Convergence and Boundedness of Cascade Algorithm In Besov Spaces and Triebel-Lizorkin Spaces: Part I *

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Abstract

In this paper, by introducing characteristic polynomial of a cascade algorithm and using the factorization technique, we give complete characterization of convergence and increment of the cascade algorithm in Besov spaces and Triebel-Lizorkin spaces.

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1 Introduction

Fix a real-valued sequence $c = \{c(n)\}_{n=0}^N$ with $c_0 c_N \neq 0$ and $\sum_{n=0}^N c(n) = 2$. Define an operator T_c associated with the sequence c on the space of compactly supported distributions by

$$T_c f = \sum_{n=0}^{N} c(n) f(2 \cdot - n). \tag{1.1}$$

For any given compactly supported nonzero distribution f_0 as an initial, define

$$f_k = T_c f_{k-1}, \quad k \geq 1 \tag{1.2}$$

inductively. The iterative scheme above is called as *cascade algorithm.*

Let ϕ be the normalized solution of such a refinement equation

$$\phi = \sum_{n=0}^{N} c(n) \phi(2 \cdot - n), \tag{1.3}$$

which means ϕ is a compactly supported distribution, and satisfies (1.3) and $\hat{\phi}(0) = 1$ (The proofs of the existence and uniqueness of the normalized solution of the refinement equation (1.3) can be found in [D] or [CDM]). Hereafter the Fourier transform $\hat{f}$ of an integrable function f is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx$$

and the one of a compactly supported distribution is understood by usual interpretation. The compactly supported distribution ϕ above is called *refinable distribution.*

Obviously ϕ is invariant under the operator T_c, i.e.,

$$T_c \phi = \phi.$$

In particular any compactly supported distribution invariant under T_c can be written as $C\phi$ for some constant C.

By taking Fourier transform at both sides of (1.3), we have

$$\hat{\phi}(\xi) = H(\frac{\xi}{2}) \hat{\phi}(\frac{\xi}{2}) \tag{1.4}$$

and hence

$$\hat{\phi}(\xi) = \prod_{j=1}^{k} H(2^{-j}\xi)\hat{\phi}(2^{-k}\xi) = \prod_{j=1}^{\infty} H(2^{-j}\xi)$$

by using (1.4) for k times and letting k tend to infinity, where trigonometrical polynomial $H(\xi)$, *symbol* of the refinement equation (1.3), is defined by

$$H(\xi) = \frac{1}{2}\sum_{n=0}^{N} c(n)e^{-in\xi}. \tag{1.5}$$

Similarly by taking Fourier transform at both sides of (1.1), we obtain

$$\hat{f}_k(\xi) = H(\frac{\xi}{2})\hat{f}_{k-1}(\frac{\xi}{2}) = \prod_{j=1}^{k} H(2^{-j}\xi) \times \hat{f}_0(2^{-k}\xi). \tag{1.6}$$

Thus $\hat{f}_k$ converges to $\hat{f}_0(0)\hat{\phi}$ uniformly on any compact set of the complex plane. Hence $f_k = T_c^k f_0, k \geq 1$ converges to $\hat{f}_0(0)\phi$ in distributional cases. So f_k may be thought as certain approximation of the normalized solution ϕ of the refinement equation (1.3) if $\hat{f}_0(0) \neq 0$. At almost all practical applications, the convergence in distributional sense is not good enough. For instance, in the application to computer graphic we are interested in the bounded solution of the refinement equation (1.3) and uniform convergence of $f_k, k \geq 1$. Unfortunely for a function space X it is not always true that f_k converges in X even if the initial $f_0 \in X$ and the refinable distribution $\phi \in X$.

Example 1 Let $X = L^p, 1 \leq p < \infty$, the sequence $c_1 = \{c_1(n)\}$ be defined by

$$c_1(n) = \begin{cases} 1, & n = 0, 1, 6, 7, \\ -1, & n = 3, 4, \\ 0, & \text{otherwise}, \end{cases}$$

and the initial f_0 be

$$f_0(x) = \begin{cases} 1 - |x|, & x \in [-1, 1], \\ 0, & \text{otherwise}, \end{cases}$$

where L^p denotes the space of all p-integrable functions. Then f_k does not converge in X (see [CDM], [J1]).

