Convergence of Cascade Algorithms and Smoothness of Refinable Distributions

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Abstract

In this paper, we at first develop a method to study convergence of the cascade algorithm in a Banach space without stable assumption on the initial (Theorem 2.7). Then we apply the previous result on the convergence to characterize compactly supported refinable distributions in fractional Sobolev spaces and Hölder continuous spaces (Theorems 3.1, 3.8, and 3.9). Finally we apply the above characterization to choose appropriate initial to guarantee the convergence of the cascade algorithm (Theorem 4.3).

Keywords: Cascade algorithm, cascade operator, refinable distribution, shift-invariant space, linear independent shifts, stable shifts, fractional Sobolev space.

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1 Introduction

Let X be a linear topological space of tempered distributions on $\mathbf{R}^d$, and let $\mathcal{D}$ denote the space of all compactly supported C^∞ functions on $\mathbf{R}^d$. We say that X has *continuous translates* if for any $y \in \mathbf{R}^d$, the shift map $\tau_y : f \longmapsto f(\cdot - y)$ is a continuous map on X, that X has *continuous dilation* if for any $a > 0$, the dilation map $D_a : f \longmapsto f(a\cdot)$ is a continuous map on X, and that X has *continuous $\mathcal{D}$-multiplication* if for any $h \in \mathcal{D}$, the multiplication map $M_h : f \longmapsto hf$ is a continuous map on X.

Take a normed linear space X of tempered distributions having continuous translates and dilation, and fix a family of $N \times N$ matrix $c(j), j \in \mathbf{Z}^d$, having finite support, i.e., $c(j) = 0$ for all but finitely many $j \in \mathbf{Z}^d$. Define a *cascade operator* T on X^N by

$$TF := \sum_{j \in \mathbf{Z}^d} c(j) F(2 \cdot - j) \quad \text{for all} \quad F \in X^N. \tag{1.1}$$

Here and hereafter, the N Cartesian product of a linear topological space X with usual topology is denoted by X^N, sometimes still by X if not confusing. The sequence $(c(j))_{j \in \mathbf{Z}^d}$ in (1.1) and the trigonometric polynomial $H(\xi) := 2^{-d} \sum_{j \in \mathbf{Z}^d} c(j) e^{-ij\xi}$ are known as the *mask* and the *symbol* of the cascade operator T respectively. Define a *cascade algorithm* with *initial* $F_0 \in X^N$ by

$$F_n := T F_{n-1}, \quad n \geq 1. \tag{1.2}$$

The cascade algorithm was first introduced to compute the refinable distribution, the fixed point of the cascade operator T, in the same way the power method computes an eigenvector of a matrix. The convergence of a cascade algorithm in some function spaces is important for some applications such as in plotting a refinable function, and in numerical computation ([3, 4, 7, 8]). There is a long list of publications on the convergence of a cascade algorithm in different function spaces (see for instance [11] for L^p spaces, [13] for Sobolev space, and [21] for one-dimensional Triebel-Lizorkin spaces and Besov spaces).

Let ℓ_0 denote the space of all sequences on $\mathbf{Z}^d$ having finite support, define the shift-invariant space generated by the initial F_0 by

$$S_0(F_0) := \Big\{ \sum_{j \in \mathbf{Z}^d} d(j)^T F_0(\cdot - j) : \ (d(j))_{j \in \mathbf{Z}^d} \in \ell_0^N \Big\},$$

and write

$$H(2^{n-1}\xi) \cdots H(\xi) =: 2^{-nd} \sum_{j \in \mathbf{Z}^d} c_n(j) e^{-ij\xi}, \quad n \geq 1. \tag{1.3}$$

By direct computation, we have

$$F_n = \sum_{j \in \mathbf{Z}^d} c_n(j) F_0(2^n \cdot -j), \quad n \geq 1. \tag{1.4}$$

Thus the dilated cascade sequence $F_n(2^{-n}\cdot)$, $n \geq 0$, is in the shift-invariant space $S_0(F_0)^N$. So we may discuss the convergence problem of a cascade algorithm within the theory of **shift-invariant spaces**.

Define the *Fourier transform* $\hat{f}$ of an integrable function f by $\hat{f}(\xi) := \int_{\mathbf{R}^d} f(x) e^{-ix\xi} dx$ and the one of a tempered distribution as usual. For any vector-valued tempered distribution $F = (f_1, \ldots, f_N)^T$ with continuous Fourier transform, we say that F has, or $f_1, \ldots, f_N$ have, *stable shifts* if there exist $k_i(\xi_0) \in \mathbf{Z}^d, 1 \leq i \leq N$, for any $\xi_0 \in \mathbf{R}^d$ such that the $N \times N$ matrix $(\widehat{F}(\xi_0 + 2k_i(\xi_0)\pi))_{1 \leq i \leq N}$ is of full rank. Let L^p, $1 \leq p \leq \infty$, be the usual spaces of all p-integrable functions on $\mathbf{R}^d$ with usual L^p norm $\|\cdot\|_p$, and let $\ell^p, 1 \leq p \leq \infty$, be the usual space of all p-summable sequences with norm $\|\cdot\|_{\ell^p}$. By (1.4), for any $1 \leq p \leq \infty$ and any compactly supported L^p function F_0 having stable shifts, there exists a positive constant C such that for all $n \geq 1$,

$$C^{-1} \|(c_n(j))_{j \in \mathbf{Z}^d}\|_{\ell^p} \leq 2^{n/p} \|F_n\|_p = \|F_n(2^{-n}\cdot)\|_p \leq C \|(c_n(j))_{j \in \mathbf{Z}^d}\|_{\ell^p} \tag{1.5}$$

(see Proposition A.4 in Appendix A or [15]). So under the assumption that the initial F_0 has **stable shifts** and belongs to certain function space, we may reduce the convergence of the cascade algorithm $F_n, n \geq 1$, in some function spaces, such as fractional Sobolev spaces and Hölder continuous spaces, to the asymptotic behavior of a quantity related to the mask $(c(j))_{j \in \mathbf{Z}^d}$.

In this paper, we consider the convergence of cascade algorithm in Banach spaces **without** stable assumption on the initial (Theorem 2.7), and apply our results on the convergence to study smoothness of refinable distributions, and to choose appropriate initial to guarantee convergence of the cascade algorithm with that initial. In fact, the applications mentioned above are our initial motivations to consider the convergence of cascade algorithm in Banach space without stable assumption on the initial, while convergence of cascade algorithm in L^p space with stable assumption on the initial have been considered in a lot of literatures on that topic (see [11] and references therein).

Let Ψ be a nonzero compactly supported distribution satisfying the *refinement equation*

$$\Psi = \sum_{j \in \mathbf{Z}^d} c(j) \Psi(2 \cdot -j). \tag{1.6}$$

The compactly supported distribution Ψ in (1.6) is called a *refinable distribution*, and the sequence $(c(j))_{j \in \mathbf{Z}^d}$ in (1.6) is known as the *mask* of the refinement equation (1.6).

For the smoothness of a refinable distribution, there are many different ways to discuss it, and publications on that topic (see for instance, [5, 6, 8, 9, 12, 18, 19, 25] for Sobolev spaces $L^{2,\gamma}$, [2, 14, 16] for Lipschitz spaces and L^p spaces, and [17] for Besov spaces and Triebel-Lizorkin spaces).

Denote the usual convolution between a Schwartz function h and a tempered distribution f by $h*f$, and for any Schwartz function h, set $h_n := 2^{nd}h(2^n\cdot), n \ge 1$. Clearly, for the refinable distribution Ψ in (1.6), $T\Psi = \Psi$, and for any Schwartz function h,

$$h_n * \Psi = T^n(h * \Psi). \tag{1.7}$$

By Proposition A.3 in Appendix A, for any $f \in L^p, 1 \le p < \infty$, we have

$$\lim_{n\to\infty} \|h_n * f\|_p = 0 \quad \text{for all} \quad h \in \mathcal{D} \quad \text{with} \quad \widehat{h}(0) = 0, \tag{1.8}$$

and conversely, if $h * f \in L^p$ for any $h \in \mathcal{D}$ and if there exist positive constants C and δ for any $h \in \mathcal{D}$ with $\widehat{h}(0) = 0$ such that

$$\|h_n * f\|_p \le C2^{-n\delta} \quad \text{for all} \quad n \ge 1, \tag{1.9}$$

then $f \in L^p$. Obviously (1.9) implies (1.8). Surprisingly, by (1.7) and Lemma 3.3, for the refinable distribution Ψ in (1.6), if (1.8) is satisfied, then (1.9) holds for some positive constants C and δ. Thus we may characterize the refinable distribution Ψ in $L^p, 1 \le p < \infty$, completely by the asymptotic behavior of $h_n * \Psi$ as n tends to infinity. Thus, the problem whether the refinable distribution Ψ in (1.6) belongs to L^p or not is closely related to the convergence in L^p of a cascade algorithm with the initial $h * \Psi$, where $h \in \mathcal{D}$ satisfies $\widehat{h}(0) = 0$. Actually, the above reduction also holds for many function spaces used to measure smoothness such as the fractional Sobolev spaces and Hölder continuous spaces discussed in this paper.

For the refinable function $\Psi \in L^p$ in (1.6), either the matrix $(\widehat{\Psi}(2j\pi))_{j\in\mathbf{Z}^d}$ is not of full rank, or $\widehat{\Psi}(0) \ne 0$ and the inner products between $\widehat{\Psi}(0)$ and $\widehat{\Psi}(2j\pi), j \in \mathbf{Z}^d\backslash\{0\}$, always equal zero. Hence, for any compactly supported refinable function Ψ in $L^p, 1 \le p < \infty$, and any $h \in \mathcal{D}$ with $\widehat{h}(0) = 0$, $h * \Psi$ always have **unstable** shifts. Thus we cannot use the estimate (1.5) directly to characterize whether refinable distribution belongs to $L^{p,\gamma}$ or not. This is our initial motivation to consider the convergence of cascade algorithm without stable assumption on the initial.

The technique developed here to study the convergence of cascade algorithm includes the decomposition of compactly supported distributions in [1], the shift-invariant sequence space in [23], and the fact $T^N F_0(2^{-n}\cdot) \in S_0(F_0)^N$ observed at the beginning of this section. The technique can be outlined as follows. In [1], for any compactly supported distributions $f_1,\ldots,f_N$, we provide the following decomposition

$$(f_1,\ldots,f_N)^T=\sum_{i=1}^{M}\sum_{j\in\mathbf{Z}^d} d_i(j)\psi_i(\cdot-j) \tag{1.10}$$

where the seqeunces $(d_i(j))_{j\in\mathbf{Z}^d}, 1\le i\le M$, have finite support, and where the compactly supported distributions $\psi_1,\cdots,\psi_M$ have stable shifts, and have almost the same regularity as the one of $f_1,\ldots,f_N$ (see Lemma 2.5 or [1] for detail). Moreover, the shift-invariant sequence space containing $(d_i(j))_{j\in\mathbf{Z}^d}, 1\le i\le M$, is the *dependent ideal* in [23]. Combining the decomposition (1.10) and the theory of shift-invariant spaces with a generator having stable shifts, we can study the convergence of a cascade algorithm in a function space without stable assumption on its initial (Theorem 2.7).

