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Wiener's Lemma for Singular Integral Operators of Bessel Potential Type

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Abstract In this paper, we introduce an algebra of singular integral operators containing Bessel potentials of positive order, and show that the corresponding unital Banach algebra is an inverse-closed Banach subalgebra of $\mathcal{B}(L^q_w)$, the Banach algebra of all bounded operators on the weighted space L^q_w, for all $1 \le q < \infty$ and Muckenhoupt A_q-weights w.

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1 Introduction

Let $L^q := L^q(\mathbb{R}^d), 1 \le q < \infty$, be the space of all q-integrable functions on $\mathbb{R}^d$ with norm denoted by $\|\cdot\|_q$. Define the Fourier transform $\hat{f}$ of an integrable function f on $\mathbb{R}^d$ by $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\xi}dx$ and understand the Fourier transform of a tempered distribution as usual. The Bessel potential operator J_γ of positive order γ,

$$\widehat{J_\gamma f}(\xi) := (1 + |\xi|^2)^{-\gamma/2}\hat{f}(\xi), \ f \in L^2, \tag{1.1}$$

is well-studied in Harmonic Analysis [5,12]. The Bessel potential $J_\gamma, 0 < \gamma < d$, is a singular integral operator,

$$J_\gamma f(x) = \int_{\mathbb{R}^d} G_\gamma(x-y)f(y)dy, \ f \in L^2, \tag{1.2}$$

whose convolution kernel G_γ has singularity of order $d-\gamma$ near the origin and fast decay at infinity,

$$\sup_{x\in\mathbb{R}^d} |x|^{d-\gamma}(1 + |x|)^N|G_\gamma(x)| < \infty \quad \text{for all } N \ge 1. \tag{1.3}$$

Denote the identity operator by I. The spectrum $\sigma_2(J_\gamma)$ of the Bessel potential J_γ on the Hilbert space L^2 is the unit interval $[0, 1]$ by (1.1), and for any complex number λ not in the spectrum

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$\sigma_2(J_\gamma)$, the resolvent operator $(\lambda I - J_\gamma)^{-1}$ can be written as $\lambda^{-1}I - T$ for some singular integral operator T whose convolution kernel has the *same* singularity near the origin and decay at infinity as the kernel G_γ of the Bessel potential J_γ has. In this paper, we consider a non-commutative extension of the above result about the resolvent operator $(\lambda I - J_\gamma)^{-1}$ associated with the Bessel potential J_γ.

For a kernel function K on $\mathbb{R}^d \times \mathbb{R}^d$, let r_K be the minimal radially decreasing function that dominates the off-diagonal decay of the kernel K,

$$r_K(z) := \sup_{|x-y|\geq|z|} |K(x,y)|, \tag{1.4}$$

and define its modified modulus of continuity by

$$\omega_\delta(K)(x,y) := \begin{cases} \sup_{|x'|,|y'|\leq\delta} |K(x+x', y+y') - K(x,y)| & \text{if } |x-y| \geq 4\delta, \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

Denote by χ_E the characteristic function on a set E, by $B(\epsilon)$ the ball of radius $\epsilon > 0$ centered at the origin, and by $p_\beta(x,y) = (1+|x-y|)^\beta$ the polynomial weight of order $\beta > 0$. Operators to be discussed in this paper are of the form $\lambda I - T$, where $\lambda \in \mathbb{C}$ and T are integral operators

$$Tf(x) := \int_{\mathbb{R}^d} K_T(x,y) f(y) dy \tag{1.6}$$

with kernels K_T satisfying

$$\sup_{\epsilon>0} (1+\epsilon^{-\alpha}) \Big(\|\chi_{B(\epsilon)} r_{K_T p_\beta}\|_1 + \sup_{0<\delta\leq 1} \delta^{-\alpha} \|\chi_{B(\epsilon)} r_{\omega_\delta(K_T) p_\beta}\|_1 \Big) < \infty \tag{1.7}$$

for some $\alpha \in (0,1]$ and $\beta > 0$. The requirement (1.7) for the kernel K_T is equivalent to the existence of a positive constant C such that

$$\int_{B(\epsilon)} r_{K_T p_\beta}(z) dz \leq C\epsilon^\alpha, \quad \int_{\mathbb{R}^d} r_{K_T p_\beta}(z) dz \leq C,$$

and

$$\int_{B(\epsilon)} \omega_{\delta(K_T)p_\beta}(z) dz \leq C\epsilon^\alpha \delta^\alpha, \quad \int_{\mathbb{R}^d} r_{\omega_\delta(K_T)p_\beta}(z) dz \leq C\delta^\alpha$$

for all $0 < \epsilon, \delta \leq 1$. So measured in L^1-norm, kernels satisfying (1.7) could be thought to have singularity of order α at the origin, decay of order β at infinity, and Hölder continuity of order α. We remark that the kernel $G_\gamma, 0 < \gamma < d$, of the Bessel potential J_γ satisfies (1.7) with $0 < \alpha < \min(1, \gamma/2)$ and $\beta > 0$.

Let $\mathcal{D}_{1,p_\beta,\alpha}$ contain all operators of the form $\lambda I - T$, where $\lambda \in \mathbb{C}$ and T are integral operators whose kernels satisfy (1.7). A subalgebra $\mathcal{A}$ of another Banach algebra $\mathcal{B}$ is said to be *inverse-closed* if $A \in \mathcal{A}$ and A has an inverse A^{-1} in $\mathcal{B}$, then $A^{-1} \in \mathcal{A}$ [6,9,13]. In this paper, we show that $\mathcal{D}_{1,p_\beta,\alpha}$ is an inverse-closed Banach subalgebra of $\mathcal{B}(L^2)$, the Banach algebra of all bounded operators on L^2, see Theorem 1 for general results. In other words, if $A = \lambda I - T$ for some nonzero complex number λ and integral operator T with kernel satisfying (1.7), and if A has bounded inverse on L^2, then $A^{-1} = \lambda^{-1}I - \tilde{T}$ for some integral operator $\tilde{T}$ whose kernel satisfies (1.7) too.

The inverse-closed subalgebra was first studied for periodic functions with absolutely convergent Fourier series [19]. The inverse-closed property (Wiener's lemma) has been established for various infinite matrices and integral (pseudo-differential) operators, and it has numerous applications in numerical analysis, time-frequency analysis, and sampling theory. We refer the reader to [2,4,6–9, 13–16,18] and references therein for extensive literature on the subject.

In this paper, the capital letter C denotes an absolute constant which may be different at each occurrence.

2 Main theorem

In this section, we extend the inverse-closedness of $\mathcal{ID}_{1,p_\beta,\alpha}$ mentioned in the introduction to more general class of operators on weighted function spaces, see Theorem 1.

A positive continuous function u on $\mathbb{R}^d \times \mathbb{R}^d$ is said to be a *weight* if it is symmetric, diagonal-normalized and slow-varying,

$$u(x,y) = u(y,x) \quad \text{for all } x, y \in \mathbb{R}^d, \tag{2.1}$$

$$u(x,y) \geq 1 \text{ and } u(x,x) = 1 \quad \text{for all } x, y \in \mathbb{R}^d, \tag{2.2}$$

and

$$C_u := \sup_{x,y\in\mathbb{R}^d} \sup_{|x'|,|y'|\leq 1} \frac{u(x+x', y+y')}{u(x,y)} < \infty. \tag{2.3}$$

A weight u on $\mathbb{R}^d \times \mathbb{R}^d$ is said to be *p-radially-submultiplicative*, $1 \leq p \leq \infty$, if there exists another weight v on $\mathbb{R}^d \times \mathbb{R}^d$ such that

$$u(x,y) \leq u(x,z)v(z,y) + v(x,z)u(z,y) \quad \text{for all } x, y, z \in \mathbb{R}^d \tag{2.4}$$

and

$$\|r_{v/u}\|_{p/(p-1)} < \infty. \tag{2.5}$$

We call the weight v satisfying (2.4) and (2.5) *a companion weight* of the p-radially-submultiplicative weight u, c.f. [13,15]. The polynomial weights $p_\beta(x,y) = (1+|x-y|)^\beta$ with $\beta > d(p-1)/p$ and (sub)exponential weights $e_{D,\delta}(x,y) = \exp(D|x-y|^\delta)$ with $D > 0$ and $\delta \in (0,1)$ are p-radially-submultiplicative.