Generally let X be a linear topological space with continuous embedding to the space of all tempered distributions. Most familiar examples of the space X are L^p with $1 \le p < \infty$, Sobolev spaces H^α, Besov spaces $B^\alpha_{p,q}$ and Triebel-Lizorkin spaces $F^\alpha_{p,q}$ (The precise definitions of Besov spaces and Triebel-Lizorkin spaces will be defined later). The problem about convergence and boundedness of the cascade algorithm may be proposed as the following:

Problem. *Let X be an appropriate function space and let $f_k, k \ge 1$ be as in (1.2). Does f_k converge in X and is f_k bounded in X? How fast does f_k converge in X when f_k converges in X and how big does f_k increase when f_k is not bounded in X?*

For the cascade algorithm, there is a much large literature about its convergence in various spaces (for instance [J1] for $L^p, 1 \le p < \infty$, [GMW], [CDM] as well as [St] for L^2, [LLS], [Sh] and [GL] for L^2 of the matrix cascade algorithm with a general dilation matrix, and [JJL] for Sobolev spaces).

In this paper, we shall consider the rate of convergence and increment of the cascade algorithm in Besov spaces and Triebel-Lizorkin spaces. To state our results, we introduce some definitions and notations.

For $-\infty < \alpha < \infty, 0 < p, q < \infty$, *Triebel-Lizorkin space* $F^\alpha_{p,q}$ is the set of tempered distribution f such that its quasi-norm $\|f\|_{F^\alpha_{p,q}}$ defined by

$$\|f\|_{F^\alpha_{p,q}} = \Big\| \Big(\hat{\tilde{\Phi}}(\cdot)\hat{f}(\cdot)\Big)^\vee \Big\|_p + \Big\| \Big(\sum_{l\ge 0} 2^{l\alpha q} \Big| \Big(\hat{\tilde{\Psi}}(2^{-l}\cdot)\hat{f}(\cdot)\Big)\Big|^q\Big)^{1/q} \Big\|_p$$

is finite, and *Besov space* $B^\alpha_{p,q}$ is the set of tempered distribution f such that its quasi-norm $\|f\|_{B^\alpha_{p,q}}$ defined by

$$\|f\|_{B^\alpha_{p,q}} = \Big\| \Big(\hat{\tilde{\Phi}}(\cdot)\hat{f}(\cdot)\Big)^\vee \Big\|_p + \Big(\sum_{l\ge 0} 2^{l\alpha q} \Big\| \Big(\hat{\tilde{\Psi}}(2^{-l}\cdot)\hat{f}(\cdot)\Big)^\vee \Big\|_p^q\Big)^{1/q}$$

is finite, where $\|f\|_p = (\int_{\mathbb{R}} |f(x)|^p dx)^{1/p}$, $f^\vee$ denotes the inverse Fourier transform of f and $\{\tilde{\Psi}, \tilde{\Phi}\}$ are Schwartz functions such that

- $\hat{\tilde{\Phi}}(\xi), \hat{\tilde{\Psi}}(\xi) \ge 0$.

- $\hat{\hat{\Psi}}$ is supported in $[-4\pi, -\pi] \cup [\pi, 4\pi]$ and $\hat{\hat{\Phi}}$ is supported in $[-2\pi, 2\pi]$.
- $\hat{\hat{\Phi}}(\xi) + \sum_{l \geq 0} \hat{\hat{\Psi}}(2^{-l}\xi) = 1, \ \forall\, \xi \in \mathbb{R}$.

The topologies of $F^\alpha_{p,q}$ and $B^\alpha_{p,q}$ are induced by the quasi-norms $\|\cdot\|_{F^\alpha_{p,q}}$ and $\|\cdot\|_{B^\alpha_{p,q}}$ respectively. The Bessel potential spaces $W^l_p = F^l_{p,2}, 1 < p < \infty, l \geq 0$ defined by

$$L^{l,p} = \{f; f, f', \cdots, f^{(l)} \in L^p\}$$

and Sobolev spaces $H^\alpha = B^\alpha_{2,2}, -\infty < \alpha < \infty$ defined by

$$H^\alpha = \{f; \|f\|_{H^\alpha} = (\int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1 + |\xi|)^\alpha d\xi)^{1/2} < \infty\}$$

are two special and very practical function spaces. The reader can refer to [T] for properties of Besov spaces and Triebel-Lizorkin spaces.

