Spectra of Bochner-Riesz means on L^p

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Abstract

The Bochner-Riesz means are shown to have either the unit interval $[0, 1]$ or the whole complex plane as their spectra on $L^p, 1 \le p < \infty$

1 Introduction and Main Results

Define Fourier transform $\hat{f}$ of an integrable function f by

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx$$

and extend its definition to all tempered distributions as usual. Consider Bochner-Riesz means $B_\delta, \delta > 0$, on $\mathbb{R}^d$,

$$\widehat{B_\delta f}(\xi) := (1 - |\xi|^2)_+^\delta \hat{f}(\xi),$$

where $t_+ := \max(t, 0)$ for $t \in \mathbb{R}$ [2, 11, 20]. A famous conjecture in Fourier analysis is that the Bochner-Riesz mean B_δ is bounded on $L^p := L^p(\mathbb{R}^d)$, the space of all p-integrable functions on $\mathbb{R}^d$ with its norm denoted by $\|\cdot\|_p$, if and only if

$$\delta > \left(d\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}\right)_+, \ 1 \le p < \infty. \tag{1.1}$$

The requirement (1.1) on the index δ is necessary for L^p boundedness of the Bochner-Riesz mean B_δ [12]. The sufficiency is completely solved only for dimension two [5] and it is still open for high dimensions, see [3, 14, 21, 27, 28, 29, 31] and references therein for recent advances.

Denote the identity operator by I. For $\lambda \notin [0, 1]$, we first show that Bochner-Reisz means B_δ is bounded on L^p if and only if its resolvents $(zI - B_\delta)^{-1}$, which are multiplier operators with symbols $(z - (1 - |\xi|^2)_+^\delta)^{-1}$, are bounded on L^p for all $z \in \mathbb{C}\backslash[0, 1]$.

Theorem 1.1. *Let $\delta > 0$ and $1 \le p < \infty$. Then the following statements are equivalent to each other.*

(i) *The Bochner-Riesz mean B_δ is bounded on L^p.*

(ii) *$(zI - B_\delta)^{-1}$ is bounded on L^p for all $z \in \mathbb{C}\backslash[0, 1]$.*

(iii) *$(z_0 I - B_\delta)^{-1}$ is bounded on L^p for some $z_0 \in \mathbb{C}\backslash[0, 1]$.*

It is obvious that $\lambda I - B_\delta, \lambda \in [0, 1]$, does not have bounded inverse on L^2, as it is a multiplier with symbol $\lambda - (1 - |\xi|^2)_+^\delta$. In the next theorem, we show that $\lambda I - B_\delta, \lambda \in [0, 1]$, does not have bounded inverse on L^p for all $1 \le p < \infty$.

Theorem 1.2. *Let $\delta > 0, 1 \le p < \infty$ and $\lambda \in [0, 1]$. Then*

$$\inf_{\|f\|_p = 1} \|(\lambda I - B_\delta) f\|_p = 0.$$

For any $\delta > 0$, define the spectra of Bochner-Riesz mean B_δ on L^p by

$$\sigma_p(B_\delta) := \mathbb{C}\backslash\{z \in \mathbb{C},\ zI - B_\delta \text{ has bounded inverse on } L^p\}.$$

As Bochner-Riesz means B_δ are multiplier operators with symbols $(1-|\xi|^2)_+^\delta$, we have

$$\sigma_p(B_\delta) = \text{closure of } \{(1-|\xi|^2)_+^\delta,\ \xi \in \mathbb{R}^d\} = [0,1] \text{ for } p = 2.$$

The equivalence in Theorem 1.1 and stability in Theorem 1.2 may not help to solve the conjecture on Bochner-Riesz means, but they imply that for any $\delta > 0$, the spectra of the Bochner-Riesz mean B_δ on L^p is invariant for different $1 \le p < \infty$ whenever it is bounded on L^p.