In order to study the smoothness of refinable distributions, we need the precise definitions of two commonly used function spaces to measure smoothness. Let $\triangle=\sum_{i=1}^d\frac{\partial^2}{\partial x_i^2}$ be the usual Laplacian. For any real number γ and $1<p<\infty$, let $L^{p,\gamma}$, which is known as a *fractional Sobolev space* or *Bessel potential space*, be the space of all tempered distributions f for which

$$\|f\|_{L^{p,\gamma}}:=\|(1-\triangle)^{\gamma/2}f\|_p=\left\|\left(\hat f(1+|\cdot|^2)^{\gamma/2}\right)^{\vee}\right\|_p$$

is finite. Here and hereafter, $f^{\vee}$ is the inverse Fourier transform of a tempered distribution f. For the fractional Sobolev space $L^{p,\gamma}, 1<p<\infty$, we have $L^{p,0}=L^p$,

$$L^{p,\gamma_2}\subset L^{p,\gamma_1} \tag{1.11}$$

for any $\gamma_1<\gamma_2$,

$$L^{p,\gamma_3}=\Big\{f:\ D^\kappa f\in L^p \quad\text{for all}\quad \kappa\in\mathbf{Z}_+^d \quad\text{with}\quad |\kappa|\le\gamma_3\Big\} \tag{1.12}$$

for any nonnegative integer γ_3, and there exists a positive constant C for any integer $k\ge-\gamma/2$ such that

$$\|(1-\triangle)^{-k}g\|_p\le C\|g\|_{L^{p,\gamma}} \tag{1.13}$$

for all $g\in L^{p,\gamma}$ ([24]).

Take a nonnegative real number α, and let α_0 be the greatest integer less than or equal to α, and set $\delta=\alpha-\alpha_0$. Let C^α, which is known as a *Hölder continuous space*, be the space of all continuous functions f on $\mathbf{R}^d$ such that f has continuous and bounded k-th derivative $D^\kappa f$ for any $\kappa\in\mathbf{Z}_+^d$ with $|\kappa|\le\alpha_0$, and

$$\|f\|_{C^\alpha}:=\sum_{|\kappa|\le\alpha_0}\|D^\kappa f\|_\infty+\sum_{|\kappa|=\alpha_0}\sup_{x_1\ne x_2}\frac{|D^\kappa f(x_1)-D^\kappa f(x_2)|}{|x_1-x_2|^\delta}<\infty.$$

Then $(C^\alpha, \|\cdot\|_{C^\alpha})$ is a Banach space for any $\alpha \geq 0$. For any nonnegative real number α, let VC^α be the space of all C^α functions f satisfying

$$\lim_{\epsilon\to 0}\sup_{0<|x_1-x_2|\leq\epsilon}\frac{|D^\kappa f(x_1)-D^\kappa f(x_1)|}{|x_1-x_2|^\delta}=0 \quad \text{for all} \quad \kappa \quad \text{with} \quad |\kappa|=\alpha_0,$$

where $\alpha = \alpha_0 + \delta$ and $0 \leq \delta < 1$. Obviously, $C^{\alpha+\epsilon} \subset VC^\alpha \subset C^\alpha$ for any positive ϵ.

The paper is organized as follows. Section 2 is devoted to the study of convergence of a cascade algorithm in Banach spaces. In Section 3, we apply the results about the convergence of a cascade algorithm in Section 2 to the study of smoothness of refinable distributions. In particular, we choose appropriate stable shift-triples, and use asymptotic behavior of a quantity related to the mask and an ideal related to Ψ to characterize whether the refinable distribution Ψ belongs to one of the following function spaces $L^{p,\gamma}, C^\alpha$ and VC^α (Theorems 3.1, 3.8 and 3.9, and Remark 3.4). Under additional assumption that the refinable distribution Ψ has linear independent shifts, the characterization in Theorem 3.1 is simplified and is stated in Theorem 3.5 and Corollary 3.6. Similar results to Theorem 3.5 and Corollary 3.6 are established in [6, 9, 12, 25] for $\gamma \geq 0$, and our assertion for $\gamma < 0$ is still new. As an application of Theorem 3.1, we show that a cascade algorithm always converges when the initial is appropriately chosen in Section 4 (Theorems 4.2 and 4.3). In the appendix A, some properties of the function spaces $L^{p,\gamma}, 1 < p < \infty$, are given. Those properties are crucial to establish a general result about the convergence of a cascade algorithm in Section 2, and our characterization of the smoothness of refinable distributions in Section 3 as well.

2 Convergence of Cascade Sequence

In order to study the convergence of cascade algorithm in Banach spaces, we introduce two new concepts: bound shift-triple and stable shift-triple, in Section 2.1. The bounded shift-triple and stable shift-triple can be thought as a natural link between continuous and discrete system in a shift-invariant space. In Section 2.1, we give some examples of bounded and stable shift-triples (Examples 2.1 – 2.4). Those examples of bounded and stable shift-triples will be used later in the study of smoothness of refinable distributions. In Section 2.2, we recall the decomposition of compactly supported distribution in [1], and the dependent ideal in [23], and also introduce a new shift-invariant sequence space $i_r(F)$ used later to study the smoothness of refinable distribution (Theorems 3.1 and 3.5), and to choose appropriate initial to guarantee the convergence of cascade algorithm (Theorems 4.2 and 4.3). The main result of this section, Theorem 2.7, is stated and proved in Section 2.3.

2.1 Bounded Shift-Triple and Stable Shift-Triple

Let ℓ_P be the space of all sequences $(d(j))_{j\in\mathbf{Z}^d}$ on $\mathbf{Z}^d$ with polynomial increase, i.e., $|d(j)| \le P(j)$ for all $j \in \mathbf{Z}^d$ and some polynomial P. For normed linear spaces $(X_1, \|\cdot\|_{X_1})$ and $(X_2, \|\cdot\|_{X_2})$ of tempered distributions, and a normed linear space $(Y, \|\cdot\|_Y)$ of sequences with polynomial increase, the triple (X_1, X_2, Y) is said to be a *bounded shift-triple* if the following conditions hold:

- $\sum_{j\in\mathbf{Z}^d} d(j) f(\cdot - j) \in X_1$ for any $f \in X_2$ and $(d(j))_{j\in\mathbf{Z}^d} \in Y$, and
- for any $f \in X_2$ there exists a positive constant C such that

$$\Big\|\sum_{j\in\mathbf{Z}^d} d(j) f(\cdot - j)\Big\|_{X_1} \le C\|(d(j))_{j\in\mathbf{Z}^d}\|_Y \qquad \text{for all} \quad (d(j))_{j\in\mathbf{Z}^d} \in Y. \tag{2.1}$$

We say that the triple (X_1, X_2, Y) is a *stable shift-triple* if the following conditions hold:

- (X_1, X_2, Y) is a bounded shift-triple, and
- for any $f_i \in X_2, 1 \le i \le N$, with $F = (f_1, \ldots, f_N)^T$ having continuous Fourier transform and stable shifts, there exists a positive constant C such that for all $(d(j))_{j\in\mathbf{Z}^d} \in Y^N$,

$$\Big\|\sum_{j\in\mathbf{Z}^d} d(j)^T F(\cdot - j)\Big\|_{X_1} \ge C\|(d(j))_{j\in\mathbf{Z}^d}\|_{Y^N}. \tag{2.2}$$

Here and hereafter, for a normed linear space $(X, \|\cdot\|_X)$, we set $\|x\|_{X^N} = \sum_{i=1}^N \|x_i\|_X$ for $x = (x_1, \ldots, x_N)^T \in X^N$, and also use $\|x\|_X$ instead of $\|x\|_{X^N}$ for $x \in X^N$ if not confusing.

Example 2.1 For $1 \le p \le \infty$, let $\mathcal{L}^p$ be the space of all measurable functions f for which $\|f\|_{\mathcal{L}^p} := \|\sum_{j\in\mathbf{Z}^d} |f(x + j)|\|_{L^p([0,1]^d)} < \infty$. Obviously $\mathcal{L}^p \subset L^p$, and any compactly supported L^p function belongs to $\mathcal{L}^p$. By the definition of L^p and $\mathcal{L}^p$, it is easy to check that both L^p and $\mathcal{L}^p$ have continuous translates and $\mathcal{D}$-multiplication. Furthermore the triple $(L^p, \mathcal{L}^p, \ell^p), 1 \le p \le \infty$, is a stable shift-triple ([15]).

Example 2.2 For any compact set K of $\mathbf{R}^d$, denote the space of all $L^{p,\gamma}$ distributions with support in K by $L^{p,\gamma}(K)$. By Proposition A.1 in Appendix A, $L^{p,\gamma}$ has continuous translates and $\mathcal{D}$-multiplication for any $p \in [1,\infty)$ and $\gamma \in \mathbf{R}$. By Proposition A.4 in Appendix A, the triple $(L^{p,\gamma}, L^{p,\gamma}(K), \ell^p)$ is a stable shift-triple for any $1 < p < \infty$ and $-\infty < \gamma < \infty$.

Example 2.3 For any compact set K of $\mathbf{R}^d$, denote the space of all C^α and VC^α functions with support in K by $C^\alpha(K)$ and $VC^\alpha(K)$ respectively. By usual procedure, it can be proved that C^α and $VC^\alpha, \alpha \geq 0$, have continuous translates and $\mathcal{D}$-multiplication, and that the triples $(C^\alpha, C^\alpha(K), \ell^\infty)$ and $(VC^\alpha, VC^\alpha(K), \ell^\infty)$ are bounded shift-triples for any $\alpha \geq 0$. Moreover, the triples $(C^\alpha, C^\alpha(K), \ell^\infty)$ and $(VC^\alpha, VC^\alpha(K), \ell^\infty)$ can be proved to be stable shift-triples for any $\alpha \geq 0$ and compact set K by using the same procedure as in the proof of Proposition A.4 in Appendix A.

Example 2.4 For $-\infty < \alpha < \infty$, let $w_\alpha(x) = (1+|x|)^\alpha$, $L^p_{w_\alpha}$ be the space of all functions f with $\|f\|_{L^p_{w_\alpha}} = \|fw_\alpha\|_p < \infty$, and let $\ell^p_{w_\alpha}$ be the space of all sequences $\{c(j)\}_{j\in\mathbf{Z}}$ with $\|\{c(j)\}_{j\in\mathbf{Z}}\|_{\ell^p_{w_\alpha}} = \|\{c(j)w_\alpha(j)\}_{j\in\mathbf{Z}}\|_{\ell^p} < \infty$. It is easy to check that $L^p_{w_\alpha}$ has continuous translates and $\mathcal{D}$-multiplication. Using Lemma A.6 in the appendix, we can show that the triple $(L^p_{w_\alpha}, L^p_{w_{|\alpha|+1}}(K), \ell^p_{w_\alpha})$ is a stable shift-triple for any $p \in [1,\infty)$, $\alpha \in \mathbf{R}$ and compact set K.

