Average Sampling in L^2

Echantillonnage moyenne dans L^2

M. Zuhair Nashed and Qiyu Sun
Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA

Wai-Shing Tang
Department of Mathematics, National University of Singapore, Singapore

Abstract

In this note, we show that any localized average sampler could not be a stable sampler for L^2, but that there is a localized determining sampler for L^2.

Résumé

Dans cet article, nous démontrons que tout échantillonneur moyen localisé ne peut pas être un échantillonneur stable pour L^2, mais quun échantillonneur déterminant localisé existe pour L^2.

1. Introduction

A goal at the heart of digital signal processing is to reconstruct continuous time signals from their available samples. The usual assumption in such problems is that the samples are ideal. For instance, in the classical band-limited model, it is well known from the Whittaker-Shannon sampling theorem that any continuous time signal $f(t) \in L^2$ bandlimited to $[-\Omega, \Omega]$ is uniquely determined and can be reconstructed in a stable way by a set of uniformly-spaced samples $f(kT), k \in \mathbf{Z}$, taken T seconds apart with $T \le \pi/\Omega$:

$$f(t) = \sum_{k\in\mathbf{Z}} f(kT)\frac{\sin \pi(t/T - k)}{\pi t/T - k}$$

Email addresses: `znashed@mail.ucf.edu` (M. Zuhair Nashed), `qsun@mail.ucf.edu` (Qiyu Sun), `mattws@nus.edu.sg` (Wai-Shing Tang).

Preprint submitted to Elsevier Science *July 9, 2009*

([2,8]). Here L^2 is the Hilbert space of all square-integrable functions on the real line with the standard L^2 inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|_2$.

Unfortunately, in practice, ideal sampling is impossible to implement. A more accurate model considers that the samples are obtained by a set of values of inner product between the continuous-time signal and the sampling functionals. More precisely, given a Hilbert space H with inner product $\langle\cdot,\cdot\rangle_H$ of time signals, the sample y_γ at the location $\gamma \in \Gamma$ is obtained by taking the inner product between a time signal f and the sampling functional ψ_γ at the location γ, i.e., $y_\gamma = \langle f, \psi_\gamma\rangle_H$. We call $\Psi = \{\psi_\gamma\}_{\gamma\in\Gamma}$ the *average sampler*. In the above setup, the sampling procedure on H via the average sampler Ψ can be interpreted as a linear operator

$$S : H \ni f \longmapsto \{\langle f, \psi_\gamma\rangle_H\}_{\gamma\in\Gamma} \in \ell^2(\Gamma), \tag{1.1}$$

and the reconstruction procedure as finding the left inverse of the linear operator S. Here $\ell^2(\Gamma)$ is the space of all square-summable sequences on Γ with norm $\|\cdot\|_{\ell^2(\Gamma)}$ (or $\|\cdot\|_2$ for short). We say that an average sampler Ψ on H is a *determining sampler* if the sampling operator S in (1.1) is one-to-one, i.e., the only time signal $f \in H$, that satisfies $\langle f, \psi_\gamma\rangle_H = 0$ for all $\gamma \in \Gamma$, is the zero signal ([3]). Similarly we say that an average sampler Ψ on H is a *stable sampler* if the sampling operator S in (1.1) has bounded left-inverse, i.e., there exist positive constants A and B such that $A\langle f, f\rangle \le \sum_{\gamma\in\Gamma} |\langle f, \psi_\gamma\rangle_H|^2 \le B\langle f, f\rangle$ for all $f \in H$ ([3]).