For $1 \leq q < \infty$, a positive locally integrable function w on $\mathbb{R}^d$ is said to be a Muckenhoupt A_q-*weight* if

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{-1/(q-1)}dx\Big)^{q-1} < \infty$$

when $1 < q < \infty$, and if

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q w(y)dy\Big)\Big(\sup_{x\in Q}\frac{1}{w(x)}\Big) < \infty$$

when $q = 1$, where the supremum is taken on all cubes Q in $\mathbb{R}^d$ [5]. Denote by $L^q_w := L^q_w(\mathbb{R}^d)$, the space of all measurable functions f on $\mathbb{R}^d$ with finite norm $\|f\|_{q,w} := (\int_{\mathbb{R}^d} |f(x)|^q w(x)dx)^{1/q} < \infty$, and let $\mathcal{B}(L^q_w)$ be the Banach algebra of all bounded linear operators on L^q_w. We remark that for any $1 \leq q < \infty$ and Muckenhoupt A_q-weight w, the Bessel potential J_γ is a bounded operator on L^q_w and hence $J_\gamma \in \mathcal{B}(L^q_w)$ by (1.3).

For $1 \leq p < \infty$, $0 < \alpha \leq 1$, and a p-radially-submultiplicative weight u on $\mathbb{R}^d \times \mathbb{R}^d$, let $\mathcal{D}_{p,u,\alpha}$ contain all integral operators $Tf(x) := \int_{\mathbb{R}^d} K_T(x,y)f(y)dy$ with finite norm

$$\|T\|_{\mathcal{D}_{p,u,\alpha}} := \sup_{\epsilon>0} (1+\epsilon^{-\alpha})\Big(\|r_{K_T u}\chi_{B(\epsilon)}\|_p + \sup_{0<\delta\leq 1} \delta^{-\alpha}\|r_{\omega_\delta(K_T)u}\chi_{B(\epsilon)}\|_p\Big) < \infty. \tag{2.6}$$

c.f. (1.7) with $p = 1$ and $u = p_\beta$. The family $\mathcal{D}_{p,u,\alpha}$ of integral operators just defined above is a Banach subalgebra of $\mathcal{B}(L^q_w)$ for all $1 \leq q < \infty$ and Muckenhoupt A_q-weights w, see Propositions 1 and 2.

For $1 \leq p < \infty$, $0 < \alpha \leq 1$, and a p-radially-submultiplicative weight u on $\mathbb{R}^d \times \mathbb{R}^d$, let

$$\mathcal{ID}_{p,u,\alpha} := \big\{\lambda I + T : \ \lambda \in \mathbb{C} \text{ and } T \in \mathcal{D}_{p,u,\alpha}\big\} \tag{2.7}$$

with

$$\|\lambda I + T\|_{\mathcal{ID}_{p,u,\alpha}} := |\lambda| + \|T\|_{\mathcal{D}_{p,u,\alpha}}. \tag{2.8}$$

Then $\mathcal{ID}_{p,u,\alpha}$ is the unital Banach algebra containing $\mathcal{D}_{p,u,\alpha}$. Furthermore, we show that it is an inverse-closed Banach subalgebra of $\mathcal{B}(L^q_w)$ for any $1 \leq q < \infty$ and Muckenhoupt A_q-weight w.

Theorem 1 *Let $1 \le p < \infty$, $0 < \alpha \le 1$, and let u be a p-radially-submultiplicative weight with companion weight v. Set $v_\tau(x,y) = v(x,y)\chi_{B(\tau)}(x-y)$ for $\tau > 0$. If there exist positive constants $D \in (0,\infty)$ and $\theta \in (0,1)$ such that*

$$\inf_{\tau\ge 1}\big(\|r_{v_\tau}\|_1 + t\|(1-\chi_{B(\tau)})r_{v/u}\|_{p/(p-1)}\big) \le Dt^\theta \quad \text{for all } t \ge 1, \tag{2.9}$$

then $\mathcal{D}_{p,u,\alpha}$ is an inverse-closed Banach subalgebra of $\mathcal{B}(L^q_w)$ for any $1 \le q < \infty$ and Muckenhoupt A_q-weight w.

The p-radially-submultiplicative polynomial weights $p_\beta(x,y) = (1+|x-y|)^\beta$ with $\beta > d(p-1)/p$ and $1 \le p < \infty$ satisfy (2.9), since in this case there exists an absolute constant C such that

$$\inf_{\tau\ge 1}\big(\|r_{v_\tau}\|_1 + t\|(1-\chi_{B(\tau)})r_{v/u}\|_{p/(p-1)}\big) \le C\inf_{\tau\ge 1}(\tau^d + t\tau^{-\beta+d(p-1)/p}) = Ct^{d/(\beta+d/p)},\ t \ge 1.$$

The p-radially-submultiplicative (sub)exponential weights $e_{D,\delta}(x,y) = \exp(D|x-y|^\delta)$ with $D > 0$ and $\delta \in (0,1)$ satisfy (2.9) too, because

$$\begin{aligned}
&\inf_{\tau\ge 1}\big(\|r_{v_\tau}\|_1 + t\|(1-\chi_{B(\tau)})r_{v/u}\|_{p/(p-1)}\big)\\
&\le C\inf_{\tau\ge 1}\Big(\tau^d\exp(D(2^\delta-1)\tau^\delta) + t\Big(\int_\tau^\infty \exp(D(2^\delta-2)p/(p-1)s^\delta)s^{d-1}ds\Big)^{(p-1)/p}\Big)\\
&\le C\inf_{\tau\ge 1}\big(\tau^d\exp(D(2^\delta-1)\tau^\delta) + t\exp(D(2^\delta-2)\tau^\delta)\tau^d\big) \le Ct^{2^\delta-1}(\ln t)^{d/\delta},\ t \ge 1,
\end{aligned}$$

where C is an absolute constant. Then we conclude from Theorem 1 that for every $\beta > 0$, $\mathcal{D}_{1,p_\beta,\alpha}$ is an inverse-closed Banach subalgebra of $\mathcal{B}(L^q_w)$ for any $1 \le q < \infty$ and Muckenhoupt A_q-weight w, which is highlighted in the introduction.

Denote by $\sigma_{q,w}(T)$ the spectrum of a bounded operator T on L^q_w. By Theorem 1, the spectrum of an operator $S \in \mathcal{D}_{p,u,\alpha}$ on L^q_w is independent on $1 \le q < \infty$ and Muckenhoupt A_q-weights w, c.f. [3].

Corollary 1 *Let p, u, α be as in Theorem 1, and let $S \in \mathcal{D}_{p,u,\alpha}$. Then*

$$\sigma_{q,w}(S) = \sigma_{q',w'}(S) \tag{2.10}$$

for all $1 \le q, q' < \infty$, Muckenhoupt A_q-weights w and Muckenhoupt $A_{q'}$-weights w'.

Applying the above corollary to the Bessel potential J_γ gives that $\sigma_{q,w}(J_\gamma) = [0,1]$ for all $1 \le q < \infty$ and Muckenhoupt A_q-weights w. The inclusion $\sigma_{q,w}(J_\gamma) \subset [0,1]$ follows directly from the kernel estimate of the operator $\lambda^{-1}I - (\lambda I - J_\gamma)^{-1}$ for $\lambda \notin [0,1]$, while the other inclusion $[0,1] \subset \sigma_{q,w}(J_\gamma)$ for arbitrary $1 \le q < \infty$ and Muckenhoupt A_q-weights w could be new to the best of the authors' knowledge.

3 Proofs

We start this section from some elementary properties for operators in $\mathcal{D}_{p,u,\alpha}$. We then show that $\mathcal{D}_{p,u,\alpha}$ is a Banach algebra in the second subsection, and establish a crucial paracompact estimate for the *fourth power* of an operator in $\mathcal{D}_{p,u,\alpha}$ in the third subsection. We devote the last subsection to the proof of Theorem 1.

3.1 Elementary properties for operators in $\mathcal{ID}_{p,u,\alpha}$

In this subsection, we provide some elementary properties for operators in $\mathcal{ID}_{p,u,\alpha}$.

Proposition 1 *Let $1 \le p < \infty$, $0 < \alpha \le 1$, and let u be a weight on $\mathbb{R}^d \times \mathbb{R}^d$. Then the following conclusions hold.*

(i) $\|S\|_{\mathcal{ID}_{p,u,\alpha'}} \le \|S\|_{\mathcal{ID}_{p,u,\alpha}}$ *for all* $S \in \mathcal{ID}_{p,u,\alpha}$ *and* $0 < \alpha' \le \alpha$.
(ii) $\|S\|_{\mathcal{ID}_{\tilde p,\tilde u,\alpha}} \le C\|S\|_{\mathcal{ID}_{p,u,\alpha}}$ *for all* $S \in \mathcal{ID}_{p,u,\alpha}$*, provided that* $1 \le \tilde p \le p$ *and* $\|r_{\tilde u/u}\|_{p\tilde p/(p-\tilde p)} < \infty$.
(iii) An operator S in $\mathcal{ID}_{p,u,\alpha}$ and its adjoint S^* have the same norm in $\mathcal{ID}_{p,u,\alpha}$, i.e., $\|S^*\|_{\mathcal{ID}_{p,u,\alpha}} = \|S\|_{\mathcal{ID}_{p,u,\alpha}}$ for all $S \in \mathcal{ID}_{p,u,\alpha}$.
(iv) $\|Sf\|_{q,w} \le C\|S\|_{\mathcal{ID}_{p,u,\alpha}}\|f\|_{q,w}$ *for all* $S \in \mathcal{ID}_{p,u,\alpha}$ *and* $f \in L^q_w$*, provided that* $1 \le q < \infty$*,* w *is a Muckenhoupt* A_q*-weight, and* $\|r_{u^{-1}}\|_{p/(p-1)} < \infty$.

