By the Riesz assumption on Φ,

$$(8.49) \quad C^{-1}\|c\|_{\ell^2(\Lambda \cap K)} \leq \Big\| \sum_{\lambda \in \Lambda \cap K} c(\lambda)\phi_\lambda \Big\|_{L^2(\mathbf{R}^d)} \leq C\|c\|_{\ell^2(\Lambda \cap K)}$$

for any sequence $c := (c(\lambda))_{\lambda \in \Lambda \cap K}$. For any $R \geq 1$, it follows the localization assumption on the generator Φ that

$$\Big\| \sum_{\lambda \in \Lambda \cap K} c(\lambda) \phi_\lambda \Big\|^2_{L^2(\mathbf{R}^d \setminus B(K,R))}$$

$$\leq \sum_{\lambda \in \Lambda \cap K} |c(\lambda)|^2 \sum_{k \in \mathbf{Z}^d \setminus B(K,R-1)} \|\phi_\lambda\|_{L^2(k+[0,1]^d)} \Big(\sum_{\lambda' \in \Lambda \cap K} \|\phi_{\lambda'}\|_{L^2(k+[0,1]^d)} \Big)$$

$$(8.50) \leq C \|\Phi\|^2_{2,p,u} \|u^{-1}\|^2_{L^{p'}(\mathbf{R}^d \setminus B(R))} \sum_{\lambda \in \Lambda \cap K} |c(\lambda)|^2.$$

We recall that

$$(8.51) \qquad \qquad \|u^{-1}\|_{L^{p'}(\mathbf{R}^d \setminus B(R))} \to 0 \text{ as } R \to \infty$$

by (2.17). Therefore by (8.49), (8.50) and (8.51), there exist positive constants C and R_1 such that

$$(8.52) \qquad C^{-1} \|c\|_{\ell^2(\Lambda \cap K)} \leq \Big\| \sum_{\lambda \in \Lambda \cap K} c(\lambda) \phi_\lambda \Big\|_{L^2(B(K,R_1))} \leq C \|c\|_{\ell^2(\Lambda \cap K)}$$

for all sequences $c := (c(\lambda))_{\lambda \in \Lambda \cap K}$.

Set $K_1 = B(K, R_1)$. For the average sampling/reconstruction process, there exists a positive constant C by Theorem 6.1 and Lemma 8.1 such that for any $R \geq 1$ and $f = \sum_{\lambda \in \Lambda \cap K} c(\lambda) \phi_\lambda$,

$$\|\tilde{f}^a_{K_1,R} - f\|_{L^2(K_1)} \leq C \|u^{-1}\|_{L^{p'}(\mathbf{R}^d \setminus B(R))} \|(\langle f, \psi_\gamma \rangle)_{\gamma \in \Gamma}\|_{\ell^2(\Gamma)}$$

$$(8.53) \qquad \qquad \leq C \|u^{-1}\|_{L^{p'}(\mathbf{R}^d \setminus B(R))} \|A_{\Psi,\Phi}\|_{\mathcal{A}_{p,u}} \Big(\sum_{\lambda \in \Lambda \cap K} |c(\lambda)|^2 \Big)^{1/2}.$$

Similarly for the ideal sampling/reconstruction procedure, there exists a positive constant C such that for any $R \geq 1$ and $f = \sum_{\lambda \in \Lambda \cap K} c(\lambda) \phi_\lambda$,

$$(8.54) \|\tilde{f}^i_{K_1,R} - f\|_{L^2(K_1)} \leq C \|u^{-1}\|_{L^{p'}(\mathbf{R}^d \setminus B(R))} \|A_{\delta_\Gamma,\Phi}\|_{\mathcal{A}_{p,u}} \Big(\sum_{\lambda \in \Lambda \cap K} |c(\lambda)|^2 \Big)^{1/2}.$$

Therefore there exists a positive constant R_2 independent of K by (8.9) and (8.51) – (8.54) that

$$(8.55) \qquad C^{-1} \Big(\sum_{\lambda \in \Lambda \cap K} |c(\lambda)|^2 \Big)^{1/2} \leq \|f^a_{K_1,R_2}\|_{L^2(K_1)} \leq C \Big(\sum_{\lambda \in \Lambda \cap K} |c(\lambda)|^2 \Big)^{1/2},$$

for the average sampling/reconstruction process, and

$$(8.56) \qquad C^{-1} \Big(\sum_{\lambda \in \Lambda \cap K} |c(\lambda)|^2 \Big)^{1/2} \leq \|f^i_{K_1,R_2}\|_{L^2(K_1)} \leq C \Big(\sum_{\lambda \in \Lambda \cap K} |c(\lambda)|^2 \Big)^{1/2},$$

for the ideal sampling/reconstruction process. Therefore the conclusion (6.9) follows by letting $R_0 = R_1 + R_2$. □