Determining and stable samplers have been studied for signals in shift-invariant spaces ([1,3,9]) and for signals with finite rate of innovation ([4,6,7]). In this paper, we consider the average sampling problem in the space L^2, particularly, the existence of localized determining samplers for L^2 (Theorem 1) and the nonexistence of localized stable samplers for L^2 (Theorem 5). Here we say that an average sampler $\Psi = \{\psi_\gamma\}_{\gamma\in\Gamma}$ is *localized* if Γ is a *relatively-separated subset* of $\mathbf{R}$, i.e.,

$$\sup_{x\in\mathbf{R}} \sum_{\gamma\in\Gamma} \chi_{\gamma+[0,1)}(x) < \infty,$$

and if there exists a function g in the *Wiener amalgam space* $\mathcal{W} := \{f|\ \sum_{k\in\mathbf{Z}} \sup_{x\in k+[0,1)} |f(x)| < \infty\}$ such that

$$|\psi_\gamma(x)| \le g(x-\gamma) \quad \text{for all } x \in \mathbf{R} \text{ and } \gamma \in \Gamma,$$

where χ_E is the characteristic function on a set E. The reasons for considering the localized sampler $\Psi = \{\psi_\gamma\}_{\gamma\in\Gamma}$ are two-fold: 1) each index $\gamma \in \Gamma$ means that there is an acquisition device located at that position, and hence it is reasonable to assume that there are finitely many such devices in any unit interval and the distribution of those devices is almost time-invariant, which in turn implies that the index set Γ is relatively-separated; 2) the sampling functional ψ_γ reflects the characteristic of the acquisition device at the location γ, and hence it should be essentially supported in a neighborhood of the sampling location γ, while the dominance of the sampling functional ψ_γ by the γ-shift of a function h in the Wiener amalgam space is a reasonable description of such a phenomenon ([6]).

2. Determining sampler for L^2

Theorem 1 *There is a localized determining sampler for L^2.*

To prove Theorem 1, we will use the following modification of a result in [5, page 2103].

Lemma 2 *Let $0 < D < 1$ and Γ be the set of all integers contained in $\cup_{n=1}^\infty [a_n, b_n)$, where the sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ satisfy the conditions $1 \le a_n < b_n < a_{n+1} < b_{n+1}$ for all $n \ge 1$, and* $\lim_{n\to\infty} a_n =$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_n - a_n = +\infty$, $\lim_{n\to\infty} \frac{b_n}{a_n} = 1$ *and* $\sum_{n=1}^\infty \left(\frac{b_n-a_n}{a_n}\right)^2 = +\infty$. *If F is an analytic function of exponential type πD, bounded on the real line, and $F(\gamma)=0$ for all $\gamma\in\Gamma$, then F is the zero function.*

Now we start to prove Theorem 1.

PROOF. Define $\Gamma_l, 0\le l\in \mathbf{Z}$, by $\Gamma_0=\cup_{n=1}^\infty\big([(1-n^{-1/2})10^n,(1+n^{-1/2})10^n]\cap\mathbf{Z}\big)$, and $\Gamma_l=\cup_{n=n_l}^\infty\big([(1+(2-2^{-l+1})n^{-1/2})10^n+1,(1+(2-2^{-l})n^{-1/2})10^n]\cap\mathbf{Z}\big)$, where the integers $n_l, l\ge 1$, are so chosen that $n^{-1/2}10^n\ge 2^{l+2}$ for all $n\ge n_l$. Define $\Gamma=\cup_{l=0}^\infty\Gamma_l$. Then Γ is a set of integers and hence a relatively-separated subset of $\mathbf{R}$.

Let h be a C^∞ function supported in $[-\pi/2,\pi/2]$ and satisfy

$$\sum_{l=0}^\infty h(x-2l\pi/3)+\sum_{l=0}^\infty h(x+(2l+1)\pi/3)\neq 0 \text{ for all } x\in\mathbf{R}. \tag{2.1}$$