Proof The first three conclusions follow from the definition of $\mathcal{ID}_{p,u,\alpha}$, while the last conclusion holds because r_{K_T} is radially decreasing and integrable on $\mathbb{R}^d$ for any integral operator $T \in \mathcal{D}_{p,u,\alpha}$ with kernel K_T [5].

3.2 Composition of two operators in $\mathcal{ID}_{p,u,\alpha}$

In this subsection, we show that composition S_1S_2 of two operators S_1, S_2 in $\mathcal{ID}_{p,u,\alpha}$ still belongs to $\mathcal{ID}_{p,u,\alpha}$.

Proposition 2 *Let $1 \le p < \infty, 0 < \alpha \le 1$, and let u be p-radially-submultiplicative. Then*

$$\|S_1S_2\|_{\mathcal{ID}_{p,u,\alpha}} \le C\|S_1\|_{\mathcal{ID}_{p,u,\alpha}}\|S_2\|_{\mathcal{ID}_{p,u,\alpha}} \quad \text{for all } S_1, S_2 \in \mathcal{ID}_{p,u,\alpha}. \tag{3.1}$$

Proof Take $S_1, S_2 \in \mathcal{ID}_{p,u,\alpha}$. Write $S_i = \lambda_i I + T_i$ for some $\lambda_i \in \mathbb{C}$ and $T_i \in \mathcal{D}_{p,u,\alpha}$ for $i = 1, 2$. Then $\|S_i\|_{\mathcal{ID}_{p,u,\alpha}} = |\lambda_i| + \|T_i\|_{\mathcal{D}_{p,u,\alpha}}$ for $i = 1, 2$, and

$$\begin{aligned}\|S_1S_2\|_{\mathcal{ID}_{p,u,\alpha}} &= |\lambda_1\lambda_2| + \|\lambda_2T_1 + \lambda_1T_2 + T_1T_2\|_{\mathcal{D}_{p,u,\alpha}}\\ &\le |\lambda_1||\lambda_2| + |\lambda_1|\,\|T_2\|_{\mathcal{D}_{p,u,\alpha}} + |\lambda_2|\,\|T_1\|_{\mathcal{D}_{p,u,\alpha}} + \|T_1T_2\|_{\mathcal{D}_{p,u,\alpha}}.\end{aligned}$$

Therefore the proof of (3.1) reduces to showing

$$\|T_1T_2\|_{\mathcal{D}_{p,u,\alpha}} \le C\|T_1\|_{\mathcal{D}_{p,u,\alpha}}\|T_2\|_{\mathcal{D}_{p,u,\alpha}} \quad \text{for all } T_1, T_2 \in \mathcal{D}_{p,u,\alpha}. \tag{3.2}$$

Let $T := T_1T_2$ be the composition of integral operators T_1 and T_2. Denote by K_1, K_2, K the kernels of operators $T_1, T_2 \in \mathcal{D}_{p,u,\alpha}$ and their composition T respectively. Observe that kernels K, K_1, K_2 are related by

$$K(x, y) = \int_{\mathbb{R}^d} K_1(x, z)K_2(z, y)dz. \tag{3.3}$$

Let v be the companion weight of the p-radially-submultiplicative weight u. Then we have the following pointwise estimate for r_{Ku}:

$$\begin{aligned}(r_{Ku}(z))^p &\le \sup_{|z_1-z_2|\ge|z|}\Big(\int_{\mathbb{R}^d}|(K_1u)(z_1,z_3)|\,|(K_2u)(z_3,z_2)| \\ &\qquad\times\big((v/u)(z_3,z_2) + (v/u)(z_1,z_3)\big)dz_3\Big)^p \\ &\le \sup_{|z_1-z_2|\ge|z|}\Big(\int_{\mathbb{R}^d}|(K_1u)(z_1,z_3)|^p|(K_2u)(z_3,z_2)|^p dz_3\Big) \\ &\qquad\times\big(\|(v/u)(\cdot,z_2)\|_{p/(p-1)} + \|(v/u)(z_1,\cdot)\|_{p/(p-1)}\big)^p \\ &\le C\sup_{|z_1-z_2|\ge|z|}\Big(\int_{|z_1-z_3|\ge|z_1-z_2|/2} + \int_{|z_2-z_3|\ge|z_1-z_2|/2}\Big) \\ &\qquad |r_{K_1u}(z_1-z_3)|^p|r_{K_2u}(z_3-z_2)|^p dz_3 \\ &\le C\|r_{K_2u}\|_p^p(r_{K_1u}(z/2))^p + C\|r_{K_1u}\|_p^p(r_{K_2u}(z/2))^p,\end{aligned} \tag{3.4}$$

where the first inequality follows from (3.3) and the p-radially-submultiplicative property for the weight u, the third inequality holds by the assumption $r_{v/u} \in L^{p/(p-1)}$, and the last inequality is true by the radially decreasing property for radial functions $r_{K_1 u}$ and $r_{K_2 u}$. Thus

$$\begin{aligned}\|\chi_{B(\epsilon)} r_{Ku}\|_p &\le C\|r_{K_2u}\|_p \|\chi_{B(\epsilon/2)} r_{K_1u}\|_p + C\|r_{K_1u}\|_p \|\chi_{B(\epsilon/2)} r_{K_2u}\|_p \\ &\le C\|T_1\|_{\mathcal{D}_{p,u,\alpha}} \|T_2\|_{\mathcal{D}_{p,u,\alpha}} \min(1,\epsilon^\alpha) \quad \text{for all } \epsilon > 0. \end{aligned} \tag{3.5}$$

By (3.3), we have that

$$\begin{aligned}\omega_\delta(K)(x,y) \le & \sup_{|x'|,|y'|\le\delta} \int_{\mathbb{R}^d} |K_1(x+x',z)||K_2(z,y+y') - K_2(z,y)|dz \\ & + \sup_{|x'|,|y'|\le\delta} \int_{\mathbb{R}^d} |K_1(x+x',z) - K_1(x,z)||K_2(z,y)|dz \end{aligned} \tag{3.6}$$

for all $x, y \in \mathbb{R}^d$ with $|x-y| \ge 4\delta$. Similar to the argument used to establish the first two inequalities in (3.4), we obtain from (3.6) and the p-radially-submultiplicative property of the weight u that

$$\begin{aligned}(r_{\omega_\delta(K)u}(z))^p \le\ & C \sup_{\substack{|z_1-z_2|\ge\max(|z|,4\delta),\\ |x'|,|y'|\le\delta}} \int_{\mathbb{R}^d} |(K_1u)(z_1+x',z_3)|^p \\ & \times |K_2(z_3,z_2+y') - K_2(z_3,z_2)|^p (u(z_3,z_2))^p dz_3 \\ & + C \sup_{\substack{|z_1-z_2|\ge\max(|z|,4\delta)\\ |x'|,|y'|\le\delta}} \int_{\mathbb{R}^d} |(K_2u)(z_3,z_2)|^p \\ & \times |K_1(z_1+x',z_3) - K_1(z_1,z_3)|^p (u(z_1,z_3))^p dz_3 \\ =:\ & I_1(z) + I_2(z). \end{aligned} \tag{3.7}$$