In this paper, we always set $X^\alpha_{p,q} = B^\alpha_{p,q}$ or $F^\alpha_{p,q}$ with $-\infty < \alpha < \infty$ and $0 < p, q < \infty$.

For a compactly supported distribution f, define its *characteristic trigonometrical polynomial* $P(f)(\xi)$ by

$$P(f)(\xi) = \prod_{\xi_0 \in (-\pi, \pi]} (e^{-i\xi} - e^{-i\xi_0})^{\kappa(f, \xi_0)} \tag{1.7}$$

where $\kappa(f, \xi_0)$ is the maximal integer such that $D^l \hat{f}(\xi_0 + 2n\pi) = 0$ holds for all $0 \leq l \leq \kappa(f, \xi_0) - 1$ and all integers n with $\xi_0 + 2n\pi \neq 0$. Observe that there is only finite $\xi_0 \in (-\pi, \pi]$ such that $\kappa(f, \xi_0) \geq 1$. Then the product at the right hand side of (1.7) is well-defined.

For the symbol $H(\xi)$ of the refinement equation (1.3), define its *characteristic trigonometrical polynomial* $P(H)(\xi)$ by

$$P(H)(\xi) = \prod_{\xi_0 \in (-\pi, \pi]} (e^{-i\xi} - e^{-i\xi_0})^{\zeta(H, \xi_0)} \tag{1.8}$$

where $\zeta(H, \xi_0)$ is the maximal integer such that $(\frac{e^{-2i\xi} - e^{-i\xi_0}}{e^{-i\xi} - e^{-i\xi_0}})^{\zeta(H, \xi_0)}$ is a factor of $H(\xi)$. It is proved in [R] (see also [JW] and [CS]) that

$$P(H)(\xi) = P(\phi)(\xi). \tag{1.9}$$

Define *characteristic trigonometrical polynomial of the cascade algorithm* by

$$P(f_0, H)(\xi) = \prod_{\xi_0 \in (-\pi,\pi]} (e^{-i\xi} - e^{-i\xi_0})^{\min(\zeta(H,\xi_0),\kappa(f_0,\xi_0))}. \qquad (1.10)$$

Obviously $P(f_0, H)$ is just a maximal common factor of $P(f_0)$ and $P(H)$.

For $-\infty < \beta < \infty$, set

$$A(\beta, f_k, X) = 2^{\beta k}\|f_k - c\phi\|_X,$$

where

$$c = \begin{cases} \hat{f}_0(0), & \beta \geq 0, \\ 0, & \beta < 0. \end{cases}$$

It is reasonable to use $A(\beta, f_k, X)$ to measure convergence and boundedness of the cascade algorithm. In particular, $A(\beta, f_k, X)$ with $\beta \geq 0$ can be interpreted as the rate of convergence of the cascade algorithm, and $A(\beta, f_k, X)$ with $\beta \leq 0$ as the rate of increment of the cascade algorithm. Recall that ϕ is invariant under the operator T_c. So we shall assume that $\hat{f}_0(0) = 0$ when $\beta \geq 0$ if unspecified.

For a trigonometrical polynomial $P(\xi) = \sum_{n\in\mathbb{Z}} d_n e^{-in\xi}$, define a quasi-norm $\|P\|_p^*, 0 < p < \infty$ by $\|P\|_p^* = (\sum_{n\in\mathbb{Z}} |d_n|^p)^{1/p}$.

In this paper, we shall prove the following characterization of convergence and boundedness of the cascade algorithm in Fourier domain. The characterization by using joint spectral radius on a finitely dimensional space, the close relationship between convergence and boundedness of the cascade algorithm and the regularity of the corresponding refinable distribution, and the application to the existence of compactly supported solutions of nonhomogeneous refinement equations in Besov spaces and Triebel-Lizorkin spaces will be given in Part II.