Theorem 1.3. *Let $\delta > 0$ and $1 \le p < \infty$. Then*

(i) $\sigma_p(B_\delta) = [0,1]$ *if the Bochner-Riesz mean B_δ is bounded on L^p; and*

(ii) $\sigma_p(B_\delta) = \mathbb{C}$ *if B_δ is unbounded on L^p.*

The above spectral invariance on different L^p spaces holds for any multiplier operator T_m with its bounded symbol m satisfying the following hypothesis,

$$|\xi|^k |\nabla^k m(\xi)| \in L^\infty,\ 0 \le k \le d/2 + 1, \tag{1.2}$$

in the classical Mikhlin multiplier theorem, because in this case,

$$\sigma_2(T_m) = \text{closure of } \{m(\xi),\ \xi \in \mathbb{R}^d\},$$

and for any $z \notin \sigma_2(T_m)$, the inverse of $zI - T_m$ is a multiplier operator with symbol $(z - m(\xi))^{-1}$ satisfying (1.2) too. Inspired by the above spectral invariance for Bochner-Riesz means and Mikhlin multipliers, we propose the following problem: Under what conditions on symbol of a multiplier, does the corresponding operator have its spectrum on L^p independent on $1 \le p < \infty$.

Spectral invariance for different function spaces is closely related to algebra of singular integral operators [4, 6, 13, 23] and Wiener's lemma for infinite matrices [10, 22, 24]. It has been established for singular integral operators with kernels being Hölder continuous and having certain off-diagonal decay [1, 6, 7, 19, 23], but it is not well studied yet for Calderon-Zygmund operators, oscillatory integrals, and many other linear operators in Fourier analysis.

In this paper, we denote by $\mathcal{S}$ and $\mathcal{D}$ the space of Schwartz functions and compactly supported C^∞ functions respectively, and we use the capital letter C to denote an absolute constant that could be different at each occurrence.

2 Proof of Theorem 1.1

Given nonnegative integers α_0 and β_0, let $\mathcal{S}_{\alpha_0,\beta_0}$ contain all functions f with

$$\|f\|_{\mathcal{S}_{\alpha_0,\beta_0}} := \sum_{|\alpha|\le\alpha_0,|\beta|\le\beta_0} \|x^\alpha\partial^\beta f(x)\|_\infty < \infty.$$

In this section, we prove the following strong version of Theorem 1.1.

Theorem 2.1. *Let* $\mathbf{B}$ *be a Banach space of tempered distributions with* $\mathcal{S}$ *being dense in* $\mathbf{B}$. *Assume that there exist nonnegative integers* α_0 *and* β_0 *such that any convolution operator with kernel* $K \in \mathcal{S}_{\alpha_0,\beta_0}$ *is bounded on* $\mathbf{B}$,

$$\|K * f\|_{\mathbf{B}} \le C\|K\|_{\mathcal{S}_{\alpha_0,\beta_0}}\|f\|_{\mathbf{B}} \ \text{ for all } f \in \mathbf{B}. \tag{2.1}$$

Then the following statements are equivalent to each other.

(i) *The Bochner-Riesz mean* B_δ *is bounded on* $\mathbf{B}$.

(ii) $(zI - B_\delta)^{-1}$ *is bounded on* $\mathbf{B}$ *for all* $z \in \mathbb{C}\backslash[0,1]$.

(iii) $(z_0I - B_\delta)^{-1}$ *is bounded on* $\mathbf{B}$ *for some* $z_0 \in \mathbb{C}\backslash[0,1]$.

Proof. (i)$\Longrightarrow$(ii). Take $z \in \mathbb{C}\backslash[0,1]$ and

$$r_0 \in (0, \min(|z/2|^{1/\delta}, 1)/2). \tag{2.2}$$

Let ψ_1 and $\psi_2 \in \mathcal{D}$ satisfy

$$\psi_1(\xi) = 1 \text{ when } |\xi| \le 1 - r_0,\ \ \psi_1(\xi) = 0 \text{ when } |\xi| \ge 1 - r_0/2; \tag{2.3}$$

and

$$\psi_2(\xi) = 1 - \psi_1(\xi) \text{ if } |\xi| \le 1 + r_0/2,\ \ \psi_2(\xi) = 0 \text{ if } |\xi| > 1 + r_0. \tag{2.4}$$