Let (X_1, X_2, Y) be a bounded shift-triple. From the definition of bounded shift-triples, we see that for any linear subspace X_2' and Y' of X_2 and Y, (X_1, X_2', Y') is also a bounded shift-triple. This together with examples of bounded shift-triples would lead to many useful bounded shift-triples. However, (X_1, X_2', Y') may not be a stable shift-triple even if (X_1, X_2, Y) is. For instance, for $p \in (1,\infty)$, set $X_1 = L^p, X_2 = L^p(K), Y = \ell^p$ and $Y' = \ell^1$. Then (X_1, X_2, Y) is a stable shift-triple, but (X_1, X_2, Y') is a **unstable** shift-triple.

2.2 Shift-Invariant Sequence Space

For any compactly supported distributions $F = (f_1, \ldots, f_N)^T$, we say that F has, or $f_1, \ldots, f_N$ have, *linearly independent shifts* if the semi-convolution map $(d(j))_{j\in\mathbf{Z}^d} \longmapsto \sum_{j\in\mathbf{Z}^d} d(j)^T F(\cdot - j)$ is one-to-one, where $d(j) \in \mathbf{C}^N$ for all $j \in \mathbf{Z}^d$. We remark that any compactly supported distribution with linear independent shifts has stable shifts. In [1], we provide a decomposition of finite many compactly supported distributions.

Lemma 2.5 ([1]) *Let $f_1, \ldots, f_N$ be finitely many compactly supported distributions. Then there exist compactly supported distributions ψ_i and sequences $D_i = (d_i(j))_{j\in\mathbf{Z}^d} \in \ell_0^N, 1 \le i \le M$, such that*

(R1) $\psi_1, \ldots, \psi_M$ have linearly independent shifts;

(R2) $\psi_1, \ldots, \psi_M$, are finite linear combinations $h_k f_s(\cdot - j)$, where $h_k \in \mathcal{D}$ and $j \in \mathbf{Z}^d$, i.e., $\psi_i = \sum_{s=1}^N \sum_{k,j} c_{i,s,k,j} h_k f_s(\cdot - j)$ for finitely many $h_k \in \mathcal{D}, j \in \mathbf{Z}^d$, and some coefficients $c_{i,s,k,j}$;

(R3) $(f_1, \ldots, f_N)^T = \sum_{i=1}^M \sum_{j\in\mathbf{Z}^d} d_i(j)\psi_i(\cdot - j)$.

We say that a linear space $\mathcal{I}$ of sequences on $\mathbf{Z}^d$ is *shift-invariant* if $(d(j-k))_{j\in\mathbf{Z}^d} \in \mathcal{I}$ for all $(d(j))_{j\in\mathbf{Z}^d} \in \mathcal{I}$ and $k \in \mathbf{Z}^d$. A shift-invariant linear subspace of ℓ_0^N is said to be an *ideal* of ℓ_0^N, or an ideal for short. A subset $\mathcal{E}$ of an ideal $\mathcal{I}$ is said to be its *generator* if $\mathcal{I}$ is the minimal ideal containing all elements in $\mathcal{E}$. For a compactly supported distribution F, there are many compactly supported distributions $\psi_1, \ldots, \psi_M$, and many sequences $D_i \in \ell_0^N, 1 \le i \le M$, such that (R1), (R2), (R3) in Lemma 2.5 hold. However, the minimal ideal containing $D_i, 1 \le i \le M$, in Lemma 2.5 is unique (see [23] for the proof), which is said to be the *dependent ideal* of F, and to be denoted by $i(F)$ in [23]. For any sequence $(d(j))_{j\in\mathbf{Z}^d} \in i(F)$, it is proved in [23] that there exists $h \in \mathcal{D}$ such that the sequence $(d(j))_{j\in\mathbf{Z}^d}$ is the same as the sampling sequence of $h * F$ on $\mathbf{Z}^d$, i.e., $d(j) = h * F(j)$ for all $j \in \mathbf{Z}^d$. Actually,

$$i(F) = \Big\{ \Big(h * F(j)\Big)_{j\in\mathbf{Z}^d} : \ h \in \mathcal{D} \Big\}. \tag{2.3}$$

For any nonnegative integer r, denote the space of all functions h in $\mathcal{D}$ satisfying $\widehat{h}(\xi) = O(|\xi|^r)$ as $\xi \to 0$ by $\mathcal{D}_r$. For a compactly supported distribution F and a nonnegative integer r, define

$$i_r(F) := \Big\{ \Big(h * F(j)\Big)_{j\in\mathbf{Z}^d} : \ h \in \mathcal{D}_r \Big\}. \tag{2.4}$$

Obviously $i_r(F)$ is an ideal of ℓ_0^N for any $r \ge 0$, and

$$i_r(F) \subset i_0(F) = i(F) \quad \text{for all} \quad r \ge 0. \tag{2.5}$$

Recall that for a compactly supported continuous function F, the family of vectors $(F(x+j))_{j\in\mathbf{Z}^d}, x \in [0,1]^d$, is a generator of the dependent ideal $i(F)$ ([23]). Therefore,

$$i(h * F) \subset i_r(F) \quad \text{for all} \quad h \in \mathcal{D}_r, \tag{2.6}$$

and

$$i_r(F) = \cup_{h\in\mathcal{D}_r} i(h*F). \tag{2.7}$$

For any $D = (d(j))_{j\in\mathbf{Z}^d} \in \ell_P$, define corresponding Fourier series by $\mathcal{F}(D)(\xi) := \sum_{j\in\mathbf{Z}^d} d(j)e^{-ij\xi}$. Then $\mathcal{F}(D)$ is a 2π-periodic distribution for any $D \in \ell_P$.

Lemma 2.6 *Let $r \geq 0$, and F be a vector-valued compactly supported distribution. Assume that there is a vector-valued trigonometric polynomial $v(\xi)$ such that*

$$v(\xi)^T \widehat{F}(\xi + 2j\pi) = O(|\xi|^r) \quad \text{as} \quad \xi \to 0 \quad \text{for all} \quad j \in \mathbf{Z}^d\backslash\{0\}, \tag{2.8}$$

and

$$v(0)^T \widehat{F}(0) \neq 0. \tag{2.9}$$

Then

$$i_r(F) = \Big\{D \in i(F):\ v(\xi)^T \mathcal{F}(D)(\xi) = O(|\xi|^r) \quad \text{as} \quad \xi \to 0\Big\}.$$

Proof. For any $h \in \mathcal{D}$, by (2.8) and Taylor expansion, there exist positive constants C_1, C_2 independent of ξ and j such that

$$\begin{aligned} &\Big|\widehat{h}(\xi + 2j\pi)v(\xi)^T \widehat{F}(\xi + 2j\pi)\Big| \leq C_1 \sum_{|\kappa|=r} |D^\kappa(\hat{h}v^T\widehat{F})(t_{\xi,j}\xi + 2j\pi)|\ |\xi|^r \\ \leq\ & C_2(1 + |j|)^{-d-1}|\xi|^r \quad \text{for all} \quad |\xi| \leq 1 \quad \text{and} \quad j \in \mathbf{Z}^d\backslash\{0\}, \end{aligned}$$

where $0 \leq t_{\xi,j} \leq 1$, and where we have used the assumption on v and the fact that $h*F \in \mathcal{D}$ to obtain the last inequality. Therefore, for any $h \in \mathcal{D}$ there exists a positive constant C such that

$$\Big|\sum_{j\in\mathbf{Z}^d\backslash\{0\}} \widehat{h}(\xi + 2j\pi)v(\xi)^T \widehat{F}(\xi + 2j\pi)\Big| \leq C|\xi|^r \quad \text{for all} \quad |\xi| \leq 1.$$

This together with (2.3), (2.9) and the Poisson formula

$$\sum_{j\in\mathbf{Z}^d} f(j)e^{-ij\xi} = \sum_{j\in\mathbf{Z}^d} \widehat{f}(\xi + 2j\pi) \quad \text{for all} \quad f \in \mathcal{D}$$

lead to the assertion. $\square$

2.3 Convergence of Cascade Algorithm

Theorem 2.7 *Let $(c(j))_{j\in\mathbf{Z}^d}$ and T be the sequence and the cascade operator in (1.1) respectively, $(c_n(j))_{j\in\mathbf{Z}^d}, n \geq 1$, be the sequences in (1.3), $F \in X_2$ and $\{(d_i(j))_{j\in\mathbf{Z}^d}$:*

$1 \le i \le M\}$ *be a generator of* $i(F)$. *Assume that* $(X_1, \|\cdot\|_{X_1})$ *and* $(X_2, \|\cdot\|_{X_2})$ *be normed linear spaces of tempered distributions having continuous translates and* $\mathcal{D}$-*multiplication, and that* $(Y, \|\cdot\|_Y)$ *be a normed linear space of sequences that satisfies* $\ell_0 \subset Y \subset \ell_P$ *and has continuous shifts, i.e., there exists a positive constant* C_k *for any* $k \in \mathbf{Z}^d$ *such that*

$$\|(y(j-k))_{j\in\mathbf{Z}^d}\|_Y \le C_k \|(y(j))_{j\in\mathbf{Z}^d}\|_Y \qquad \text{for all} \quad (y(j))_{j\in\mathbf{Z}^d} \in Y. \tag{2.10}$$

Then we have

(i) There exists a positive constant C_0 *independent of* n *such that*

$$\|T^n F(2^{-n}\cdot)\|_{X_1} \le C_0 \sum_{i=1}^{M} \Big\| \Big(\sum_{j'\in\mathbf{Z}^d} c_n(j-j') d_i(j') \Big)_{j\in\mathbf{Z}^d} \Big\|_{Y^N} \qquad \text{for all} \quad n \ge 1 \tag{2.11}$$

if (X_1, X_2, Y) *is a bounded shift-triple.*

(ii) There exists a positive constant C_1 *independent of* n *such that*

$$\|T^n F(2^{-n}\cdot)\|_{X_1} \ge C_1 \sum_{i=1}^{M} \Big\| \Big(\sum_{j'\in\mathbf{Z}^d} c_n(j-j') d_i(j') \Big)_{j\in\mathbf{Z}^d} \Big\|_{Y^N} \quad \text{for all} \quad n \ge 1 \tag{2.12}$$

if (X_1, X_2, Y) *is a stable shift-triple.*

Proof. Let $D_i = (d_i(j))_{j\in\mathbf{Z}^d} \in i(F), 1 \le i \le M$, and compactly supported distributions $\psi_i, 1 \le i \le M$, be as in Lemma 2.5. Then $\psi_1, \ldots, \psi_M$ have stable shifts by (R1) in Lemma 2.5, and belong to X_2 by (R2) in Lemma 2.5 and by the assumptions that $F \in X_2$ and that X_2 has continuous translates and $\mathcal{D}$-multiplication. Still by Lemma 2.5,

$$F = \sum_{i=1}^{M} \sum_{j\in\mathbf{Z}^d} d_i(j) \psi_i(\cdot - j).$$

This together with (1.4) lead to

$$T^n F(2^{-n}\cdot) = \sum_{i=1}^{M} \sum_{j,j'\in\mathbf{Z}^d} c_n(j-j') d_i(j') \psi_i(\cdot - j).$$