Thus for $|z| \ge 4\delta$,

$$\begin{aligned} I_1(z) \le\ & C \sup_{\substack{|z_1-z_2|\ge|z|\\ |x'|,|y'|\le\delta}} \int_{\mathbb{R}^d} (r_{K_1u}(z_1+x'-z_3))^p \Big((r_{\omega_\delta(K_2)u}(z_3-z_2))^p \big(1-\chi_{B(4\delta)}(z_3-z_2)\big) \\ & + (r_{K_2u}(z_3-z_2-y'))^p \chi_{B(4\delta)}(z_3-z_2) + (r_{K_2u}(z_3-z_2))^p \chi_{B(4\delta)}(z_3-z_2) \Big) dz_3 \\ & \quad \big(\text{by the definitions of } \ \omega_\delta(K_2)u, r_{K_2u}, \text{and } r_{\omega_\delta(K_2)u}\big) \\ \le\ & C \sup_{\substack{|z_1-z_2|\ge|z|\\ |x''|\le 2\delta}} \Big(\int_{|z_1-z_3|\ge 2|z_1-z_2|/3} + \int_{|z_2-z_3|\ge|z_1-z_2|/3} \Big) (r_{K_1u}(z_1+x''-z_3))^p \\ & \times \Big((r_{\omega_\delta(K_2)u}(z_3-z_2))^p + (r_{K_2u}(z_3-z_2))^p \chi_{B(5\delta)}(z_3-z_2) \Big) dz_3 \\ & \quad \big(\text{by changing variable and splitting } \mathbb{R}^d \text{ into two domains}\big) \\ \le\ & C\|r_{\omega_\delta(K_2)u}\|_p^p (r_{K_1u}(z/6))^p + C\|r_{K_1u}\|_p^p (r_{\omega_\delta(K_2)u}(z/3))^p \\ & + C\|\chi_{B(5\delta)} r_{K_2u}\|_p^p (r_{K_1u}(z/6))^p + C(r_{K_2u}(z/3))^p \\ & \times \Big(\sup_{|z_1-z_2|\ge|z|, |x''|\le 2\delta} \int_{|z_2-z_3|\le 5\delta} (r_{K_1u}(z_1+x''-z_3))^p dz_3 \Big) \\ & \quad \big(\text{because } r_{K_1u}, r_{K_2u} \text{ and } r_{\omega_\delta(K_2)u} \text{ are radially decreasing functions}\big) \\ \le\ & C\|T_2\|_{\mathcal{D}_{p,u,\alpha}}^p \delta^{\alpha p} (r_{K_1u}(z/6))^p + C\|T_1\|_{\mathcal{D}_{p,u,\alpha}}^p (r_{\omega_\delta(K_2)u}(z/6))^p \\ & + C\|T_1\|_{\mathcal{D}_{p,u,\alpha}}^p \delta^{\alpha p} (r_{K_2u}(z/6))^p, \end{aligned} \tag{3.8}$$

where the last inequality follows from the following estimate:

$$\begin{aligned} \int_{|z-z_0|\le 5\delta} (r_{K_1u}(z))^p dz \le & \int_{\Omega(z_0)} (r_{K_1u}(z))^p dz + (r_{K_1u}(z_1))^p \big(|\{z : |z-z_0| \le 5\delta\}| - |\Omega(z_0)| \big) \\ \le & \int_{|z|\le 5\delta} (r_{K_1u}(z))^p dz \quad \text{for all } z_0 \in \mathbb{R}^d, \end{aligned} \tag{3.9}$$

where $z_1 \in \mathbb{R}^d$ with $|z_1| = 5\delta$, $\Omega(z_0) = \{z : |z - z_0| \le 5\delta \text{ and } |z| \le 5\delta\}$ and $|\Omega(z_0)|$ is the Lebesgue measure of the set $\Omega(z_0)$. The inequalities in the above estimate hold due to the radially decreasing property of the function $r_{K_1 u}$. Applying similar argument used to establish (3.8), we obtain that

$$I_2(z) \le C\|T_1\|^p_{\mathcal{D}_{p,u,\alpha}} \delta^{\alpha p} (r_{K_2 u}(z/6))^p + C\|T_2\|^p_{\mathcal{D}_{p,u,\alpha}} (r_{\omega_\delta(K_1)u}(z/6))^p$$
$$+ C\|T_2\|^p_{\mathcal{D}_{p,u,\alpha}} \delta^{\alpha p} (r_{K_1 u}(z/6))^p \quad \text{for all } z \in \mathbb{R}^d \text{ with } |z| \ge 4\delta. \tag{3.10}$$

Substituting (3.8) and (3.10) into (3.7) and then using the radially decreasing property of the function $r_{K_i u}, r_{\omega_\delta(K_i)u}, i = 1, 2$, we obtain the following pointwise estimate of $r_{\omega_\delta(K)u}$ for all $z \in \mathbb{R}^d$:

$$r_{\omega_\delta(K)u}(z) \le C\|T_1\|_{\mathcal{D}_{p,u,\alpha}} \delta^\alpha r_{K_2 u}(z/6) + C\|T_2\|_{\mathcal{D}_{p,u,\alpha}} \delta^\alpha r_{K_1 u}(z/6)$$
$$+ C\|T_2\|_{\mathcal{D}_{p,u,\alpha}} r_{\omega_\delta(K_1)u}(z/6) + C\|T_1\|_{\mathcal{D}_{p,u,\alpha}} r_{\omega_\delta(K_2)u}(z/6). \tag{3.11}$$

Integrating both sides of the above estimate (3.11) yields

$$\|\chi_{B(\epsilon)} r_{\omega_\delta(K)u}\|_p \le C\|T_1\|_{\mathcal{D}_{p,u,\alpha}} \|T_2\|_{\mathcal{D}_{p,u,\alpha}} \delta^\alpha \min(1, \epsilon^\alpha) \tag{3.12}$$

for all $\delta \in (0, 1)$ and $\epsilon \in (0, \infty)$. Combining (3.5) and (3.12) proves (3.2), and hence the desired conclusion (3.1).

3.3 Paracompact estimates for operators in $\mathbb{D}_{p,u,\alpha}$

By Proposition 2, the composition of four operators in $\mathbb{D}_{p,u,\alpha}$ still lives in $\mathbb{D}_{p,u,\alpha}$. In this subsection, we establish a paracompact estimate for the fourth power of an operator in $\mathbb{D}_{p,u,\alpha}$.

Proposition 3 *Let $1 \le p < \infty$, $0 < \alpha \le 1$, and let u be a p-radially-submultiplicative weight on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying* (2.9) *for some $\theta \in (0, 1)$ and $D \in (0, \infty)$. Then*

$$\|S^4\|_{\mathbb{D}_{p,u,\alpha}} \le C\|S\|^{4-\tilde\theta}_{\mathbb{D}_{p,u,\alpha}} \|S\|^{\tilde\theta}_{\mathcal{B}^2} \quad \text{for all } S \in \mathbb{D}_{p,u,\alpha}, \tag{3.13}$$

where $\tilde\theta = \frac{\alpha(1-\theta)}{\alpha + d(1-\theta)}$.

For $1 \le p < \infty, 0 < \alpha \le 1$ and a weight u on $\mathbb{R}^d \times \mathbb{R}^d$, let $\mathcal{D}^0_{p,u,\alpha}$ contain all integral operators T with

$$\|T\|_{\mathcal{D}^0_{p,u,\alpha}} := \sup_{\epsilon > 0} (1 + \epsilon^{-\alpha}) \|r_{K_T u} \chi_{B(\epsilon)}\|_p < \infty, \tag{3.14}$$

where K_T is the kernel of the integral operator T. Comparing with integral operators in $\mathcal{D}_{p,u,\alpha}$ and $\mathcal{D}^0_{p,u,\alpha}$, we see that their kernels have singularity of same order at the origin and decay of same order at infinity. On the other hand, we also observe that modulus of kernels of integral operators in $\mathcal{D}_{p,u,\alpha}$ have certain singularity at the origin and decay at infinity, while there is no restriction on modulus of kernels of integral operators in $\mathcal{D}^0_{p,u,\alpha}$. Thus

$$\mathcal{D}_{p,u,\alpha} \subset \mathcal{D}^0_{p,u,\alpha}. \tag{3.15}$$

To prove Proposition 3, we need the following result about composition of three integral operators with the first and last operators belonging to $\mathcal{D}_{p,u,\alpha}$ and the middle integral operator living in $\mathcal{D}^0_{p,u,\alpha}$, a superset of $\mathcal{D}_{p,u,\alpha}$ by (3.15).

Lemma 1 *Let $1 \le p < \infty$, $0 < \alpha \le 1$, and u be p-radially-submultiplicative. Then*

$$\|T_1 T_2 T_3\|_{\mathcal{D}_{p,u,\alpha}} \le C\|T_1\|_{\mathcal{D}_{p,u,\alpha}} \|T_2\|_{\mathcal{D}^0_{p,u,\alpha}} \|T_3\|_{\mathcal{D}_{p,u,\alpha}} \tag{3.16}$$

for all $T_1, T_3 \in \mathcal{D}_{p,u,\alpha}$ and $T_2 \in \mathcal{D}^0_{p,u,\alpha}$.