Define $m(\xi) := (z - (1 - |\xi|^2)_+^\delta)^{-1}$, $m_1(\xi) := m(\xi)\psi_1(\xi)$ and $m_2(\xi) := m(\xi)\psi_2(\xi)$. Then $m(\xi)$ is the symbol of the multiplier operator $(zI - B_\delta)^{-1}$ and

$$m(\xi) = m_1(\xi) + m_2(\xi) + z^{-1}(1 - \psi_1(\xi) - \psi_2(\xi)).$$

As $m_1, \psi_1, \psi_2 \in \mathcal{D}$, multiplier operators with symbols m_1 and $\psi_1 + \psi_2$ are bounded on $\mathbf{B}$ by (2.1). Therefore the proof reduces to establishing the boundedness of the multiplier operator with symbol m_2,

$$\|(m_2\hat{f})^\vee\|_{\mathbf{B}} \le C\|f\|_{\mathbf{B}},\ \ f \in \mathbf{B}, \tag{2.5}$$

where $f^\vee$ is the inverse Fourier transform of f.

Take an integer $N_0 > \alpha_0/\delta$. Write

$$m_2(\xi) = z^{-1}\Big(\sum_{n=0}^{N_0} + \sum_{n=N_0+1}^{\infty}\Big)(z^{-1})^n((1-|\xi|^2)_+^\delta)^n\psi_2(\xi) =: m_{21}(\xi) + m_{22}(\xi),$$

and denote multiplier operators with symbols m_{21} and m_{22} by T_{21} and T_{22} respectively. Observe that

$$T_{21} = z^{-1}\Psi_2 + \sum_{n=1}^{N_0} z^{-n-1}(B_\delta)^n\Psi_2,$$

where Ψ_2 is the multiplier operator with symbol ψ_2. Then T_{21} is bounded on $\mathbf{B}$ by (2.1) and the boundedness assumption (i),

$$\|T_{21}f\|_{\mathbf{B}} \le C\|f\|_{\mathbf{B}} \quad \text{for all } f \in \mathbf{B}. \tag{2.6}$$

Recall that $\psi_2 \in \mathcal{D}$ is supported on $\{\xi, 1-r_0 \le |\xi| \le 1+r_0\}$. Then the inverse Fourier transform K_n of $(1-|\xi|^2)_+^{n\delta}\psi_2(\xi)$ satisfies

$$\|K_n\|_{\mathcal{S}_{\alpha_0,\beta_0}} \le Cn^{\alpha_0}(2r_0)^{n\delta},\ n \ge N_0+1.$$

Therefore the convolution kernel

$$K(x) := z^{-1}\sum_{n=N_0+1}^{\infty} z^{-n}K_n(x)$$

of T_{22} belongs to $\mathcal{S}_{\alpha_0,\beta_0}$. This together with (2.1) proves

$$\|T_{22}f\|_{\mathbf{B}} \le C\|f\|_{\mathbf{B}} \quad \text{for all } f \in \mathbf{B}. \tag{2.7}$$

Combining (2.6) and (2.7) proves (2.5) and hence completes the proof of the implication (i)$\Longrightarrow$(ii).

(ii)$\Longrightarrow$(iii). The implication is obvious.

(iii)$\Longrightarrow$(i). Let $z_0 \in \mathbb{C}\backslash[0,1]$ so that $(z_0I - B_\delta)^{-1}$ is bounded on $\mathbf{B}$, r_0 be as in (2.2) with z replaced by z_0, and let ψ_1, ψ_2 be given in (2.3) and (2.4) respectively. Following the argument used in the proof of the implication (i)$\Longrightarrow$(ii), we see that it suffices to prove the operator T_3 associated with the multiplier $m_3(\xi) := (1-|\xi|^2)_+^\delta \psi_2(\xi)$ is bounded on $\mathbf{B}$. Take an integer $N_0 > \alpha_0/\delta$ and write