Therefore, there exist positive constants C_0 and C_1 independent of n such that

$$\|T^n F(2^{-n}\cdot)\|_{X_1} \le C_0 \sum_{i=1}^{M} \Big\| \Big(\sum_{j'\in\mathbf{Z}^d} c_n(j') d_i(j-j') \Big)_{j\in\mathbf{Z}^d} \Big\|_{Y^N} \quad \text{for all} \quad n \ge 1 \tag{2.13}$$

if (X_1, X_2, Y) is a bounded shift-triple, and

$$\|T^n F(2^{-n}\cdot)\|_{X_1} \ge C_1 \sum_{i=1}^{M} \Big\| \Big(\sum_{j'\in\mathbf{Z}^d} c_n(j') d_i(j-j') \Big)_{j\in\mathbf{Z}^d} \Big\|_{Y^N} \quad \text{for all} \quad n \ge 1 \tag{2.14}$$

if (X_1, X_2, Y) is a stable shift-triple, where we have used the assumption that $\ell_0 \subset Y$ in (2.13) and (2.14). By (2.13) and (2.14), it suffices to prove that, for two generators $\{(d_{i,1}(j))_{j\in\mathbf{Z}^d}, 1 \le i \le M_1\}$ and $\{(d_{i,2}(j))_{j\in\mathbf{Z}^d}, 1 \le i \le M_2\}$ of the dependent ideal $i(F)$, there exists a positive constant C such that

$$\sum_{i=1}^{M_1} \Big\| \Big(\sum_{j'\in\mathbf{Z}^d} y(j')^T d_{i,1}(j-j') \Big)_{j\in\mathbf{Z}^d} \Big\|_{Y} \le C \sum_{i=1}^{M_2} \Big\| \Big(\sum_{j'\in\mathbf{Z}^d} y(j')^T d_{i,2}(j-j') \Big)_{j\in\mathbf{Z}^d} \Big\|_{Y} \tag{2.15}$$

for all $(y(j))_{j\in\mathbf{Z}^d} \in \ell_0^N$. By the definition of a generator of an ideal, there exist sequences $(g_{ii'}(k))_{k\in\mathbf{Z}^d} \in \ell_0, 1 \le i \le M_1, 1 \le i' \le M_2$, such that

$$d_{i,1}(j) = \sum_{i'=1}^{M_2} \sum_{k\in\mathbf{Z}^d} g_{ii'}(k) d_{i',2}(j-k) \qquad \text{for all} \quad j \in \mathbf{Z}^d \quad \text{and} \quad 1 \le i \le M_1. \tag{2.16}$$

Combining (2.10) and (2.16) leads to

$$\begin{aligned}
& \sum_{i=1}^{M_1} \Big\| \Big(\sum_{j'\in\mathbf{Z}^d} y(j')^T d_{i,1}(j-j') \Big)_{j\in\mathbf{Z}^d} \Big\|_{Y} \\
\le\ & \sum_{i=1}^{M_1} \sum_{i'=1}^{M_2} \sum_{k\in\mathbf{Z}^d} |g_{ii'}(k)| \times \Big\| \Big(\sum_{j'\in\mathbf{Z}^d} y(j')^T d_{i',2}(j-j'-k) \Big)_{j\in\mathbf{Z}^d} \Big\|_{Y} \\
\le\ & \sum_{i=1}^{M_1} \sum_{i'=1}^{M_2} \sum_{k\in\mathbf{Z}^d} C_k |g_{ii'}(k)| \times \Big\| \Big(\sum_{j'\in\mathbf{Z}^d} y(j')^T d_{i',2}(j-j') \Big)_{j\in\mathbf{Z}^d} \Big\|_{Y} \\
\le\ & C \sum_{i'=1}^{M_2} \Big\| \Big(\sum_{j'\in\mathbf{Z}^d} y(j')^T d_{i',2}(j-j') \Big)_{j\in\mathbf{Z}^d} \Big\|_{Y},
\end{aligned}$$

where we have used $(g_{ii'}(k))_{k\in\mathbf{Z}^d} \in \ell_0$ to obtain the last inequality. This proves (2.15), and completes the proof of Theorem 2.7. □

3 Smoothness of Refinable Distributions

This section is devoted to the characterization of refinable distributions in fractional Sobolev spaces and Hölder continuous spaces.

Theorem 3.1 *Let $1 < p < \infty$, $-\infty < \gamma < r < \infty$, $(c_n(j))_{j\in\mathbf{Z}^d}, n \ge 1$, be the sequences in (1.3), and Ψ be the refinable distribution in (1.6). Then $\Psi \in L^{p,\gamma}$ if and only if*

$$\lim_{n\to\infty} 2^{n\gamma-nd/p}\Big(\sum_{j\in\mathbf{Z}^d}\Big|\sum_{j'\in\mathbf{Z}^d} c_n(j')d(j-j')\Big|^p\Big)^{1/p} = 0 \tag{3.1}$$

for all $(d(j))_{j\in\mathbf{Z}^d} \in i_r(\Psi)$.

The proof of Theorem 3.1 in turn depends on the following two lemmas.

Lemma 3.2 *Let $p, \gamma, r, (c_n(j))_{j\in\mathbf{Z}^d}$ and Ψ be as in Theorem 3.1. Then (3.1) holds for any $\Psi \in L^{p,\gamma}$. Conversely, if there exist positive constant δ independent of n such that*

$$2^{n\gamma-nd/p+\delta n}\Big(\sum_{j\in\mathbf{Z}^d}\Big|\sum_{j'\in\mathbf{Z}^d} c_n(j')d(j-j')\Big|^p\Big)^{1/p}, \quad n \ge 1, \tag{3.2}$$

is a bounded sequence for any $(d(j))_{j\in\mathbf{Z}^d} \in i_r(\Psi)$, then $\Psi \in L^{p,\gamma}$.

Proof. At first we prove (3.1) under the assumption that $\Psi \in L^{p,\gamma}$. By Proposition A.3, for any $h \in \mathcal{D}_r$ with $r > \gamma$, we have

$$\lim_{n\to\infty} 2^{n\gamma}\|h_n * \Psi\|_p = 0. \tag{3.3}$$

This together with (1.7) leads to

$$\lim_{n\to\infty} 2^{n(\gamma-d/p)}\|T^n(h*\Psi)(2^{-n}\cdot)\|_p = \lim_{n\to\infty} 2^{n\gamma}\|h_n * \Psi\|_p = 0. \tag{3.4}$$

Recall that $(L^p, L^p(K), \ell^p)$ is a stable shift-triple for any $1 < p < \infty$ and any compact set K of $\mathbf{R}^d$ by Proposition A.8, and that $h * \Psi \in \mathcal{D}$. Thus, for any generator $\{(d_s(j))_{j\in\mathbf{Z}^d} :\ 1 \le s \le M\}$ of $i(h * \Psi)$, by Theorem 2.7 there exists a positive constant C independent of n such that

$$\begin{aligned} C^{-1}\|T^n(h*\Psi)(2^{-n}\cdot)\|_p &\le \sum_{s=1}^M \Big\|\Big(\sum_{j'\in\mathbf{Z}^d} c_n(j-j')d_s(j')\Big)_{j\in\mathbf{Z}^d}\Big\|_{\ell^p} \\ &\le C\|T^n(h*\Psi)(2^{-n}\cdot)\|_p \quad \text{for all} \quad n \ge 1. \end{aligned} \tag{3.5}$$

For any $D = (d(j))_{j\in\mathbf{Z}^d} \in i(h * \Psi)$ and any $h \in \mathcal{D}_r$ with $r > \gamma$, it follows from (3.4) and (3.5) that

$$\lim_{n\to\infty} 2^{n(\gamma-d/p)}\Big\|\Big(\sum_{j'\in\mathbf{Z}^d} c_n(j-j')d(j')\Big)_{j\in\mathbf{Z}^d}\Big\|_{\ell^p} = 0. \tag{3.6}$$

Then combining (2.7) and (3.6) leads to (3.1).

Next we prove $\Psi \in L^{p,\gamma}$ under the assumption (3.2). By (2.6), (3.2) and (3.5), there exists a positive constant C for any $h \in \mathcal{D}_r$ with $r > \gamma$ such that

$$\|T^n(h * \Psi)(2^{-n}\cdot)\|_p \le C 2^{-(\gamma - d/p + \delta)n}.$$

This together with (1.7) imply that $\|h_n * \Psi\|_p \le C 2^{-n\gamma - n\delta}$. Thus $\Psi \in L^{p,\gamma}$ by Proposition A.3. □

Lemma 3.3 *Let $-\infty < \beta < \infty, 1 \le p \le \infty$, $(c_n(j))_{j \in \mathbf{Z}^d}$ be as in (1.3), and let $\mathcal{I}$ be an ideal of ℓ_0^N. Then the following two statements are equivalent:*

(i) $\lim_{n\to\infty} 2^{n\beta} \left\| \left(\sum_{j' \in \mathbf{Z}^d} c_n(j') d(j - j') \right)_{j \in \mathbf{Z}^d} \right\|_{\ell^p} = 0$ *for any* $(d(j))_{j \in \mathbf{Z}^d} \in \mathcal{I}$.

(ii) There exist positive constants C and δ independent of n for any $(d(j))_{j \in \mathbf{Z}^d} \in \mathcal{I}$ such that

$$\Big\| \Big(\sum_{j' \in \mathbf{Z}^d} c_n(j') d(j - j') \Big)_{j \in \mathbf{Z}^d} \Big\|_{\ell^p} \le C 2^{-n\beta - n\delta} \quad \text{for all} \quad n \ge 1.$$

We shall use the same method as in [17] to give a proof of Lemma 3.3.