Theorem 1.1 *Let* $0 < p, q < \infty$, $-\infty < \alpha, \beta < \infty$ *and let* f_0, ϕ *and* $P(f_0, H)$ *be defined as above. Suppose that the compactly supported distribution* $f_0 \in X^\alpha_{p,q}$ *satisfies* $\hat{f}_0(0) = 0$ *if* $\beta \geq 0$. *Then we have*

(1) $\lim_{k\to\infty} A(\beta, f_k, X^{\alpha}_{p,q}) = 0$ *if and only if*

$$D^{\gamma}\hat{f}_0(0) = 0, \quad \forall\ 0 \le \gamma \le \beta \tag{1.11}$$

and

$$\lim_{k\to\infty} 2^{(\alpha+\beta+1-1/p)k} \|P(f_0, H)(\xi) \prod_{j=0}^{k-1} H(2^j\xi)\|_p^* = 0. \tag{1.12}$$

(2) Suppose that $D^{\beta}\hat{f}_0(0) = 0$ *if* β *is a positive integer. Then* $A(\beta, f_k, X^{\alpha}_{p,q})$ *is bounded if and only if* $D^{\gamma}\hat{f}_0(0) = 0$ *holds for all* $0 \le \gamma < \beta$ *and*

$$2^{(\alpha+\beta+1-1/p)k} \|P(f_0, H)(\xi) \prod_{j=0}^{k-1} H(2^j\xi)\|_p^*$$

is bounded.

(3) Suppose that β *is a positive integer and* $D^{\beta}\hat{f}_0(0) \neq 0$*. Then* $A(\beta, f_k, X^{\alpha}_{p,q})$ *is bounded if and only if* $\phi \in X^{\alpha+\beta}_{p,q}$*,* $D^{\gamma}\hat{f}_0(0) = 0$ *holds for all* $0 \le \gamma < \beta$ *and*

$$2^{(\alpha+\beta+1-1/p)k} \|P(f_0, H)(\xi) \prod_{j=0}^{k-1} H(2^j\xi)\|_p^*$$

is bounded.

Example 1 (continued) By computation, we have

$$H(\xi) = \frac{1+e^{-i\xi}}{2}(1 - e^{-3i\xi} + e^{-6i\xi})$$

and $P(f_0, H)(\xi) = (e^{-i\xi} - 1)$. Hence there exists a constant C such that

$$2^{-k}C^{-1}\| \prod_{j=0}^{k-1} \tilde{H}(2^j\xi)\|_p^* \le \|P(f_0, H)(\xi) \prod_{j=0}^{k-1} H(2^j\xi)\|_p^* \le 2^{-k}C\| \prod_{j=0}^{k-1} \tilde{H}(2^j\xi)\|_p^*,$$

where $\tilde{H}(\xi) = 1 - e^{-i\xi} + e^{-2i\xi} = (1+e^{-3i\xi})/(1+e^{-i\xi})$. By computation, we get

$$\prod_{j=0}^{k-1} \tilde{H}(2^j\xi) = \frac{1 + e^{-2^k i\xi} + e^{-2^{k+1} i\xi}}{1 + e^{-i\xi} + e^{-2i\xi}} = \sum_{l=0}^{2^{k+1}-2} a_l e^{-il\xi},$$

where

$$a_l = \begin{cases} 1, & l \in 3\mathbb{Z},\ 0 \leq \mathrm{l} \leq 2^{\mathrm{k}} - 1, \\ -1, & l - 1 \in 3\mathbb{Z},\ 0 \leq \mathrm{l} \leq 2^{\mathrm{k}} - 1, \\ 0, & l - 2 \in 3\mathbb{Z},\ 0 \leq \mathrm{l} \leq 2^{\mathrm{k}} - 1, \\ a_{2^{k+1}-2-l}, & 2^k \leq l \leq 2^{k+1} - 2. \end{cases}$$

Thus

$$(4/3)^{1/p}(2^k - 1)^{1/p} \leq \| \prod_{j=0}^{k-1} \tilde{H}(2^j\xi)\|_p^* \leq 2^{1/p}(2^k - 1)^{1/p}.$$

Hence for $1 < p < \infty$, f_k converges in $X_{p,q}^\alpha$ with $\alpha < 0$, is bounded in L^p, but does not converge in L^p by Theorem 1.1.

Observe that

$$\| \prod_{j=0}^{k} H(2^j\xi)\|_p^* \leq C 2^{k/p}(\sup_{\xi\in\mathbb{R}} |H(\xi)|)^k.$$

Then by Theorem 1.1 we have

Corollary 1.2 *Let $0 < p, q < \infty, -\infty < \alpha < \infty$. Then we have*

(1) If f_k is bounded in $X_{p,q}^\alpha$, then f_k converges to $\hat{f}_0(0)\phi$ in $X_{p,q}^{\alpha'}$ for all $\alpha' < \alpha$.