$$m_3(\xi) = -z_0\Big(\sum_{n=1}^{N_0} + \sum_{n=N_0+1}^{\infty}\Big)\big(1-z_0(z_0-(1-|\xi|^2)_+^\delta)^{-1}\big)^n\psi_2(\xi) =: m_{31}(\xi)+m_{32}(\xi),$$

where the series is convergent since

$$|1-z_0(z_0-(1-|\xi|^2)_+^\delta)^{-1}| \le \sum_{n=1}^{\infty}(|z_0|^{-1}(1-|\xi|^2)_+^\delta)^n \le \sum_{n=1}^{\infty}(|z_0|^{-1}(2r_0)^\delta)^n < 1.$$

Denote by T_{31} and T_{32} the operators associated with multiplier m_{31} and m_{32} respectively. As T_{31} is a linear combination of Ψ_2 and $(z_0 I - B_\delta)^{-n}\Psi_2, 1 \le n \le N_0$, it is bounded on $\mathbf{B}$,

$$\|T_{31}f\|_{\mathbf{B}} \le C\|f\|_{\mathbf{B}} \quad \text{for all } f \in \mathbf{B}, \tag{2.8}$$

by (2.1) and the boundedness assumption (iii).

Define the inverse Fourier transform of $\big(1-z_0(z_0-(1-|\xi|^2)_+^\delta)^{-1}\big)^n\psi_2(\xi), n > N_0$, by $\tilde{K}_n$. One may verify that

$$\|\tilde{K}_n\|_{\mathcal{S}_{\alpha_0,\beta_0}} \le C n^{\alpha_0}(2r_0)^{n\delta}|z_0|^{-n}, \ n \ge N_0+1.$$

Therefore

$$\|T_{32}f\|_{\mathbf{B}} \le C\sum_{n=N_0+1}^{\infty}\|\tilde{K}_n\|_{\mathcal{S}_{\alpha_0,\beta_0}}\|f\|_{\mathbf{B}} \le C\Big(\sum_{n=N_0+1}^{\infty} n^{\alpha_0}(2r_0)^{n\delta}|z_0|^{-n}\Big)\|f\|_{\mathbf{B}} \tag{2.9}$$

for all $f \in \mathbf{B}$. Combining (2.8) and (2.9) completes the proof. □

3 Proof of Theorem 1.2

Let f and K be Schwartz functions with $f(0) = 1$ and $\hat{K}(0) = 0$, and set $f_N(x) = N^{-d} f(x/N), N \ge 1$. Then for any positive integer $\alpha \ge d+1$ there exists a constant C_α such that

$$\begin{aligned} |K * f_N(x)| &\le \Big(\int_{|x-y|>\sqrt{N}} + \int_{|x-y|\le\sqrt{N}}\Big)|K(x-y)|\ |f_N(y)-f_N(x)|dy \\ &\le \int_{|x-y|>\sqrt{N}} |K(x-y)|\ |f_N(y)|dy + C_\alpha N^{-d-1/2}(1+|x/N|)^{-\alpha} \\ &\le C_\alpha N^{-1/2}\Big(\int_{\mathbb{R}^d}(1+|x-y|)^{-\alpha}|f_N(y)|dy + N^{-d}(1+|x/N|)^{-\alpha}\Big). \end{aligned} \tag{3.1}$$

This implies that

$$\lim_{N\to\infty}\frac{\|K * f_N\|_p}{\|f_N\|_p} = 0,\ 1 \le p < \infty.$$

Therefore the following is a strong version of Theorem 1.2.