Proof. Obviously (ii) implies (i). Then it suffices to prove (ii) under the assumption (i). Let $\{D_s = (d_s(j))_{j \in \mathbf{Z}^d} : 1 \le s \le M\}$ be a generator of $\mathcal{I}$. Such a generator exists since every ideal of ℓ_0^N is finite generated ([23]). Let $\mathcal{E} = \{0,1\}^d$, and let K_0 be a finite set of $\mathbf{Z}^d$ chosen so that

$$d_s(j) = 0 \quad \text{for all} \quad j \notin K_0 \quad \text{and} \quad 1 \le s \le M, \tag{3.7}$$

and

$$c(2i - j + \epsilon) = 0 \quad \text{for all} \quad i \notin K_0, \ j \in K_0 \quad \text{and} \quad \epsilon \in \mathcal{E}. \tag{3.8}$$

Such a finite set K_0 exists since the sequence $(c(j))_{j \in \mathbf{Z}^d}$ has finite support. For any $\epsilon \in \mathcal{E}$, define

$$B_\epsilon = \Big(c(2i - j + \epsilon) \Big)_{i,j \in K_0}.$$

For any finite set $K \subset \mathbf{Z}^d$, denote the space of all sequences supported in K by $\ell(K)$. Then $D_s \in \ell(K_0)^N, 1 \le s \le M$, by (3.7), and $B_\epsilon, \epsilon \in \mathcal{E}$, is a linear transform from $\ell(K_0)^N$ to $\ell(K_0)^N$ by (3.8). Let V be the minimal linear subspace of $\ell(K_0)^N$ such that $D_s \in V$ for all $1 \le s \le M$, and $B_\epsilon V \subset V$ for all $\epsilon \in \mathcal{E}$. Then the restrictions of the matrices $B_\epsilon, \epsilon \in \mathcal{E}$, on V are well-defined. By direct computation, and using (3.7) and (3.8),

$$\Big\| \Big(\sum_{j' \in \mathbf{Z}^d} c_n(j - j') d(j') \Big) \Big\|_{\ell^p} = \Big(\sum_{\epsilon_1, \ldots, \epsilon_n \in \mathcal{E}} \|B_{\epsilon_1} \cdots B_{\epsilon_n} D\|^p \Big)^{1/p} \tag{3.9}$$

for $1 \le p < \infty$, and

$$\Big\|\Big(\sum_{j'\in\mathbf{Z}^d} c_n(j-j')d(j')\Big)\Big\|_{\ell^\infty} = \sup_{\epsilon_1,\ldots,\epsilon_n\in\mathcal{E}} \|B_{\epsilon_1}\cdots B_{\epsilon_n} D\| \tag{3.10}$$

for $p=\infty$, where $D=(d(j))_{j\in\mathbf{Z}^d}\in V$. Then by (3.9), (3.10), Lemma 4 in [17] and the assumption (i), there exists an integer n_0 such that

$$2^{n_0\beta}\Big(\sum_{\epsilon_1,\ldots,\epsilon_{n_0}\in\mathcal{E}} \|B_{\epsilon_1}\cdots B_{\epsilon_{n_0}} u\|^p\Big)^{1/p} \le \frac{1}{2}\|u\| \qquad \text{for all} \quad u\in V \tag{3.11}$$

if $1\le p<\infty$, and

$$2^{n_0\beta}\sup_{\epsilon_1,\ldots,\epsilon_{n_0}\in\mathcal{E}} \|B_{\epsilon_1}\cdots B_{\epsilon_{n_0}} u\| \le \frac{1}{2}\|u\| \qquad \text{for all} \quad u\in V \tag{3.12}$$

if $p=\infty$. For any $n\ge n_0$, write $n=kn_0+m$ for some integers $0<m\le n_0$ and $k\ge 0$. Then for any $1\le p<\infty$, it follows from (3.9) and (3.11) that

$$\begin{aligned} & 2^{np\beta}\Big\|\Big(\sum_{j'\in\mathbf{Z}^d} c_n(j-j')d_s(j')\Big)_{j\in\mathbf{Z}^d}\Big\|_{\ell^p}^p = 2^{np\beta}\sum_{\epsilon_1,\ldots,\epsilon_n\in\mathcal{E}} \|B_{\epsilon_1}\cdots B_{\epsilon_n} D_s\|^p \\ \le\ & 2^{(n-n_0)p\beta-p}\sum_{\epsilon_{n_0+1},\cdots,\epsilon_n\in\mathcal{E}} \|B_{\epsilon_{n_0+1}}\cdots B_{\epsilon_n} D_s\|^p \le \cdots \\ \le\ & 2^{-kp+mp\beta}\sum_{\epsilon_{kn_0+1},\cdots,\epsilon_n\in\mathcal{E}} \|B_{\epsilon_{kn_0+1}}\cdots B_{\epsilon_n} D_s\|^p \le C2^{-pn/n_0}, \end{aligned} \tag{3.13}$$

and, similarly for $p=\infty$ it follows from (3.10) and (3.12) that

$$2^{n\beta}\Big\|\Big(\sum_{j'\in\mathbf{Z}^d} c_n(j-j')d_s(j')\Big)_{j\in\mathbf{Z}^d}\Big\|_{\ell^\infty} \le C2^{-n/n_0}. \tag{3.14}$$

Then by letting $\delta=1/n_0$, (ii) follows from (3.13) and (3.14). $\square$

Remark 3.4 Let $1\le p\le\infty$, and $B_\epsilon, \epsilon\in\mathcal{E}$, and V be as in the proof of Lemma 3.3. Define *joint spectral radius* of $B_\epsilon, \epsilon\in\mathcal{E}$, on the linear space V by

$$\rho_p(B_\epsilon, V) := \begin{cases} \liminf_{n\to\infty}\Big(\sum_{\epsilon_1,\ldots,\epsilon_n\in\mathcal{E}} \|B_{\epsilon_1}\cdots B_{\epsilon_n}|_V\|^p\Big)^{1/(np)} & \text{if } p\in[1,\infty) \\ \liminf_{n\to\infty}\Big(\sup_{\epsilon_1,\ldots,\epsilon_n\in\mathcal{E}} \|B_{\epsilon_1}\cdots B_{\epsilon_n}|_V\|\Big)^{1/n} & \text{if } p=\infty, \end{cases}$$

where $\|A|_V\|$ is a norm of the operator A restricted to V. By usual procedure used in the theory of joint spectral radius (see [17] for instance),

$$\begin{aligned} \rho_p(B_\epsilon, V) &= \inf_{n\ge 1}\Big(\sum_{\epsilon_1,\ldots,\epsilon_n\in\mathcal{E}} \|B_{\epsilon_1}\cdots B_{\epsilon_n}|_V\|^p\Big)^{1/(np)} \\ &= \inf_{n\ge 1}\Big(\sup_{u\in V,\|u\|_*=1}\sum_{\epsilon_1,\ldots,\epsilon_n\in\mathcal{E}} \|B_{\epsilon_1}\cdots B_{\epsilon_n}u\|_*^p\Big)^{1/(np)} \end{aligned}$$

for $1 \le p < \infty$, and

$$\rho_\infty(B_\epsilon, V) = \inf_{n\ge 1}\Big(\sup_{u\in V, \|u\|_*=1}\ \sup_{\epsilon_1,\ldots,\epsilon_n\in\mathcal{E}} \|B_{\epsilon_1}\cdots B_{\epsilon_n} u\|_*\Big)^{1/n},$$

where $\|u\|_*$ is a norm on V. By the proof of Lemma 3.3, we can interpreted the condition (i) in Lemma 3.3 as

$$\rho_p(B_\epsilon, V) < 2^{-\beta}. \tag{3.15}$$

By Theorem 3.1, we have the following characterization of refinable distributions in $L^{p,\gamma}$ under the linear independence assumption.

Theorem 3.5 *Let $p, \gamma, (c_n(j))_{j\in\mathbf{Z}^d}$ and Ψ be as in Theorem 3.1. Assume that $N = 1$, Ψ has linearly independent shifts and satisfies $\widehat{\Psi}(0) \neq 0$. Then $\Psi \in L^{p,\gamma}$ if and only if*

$$\lim_{n\to\infty} 2^{n\gamma - nd/p}\Big(\sum_{j\in\mathbf{Z}^d}\Big|\sum_{j'\in\mathbf{Z}^d} c_n(j')d(j-j')\Big|^p\Big)^{1/p} = 0$$

for all sequence $D = (d(j))_{j\in\mathbf{Z}^d} \in \ell_0^N$ with $\mathcal{F}(D)(\xi) = O(|\xi|^{\gamma_1})$ as $\xi \to 0$, where γ_1 is the smallest nonnegative integer strictly larger than γ.

By Theorem 3.5 and Remark 3.4, we can use the joint spectral radius on an explicit linear space of sequences to characterize all scale-valued refinable distributions in $L^{p,\gamma}$ under the additional assumption that they have linearly independent shifts.

Corollary 3.6 *Let $p, \gamma, \gamma_1, \Psi$ be as in Theorem 3.5, and let $B_\epsilon, \epsilon \in \mathcal{E}$ and K_0 be as in Remark 3.4 with $\mathcal{I} = i_{\gamma_1}(\Psi)$. Assume that $N = 1$, Ψ has linearly independent shifts and satisfies $\widehat{\Psi}(0) \neq 0$. Then $\Psi \in L^{p,\gamma}$ if and only if $\rho_p(B_\epsilon, V) < 2^{-\gamma+d/p}$, where*

$$V = \Big\{(d(j))_{j\in\mathbf{Z}^d} \in \ell(K_0):\ \sum_{j\in K_0} d(j)p(j) = 0 \quad \text{for all} \quad p \in \Pi_\gamma\Big\}$$

and Π_γ is the set of all polynomials with their degrees less than γ.

By Theorem 3.1, the proof of Theorem 3.5 reduces to

$$i_{\gamma_1}(\Psi) = \Big\{D \in \ell_0:\ \mathcal{F}(D)(\xi) = O(|\xi|^{\gamma_1}) \quad \text{as} \quad \xi \to 0\Big\}. \tag{3.16}$$

Recall that $i(F) = \ell_0^N$ if F has compact support and linearly independent shifts (see [23]). Then by Lemma 2.6, the equality in (3.16) in turn depends on the following result about moment conditions of refinable distributions, which was given in [10].

Proposition 3.7 *Let $N = 1$, and Ψ be the compactly supported distribution in (1.6) with $\widehat{\Psi}(0) \neq 0$. Assume that $\Psi \in L^{p,\gamma}$ for some $1 < p < \infty$ and $\gamma \geq 0$. Then $D^\kappa \Psi(2j\pi) = 0$ for all $j \in \mathbf{Z}^d \backslash \{0\}$ and $|\kappa| \leq \gamma$.*

For any $\alpha \geq 0$ and $f \in C^\alpha$, define

$$\omega_\alpha(f, t) = \sup_{0<|x_1-x_2|<t} \sum_{|\kappa|=\alpha_0} \frac{|D^\kappa f(x_1) - D^\kappa f(x_2)|}{|x_1 - x_2|^\delta},$$

where $\alpha = \alpha_0 + \delta$ and $0 \leq \delta < 1$. Then $\omega_\alpha(f, t)$ is a bounded function of t for any $f \in C^\alpha$, and $\lim_{t\to 0} \omega_\alpha(f, t) = 0$ for any $f \in VC^\alpha$. By Taylor expansion, there exists a positive constant C independent of n for any $h \in \mathcal{D}_r$ with $r > \alpha$ and any compactly supported $f \in C^\alpha$ such that

$$\|h_n * f\|_\infty \leq C 2^{-n\alpha} \omega_\alpha(f, 2^{-n}) \quad \text{for all} \quad n \geq 1,$$

where $h_n = 2^{nd} h(2^n \cdot), n \geq 0$. Therefore, the sequence $2^{n\alpha}\|h_n * f\|_\infty$, $n \geq 1$, is a bounded sequence if $f \in C^\alpha$ and $h \in \mathcal{D}_r$ with $r > \alpha$, and the sequence $2^{n\alpha}\|h_n * f\|_\infty, n \geq 1$, converges to zero if $f \in VC^\alpha$ and $h \in \mathcal{D}_r$ with $r > \alpha$. Recall that $(C^\alpha, C^\alpha(K), \ell^\infty)$ and $(VC^\alpha, VC^\alpha(K), \ell^\infty)$ are stable shift-triples for any compact set K and $\alpha \geq 0$ (see Example 2.3). Then by using the same procedure as in the proof of Theorem 3.1, we obtain the following result about refinable distributions in C^α and VC^α.