(2) If f_k converges to $\hat{f}_0(0)\phi$ in $X_{p,q}^\alpha$ and $f_0 \in X_{p,q}^{\alpha'}$ for some $\alpha' > \alpha$, then there exists $\delta > 0$ such that f_k converges to $\hat{f}_0(0)\phi$ in $X_{p,q}^{\alpha+\delta}$.

(3) If $f_0 \in X_{p,q}^\alpha$ and $\alpha < -1 - \ln_2 \sup_{\xi\in\mathbb{R}} |H(\xi)|$. Then f_k always converges to $\hat{f}_0(0)\phi$ in $X_{p,q}^\alpha$.

Hence boundedness of the cascade algorithm in a Besov space or a Triebel-Lizorkin space implies convergence of the cascade algorithm in Besov spaces or Triebel-Lizorkin spaces with lower index of regularity, and convergence of the cascade algorithm in a Besov space or a Triebel-Lizorkin space implies convergence of the cascade algorithm in Besov spaces or Triebel-Lizorkin spaces with little higher index of regularity provided that the initial is regular enough. By Corollary 1.2. we also see that the cascade algorithm always converges when the index of regularity of Besov spaces and Triebel-Lizorkin spaces is sufficiently small. Of course convergence of the cascade algorithm in such a function space is better than the one in distributional sense, though it seems still not good enough for practical application.

The paper is organized as follows. In Section 2, some necessary conditions on the initial f_0 and the refinable distribution ϕ of the cascade algorithm, and on the rate of increment of the term $\|P(f_0, H)(\xi)\prod_{j=0}^{k} H(2^j\xi)\|_p^*$ are given (Theorems 2.1 and 2.2). From Theorem 2.1, we see that certain regularity of the refinable distribution ϕ is always necessary for us to study convergence and boundedness of the cascade algorithm even if f_k tends to zero in distributional sense. This result is still new even for $X_{p,q}^{\alpha} = L^p$.

In Section 3, we prove some sufficient conditions to the convergence and boundedness of the cascade algorithm (Theorem 3.1) and give the proof of Theorem 1.1. The main ideas to prove Theorem 3.1 are the estimates of $P_0(f_k)$ and $Q_0(f_k)$ in Lemma 3.2 and the identities

$$\begin{cases} Q_l(f_k) = T_c^l(Q_0(f_{k-l})), & l \le k, \\ Q_l(f_k) = T_c^k(Q_{l-k}(f_0)), & l \ge k, \end{cases}$$

where $Q_l, l \ge 0$ and P_0 are projection operators of a multiresolution (see Lemma 3.2 for precise statement).

2 Necessary Conditions

In this section, we shall give some necessary conditions to the convergence and boundedness of the cascade algorithm. In particular, we shall prove the following results.

Theorem 2.1 *Let p, q, α, β, f_0 be as in Theorem 1.1. Suppose that $\hat{f}_0(0) = 0$ if $\beta \ge 0$. Then we have*

(1) If $\lim_{k\to\infty} A(\beta, f_k, X_{p,q}^{\alpha}) = 0$, then $\phi \in X_{p,q}^{\alpha+\beta}$ and

$$D^{\gamma}\hat{f}_0(0) = 0, \quad \forall\ 0 \le \gamma \le \beta. \tag{2.1}$$

(2) If $A(\beta, f_k, X^{\alpha}_{p,q})$ is bounded, then $\phi \in X^{\alpha+\beta}_{p,\infty}$ and

$$D^{\gamma}\hat{f}_0(0) = 0, \quad \forall\ 0 \leq \gamma < \beta. \tag{2.2}$$

(3) If $A(\beta, f_k, X^{\alpha}_{p,q})$ is bounded, β is a positive integer and $D^{\beta}\hat{f}_0(0) \neq 0$, then $\phi \in X^{\alpha+\beta}_{p,q}$.