Theorem 3.1. *Let* $\mathbf{B}$ *be a Banach space of tempered distributions with* $\mathcal{S}$ *being dense in* $\mathbf{B}$*. Assume that* (2.1) *holds for some* $\alpha_0, \beta_0 \ge 0$ *and that for any* $\xi_0 \in \mathbb{R}^d$ *there exists* $\varphi_0 \in \mathcal{D}$ *such that* $\widehat{\varphi}_0(0) = 1$ *and*

$$\lim_{N\to\infty}\frac{\|(m\widehat{f_{N,\xi_0}})^\vee\|_{\mathbf{B}}}{\|f_{N,\xi_0}\|_{\mathbf{B}}} = 0 \tag{3.2}$$

for all Schwartz functions m *with* $m(\xi_0) = 0$*, where* $\widehat{f_{N,\xi_0}}(\xi) = \varphi_0(N(\xi-\xi_0))$*. Then*

$$\inf_{f\ne 0}\frac{\|(\lambda I - B_\delta)f\|_{\mathbf{B}}}{\|f\|_{\mathbf{B}}} = 0 \quad \text{for all } \lambda \in [0,1]. \tag{3.3}$$

Proof. The infimum in (3.3) is obvious for $\lambda = 0$. So we assume that $\lambda \in (0,1]$ from now on. Select $\xi_0 \in \mathbb{R}^d$ so that $(1-|\xi_0|^2)_+^\delta = \lambda$. Then for sufficiently large $N \ge 1$,

$$(\lambda I - B_\delta)f_{N,\xi_0} = (m_{\xi_0}\widehat{f_{N,\xi_0}})^\vee, \tag{3.4}$$

where $m_{\xi_0}(\xi) = (\lambda - (1-|\xi|^2)_+^\delta)\psi(\xi-\xi_0)$ and $\psi \in \mathcal{D}$ is so chosen that $\psi(\xi) = 1$ for $|\xi| \le (1-|\xi_0|)/2$ and $\psi(\xi) = 0$ for $|\xi| \ge 1-|\xi_0|$. Observe that $m_{\xi_0} \in \mathcal{D}$ satisfies $m_{\xi_0}(\xi_0) = 0$. This together with (3.2) and (3.4) proves that

$$\lim_{N\to\infty}\frac{\|(\lambda I - B_\delta)f_{N,\xi_0}\|_{\mathbf{B}}}{\|f_{N,\xi_0}\|_{\mathbf{B}}} = 0.$$

Hence (3.3) is proved for $\lambda \in (0,1]$. □

Remark 3.2. For $\xi \in \mathbb{R}^d$, define modulation operator M_ξ by

$$M_\xi f(x) = e^{ix\xi} f(x).$$

We say that a Banach space $\mathbf{B}$ is *modulation-invariant* if for any $\xi \in \mathbb{R}^d$ there exists a positive constant C_ξ such that

$$\|M_\xi f\|_{\mathbf{B}} \le C_\xi \|f\|_{\mathbf{B}}, \quad f \in \mathbf{B}.$$

Such a Banach space with modulation bound C_ξ being dominated by a polynomial of ξ was introduced in [26] to study oscillatory integrals and Bochner-Riesz means. Modulation-invariant Banach spaces include weighted L^p spaces, Triebel-Lizorkin spaces $F^\alpha_{p,q}$, Besov spaces $B^\alpha_{p,q}$, Herz spaces $K^{\alpha,q}_p$, and modulation spaces $M^{p,q}$, where $\alpha \in \mathbb{R}$ and $1 \le p, q < \infty$ [9, 11, 16, 25]. For functions $f_{N,\xi_0}, N \ge 1$, in Theorem 3.1,

$$f_{N,\xi_0}(x) = e^{ix\xi_0}(\varphi_0(N\cdot))^\vee(x)$$

and

$$(m\widehat{f_{N,\xi_0}})^\vee(x) = e^{ix\xi_0}(m_{\xi_0}\varphi_0(N\cdot))^\vee(x),$$

where $m_{\xi_0}(\xi) = m(\xi + \xi_0)$ satisfies $m_{\xi_0}(0) = 0$. Then for a modulation-invariant space $\mathbf{B}$, the limit (3.2) holds for any $\xi_0 \in \mathbb{R}^d$ if and only if it is true for $\xi_0 = 0$. Therefore we obtain the following result from Theorems 2.1 and 3.1.