Define the Fourier transform $\hat f$ of an integrable function f by $\hat f(\xi)=\int_{\mathbf{R}} f(x)e^{-ix\xi}dx$, and define $\Psi=\{\phi_\gamma(\cdot-\gamma)|\ \gamma\in\Gamma\}$ with the help of the Fourier transform by

$$\widehat{\phi}_\gamma(\xi)=\begin{cases} h(\xi-l\pi/3) \text{ if } \gamma\in\Gamma_l \text{ and } l\in 2\mathbf{Z},\\ h(\xi+l\pi/3) \text{ if } \gamma\in\Gamma_l \text{ and } l\in 2\mathbf{Z}+1.\end{cases} \tag{2.2}$$

Here the average sampler Ψ is well-defined because $\Gamma_l\cap\Gamma_{l'}=\emptyset$ for all nonnegative integers $l\neq l'$. From the above definition of the average sampler Ψ, we have that $|\phi_\gamma(x)|\le|\widehat{h}(x)|$ for all $\gamma\in\Gamma$. This shows that $\phi_\gamma,\gamma\in\Gamma$, are uniformly dominated by a function in the Wiener amalgam space, and hence Ψ in (2.2) is a localized sampler.

Now we prove that Ψ in (2.2) is a determining sampler. Take any function $f\in L^2$ such that $\langle f,\phi_\gamma(\cdot-\gamma)\rangle=0$ for all $\gamma\in\Gamma$. Define $F_l, 0\le l\le\mathbf{Z}$, by

$$\widehat{F_l}(\xi)=\begin{cases}\hat f(\xi+l\pi/3)h(\xi) \text{ if } l\in 2\mathbf{Z},\\ \hat f(\xi-l\pi/3)h(\xi) \text{ if } l\in 2\mathbf{Z}+1.\end{cases} \tag{2.3}$$

Then for any $0\le l\in\mathbf{Z}$, $\widehat{F_l}$ is supported in $[-\pi/2,\pi/2]$ and belongs to $L^1\cap L^2$, and

$$|F_l(\gamma)|=\frac{1}{2\pi}\Big|\int_{-\infty}^{\infty}\hat f(\xi)\widehat{\phi}_\gamma(\xi)e^{i\gamma\xi}d\xi\Big|=|\langle f,\phi_\gamma(\cdot-\gamma)\rangle|=0 \quad \text{for all} \quad \gamma\in\Gamma_l.$$

Then it follows from Lemma 2 that $F_l\equiv 0$ for all $0\le l\in\mathbf{Z}$. This together with (2.1) and (2.3) yields $f\equiv 0$. Therefore Ψ in (2.2) is a localized determining sampler for L^2.

Remark 3 *The functions $\phi_\gamma,\gamma\in\Gamma$, in the average sampler Ψ constructed in the proof of Theorem 1 are dominated by a function in the Wiener amalgam space, but their derivatives are not. Define $\tilde\Psi=\{\tilde\phi_\gamma(\cdot-\gamma)|\ \gamma\in\Gamma\}$ by*

$$\widehat{\tilde\phi}_\gamma(\xi)=h(\xi)+e^{-l}\widehat{\phi}_\gamma(\xi) \quad \text{if } \gamma\in\Gamma_l \text{ and } l\ge 0.$$

Then $\tilde\phi_\gamma,\gamma\in\Gamma$, are in a bounded set of the Schwartz class $\mathcal{S}$. Moreover, one may verify that $\tilde\Psi$ is a determining sampler for L^2 too.

Remark 4 *One may easily verify that for any relatively-separated subset* Γ *of* $\mathbf{R}$, $\Psi = \{\phi_\gamma(\cdot - \gamma) | \ \gamma \in \Gamma\}$ *is not a determining sampler for* L^2 *if all the functions* $\phi_\gamma, \gamma \in \Gamma$, *are supported in a compact set* K, *or if all* $\widehat{\phi}_\gamma, \gamma \in \Gamma$, *are supported in a compact set* Ω. *We do not know whether there is a determining sampler* $\Psi = \{\phi_\gamma(\cdot - \gamma) | \ \gamma \in \Gamma\}$ *such that* $\|\phi_\gamma\|_2 = 1$ *and* $|\phi_\gamma(x)| \le C \exp(-\epsilon|x|)$ *for some positive constants* C, ϵ *and a relatively-separated subset* Γ *of* $\mathbf{R}$.