Proof Take $T_1, T_3 \in \mathcal{D}_{p,u,\alpha}$ and $T_2 \in \mathcal{D}^0_{p,u,\alpha}$. Denote kernels of integral operators $T_1, T_2, T_3, T_1T_2, T_2T_3, T_1T_2T_3$ by K_1, K_2, K_3, K_{12}, K_{23} and K respectively. Then

$$K_{12}(x,y) = \int_{\mathbb{R}^d} K_1(x,z)K_2(z,y)dz, \quad K_{23}(x,y) = \int_{\mathbb{R}^d} K_2(x,z)K_3(z,y)dz \tag{3.17}$$

and

$$\begin{aligned} K(x,y) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_1(x,z_1)K_2(z_1,z_2)K_3(z_2,y)dz_1dz_2 \\ &= \int_{\mathbb{R}^d} K_1(x,z)K_{23}(z,y)dz = \int_{\mathbb{R}^d} K_{12}(x,z)K_3(z,y)dz. \end{aligned} \tag{3.18}$$

By (3.4),

$$r_{K_{12}u}(z) \le C\|r_{K_1u}\|_p r_{K_2u}(z/2) + C\|r_{K_2u}\|_p r_{K_1u}(z/2). \tag{3.19}$$

Applying (3.4) with kernels K_1 and K_2 replaced by K_2 and K_3 respectively, we obtain

$$r_{K_{23}u}(z) \le C\|r_{K_2u}\|_p r_{K_3u}(z/2) + C\|r_{K_3u}\|_p r_{K_2u}(z/2). \tag{3.20}$$

Integrating both sides of (3.19) and (3.20) yields

$$\|r_{K_{12}u}\|_p \le C\|r_{K_1u}\|_p\|r_{K_2u}\|_p \quad \text{and} \quad \|r_{K_{23}u}\|_p \le C\|r_{K_2u}\|_p\|r_{K_3u}\|_p. \tag{3.21}$$

Applying (3.4) again with kernels K_2 replaced by K_{23}, and then using (3.18), (3.20) and (3.21), we get the following pointwise estimate for r_{Ku}:

$$\begin{aligned} r_{Ku}(z) &\le C\|r_{K_1u}\|_p r_{K_{23}u}(z/2) + C\|r_{K_{23}u}\|_p r_{K_1u}(z/2) \\ &\le C\|r_{K_2u}\|_p\|r_{K_3u}\|_p r_{K_1u}(z/4) + C\|r_{K_1u}\|_p\|r_{K_3u}\|_p r_{K_2u}(z/4) \\ &\quad + C\|r_{K_1u}\|_p\|r_{K_2u}\|_p r_{K_3u}(z/4), \ z \in \mathbb{R}^d. \end{aligned} \tag{3.22}$$

Therefore

$$\|\chi_{B(\epsilon)} r_{Ku}\|_p \le C\|T_1\|_{\mathcal{D}_{p,u,\alpha}}\|T_2\|_{\mathcal{D}^0_{p,u,\alpha}}\|T_3\|_{\mathcal{D}_{p,u,\alpha}} \min(1,\epsilon^\alpha) \quad \text{for all} \ \ \epsilon > 0. \tag{3.23}$$

By (3.18),

$$\begin{aligned} \omega_\delta(K)(x,y) \le & \sup_{|x'|,|y'|\le\delta} \Big|\int_{\mathbb{R}^d} K_{12}(x+x',z)\big(K_3(z,y+y') - K_3(z,y)\big)dz\Big| \\ & + \sup_{|x'|\le\delta}\Big|\int_{\mathbb{R}^d} \big(K_1(x+x',z) - K_1(x,z)\big)K_{23}(z,y)dz\Big| \\ =: & \, I_{1,\delta}(x,y) + I_{2,\delta}(x,y) \end{aligned} \tag{3.24}$$

for all $x, y \in \mathbb{R}^d$ with $|x-y| \ge 4\delta$. Then for $z \in \mathbb{R}^d$ with $|z| \ge 4\delta$,

$$\begin{aligned} & \sup_{|z_1-z_2|\ge|z|} I_{1,\delta}(z_1,z_2)u(z_1,z_2) \\ & \le C\|T_3\|_{\mathcal{D}_{p,u,\alpha}}\delta^\alpha r_{K_{12}u}(z/6) + C\|T_1T_2\|_{\mathcal{D}^0_{p,u,\alpha}} r_{\omega_\delta(K_3)u}(z/6) \\ & \quad + C\|T_1T_2\|_{\mathcal{D}^0_{p,u,\alpha}}\delta^\alpha r_{K_3u}(z/6) \\ & \le C\|T_2\|_{\mathcal{D}^0_{p,u,\alpha}}\|T_3\|_{\mathcal{D}_{p,u,\alpha}}\delta^\alpha r_{K_1u}(z/12) + C\|T_1\|_{\mathcal{D}_{p,u,\alpha}}\|T_3\|_{\mathcal{D}_{p,u,\alpha}}\delta^\alpha r_{K_2u}(z/12) \\ & \quad + C\|T_1\|_{\mathcal{D}_{p,u,\alpha}}\|T_2\|_{\mathcal{D}^0_{p,u,\alpha}}\delta^\alpha r_{K_3u}(z/12) + C\|T_1\|_{\mathcal{D}_{p,u,\alpha}}\|T_2\|_{\mathcal{D}^0_{p,u,\alpha}} r_{\omega_\delta(K_3)u}(z/12), \end{aligned} \tag{3.25}$$

where the first inequality is obtained from (3.8) with T_1 and T_2 replaced by T_1T_2 and T_3 respectively, and the second inequality follows from (3.19). Similar to (3.25), we get

$$\begin{aligned}&\sup_{|z_1-z_2|\ge|z|} I_{2,\delta}(z_1,z_2)u(z_1,z_2)\\ &\le C\|T_2\|_{\mathcal{D}^0_{p,u,\alpha}}\|T_3\|_{\mathcal{D}_{p,u,\alpha}}\delta^\alpha r_{K_1u}(z/12)+C\|T_1\|_{\mathcal{D}_{p,u,\alpha}}\|T_3\|_{\mathcal{D}_{p,u,\alpha}}\delta^\alpha r_{K_2u}(z/12)\\ &+C\|T_1\|_{\mathcal{D}_{p,u,\alpha}}\|T_2\|_{\mathcal{D}^0_{p,u,\alpha}}\delta^\alpha r_{K_3u}(z/12)+C\|T_2\|_{\mathcal{D}^0_{p,u,\alpha}}\|T_3\|_{\mathcal{D}_{p,u,\alpha}}r_{\omega_\delta(K_1)u}(z/12)\end{aligned}\tag{3.26}$$

for all $z\in\mathbb{R}^d$ with $|z|\ge 4\delta$. Combining (3.24), (3.25) and (3.26) yields the following pointwise estimate for the radial function $r_{\omega_\delta(K)u}$:

$$\begin{aligned}r_{\omega_\delta(K)u}(z)\le& C\|T_2\|_{\mathcal{D}^0_{p,u,\alpha}}\|T_3\|_{\mathcal{D}_{p,u,\alpha}}\delta^\alpha r_{K_1u}(z/12)+C\|T_1\|_{\mathcal{D}_{p,u,\alpha}}\|T_3\|_{\mathcal{D}_{p,u,\alpha}}\delta^\alpha r_{K_2u}(z/12)\\ &+C\|T_1\|_{\mathcal{D}_{p,u,\alpha}}\|T_2\|_{\mathcal{D}^0_{p,u,\alpha}}\delta^\alpha r_{K_3u}(z/12)+C\|T_2\|_{\mathcal{D}^0_{p,u,\alpha}}\|T_3\|_{\mathcal{D}_{p,u,\alpha}}r_{\omega_\delta(K_1)u}(z/12)\\ &+C\|T_1\|_{\mathcal{D}_{p,u,\alpha}}\|T_2\|_{\mathcal{D}^0_{p,u,\alpha}}r_{\omega_\delta(K_3)u}(z/12),\ z\in\mathbb{R}^d.\end{aligned}\tag{3.27}$$

Integrating the above estimate for $r_{\omega_\delta(K)u}$ at both sides leads to

$$\|\chi_{B(\epsilon)}r_{\omega_\delta(K)u}\|_p\le C\|T_1\|_{\mathcal{D}_{p,u,\alpha}}\|T_2\|_{\mathcal{D}^0_{p,u,\alpha}}\|T_3\|_{\mathcal{D}_{p,u,\alpha}}\delta^\alpha\min(1,\epsilon^\alpha)\tag{3.28}$$

for all $\delta\in(0,1)$ and $\epsilon\in(0,\infty)$. The desired conclusion (3.16) then follows from (3.23) and (3.28).

To prove Proposition 3, we also need the following paracompact estimate for the composition of two integral operators in $\mathcal{D}_{p,u,\alpha}$.