8.12. Proof of Theorem 7.1. Let $A_{\Psi,\Phi}$ be as in (4.2), and the matrix $B \in \mathcal{A}_{p,u}(\Lambda, \Lambda)$ be as in (8.39) and (8.40). Write the sequence $(a_\gamma)_{\gamma \in \Gamma}$ as a vector, to be denoted by a, and the family of functions Φ as a vector, which is still denoted by Φ. We claim that

$$(8.57) \qquad f_n = \sum_{k=0}^n a^T A_{\Psi,\Phi} B^k \Phi, \ n \geq 0,$$

where a^T denotes the transpose of the vector a. The above claim is obviously true for $n = 0$. Inductively we assume that the claim is true for n. By (7.1), (8.39), and the inductive hypothesis, we have

$$f_{n+1} = a^T A_{\Psi,\Phi} \Phi + \sum_{k=0}^n a^T A_{\Psi,\Phi} B^k \Phi - A^{-2} \sum_{k=0}^n a^T A_{\Psi,\Phi} B^k A_{\Psi,\Phi}^* A_{\Psi,\Phi} \Phi$$

$$= a^T A_{\Psi,\Phi} \Phi + \sum_{k=0}^n a^T A_{\Psi,\Phi} B^k \Phi - \sum_{k=0}^n a^T A_{\Psi,\Phi} B^k (I - B) \Phi$$

$$= \sum_{k=0}^{n+1} a^T A_{\Psi,\Phi} B^k \Phi.$$

This proves the claim (8.57) by induction.

By (8.9), (8.41), (8.57) and Proposition 2.1, we have

$$\|f_{n+1} - f_n\|_r = \|a^T A_{\Psi,\Phi} B^{n+1} \Phi\|_r$$

$$\leq C \|a\|_{\ell^r(\Gamma)} \|A_{\Psi,\Phi}\|_{\mathcal{A}_{p,u}} \|B^{n+1}\|_{\mathcal{A}_{p,u}} \|\Phi\|_{q,p,u}$$

$$(8.58) \qquad \leq C \Big(\frac{\|B\|_{\mathcal{B}^2} + 1}{2} \Big)^n \|a\|_{\ell^r(\Gamma)} \quad \text{for all } n \geq 0,$$

where $a := (a_\gamma)_{\gamma \in \Gamma}$. The first conclusion then follows from (8.58).

Now we assume that the initial data $a := (a_\gamma)_{\gamma \in \Gamma}$ are obtained from average sampling a function $f \in V_r(\Phi, \Lambda)$. Taking limit at both side of $f_n = f_0 + f_{n-1} - A^{-2} T_{as} f_{n-1}$ and using the Riesz property of Φ, we obtain

$$(8.59) \qquad \sum_{\gamma \in \Gamma} \langle f - f_\infty, \psi_\gamma \rangle \langle \psi_\gamma, \phi_\lambda \rangle = 0 \ \text{ for all } \lambda \in \Lambda.$$

Write

$$(8.60) \qquad f - f_\infty = \sum_{\lambda \in \Lambda} d_\lambda \phi_\lambda$$

for some ℓ^r sequence $d = (d_\lambda)_{\lambda \in \Lambda}$. We then may write (8.59) as

$$(8.61) \qquad d^T A_{\Psi,\Phi}^* A_{\Psi,\Phi} = 0.$$

Combining (8.13), (8.60), and (8.61) leads to the second conclusion of the theorem that the limit function f_∞ agrees with the original function f. $\quad \square$

8.13. Proof of Theorem 7.2. We may use the same argument as in the proof of Theorem 7.1 with standard modification, for instance, the matrix $A_{\Psi,\Phi}$ in the proof of Theorem 7.1 by the matrix $A_{\delta_\Gamma, \Phi} := (\phi_\lambda(\gamma))_{\gamma \in \Gamma, \lambda \in \Lambda}$. We omit the details of the proof here. $\quad \square$

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