3. Stable sampler for L^2

Theorem 5 *Any localized average sampler is not a stable sampler for* L^2.

PROOF. Take a localized average sampler $\Psi = \{\phi_\gamma(\cdot - \gamma) | \ \gamma \in \Gamma\}$, where Γ is a relatively-separated subset of $\mathbf{R}$. Assume that h is a function in the Wiener amalgam space $\mathcal{W}$ that dominates all $\phi_\gamma, \gamma \in \Gamma$, i.e., $|\phi_\gamma(x)| \le h(x)$ for all $x \in \mathbf{R}$ and $\gamma \in \Gamma$.

For any $R > 1$, let g_R be a function in L^2 such that $\|g_R\|_2 = 1$, g_R is supported in $[0, 1]$, and $\langle g_R, \phi_\gamma(\cdot - \gamma)\rangle = 0$ for all $\gamma \in (-R, R) \cap \Gamma$. The existence of such a function g_R follows from the facts that $L^2([0, 1])$ is an infinite-dimensional space and that $(-R, R) \cap \Gamma$ is a finite set. Then

$$\begin{aligned}\sum_{\gamma\in\Gamma} |\langle g_R, \phi_\gamma(\cdot - \gamma)\rangle|^2 &\le \sum_{\gamma\in\Gamma\setminus(-R,R)} \Big(\int_0^1 |g_R(x)|^2 |\phi_\gamma(x-\gamma)| dx\Big) \times \int_0^1 |\phi_\gamma(x-\gamma)| dx \\ &\le C\|h\|_1 \Big(\sum_{\gamma\in\Gamma\setminus(-R,R)} \sup_{x\in[0,1]} |h(x-\gamma)|\Big) \to 0 \quad \text{as } R \to \infty, \end{aligned} \tag{3.4}$$

where C is a positive constant independent of R. This proves that Ψ is not a stable sampler for L^2.

Acknowledgement. The authors thank the reviewer for the suggestion to improve the presentation. The third author wishes to thank the hospitality of the Department of Mathematics, University of Central Florida, during his visit in 2008-09.

References

[1] A. Aldroubi, Non-uniform weighted average sampling and reconstruction in shift-invariant and wavelet spaces, *Appl. Comput. Harmon. Anal.*, **13**(2002), 151–161.

[2] A. Aldroubi and K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant space, *SIAM Review*, **43**(2001), 585–620.

[3] A. Aldroubi, Q. Sun, and W.-S. Tang, Convolution, average sampling and a Calderon resolution of the identity for shift-invariant spaces, *J. Fourier Anal. Appl.*, **11**(2005), 215–244.

[4] N. Bi, M. Z. Nashed and Q. Sun, Reconstructing signals with finite rate of innovation from noisy samples, *Acta Appl. Math.*, **107**(2009), 339–372.

[5] P. Koosis, Sur la totalite des systemes d'exponentielles imaginaries, *C. R. Acad. Sci. Paris*, **250**(1960), 2102–2103.

[6] Q. Sun, Non-uniform average sampling and reconstruction of signals with finite rate of innovation, *SIAM J. Math. Anal.*, **38**(2006), 1389–1422.

[7] M. Vetterli, P. Marziliano, and T. Blu, Sampling signals with finite rate of innovation, *IEEE Trans. Signal Proc.*, **50**(2002), 1417–1428.

[8] M. Unser, Sampling – 50 years after Shannon, *Proceedings of the IEEE*, **88**(2000), 569–587.

[9] M. Unser and A. Aldroubi, A general sampling theory for non-ideal acquisition devices, *IEEE Trans. Signal Process*, **42**(1994), 2915-2925.