Corollary 3.3. *Let* $\mathbf{B}$ *be a modulation-invariant Banach space of tempered distributions with* $\mathcal{S}$ *being dense in* $\mathbf{B}$*. Assume that* (2.1) *holds for some* $\alpha_0, \beta_0 \ge 0$ *and that there exists* $\varphi_0 \in \mathcal{D}$ *such that*

$$\hat{\varphi}_0(0) = 1 \text{ and } \lim_{N\to\infty} \|(m\varphi_0(N\cdot))^\vee\|_{\mathbf{B}} / \|(\varphi_0(N\cdot))^\vee\|_{\mathbf{B}} = 0$$

for all Schwartz functions $m(\xi)$ *with* $m(0) = 0$*. If the Bochner-Riesz mean* B_δ *is bounded on* $\mathbf{B}$*, then its spectrum on* $\mathbf{B}$ *contains the unit interval* $[0, 1]$.

4 Remarks

In this section, we extend conclusions in Theorem 1.3 to weighted L^p spaces, Triebel-Lizorkin spaces, Besov spaces, and Herz spaces.

4.1 Spectra on weighted L^p spaces

Let $1 \le p < \infty$ and $\mathcal{Q}$ contain all cubes $Q \subset \mathbb{R}^d$. A positive function w is said to be a *Muckenhoupt A_p-weight* if

$$\Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\ \Big(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\Big)^{p-1} \le C,\ Q\in\mathcal{Q}$$

for $1 < p < \infty$, and

$$\frac{1}{|Q|}\int_Q w(x)dx \le C\inf_{x\in Q} w(x),\ Q\in\mathcal{Q}$$

for $p = 1$ [8, 17]. For $\delta > (d-1)/2$, convolution kernel of the Bochner-Riesz mean B_δ is dominated by a multiple of $(1+|x|)^{-\delta-(d+1)/2}$ and hence it is bounded on weighted L^p space L^p_w for all $1 \le p < \infty$ and Muckenhoupt A_p-weights w. For $\delta = (d-1)/2$, complex interpolation method was introduced in [18] to establish L^p_w-boundedness of B_δ for all $1 < p < \infty$ and Muckenhoupt A_p-weights w. The reader may refer to [8, 15] and references therein for L^p_w-boundedness of Bochner-Riesz means with various weights w. In this subsection, we consider spectra of Bochner-Riesz means on L^p_w.

Theorem 4.1. *Let $\delta > 0, 1 \le p < \infty$, and w be a Muckenhoupt A_p-weight. If the Bochner-Riesz mean B_δ is bounded on L^p_w, then its spectrum on L^p_w is the unit interval* $[0, 1]$.

Proof. Denote the norm on L^p_w by $\|\cdot\|_{p,w}$. By Theorems 2.1 and 3.1, and modulation-invariance of L^p_w, it suffices to prove

$$\|K*f\|_{p,w} \le C\|K\|_{\mathcal{S}_{d+1,0}}\|f\|_{p,w} \ \text{ for all } f\in L^p_w, \tag{4.1}$$

and

$$\lim_{N\to\infty}\frac{\|(m\varphi_0(N\cdot))^\vee\|_{p,w}}{\|(\varphi_0(N\cdot))^\vee\|_{p,w}} = 0 \tag{4.2}$$

for all Schwartz functions φ_0 and m with $\widehat{\varphi}_0(0) = 1$ and $m(0) = 0$.

Observe that $|K(x)| \le C\|K\|_{\mathcal{S}_{d+1,0}}(1+|x|)^{-d-1}$. Then (4.1) follows from the standard argument for weighted norm inequalities [8].

Recall that any A_p-weight is a doubling measure [8]. This doubling property together with (3.1) leads to

$$\|(m\varphi_0(N\cdot))^\vee\|^p_{p,w} \le CN^{-p/2}\|(\varphi_0(N\cdot))^\vee\|^p_{p,w} + CN^{-(d+1/2)p}w([-N,N]^d). \tag{4.3}$$

On the other hand, there exists $\epsilon_0 > 0$ such that $|\varphi_0^\vee(x)| \geq |\varphi_0^\vee(0)|/2 \neq 0$ for all $|x| \leq \epsilon_0$. This implies that

$$\|(\varphi_0(N\cdot))^\vee\|_{p,w}^p \geq CN^{-dp} w([-\epsilon_0 N, \epsilon_0 N]^d). \tag{4.4}$$