Theorem 3.8 *Let $\alpha \geq 0$, r be the smallest integer strictly larger than α, and Ψ be the refinable function in (1.6). If $\Psi \in C^\alpha$, then for any $(d(j))_{j\in\mathbf{Z}^d} \in i_r(\Psi)$, there exists a positive constant C such that*

$$2^{n\alpha} \sup_{j\in\mathbf{Z}^d} |\sum_{j'\in\mathbf{Z}^d} c_n(j') d(j - j')| \leq C \qquad \text{for all} \quad n \geq 1.$$

Theorem 3.9 *Let $\alpha \geq 0$, r be the smallest integer strictly larger than α, and Ψ be the refinable function in (1.6). Then $\Psi \in VC^\alpha$ if and only if*

$$\lim_{n\to\infty} 2^{n\alpha} \sup_{j\in\mathbf{Z}^d} \Big| \sum_{j'\in\mathbf{Z}^d} c_n(j') d(j - j') \Big| = 0 \quad \text{for all} \quad (d(j))_{j\in\mathbf{Z}^d} \in i_r(\Psi).$$

4 Initial of Cascade Algorithm

In this section, we discuss the problem how to choose the initial appropriately such that the cascade algorithm always converges. First by Theorems 2.7 and 3.1, Propositions A.2 and A.4, we have the following result.

Corollary 4.1 *Let $1 < p < \infty, \gamma \geq 0, \Psi \in L^{p,\gamma}$ be the refinable function in (1.6). If the initial G is so chosen that G is a compactly supported $L^{p,\beta}$ function, and satisfies*

$$i(G) \subset i_{\gamma_1}(\Psi), \tag{4.1}$$

then the cascade algorithm $T^n G$ converges in $L^{p,\beta}$ with rate $\gamma - \max(0, \beta)$, i.e.,

$$\lim_{n\to\infty} 2^{n(\gamma-\max(0,\beta))} \|T^n G\|_{L^{p,\beta}} = 0, \tag{4.2}$$

where $\beta \in \mathbf{R}$, and γ_1 be the smallest nonnegative integer strictly larger than γ.

We emphasize that β in Corollary 4.1 can be chosen that $\beta \geq \gamma$ or $\beta \leq 0$. So the essential condition for the convergence of a cascade algorithm is the one about the dependent ideal of the initial. For $N = 1$, we may use Lemma 2.6 and Proposition 3.7 to simplify the dependent ideal condition (4.1) on the initial G.

Theorem 4.2 *Let $N = 1$, Ψ and G be compactly supported distributions. Assume that Ψ satisfies the refinement equation (1.6), $\widehat{\Psi}(0) \neq 0$ and $\Psi \in L^{p,\gamma}$ for some $1 < p < \infty$ and $\gamma \geq 0$, and that $G \in L^p, i(G) \subset i(\Psi)$ and*

$$D^\kappa G(2j\pi) = 0 \quad \text{for all} \quad |\kappa| \leq \gamma \quad \text{and} \quad j \in \mathbf{Z}^d.$$

Then $\lim_{n\to\infty} 2^{n\gamma} \|T^n G\|_p = 0$.

Proof. Obviously it suffices to prove (4.1). From Lemma 2.6 and Proposition 3.7, we have

$$i_{\gamma_1}(\Psi) = \Big\{D \in i(\Psi) : \ \mathcal{F}(D)(\xi) = O(|\xi|^{\gamma_1}) \quad \text{as} \quad \xi \to 0\Big\}. \tag{4.3}$$

Thus, by (4.3) and the assumption $i(G) \subset i(\Psi)$, the proof of (4.1) reduces to proving

$$\sum_{j\in\mathbf{Z}^d} d(j)e^{-ij\xi} = O(|\xi|^{\gamma_1}) \quad \text{as} \quad \xi \to 0 \quad \text{for all} \quad (d(j))_{j\in\mathbf{Z}^d} \in i(G). \tag{4.4}$$

Let $(d(j))_{j\in\mathbf{Z}^d}$ be any sequence in $i(G)$. Then there exists $h_1 \in \mathcal{D}$ by (2.3) such that $d(j) = (h_1 * G)(j)$ for all $j \in \mathbf{Z}^d$. By using the same procedure in the proof of Lemma 2.6, we obtain

$$\sum_{j\in\mathbf{Z}^d} \widehat{h}_1(\xi + 2j\pi)\widehat{G}(\xi + 2j\pi) = O(|\xi|^{\gamma_1}) \quad \text{as} \quad \xi \to 0. \tag{4.5}$$

Hence,

$$\sum_{j\in\mathbf{Z}^d} d(j)e^{-ij\xi} = \sum_{j\in\mathbf{Z}^d} \widehat{h}_1(\xi + 2j\pi)\widehat{G}(\xi + 2j\pi) = O(|\xi|^{\gamma_1}) \quad \text{as} \quad \xi \to 0$$

by (4.5) and the Poisson formula. This proves (4.4). $\square$

As we know, the refinable function in (1.6) is a fixed point of the cascade operator T. When the initial g is chosen appropriately, $T^N g$ converges to the limit Ψ in L^p norm very fast.

Theorem 4.3 *Let $\gamma \geq 0, 1 < p < \infty$ and $N = 1$. Assume that $\Psi \in L^{p,\gamma}$ satisfies the refinement equation (1.6) and $\widehat{\Psi}(0) = 1$. Let g be a compactly supported L^p function that satisfies the following three conditions:*

(i) $D^\kappa \widehat{g}(2j\pi) = 0$ for all $j \in \mathbf{Z}^d \backslash \{0\}$ and $\kappa \in \mathbf{Z}_+^d$ with $|\kappa| \leq \gamma$, and $\hat{g}(0) = 1$.

(ii) $\widehat{\Psi} - \widehat{g} = O(|\cdot|^\gamma)$ near the origin.

(iii) $i(g) \subset i(\Psi)$.

Then there exists a positive constant C such that

$$\|T^n g - \Psi\|_p \leq C 2^{-n\gamma} \qquad \text{for all} \qquad n \geq 1.$$

Proof. Set $G_\gamma = g - \Psi - \sum_{|\alpha|=\gamma} a_\alpha D^\alpha \Psi$ if γ is a nonnegative integer, and set $G_\gamma = g - \Psi$ otherwise, where the coefficients a_α with $\alpha \in \mathbf{Z}_+^d$ and $|\alpha| = \gamma$ are chosen so that $\widehat{G}_\gamma = O(|\cdot|^{\gamma+1})$ near the origin. The existence and uniqueness of those coefficients follow from (ii), $\widehat{\Psi}(0) = 1$ and the fact that $\widehat{\Psi}$ and $\hat{g}$ are analytic functions. Then, $G_\gamma \in L^p$ since $g \in L^p$ and $\Psi \in L^{p,\gamma}$ for some $\gamma \geq 0$. Moreover, $D^\kappa G_\gamma(2j\pi) = 0$ for all $j \in \mathbf{Z}^d$ and $\kappa \in \mathbf{Z}_+^d$ with $|\kappa| \leq \gamma$ because of Proposition 3.7 and the assumptions on $\hat{g}$, and the assertion $i(G_\gamma) \subset i(\Psi)$ follows from the assumption (iii) and the observation $i(D^\alpha \Psi) \subset i(\Psi)$, which is obvious by (2.3). So G_γ satisfies the required conditions in Theorem 4.2. Therefore there exists a positive constant C by Theorem 4.2 such that

$$\|T^n G_\gamma\|_p \leq C 2^{-n\gamma} \quad \text{for all} \quad n \geq 1. \tag{4.6}$$

By direct computation,

$$T^n g - \Psi = T^n G_\gamma \tag{4.7}$$

if γ is not a nonnegative integer, and

$$T^n g - \Psi = T^n G_\gamma + 2^{-n\gamma} \sum_{|\alpha|=\gamma} a_\alpha D^\alpha \Psi \tag{4.8}$$

if γ is a nonnegative integer. Hence combining (4.6), (4.7) and (4.8) leads to the desired assertion. $\square$

A Properties of Fractional Sobolev Spaces

In this appendix, we give some basic properties of the fractional Sobolev space $L^{p,\gamma}$.

Proposition A.1 *Let $1 < p < \infty$ and $-\infty < \gamma < \infty$. Then $L^{p,\gamma}$ has continuous translates and $\mathcal{D}$-multiplication.*

Proof. From the definition of the space $L^{p,\gamma}$, we have $\|f(\cdot - y)\|_{L^{p,\gamma}} = \|f\|_{L^{p,\gamma}}$. Hence $L^{p,\gamma}, 1 < p < \infty$, has continuous translates.

For nonnegative integer γ, the assertion about continuous $\mathcal{D}$-multiplication of the fractional Sobolev space $L^{p,\gamma}$ follows easily from (1.12). For general real number γ, we need use the classical multiplier theorem in [20]. Set $m_{\eta,\gamma}(\xi) = (1 + |\xi + \eta|^2)^{\gamma/2}(1 + |\xi|^2)^{-\gamma/2}$ for any $\eta \in \mathbf{R}^d$, and let

$$\Sigma_t(\alpha) = \Big\{(\alpha_1, \ldots, \alpha_t) : \sum_{i=1}^{t} \alpha_i = \alpha \quad \text{and} \quad 0 \neq \alpha_i \in \mathbf{Z}_+^d \quad \text{for all } 1 \leq i \leq t\Big\} \tag{A.1}$$

for any $\alpha \in \mathbf{Z}_+^d$. By direct computation,

$$\begin{aligned}
& |D^\alpha m_{\eta,\gamma}(\xi)| \\
\leq\ & C \sum_{1\leq t\leq|\alpha|} |m_{\eta,\gamma-2t}(\xi)| \sum_{(\alpha_1,\ldots,\alpha_t)\in\Sigma_t(\alpha)} \prod_{i=1}^{t} |D^{\alpha_i} m_{\eta,2}(\xi)| \\
\leq\ & C \sum_{1\leq t\leq|\alpha|} |m_{\eta,\gamma}(\xi)| \sum_{(\alpha_1,\ldots,\alpha_t)\in\Sigma_t(\alpha)} \prod_{i=1}^{t} \Big(\sum_{\beta_i\leq\alpha_i} (1 + |\xi + \eta|)^{-|\alpha_i|+|\beta_i|}(1 + |\xi|)^{-|\beta_i|}\Big) \\
\leq\ & C(1 + |\eta|)^{|\gamma|+|\alpha|}(1 + |\xi|)^{-|\alpha|}.
\end{aligned}$$

Therefore, $m_{\eta,\gamma}$ is an L^p multiplier by classical multiplier theorem for any $1 < p < \infty$ ([20, p. 96]). Moreover there exists a positive constant C independent of η such that

$$\|(m_{\eta,\gamma}\hat{f})^\vee\|_p \leq C(1 + |\eta|)^{|\gamma|+d}\|f\|_p \quad \text{for all} \quad f \in L^p \quad \text{and} \quad \eta \in \mathbf{R}^d. \tag{A.2}$$