Lemma 2 *Let $1\le p<\infty$, $0<\alpha\le 1$, and let u be a p-radially-submultiplicative weight with companion weight v satisfying* (2.9) *for some $\theta\in(0,1)$ and $D\in(0,\infty)$. Then*

$$\|T_1T_2\|_{D^0_{p,u,\alpha}}\le C\|T_1\|_{\mathcal{D}_{p,u,\alpha}}\|T_2\|_{\mathcal{D}_{p,u,\alpha}}\Big(\Big(\frac{\|T_1\|_{\mathcal{B}^2}}{\|T_1\|_{\mathcal{D}_{p,u,\alpha}}}\Big)^{\tilde\theta}+\Big(\frac{\|T_2\|_{\mathcal{B}^2}}{\|T_2\|_{\mathcal{D}_{p,u,\alpha}}}\Big)^{\tilde\theta}\Big)\tag{3.29}$$

for all $T_1,T_2\in\mathcal{D}_{p,u,\alpha}$, where $\tilde\theta=\frac{\alpha(1-\theta)}{\alpha+d(1-\theta)}$.

Proof Without loss of generality, we assume that

$$\|T_1\|_{\mathcal{B}^2}\le\|T_1\|_{\mathcal{D}_{p,u,\alpha}}\quad\text{and}\quad\|T_2\|_{\mathcal{B}^2}\le\|T_2\|_{\mathcal{D}_{p,u,\alpha}}\tag{3.30}$$

as otherwise the conclusion (3.29) follows from Proposition 2 and the trivial inequality $\|T_1T_2\|_{\mathcal{D}^0_{p,u,\alpha}}\le\|T_1T_2\|_{\mathcal{D}_{p,u,\alpha}}$. Denote by K_1,K_2 and $\tilde K$ kernels of T_1,T_2 and T_1T_2 respectively. Then those three kernels are related by

$$\tilde K(x,y)=\int_{\mathbb{R}^d}K_1(x,z)K_2(z,y)dz.$$

This together with the p-radially-submultiplicative property of the weight u implies that

$$|(\tilde Ku)(x,y)|\le\int_{\mathbb{R}^d}|(K_1u)(x,z)||(K_2v)(z,y)|dz+\int_{\mathbb{R}^d}|(K_1v)(x,z)||(K_2u)(z,y)|dz,\tag{3.31}$$

where v is the companion weight of u.

Let $\epsilon_0=(\frac{\|T_2\|_{\mathcal{B}^2}}{\|T_2\|_{\mathcal{D}_{p,u,\alpha}}})^{\frac{1-\theta}{\alpha+d(1-\theta)}}\in(0,1]$, and $\tau\ge1$ be so chosen that

$$\|r_{v_\tau}\|_1+(\epsilon_0)^{-\alpha/(1-\theta)}\|(1-\chi_{B(\tau)})r_{v/u}\|_{p/(p-1)}\le D(\epsilon_0)^{-\alpha\theta/(1-\theta)},\tag{3.32}$$

where $v_\tau(x,y)=v(x,y)\chi_{B(\tau)}(x-y)$. The existence of such a positive number $\tau\ge1$ follows from (2.9). For $x,y\in\mathbb{R}^d$ with $|x-y|\ge 4\epsilon_0$,

$$\begin{aligned}|K_2(x,y)|&\le\omega_{\epsilon_0}(K_2)(x,y)+\epsilon_0^{-2d}\Big|\int_{|x'|,|y'|\le\epsilon_0}K_2(x+x',y+y')dx'dy'\Big|\\ &\le\omega_{\epsilon_0}(K_2)(x,y)+\epsilon_0^{-3d/2}\|T_2\chi_{B(\epsilon_0)}(\cdot-y)\|_2\le\omega_{\epsilon_0}(K_2)(x,y)+\epsilon_0^{-d}\|T_2\|_{\mathcal{B}^2}\end{aligned}\tag{3.33}$$

c.f. [14, Lemma 3.10]. Thus

$$
\begin{aligned}
&\int_{\mathbb{R}^d} |(K_1u)(x,z)||(K_2v)(z,y)|dz \\
&\le \Big(\int_{|z-y|<4\epsilon_0} + \int_{|z-y|\ge\tau}\Big)|(K_1u)(x,z)||(K_2v)(z,y)|dz \\
&\quad + \int_{4\epsilon_0\le|z-y|\le\tau} |(K_1u)(x,z)|\omega_{\epsilon_0}(K_2)(z,y)v(z,y)dz \\
&\quad + \epsilon_0^{-d}\|T_2\|_{\mathcal{B}^2}\int_{4\epsilon_0\le|z-y|\le\tau}|(K_1u)(x,z)|v(z,y)dz \\
&=: I_1(x,y)+I_{2,\tau}(x,y)+I_{3,\tau}(x,y)+I_{4,\tau}(x,y). \qquad (3.34)
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\sup_{|z_1-z_2|\ge|z|} I_1(z_1,z_2) &\le C \sup_{|z_1-z_2|\ge|z|}\Big(\big(\int_{|z_1-z_3|\ge|z_1-z_2|/2}+\int_{|z_2-z_3|\ge|z_1-z_2|/2}\big) \\
&\qquad |(K_1u)(z_1,z_3)|^p|(K_2u)(z_3,z_2)|^p\chi_{B(4\epsilon_0)}(z_3-z_2)dz_3\Big)^{1/p} \\
&\le Cr_{K_1u}(z/2)\|\chi_{B(4\epsilon_0)}r_{K_2u}\|_p + Cr_{K_2u}(z/2) \\
&\qquad \times \sup_{|z_1-z_2|\ge|z|}\Big(\int_{|z_3-z_2|\le4\epsilon_0}|r_{K_1u}(z_1-z_3)|^pdz_3\Big)^{1/p} \\
&\le C(\epsilon_0)^\alpha\big(\|T_2\|_{\mathcal{D}_{p,u,\alpha}}r_{K_1u}(z/2)+\|T_1\|_{\mathcal{D}_{p,u,\alpha}}r_{K_2u}(z/2)\big), \qquad (3.35)
\end{aligned}
$$

where the last inequality follows from (3.9). Similarly,

$$
\begin{aligned}
&\sup_{|z_1-z_2|\ge|z|} I_{2,\tau}(z_1,z_2) \\
&\le C\|(1-\chi_{B(\tau)})r_{v/u}\|_{p/(p-1)}\big(\|T_2\|_{\mathcal{D}_{p,u,\alpha}}r_{K_1u}(z/2)+\|T_1\|_{\mathcal{D}_{p,u,\alpha}}r_{K_2u}(z/2)\big) \\
&\le C(\epsilon_0)^\alpha\big(\|T_2\|_{\mathcal{D}_{p,u,\alpha}}r_{K_1u}(z/2)+\|T_1\|_{\mathcal{D}_{p,u,\alpha}}r_{K_2u}(z/2)\big) \qquad (3.36)
\end{aligned}
$$

by (3.32) and the radially decreasing property for functions r_{Ku} and $r_{\omega_\delta(K)u}$,

$$
\begin{aligned}
\sup_{|z_1-z_2|\ge|z|} I_{3,\tau}(z_1,z_2) &\le C\sup_{|z_1-z_2|\ge|z|}\Big(\int_{\mathbb{R}^d}|(K_1u)(z_1,z_3)|^p|(\omega_{\epsilon_0}(K_2)u)(z_3,z_2)|^pdz_3\Big)^{1/p} \\
&\le C\|T_1\|_{\mathcal{D}_{p,u,\alpha}}r_{\omega_{\epsilon_0}(K_2)u}(z/2)+C(\epsilon_0)^\alpha\|T_2\|_{\mathcal{D}_{p,u,\alpha}}r_{K_1u}(z/2), \qquad (3.37)
\end{aligned}
$$

and

$$
\begin{aligned}
\sup_{|z_1-z_2|\ge|z|} I_{4,\tau}(z_1,z_2) &\le \epsilon_0^{-d}\|T_2\|_{\mathcal{B}^2}\sup_{|z_1-z_2|\ge|z|}\Big(\big(\int_{|z_1-z_3|\ge|z|/2}+\int_{|z_1-z_3|\le|z|/2}\big) \\
&\qquad |(K_1u)(z_1,z_3)|v_\tau(z_3,z_2)dz_3\Big) \\
&\le C\epsilon_0^{-d}\|T_2\|_{\mathcal{B}^2}\|r_{v_\tau}\|_1\, r_{K_1u}(z/2)+C\epsilon_0^{-d}\|T_2\|_{\mathcal{B}^2}\|\chi_{B(|z|/2)}r_{K_1u}\|_p r_{v_\tau}(z/2) \\
&\le C(\epsilon_0)^\alpha\|T_2\|_{\mathcal{D}_{p,u,\alpha}}r_{K_1u}(z/2) \\
&\quad +C(\epsilon_0)^{\alpha/(1-\theta)}\|T_1\|_{\mathcal{D}_{p,u,\alpha}}\|T_2\|_{\mathcal{D}_{p,u,\alpha}}\min(1,|z|^\alpha)r_{v_\tau}(z/2). \qquad (3.38)
\end{aligned}
$$