Combining (4.3), (4.4) and the doubling property for the weight w, we establish the limit (4.2) and complete the proof. □

4.2 Spectra on Triebel-Lizorkin spaces and Besov spaces

Let ϕ_0 and $\psi \in \mathcal{S}$ be so chosen that $\widehat{\phi_0}$ is supported in $\{\xi, |\xi| \leq 2\}$, $\hat{\psi}$ supported in $\{\xi, 1/2 \leq |\xi| \leq 2\}$, and

$$\widehat{\phi_0}(\xi) + \sum_{l=1}^{\infty} \hat{\psi}(2^{-l}\xi) = 1, \quad \xi \in \mathbb{R}^d.$$

For $\alpha \in \mathbb{R}$ and $1 \leq p, q < \infty$, let Triebel-Lizorkin space $F_{p,q}^\alpha$ contain all tempered distributions f with

$$\|f\|_{F_{p,q}^\alpha} := \|\phi_0 * f\|_p + \Big\| \Big(\sum_{l=1}^{\infty} 2^{l\alpha q} |\psi_l * f|^q \Big)^{1/q} \Big\|_p < \infty,$$

where $\psi_l = 2^{ld}\psi(2^l\cdot), l \geq 1$. Similarly, let Besov space $B_{p,q}^\alpha$ be the space of tempered distributions f with

$$\|f\|_{B_{p,q}^\alpha} := \|\phi_0 * f\|_p + \Big(\sum_{l=1}^{\infty} 2^{l\alpha q} \|\psi_l * f\|_p^q \Big)^{1/q} < \infty.$$

Next is our results about spectra of Bochner-Riesz means on Triebel-Lizorkin spaces and on Besov spaces.

Theorem 4.2. *Let $\delta > 0, \alpha \in \mathbb{R}$ and $1 \leq p, q < \infty$. If the Bochner-Riesz mean B_δ is bounded on $F_{p,q}^\alpha$ (resp. $B_{p,q}^\alpha$), then its spectrum on $F_{p,q}^\alpha$ (resp. on $B_{p,q}^\alpha$) is the unit interval $[0,1]$.*

Proof. For $z \notin [0,1]$ and $\delta > 0$, both B_δ and $(zI - B_\delta)^{-1} - z^{-1}I$ are multiplier operators with compactly supported symbols. Therefore B_δ (resp. $(zI - B_\delta)^{-1}$) is bounded on the Triebel-Lizorkin space $F_{p,q}^\alpha$ if and only if it is bounded on the Besov space $B_{p,q}^\alpha$ if and only if it is bounded on L^p. The above equivalence together with Theorem 1.1 yields our desired conclusions. □

4.3 Spectra on Herz spaces

For $\alpha \in \mathbb{R}$ and $1 \leq p, q < \infty$, let Herz space $K_p^{\alpha,q}$ contain all locally p-integrable functions f with

$$\|f\|_{K_p^{\alpha,q}} := \|f\chi_{|\cdot|\leq 1}\|_p + \Big(\sum_{l=1}^{\infty} 2^{l\alpha q}\|f\chi_{2^{l-1}<|\cdot|\leq 2^l}\|_p^q\Big)^{1/q} < \infty,$$

where χ_E is the characteristic function on a set E. The boundedness of Bochner-Riesz means on Herz spaces is well studied, see for instance [16, 30]. Following the argument used in the proof of Theorem 4.1, we have

Theorem 4.3. *Let $\delta > 0, 1 \leq p, q < \infty$ and $\alpha > -d/p$. If the Bochner-Riesz mean B_δ is bounded on $K_p^{\alpha,q}$, then its spectrum on $K_p^{\alpha,q}$ is $[0, 1]$.*

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References

[1] B. A. Barnes, When is the spectrum of a convolution operator on L_p independent of p? Proc. Edinburgh Math. Soc., 33(1990), 327-332.