Recall that $\widehat{hf} = \widehat{h} * \widehat{f}$. This together with (A.2) leads to

$$\begin{aligned}
\|hf\|_{L^{p,\gamma}} &\leq C \int_{\mathbf{R}^d} |\widehat{h}(\eta)| \times \Big\|\Big(m_{\eta,\gamma}\hat{f}(1 + |\cdot|^2)^{\gamma/2}\Big)^\vee\Big\|_p d\eta \\
&\leq C \int_{\mathbf{R}^d} |\widehat{h}(\eta)|(1 + |\eta|)^{|\gamma|+|d|} d\eta \times \|f\|_{L^{p,\gamma}} < \infty
\end{aligned}$$

for any $h \in \mathcal{D}$. Hence $L^{p,\gamma}$ has continuous $\mathcal{D}$-multiplication. □

Proposition A.2 *Let $-\infty < \gamma < \infty$, $\lambda \geq 1$ and $1 < p < \infty$. Therefore, there exists a positive constant independent of λ such that*

$$\|f(\lambda\cdot)\|_{L^{p,\gamma}} \leq C\lambda^{\max(0,\gamma)-d/p}\|f\|_{L^{p,\gamma}} \quad \text{for all} \quad f \in L^{p,\gamma}.$$

Proof. Set $m_{\lambda,\gamma}(\xi) = (1+\lambda^2|\xi|^2)^{\gamma/2}(1+|\xi|^2)^{-\gamma/2}$. Recall that for any $\alpha \in \mathbf{Z}_+^d$ there exists a positive constant C_α such that $|D^\alpha(1+|\xi|^2)^{-1}| \le C_\alpha(1+|\xi|)^{-2-|\alpha|}$ for all $\xi \in \mathbf{R}^d$. Therefore,

$$\begin{aligned}
|D^\alpha m_{\lambda,2}(\xi)| &\le C_1 \sum_{\beta\le\alpha} |D^\beta(1+\lambda^2|\xi|^2)| \times |D^{\alpha-\beta}(1+|\xi|^2)^{-1}| \\
&\le C_2 \sum_{\beta\le\alpha} \lambda^{|\beta|}(1+\lambda|\xi|)^{2-|\beta|}(1+|\xi|)^{-2-|\alpha|+|\beta|} \\
&\le \begin{cases} C_3\lambda^2|\xi|^{-|\alpha|} & \text{if } |\xi|\ge 1 \\ C_3\lambda^2|\xi|^{2-|\alpha|} & \text{if } \lambda^{-1}\le|\xi|\le 1 \\ C_3\lambda^{|\alpha|} & \text{if } |\xi|\le\lambda^{-1}, \end{cases}
\end{aligned} \tag{A.3}$$

where C_1, C_2, C_3 are positive constants independent of n and ξ. Let $\Sigma_t(\alpha)$ be as in (A.1). By induction,

$$D^\alpha m_{\lambda,\gamma}(\xi) = \sum_{1\le t\le|\alpha|} m_{\lambda,\gamma-2t}(\xi) \sum_{(\alpha_1,\ldots,\alpha_t)\in\Sigma_t(\alpha)} C(\alpha;\alpha_1,\ldots,\alpha_t) \prod_{i=1}^t D^{\alpha_i} m_{\lambda,2}(\xi) \tag{A.4}$$

for some coefficients $C(\alpha;\alpha_1,\ldots,\alpha_t)$. Combining (A.3) and (A.4) leads to

$$\begin{aligned}
|D^\alpha m_{\lambda,\gamma}(\xi)| &\le C_1 \sum_{1\le t\le|\alpha|} |m_{\lambda,\gamma-2t}(\xi)| \sum_{(\alpha_1,\ldots,\alpha_t)\in\Sigma_t(\alpha)} \prod_{i=1}^t |D^{\alpha_i} m_{\lambda,2}(\xi)| \\
&\le \begin{cases} C_2\lambda^\gamma|\xi|^{-|\alpha|} & \text{if } |\xi|\ge 1 \\ C_2\lambda^\gamma|\xi|^{\gamma-|\alpha|} & \text{if } \lambda^{-1}\le|\xi|\le 1 \\ C_2\lambda^{|\alpha|} & \text{if } |\xi|\le\lambda^{-1} \end{cases} \\
&\le C_3\lambda^{\max(0,\gamma)}|\xi|^{-|\alpha|},
\end{aligned} \tag{A.5}$$

where C_1, C_2, C_3 are positive constants independent of n and ξ. Then, by (A.5) and the classical multiplier theorem ([20], p. 96),

$$\|(m_{\lambda,\gamma}\widehat{f})^\vee\|_p \le C\lambda^{\max(0,\gamma)}\|f\|_p \quad \text{for all} \quad f\in L^p \quad \text{and} \quad n\ge 1, \tag{A.6}$$

where C is a positive constant independent of λ and f. This implies that

$$\begin{aligned}
\|f\|_{L^{p,\gamma}} &= \lambda^{-d/p}\|(\lambda^d\widehat{f}(\lambda\cdot)(1+\lambda^2|\cdot|^2)^{\gamma/2})^\vee\|_p \\
&\le C\lambda^{-d/p+\max(0,\gamma)}\|(\lambda^d\widehat{f}(\lambda\cdot)(1+|\cdot|^2)^{\gamma/2})^\vee\|_p \\
&= C\lambda^{-d/p+\max(0,\gamma)}\|f(\lambda^{-1}\cdot)\|_{L^{p,\gamma}} \quad \text{for any} \quad f\in L^{p,\gamma},
\end{aligned}$$

where C is a positive constant independent of f and λ. $\square$

Proposition A.3 *Let $1 < p < \infty$ and $-\infty < \gamma < r < \infty$. Then*

$$\lim_{n\to\infty} 2^{n\gamma}\|h_n * f\|_p = 0 \tag{A.7}$$

for any $h \in \mathcal{D}_r$ and $f \in L^{p,\gamma}$, where $h_n = 2^{nd}h(2^n\cdot), n \ge 0$. Conversely, if

$$h_{-1} * f \in L^p \tag{A.8}$$

for any function $h_{-1} \in \mathcal{D}$, and if

$$2^{n(\gamma+\delta)}\|h_n * f\|_p, \quad n \ge 1, \tag{A.9}$$

is a bounded sequence for some positive constant δ and any $h \in \mathcal{D}_r$, then $f \in L^{p,\gamma}$.

Proof. For any $h \in \mathcal{D}_r$ and $g \in \mathcal{D}$, set $m_{n,1}(\xi) = \widehat{h}(2^{-n}\xi)(1+|\xi|^2)^{-\gamma/2}$ and $m_{n,2}(\xi) = \widehat{h}(2^{-n}\xi)g(\xi)$. Let N be a sufficiently large constant chosen later. By direct computation, for any $\alpha \in \mathbf{Z}_+^d$ and some positive constants C_α, we have

$$|D^\alpha \widehat{h}(\xi)| \le \begin{cases} C_\alpha \min(1, |\xi|^{r-|\alpha|}) & \text{if } |\xi| \le 1 \\ C_\alpha |\xi|^{-N} & \text{if } |\xi| \ge 1. \end{cases}$$

Let C_g be a positive constant chosen so that $g(\xi) = 0$ for all ξ with $|\xi| \ge C_g$. Therefore, for all $\xi \in \mathbf{R}^d$ and $\alpha \in \mathbf{Z}_+^d$,

$$\begin{aligned} & |D^\alpha m_{n,1}(\xi)| \\ \le\ & C_{\alpha,1} \sum_{\beta\le\alpha} 2^{-n|\beta|} |(D^\beta \widehat{h})(2^{-n}\xi)|(1+|\xi|)^{-\gamma-|\alpha|+|\beta|} \\ \le\ & \begin{cases} C_{\alpha,2} 2^{-n\gamma}|\xi|^{-|\alpha|} \sum_{\beta\le\alpha} (2^{-n}|\xi|)^{-N+|\beta|-\gamma} & \text{if } |\xi| \ge 2^n \\ C_{\alpha,2} 2^{-n\gamma}|\xi|^{-|\alpha|} \sum_{\beta\le\alpha} \min(1, (2^{-n}|\xi|)^{r-|\beta|})(2^{-n}|\xi|)^{|\beta|-\gamma} & \text{if } 1 \le |\xi| \le 2^n \\ C_{\alpha,2} \sum_{\beta\le\alpha} 2^{-n|\beta|} \min(1, (2^{-n}|\xi|)^{r-|\beta|}) & \text{if } |\xi| \le 1 \end{cases} \\ \le\ & C_{\alpha,3} 2^{-n\gamma}(1+|\xi|)^{-|\alpha|}, \end{aligned}$$

and

$$\begin{aligned} |D^\alpha m_{n,2}(\xi)| &\le C_{\alpha,4} \sum_{\beta\le\alpha} 2^{-n|\beta|} |(D^\beta \widehat{h})(2^{-n}\xi)| \times |D^{\alpha-\beta} g(\xi)| \\ &\le \begin{cases} C_{\alpha,5} \sum_{\beta\le\alpha} 2^{-n|\beta|} \min(1, (2^{-n}|\xi|)^{r-|\beta|}) & \text{if } |\xi| \le C_g \\ 0 & \text{if } |\xi| \ge C_g \end{cases} \\ &\le C_{\alpha,6} 2^{-nr}(1+|\xi|)^{-|\alpha|}, \end{aligned}$$

where $C_{\alpha,i}, 1 \le i \le 6$, are positive constants independent of n and ξ. Thus, by the classical multiplier theorem ([20]), there exists a positive constant C such that

$$\|(m_{n,1}\hat{f})^{\vee}\|_p \le C2^{-n\gamma}\|f\|_p \tag{A.10}$$

and

$$\|(m_{n,2}\hat{f})^{\vee}\|_p \le C2^{-nr}\|f\|_p \quad \text{for all} \quad f \in L^p \quad \text{and} \quad n \ge 0. \tag{A.11}$$

For any $f \in L^{p,\gamma}$ and any positive constant ϵ less than one, we may write

$$f = f_{1,\epsilon} + f_{2,\epsilon} \tag{A.12}$$

such that $\widehat{f}_{1,\epsilon}$ has compact support and $\|f_{2,\epsilon}\|_{L^{p,\gamma}} \le \epsilon\|f\|_{L^{p,\gamma}}$. Let H be a Schwartz function chosen so that $\widehat{H}$ has compact support and $\widehat{H}(\xi) = 1$ on $\{\xi : \ |\xi| \le 1\}$, and set $H_\delta = \delta^{-d}H(\delta^{-1}\cdot)$. Then $\lim_{\delta\to 0}\|H_\delta * f - f\|_{L^{p,\gamma}} = 0$ for any $f \in L^{p,\gamma}$. So we need only choose $f_{1,\epsilon} = H_\delta * f$ for some sufficiently small positive constant δ. By letting δ_1 be chosen so that $f_{1,\epsilon}$ is supported in $|\xi| \le \delta_1^{-1}$, and using (A.11) with $g(\xi) = (1+|\xi|^2)^{-\gamma/2}\widehat{H}(\xi/\delta_1)$,

$$\|h_n * f_{1,\epsilon}\|_{L^{p,\gamma}} = \|(m_{n,2}\widehat{f}_{1,\epsilon}(1+|\cdot|^2)^{\gamma/2})^{\vee}\|_p \le C_\epsilon 2^{-nr}\|f\|_{L^{p,\gamma}} \tag{A.13}$$

for some positive constant C_ϵ independent of n. By (A.10), there exists a positive constant C independent of ϵ and n such that

$$\|h_n * f_{2,\epsilon}\|_{L^{p,\gamma}} = \|(m_{n,1}\widehat{f}_{2,\epsilon}(1+|\cdot|^2)^{\gamma/2})^{\vee}\|_p \le C2^{-n\gamma}\|f_{2,\epsilon}\|_{L^{p,\gamma}} \le C2^{-n\gamma}\epsilon\|f\|_{L^{p,\gamma}}. \tag{A.14}$$

Recall that ϵ can be chosen arbitrary small. Then (A.7) follows from (A.12), (A.13) and (A.14).