Combining (3.34)–(3.38) leads to the pointwise estimate for the first term on the right hand side of the inequality (3.31):

$$
\begin{aligned}
&\sup_{|z_1-z_2|\ge|z|}\int_{\mathbb{R}^d}|(K_1u)(z_1,z_2)||(K_2v)(z_3,z_2)|dz_3 \\
&\le C\|T_1\|_{\mathcal{D}_{p,u,\alpha}}r_{\omega_{\epsilon_0}(K_2)u}(z/2)+C(\epsilon_0)^\alpha\big(\|T_2\|_{\mathcal{D}_{p,u,\alpha}}r_{K_1u}(z/2)+\|T_1\|_{\mathcal{D}_{p,u,\alpha}}r_{K_2u}(z/2)\big) \\
&\quad +C(\epsilon_0)^{\alpha/(1-\theta)}\|T_1\|_{\mathcal{D}_{p,u,\alpha}}\|T_2\|_{\mathcal{D}_{p,u,\alpha}}\min(1,|z|^\alpha)r_{v_\tau}(z/2). \qquad (3.39)
\end{aligned}
$$

Observe that for $\tau \geq 1$ and $1 \leq p < \infty$,

$$\begin{aligned}\|r_{v_\tau}\|_p &\leq C\Big(\sum_{k\in\mathbb{Z}^d}\big(\sup_{\substack{|m-n|\geq|k|\\ m,n\in\mathbb{Z}^d}} r_{v_\tau}(m,n)\big)^p\Big)^{1/p}\\ &\leq C\sum_{k\in\mathbb{Z}^d}\big(\sup_{\substack{|m-n|\geq|k|\\ m,n\in\mathbb{Z}^d}} r_{v_\tau}(m,n)\big) \leq C\|r_{v_\tau}\|_1\end{aligned}$$

by the slow-varying property (2.3) for the companion weight v. This together with (3.32) and (3.39) proves that

$$\begin{aligned}&\Big\|\big(\sup_{|z_1-z_2|\geq|\cdot|}\int_{\mathbb{R}^d}|(K_1u)(z_1,z_3)||(K_2v)(z_3,z_2)|dz_3\big)\chi_{B(\epsilon)}\Big\|_p\\ &\leq C\big((\epsilon_0)^\alpha+(\epsilon_0)^{\alpha/(1-\theta)}\|r_{v_\tau}\|_p\big)\|T_1\|_{\mathcal{D}_{p,u,\alpha}}\|T_2\|_{\mathcal{D}_{p,u,\alpha}}\min(1,\epsilon^\alpha)\\ &\leq C\|T_1\|_{\mathcal{D}_{p,u,\alpha}}\|T_2\|^{1-\tilde\theta}_{\mathcal{D}_{p,u,\alpha}}\|T_2\|^{\tilde\theta}_{\mathcal{B}^2}\min(1,\epsilon^\alpha). \qquad (3.40)\end{aligned}$$

Let $\tilde\epsilon_0 = \big(\frac{\|T_1\|_{\mathcal{B}^2}}{\|T_1\|_{\mathcal{D}_{p,u,\alpha}}}\big)^{\frac{1-\theta}{\alpha+d(1-\theta)}} \in (0,1]$, and $\tilde\tau \geq 1$ be so chosen that

$$\|r_{v_{\tilde\tau}}\|_1 + (\tilde\epsilon_0)^{-\alpha/(1-\theta)}\|(1-\chi_{B(\tilde\tau)})r_{v/u}\|_{p/(p-1)} \leq D(\tilde\epsilon_0)^{-\alpha\theta/(1-\theta)}.$$

Applying similar argument to establish (3.40), we have that

$$\begin{aligned}&\Big\|\big(\sup_{|z_1-z_2|\geq|\cdot|}\int_{\mathbb{R}^d}|(K_1v)(z_1,z_3)||(K_2u)(z_3,z_2)|dz_3\big)\chi_{B(\epsilon)}\Big\|_p\\ &\leq C\|T_2\|_{\mathcal{D}_{p,u,\alpha}}\|T_1\|^{1-\tilde\theta}_{\mathcal{D}_{p,u,\alpha}}\|T_1\|^{\tilde\theta}_{\mathcal{B}^2}\min(1,\epsilon^\alpha). \qquad (3.41)\end{aligned}$$

Therefore the desired conclusion (3.29) follows from (3.31), (3.40) and (3.41).

Lemma 3 *Let $1 \leq p < \infty$, $0 \leq \alpha \leq 1$, and u be a p-radially-submultiplicative weight on $\mathbb{R}^d \times \mathbb{R}^d$. Take $S = \lambda I + T \in \mathbb{D}_{p,u,\alpha}$ for some $\lambda \in \mathbb{C}$ and $T \in \mathcal{D}_{p,u,\alpha}$. Then*

$$|\lambda| \leq \|S\|_{\mathcal{B}(L^q_w)} \qquad (3.42)$$

for all $1 \leq q < \infty$ and Muckenhoupt A_q-weights w.

Proof We mimic the argument used in [14]. Denote the kernel of the operator T by K. Let ϕ be a nonzero smooth function supported on $[-1/2, 1/2]^d$ and define $\phi_z(x) = \phi(x)e^{izx}$, $z \in \mathbb{R}^d$. Then for $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^d$,

$$\|\phi_{t\mathbf{1}}\|_{q,w} = \|\phi\|_{q,w}, \ t \in \mathbb{R}, \qquad (3.43)$$

and

$$\lim_{t\to+\infty}\|T\phi_{t\mathbf{1}}\|_{q,w} = 0 \qquad (3.44)$$

because $T\phi_{t\mathbf{1}}$ is dominated by $g = \int_{\mathbb{R}^d}|K(\cdot,y)||\phi(y)|dy$, which belongs to L^q_w by Proposition 1, and $\lim_{t\to+\infty} T\phi_{t\mathbf{1}}(x) = 0$ for almost all $x \in \mathbb{R}^d$, which follows from

$$\begin{aligned}|T\phi_{t\mathbf{1}}(x)| \leq &\int_{\mathbb{R}^d}|r_K(x-y)||\phi(y+(dt)^{-1}\pi\mathbf{1})-\phi(y)|dy\\ &+\int_{|x-y|\leq 4/t}(|K(x,y+(dt)^{-1}\pi\mathbf{1})|+|K(x,y)|)|\phi(y+(dt)^{-1}\pi\mathbf{1})|dy\\ &+\int_{|x-y|\geq 4/t}\omega_{t^{-1}}(K)(x,y)|\phi(y+(dt)^{-1}\pi\mathbf{1})|dy\\ \to 0 \quad &\text{as} \quad t\to+\infty \quad \text{for almost all} \quad x\in\mathbb{R}^d.\end{aligned}$$

Combining (3.43) and (3.44) leads to

$$|\lambda|\|\phi\|_{q,w} = \lim_{t\to\infty} \|S\phi_{t\mathbf{1}}\|_{q,w} \le \|S\|_{\mathcal{B}(L^q_w)} \lim_{t\to\infty} \|\phi_{t\mathbf{1}}\|_{q,w} = \|S\|_{\mathcal{B}(L^q_w)} \|\phi\|_{q,w}.$$

Hence (3.42) is proved.

We finish this subsection with the proof of Proposition 3.