[2] S. Bochner, Summation of multiple Fourier series by spherical means, Trans. Amer. Math. Soc., 40(1936), 175-207.

[3] J. Bourgain, Besicovitch type maximal operators and applications to Fourier analysis, Geom. Funct. Anal., 1(1991), 147-187.

[4] A. P. Calderon and A. Zygmund, Algebras of certain singular integral operators, Amer. J. Math., 78(1956), 310-320.

[5] L. Carleson and P. Sjölin, Oscillatory integrals and a multiplier problem for the disc, Studia Math., 44(1972), 287-299.

[6] B. Farrell and T. Strohmer, Inverse-closedness of a Banach algebra of integral operators on the Heisenberg group, J. Operator Theory, 64(2010), 189-205.

[7] Q. Fang, C. E. Shin and Q. Sun, Wiener's lemma for singular integral operators of Bessel potential type, Monat. Math., 173(2014), 35-54.

[8] J. Garcia-Cuerva and J. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam, 1985.

[9] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhauser, 2000.

[10] K. Gröchenig and M. Leinert, Symmetry and inverse-closedness of matrix algebras and functional calculus for infinite matrices, Trans. Amer. Math. Soc., 358(2006), 2695-2711.

[11] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson, 2004.

[12] C. S. Herz, On the mean inversion of Fourier and Hankel transforms, Proc. Nat. Acad. Sci. U.S.A., 40(1954), 996-999.

[13] V. G. Kurbatov, Some algebras of operators majorized by a convolution, Funct. Differ. Equ., 8(2001), 323-333.

[14] S. Lee, Improved bounds for Bochner-Riesz and maximal Bochner-Riesz operators, Duke Math. J., 122(2004), 205-232.

[15] K. Li and W. Sun, Sharp bound of the maximal Bochner-Riesz operator in weighted Lebesgue spaces, J. Math. Anal. Appl., 395(2012), 385-392.

[16] S. Lu, D. Yang and G. Hu, *Herz Type Spaces and Their Applications*, Science Press, Beijing, 2008.

[17] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc., 165(1972), 207-226.

[18] X. Shi and Q. Sun, Weighted norm inequalities for Bochner-Riesz operators and singular integral operators, Proc. Amer. Math. Soc., 116(1992), 665-673.

[19] C. E. Shin and Q. Sun, Stability of localized operators, J. Funct. Anal., 256(2009), 2417-2439.

[20] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, 1993.

[21] E. M. Stein, Localization and summability of multiple Fourier series, Acta Math., 100(1958), 93-147.

[22] Q. Sun, Wiener's lemma for infinite matrices II, Constr. Approx., 34(2011), 209-235.

[23] Q. Sun, Wiener's lemma for localized integral operators, Appl. Comput. Harmonic Anal., 25(2008), 148-167.

[24] Q. Sun, Wiener's lemma for infinite matrices, Trans. Amer. Math. Soc., 359(2007), 3099-3123.

[25] H. Triebel, *Theory of Function Spaces*, Birkhauser, 1983; Theory of Function Spaces II, Birksauser, 1992.

[26] T. Tao, On the maximal Bochener-Riesz conjecture in the plane for $p < 2$, Trans. Amer. Math. Soc., 354(2002), 1947-1959.

[27] T. Tao and A. Vargas, A bilinear approach to cone multipliers I: restriction estimates, Geom. Funct. Anal., 10(2000), 185-215.

[28] T. Tao and A. Vargas, A bilinear approach to cone multipliers II: application, Geom. Funct. Anal., 10(2000), 216-258.

[29] T. Tao, A. Vargas and L. Vega, A bilinear approach to the restriction and Kakeya conjectures, J. Amer. Math. Soc., 11(1998), 967-1000.

[30] H. Wang, Some estimates for Bochner-Riesz operators on the weighted Herz-type Hardy spaces, J. Math. Anal. Appl., 381(2011), 134-145.

[31] T. Wolff, An improved bound for Kakeya type maximal functions, Rev. Mat. Iberoamericana, 11(1995), 651-674.