Let ψ_{-1} and ψ_0 be Schwartz functions such that $\widehat{\psi_{-1}}$ is supported in $\{\xi : |\xi| \le 1\}$, $\widehat{\psi_0}$ is supported in $\{\xi : 1/2 \le |\xi| \le 2\}$, and

$$\sum_{n=-1}^{\infty} \widehat{\psi_n} \equiv 1 \quad \text{on} \quad \mathbf{R}^d, \tag{A.15}$$

where $\psi_n = 2^{nd}\psi_0(2^n\cdot), \ n \ge 0$. Let $h_{-1} \in \mathcal{D}$ and $h_0 \in \mathcal{D}_r$ be chosen so that $\widehat{h_{-1}}(\xi) \ne 0$ on $\{\xi : |\xi| \le 2\}$ and $\widehat{h_0}(\xi) \ne 0$ on $\{\xi : 1/4 \le |\xi| \le 8\}$. Define $\psi_{n,\gamma}, n \ge -1$, by $\widehat{\psi_{n,\gamma}} = \widehat{\psi}_n(1+|\cdot|^2)^{\gamma/2}$. By direct computation, for all $f \in L^p$ and $n \ge 0$, we have

$$\|\psi_{-1,\gamma} * f\|_p \le \Big\|\Big(\widehat{\psi_{-1}}(\widehat{h_{-1}})^{-1}(1+|\cdot|^2)^{\gamma/2}\Big)^{\vee}\Big\|_1 \|h_{-1} * f\|_p \le C\|h_{-1} * f\|_p \tag{A.16}$$

and

$$\|\psi_{n,\gamma} * f\|_p \le \Big\|\Big(\widehat{\psi_0}(2^{-n}\cdot)h_0(2^{-n}\cdot)^{-1}(1+|\cdot|^2)^{\gamma/2}\Big)^{\vee}\Big\|_1 \|h_n * f\|_p \le C2^{n\gamma}\|h_n * f\|_p, \tag{A.17}$$

where C is a positive constant independent of n and f. Combining (A.8), (A.9), (A.15), (A.16), and (A.17) leads to

$$\|f\|_{L^{p,\gamma}} \le C_1 \sum_{n=-1}^{\infty} \|\psi_{n,\gamma} * f\|_p \le C_2 \sum_{n=-1}^{\infty} 2^{-n\delta} < \infty,$$

where C_1 and C_2 are positive constants. This proves that $f \in L^{p,\gamma}$ under the assumption that (A.8) and (A.9) hold. □

Proposition A.4 *Let $1 < p < \infty$, $-\infty < \gamma < \infty$, and let K be a compact set of $\mathbf{R}^d$. Then $(L^{p,\gamma}, L^{p,\gamma}(K), \ell^p)$ is a stable shift-triple.*

Define the usual Kronecker symbol δ by $\delta_{st} = 1$ if $s = t$ and $\delta_{st} = 0$ otherwise. To prove Proposition A.4, we need another definition of the fractional Sobolev space $L^{p,\gamma}$, which follows easily from the classical Littlewood-Paley theory ([24]), and a result about functions having stable shifts ([22]).

Lemma A.5 ([24]) *Let $1 < p < \infty, -\infty < \gamma < \infty$ and $r > |\gamma| + d$. Let $\psi_{-1} \in \mathcal{D}$ and $\psi_0 \in \mathcal{D}_r$ satisfy $\sum_{n=-1}^{\infty} |\widehat{\psi_n}(\xi)|^2 \ge C_0$ for all $\xi \in \mathbf{R}^d$, where $\psi_n = 2^{nd}\psi(2^n\cdot),\ n \ge 0$. Then there exists a positive constant C such that*

$$C^{-1}\|f\|_{L^{p,\gamma}} \le \Big\|\Big(\sum_{n=-1}^{\infty} 2^{2n\gamma}|\psi_n * f|^2\Big)^{1/2}\Big\|_p \le C\|f\|_{L^{p,\gamma}}.$$

Lemma A.6 ([22]) *Let $\phi_1, \ldots, \phi_M$ be compactly supported distributions having stable shifts. Then there exist functions $\psi_s \in \mathcal{D}, 1 \le s \le M$, and a sequence $P = (p_s(j))_{j\in\mathbf{Z}^d} \in \ell_0$ such that $\mathcal{F}(P)(\xi) \neq 0$ for all $\xi \in \mathbf{R}^d$, and*

$$\langle \phi_s, \psi_t(\cdot - j)\rangle = \delta_{st} p(j) \quad \text{for all} \quad j \in \mathbf{Z}^d.$$

Proof of Proposition A.4. For any $f \in L^{p,\gamma}(K)$ and any $D = (d(j))_{j\in\mathbf{Z}^d} \in \ell^p$, set $g = \sum_{j\in\mathbf{Z}^d} d(j) f(\cdot - j)$. Let r, ψ_{-1} and ψ_0 be chosen as in Lemma A.5. Then $\psi_n * f, n \ge -1$, are supported in a compact set independent of n. This implies that $(\sum_{n=-1}^{\infty} 2^{2n\gamma}|\psi_n * f|^2)^{1/2}$ is compactly supported. Therefore,

$$\begin{aligned}
\|g\|_{L^{p,\gamma}} &\le C_1\Big\|\Big(\sum_{n=-1}^{\infty} 2^{2n\gamma}|\psi_n * g|^2\Big)^{1/2}\Big\|_p \\
&\le C_2\Big\|\sum_{j\in\mathbf{Z}^d} |d(j)|\Big(\sum_{n=-1}^{\infty} 2^{2n\gamma}|\psi_n * f(\cdot - j)|^2\Big)^{1/2}\Big\|_p \\
&\le C_3\|D\|_{\ell^p}\Big\|\Big(\sum_{n=-1}^{\infty} 2^{2n\gamma}|\psi_n * f|^2\Big)^{1/2}\Big\|_p \\
&\le C_4\|D\|_{\ell^p}\|f\|_{L^{p,\gamma}},
\end{aligned}$$

where $C_i, i = 1, 2, 3, 4$, are positive constants independent of $D \in \ell^p$. This proves that $(L^{p,\gamma}, L^{p,\gamma}(K), \ell^p)$ is a bounded shift-triple.

Let $f_1, \ldots, f_M \in L^{p,\gamma}(K)$ have stable shifts, and $D_s = (d_s(j))_{j\in\mathbf{Z}^d} \in \ell^p, 1 \le s \le M$. Then

$$g = \sum_{s=1}^{M} \sum_{j\in\mathbf{Z}^d} d_s(j) f_s(\cdot - j) \in L^{p,\gamma} \tag{A.18}$$

because $(L^{p,\gamma}, L^{p,\gamma}(K), \ell^p)$ is a bounded shift-triple. By Lemma A.6, there exist $\psi_s \in \mathcal{D}, 1 \le s \le M$, and $P = (p(j))_{j\in\mathbf{Z}^d} \in \ell_0$ such that

$$\langle \psi_s(\cdot - j), f_t\rangle = \delta_{st} p(j) \quad \text{for all} \quad 1 \le s, t \le M \quad \text{and} \quad j \in \mathbf{Z}^d, \tag{A.19}$$

and

$$\mathcal{F}(P)(\xi) \ne 0 \quad \text{on} \quad \xi \in \mathbf{R}^d. \tag{A.20}$$

Let $\tilde\gamma$ be the minimal nonnegative integer larger than or equal to $-\gamma/2$. By (A.19),

$$\sum_{j'\in\mathbf{Z}^d} d_s(j') p(j - j') = \langle \psi_s(\cdot - j), g\rangle = \langle (1 - \triangle)^{\tilde\gamma} \psi_s(\cdot - j), (1 - \triangle)^{-\tilde\gamma} g\rangle. \tag{A.21}$$

Hence,

$$\Big(\sum_{j\in\mathbf{Z}^d} \Big| \sum_{j'\in\mathbf{Z}^d} d_s(j') p(j - j')\Big|^p\Big)^{1/p} \le \|(1 - \triangle)^{-\tilde\gamma} g\|_p \|(1 - \triangle)^{\tilde\gamma}\psi_s\|_{\mathcal{L}^\infty} < \infty, \tag{A.22}$$

where the last inequality holds because of (1.13) and the fact that $(1 - \triangle)^{\tilde\gamma} f \in \mathcal{D}$ for any $f \in \mathcal{D}$. Write $(\mathcal{F}(P)(\xi))^{-1} = \sum_{j\in\mathbf{Z}^d} r(j) e^{-ij\xi}$. Then the sequence $(r(j))_{j\in\mathbf{Z}^d}$ decays exponentially by (A.20), i.e., there exist positive constants C and δ independent of j such that

$$|r(j)| \le C e^{-\delta|j|} \quad \text{for all} \quad j \in \mathbf{Z}^d. \tag{A.23}$$

Moreover, from the definition of the sequences $(r(j))_{j\in\mathbf{Z}^d}$ it follows that

$$\sum_{j'\in\mathbf{Z}^d} r(j - j') p(j') = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j \ne 0. \end{cases} \tag{A.24}$$

By (A.23) and the assumption that $(d_s(j))_{j\in\mathbf{Z}^d} \in \ell^p$,

$$\sum_{j',j''\in\mathbf{Z}^d} |r(j - j') d_s(j'') p(j' - j'')| < \infty \quad \text{for all} \quad j \in \mathbf{Z}^d \quad \text{and} \quad 1 \le s \le M.$$

This together with (A.24) lead to

$$\begin{aligned} &\sum_{j'\in\mathbf{Z}^d} r(j - j')\Big(\sum_{j''\in\mathbf{Z}^d} d_s(j'') p(j' - j'')\Big) \\ = \;& \sum_{j''\in\mathbf{Z}^d} d_s(j'') \sum_{j'\in\mathbf{Z}^d} r(j - j'' - j') p(j') = d_s(j) \end{aligned} \tag{A.25}$$

for all $j \in \mathbf{Z}^d$ and $1 \leq s \leq M$. Hence by (A.22), (A.23) and (A.25), there exists a positive constant C independent of $D_s \in \ell^p, 1 \leq s \leq M$, such that $\|g\|_{L^{p,\gamma}} \geq C \sum_{s=1}^{M} \|D_s\|_{\ell^p}$. This proves that $(L^{p,\gamma}, L^{p,\gamma}(K), \ell^p)$ is a stable shift-triple. □

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