Proof (Proof of Proposition 3) Take $S \in \mathbb{D}_{p,u,\alpha}$. Write $S = \lambda I + T$ for some $\lambda \in \mathbb{C}$ and $T \in \mathcal{D}_{p,u,\alpha}$. Then

$$\|S\|_{\mathcal{B}^2} \le C\|S\|_{\mathbb{D}_{p,u,\alpha}} \tag{3.45}$$

by Proposition 1, and

$$|\lambda| \le \|S\|_{\mathcal{B}^2} \tag{3.46}$$

by (3.42) in Lemma 3. Therefore

$$\begin{aligned} \|S^4\|_{\mathbb{D}_{p,u,\alpha}} &\le |\lambda|^4 + 4|\lambda|^3\|T\|_{\mathcal{D}_{p,u,\alpha}} + 6|\lambda|^2\|T^2\|_{\mathcal{D}_{p,u,\alpha}} + 4|\lambda|\|T^3\|_{\mathcal{D}_{p,u,\alpha}} + \|T^4\|_{\mathcal{D}_{p,u,\alpha}} \\ &\le C\|S\|^4_{\mathcal{B}^2} + C\|S\|^3_{\mathcal{B}^2}\|T\|_{\mathcal{D}_{p,u,\alpha}} + C\|S\|^2_{\mathcal{B}^2}\|T\|^2_{\mathcal{D}_{p,u,\alpha}} \\ &\quad + C\|S\|_{\mathcal{B}^2}\|T\|^3_{\mathcal{D}_{p,u,\alpha}} + C\|T\|^2_{\mathcal{D}_{p,u,\alpha}}\|T^2\|_{\mathcal{D}^0_{p,u,\alpha}} \\ &\le C\|S\|_{\mathcal{B}^2}\|S\|^3_{\mathbb{D}_{p,u,\alpha}} + C\|S\|^{4-\tilde\theta}_{\mathbb{D}_{p,u,\alpha}}\|S-\lambda I\|^{\tilde\theta}_{\mathcal{B}^2} \le C\|S\|^{\tilde\theta}_{\mathcal{B}^2}\|S\|^{4-\tilde\theta}_{\mathbb{D}_{p,u,\alpha}}, \end{aligned} \tag{3.47}$$

where the second inequality follows from (3.46) and Lemma 1, the third inequality holds by (3.45) and Lemma 2, and the last inequality is true by (3.45) and (3.46). This proves (3.13) and hence completes the proof.

3.4 Proof of Theorem 1

An operator $T \in \mathcal{B}(L^q_w)$ is said to have L^q_w*-stability* if there exists a positive constant A such that

$$A\|f\|_{q,w} \le \|Tf\|_{q,w} \quad \text{for all } f \in L^q_w$$

[1,10,11,17]. To prove Theorem 1, we recall a result in [10] on equivalence among L^q_w-stability of an operator in $\mathcal{D}_{p,u,\alpha}$ for different exponents $1 \le q < \infty$ and Muckenhoupt A_q-weights w.

Lemma 4 ([10]) *Let $0 < \alpha \le 1, \lambda \in \mathbb{C}$, and let T be an integral operator with its kernel K satisfying*

$$\|r_K\|_1 + \sup_{0<\delta\le 1} \delta^{-\alpha}\|r_{\omega_\delta(K)}\|_1 + \sup_{0<\delta\le 1} \delta^{-\alpha}\|r_K\chi_{|\cdot|\le\delta}\|_1 < \infty. \tag{3.48}$$

If the operator $\lambda I - T$ has stability on L^q_w for some $1 \le q < \infty$ and Muckenhoupt A_q-weight w, then it has stability on $L^{q'}_{w'}$ for all $1 \le q' < \infty$ and Muckenhoupt $A_{q'}$-weights w'.

We have all ingredients to prove Theorem 1.

Proof (Proof of Theorem 1) Let $1 \le q < \infty$, w be a Muckenhoupt A_q-weight, and let $S \in \mathbb{D}_{p,u,\alpha}$ have bounded inverse $\tilde S \in \mathcal{B}(L^q_w)$. Then

$$\|\tilde S f\|_{q,w} \le \|\tilde S\|_{\mathcal{B}(L^q_w)}\|f\|_{q,w} \tag{3.49}$$

and

$$S\tilde S f = \tilde S S f = f \quad \text{for all } f \in L^q_w \tag{3.50}$$

by the definition of the bounded operator $\tilde S$ on L^q_w, and

$$\|Sf\|_{q,w} \le \|S\|_{\mathcal{B}(L^q_w)}\|f\|_{q,w} \le C\|S\|_{\mathbb{D}_{p,u,\alpha}}\|f\|_{q,w} \quad \text{for all } f \in L^q_w \tag{3.51}$$

by Proposition 1. From (3.49)–(3.51) it follows that S has L^q_w-stability,

$$\|f\|_{q,w} = \|\tilde{S}Sf\|_{q,w} \le \|\tilde{S}\|_{\mathcal{B}(L^q_w)}\|Sf\|_{q,w} \quad \text{for all } f \in L^q_w.$$

Observe that the kernel of an integral operator in $\mathcal{D}_{p,u,\alpha}$ satisfies (3.48) when the weight u has the p-radially-submultiplicative property. Therefore the operator $S \in \mathcal{D}_{p,u,\alpha}$ has L^2-stability by Lemma 4. So there are positive constants A and B such that

$$A\|f\|_2^2 \le \langle S^*Sf, f\rangle \le B\|f\|_2^2 \quad \text{for all } f \in L^2 \tag{3.52}$$

by the L^2-stability and Proposition 1. Set $R = \frac{2}{A+B}S^*S - I$. Then R belongs to $\mathcal{D}_{p,u,\alpha}$ by Propositions 1 and 2, and

$$\|R\|_{\mathcal{B}^2} \le \frac{B-A}{B+A} < 1 \tag{3.53}$$

by (3.52). Moreover, there exists an absolute constant C_1 by Propositions 2 and 3 such that

$$\|R^{n+1}\|_{\mathcal{D}_{p,u,\alpha}} \le C_1\|R\|_{\mathcal{D}_{p,u,\alpha}}\|R^n\|_{\mathcal{D}_{p,u,\alpha}} \tag{3.54}$$

and

$$\|R^{4n}\|_{\mathcal{D}_{p,u,\alpha}} \le C_1\|R^n\|^{4-\tilde{\theta}}_{\mathcal{D}_{p,u,\alpha}}\|R^n\|^{\tilde{\theta}}_{\mathcal{B}^2} \le C_1\|R^n\|^{4-\tilde{\theta}}_{\mathcal{D}_{p,u,\alpha}}(\|R\|_{\mathcal{B}^2})^{n\tilde{\theta}} \tag{3.55}$$

for all $n \ge 1$. Write $n = \sum_{j=0}^k \epsilon_j 4^j$ with $\epsilon_j \in \{0,1,2,3\}, 0 \le j \le k$, and $k \ge 0$ being so chosen that $\epsilon_k \ne 0$. Applying (3.54) and (3.55) iteratively we obtain that

$$\begin{aligned}
\|R^n\|_{\mathcal{D}_{p,u,\alpha}} &\le (C_1\|R\|_{\mathcal{D}_{p,u,\alpha}})^{\epsilon_0}\|R^{n-\epsilon_0}\|_{\mathcal{D}_{p,u,\alpha}} \\
&\le (C_1\|R\|_{\mathcal{D}_{p,u,\alpha}})^{\epsilon_0}(\|R\|_{\mathcal{B}^2})^{(n-\epsilon_0)\tilde{\theta}/4}(\|R^{(n-\epsilon_0)/4}\|_{\mathcal{D}_{p,u,\alpha}})^{4-\tilde{\theta}} \\
&\le \cdots \\
&\le (C_1\|R\|_{\mathcal{D}_{p,u,\alpha}})^{\sum_{j=0}^k \epsilon_j(4-\tilde{\theta})^j}(\|R\|_{\mathcal{B}^2})^{\sum_{i=1}^k\sum_{j=i}^k \epsilon_j 4^{j-i}(4-\tilde{\theta})^{i-1}\tilde{\theta}} \\
&\le (\|R\|_{\mathcal{B}^2})^n\Big(\frac{C_1\|R\|_{\mathcal{D}_{p,u,\alpha}}}{\|R\|_{\mathcal{B}^2}}\Big)^{(4-\tilde{\theta})n^{\log_4(4-\tilde{\theta})}}.
\end{aligned} \tag{3.56}$$

Combining (3.53) and (3.56) proves the exponential decay property for the operator norm of $R^n \in \mathcal{D}_{p,u,\alpha}$, i.e., $\|R^n\|_{\mathcal{D}_{p,u,\alpha}} \le Cr^n$ for some $r \in (0,1)$. This implies that

$$\sum_{n=0}^{\infty}(-1)^n R^n \in \mathcal{D}_{p,u,\alpha}. \tag{3.57}$$

Set $F := \frac{2}{A+B}\big(\sum_{n=0}^\infty (-1)^n R^n\big)S^*$. Then

$$FS = \Big(\sum_{n=0}^{\infty}(-1)^n R^n\Big)(I+R) = I, \tag{3.58}$$

and

$$F \in \mathcal{D}_{p,u,\alpha} \tag{3.59}$$

(and it is a bounded operator on L^q_w) by (3.57) and Propositions 1 and 2. This together with (3.50) and (3.58) implies that

$$(\tilde{S} - F)g = (\tilde{S} - F)S\tilde{S}g = (\tilde{S}S - FS)\tilde{S}g = \tilde{S}g - \tilde{S}g = 0 \quad \text{for all } g \in L^q_w. \tag{3.60}$$

Hence the inverse $\tilde{S}$ of the operator S on L^q_w is equal to F and thus belongs to $\mathcal{D}_{p,u,\alpha}$ by (3.59).

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