Theorem 2.2 *Let $p, q, \alpha, \beta, f_0, P(f_0, H)$ be as in Theorem 1.1. Suppose that $\hat{f}_0(0) = 0$ if $\beta \geq 0$. Then we have*

(1) If $\lim_{k\to\infty} A(\beta, f_k, X^{\alpha}_{p,q}) = 0$, then

$$\lim_{k\to\infty} 2^{(\alpha+\beta+1-1/p)k} \|P(f_0, H)(\xi) \prod_{j=0}^{k-1} H(2^j\xi)\|^*_p = 0. \tag{2.3}$$

(2) If $A(\beta, f_k, X^{\alpha}_{p,q})$ is bounded, then there exists a constant C independent of $k \geq 1$ such that

$$\|P(f_0, H)(\xi) \prod_{j=0}^{k-1} H(2^j\xi)\|^*_p \leq C 2^{-(\alpha+\beta+1-1/p)k}. \tag{2.4}$$

Remark. The second assertion in Theorem 2.1 can not be improved in general. For example, $\phi = \delta(\cdot) - \delta(\cdot - 1) + \delta(\cdot - 2)$ is the normalized solution of the refinement equation

$$\phi = 2\phi(2\cdot) - 2\phi(2\cdot -1) + 2\phi(2\cdot -2),$$

where $\delta(\cdot)$ is the delta distribution. Obviously $\phi \in B^{1/p-1}_{p,\infty}$ and $\phi \notin B^{1/p-1}_{p,q}$ for any $0 < q < \infty$. By choosing the initial $f_0 = \chi_{[0,1]}(x) - \chi_{[0,1]}(x-3)$, we obtain

$$\begin{aligned} f_k(x) \ &= \ 2^k\chi_{[0,2^{-k}]}(x) - 2^k\chi_{[2^{-k},2^{-k+1}]}(x) + 2^k\chi_{[0,2^{-k}]}(x-1) \\ &\quad -2^k\chi_{[2^{-k},2^{-k+1}]}(x-1) + 2^k\chi_{[0,2^{-k}]}(x-2) - 2^k\chi_{[2^{-k},2^{-k+1}]}(x-2) \end{aligned}$$

and

$$\|f_k\|_{B^{1/p-1}_{p,q}} \ \leq \ C\|2^k\chi_{[0,2^{-k}]}(x) - 2^k\chi_{[2^{-k},2^{-k+1}]}(x)\|_{B^{1/p-1}_{p,q}}$$

$$
\begin{aligned}
&\leq C\|\chi_{[0,1]}(x)-\chi_{[0,1]}(x-1)\|_{B^{1/p-1}_{p,q}} \\
&\quad +C\Big(\sum_{l=0}^{k}2^{(1/p-1)lq}\Big\|\Big(\frac{\hat{\tilde{\Psi}}(2^{-l}\xi)(1-e^{-i2^{-k}\xi})^2}{i2^{-k}\xi}\Big)^{\vee}\Big\|_p^q\Big)^{1/q} \\
&\leq C\|\chi_{[0,1]}(x)\|_{B^{1/p-1}_{p,q}}+\Big(\sum_{l=0}^{k}2^{-(1/p-1)lq}\Big\|\Big(\hat{\tilde{\Psi}}(2^{l}\xi)\times\frac{(1-e^{-i\xi})^2}{i\xi}\Big)^{\vee}\Big\|_p^q\Big)^{1/q} \\
&\leq C+C(\sum_{l=0}^{k}2^{-lq})^{1/q}<\infty
\end{aligned}
$$

where the last inequality follows from the fact that there exists a constant C_K for any $K>0$ such that

$$\Big|\Big(\hat{\tilde{\Psi}}(2^l\cdot)(1-e^{-i\cdot})^2/(i\cdot)\Big)^{\vee}(x)\Big|\leq C_K 2^{-2l}(1+2^{-l}\mid x\mid)^{-K}.$$

Hence $A(0,f_k,B^{1/p-1}_{p,q})$ is bounded.

2.1 Proof of Theorem 2.1

To prove Theorem 2.1, we need a multiplier theorem and a characterization of Besov spaces and Triebel-Lizorkin spaces by using projection operators $Q_l, l\geq 0$ and P_0 of a multiresolution in [MS]. The advantage to use projection operators P_0f and $Q_lf, l\geq 0$ instead of $(\hat{\tilde{\Phi}}(\cdot)\hat{f}(\cdot))^{\vee}$ and $(\hat{\tilde{\Psi}}(2^{-k}\cdot)\hat{f}(\cdot))^{\vee}, k\geq 0$ is that $Q_lf, l\geq 0$ and P_0f are also compactly supported if f is and their supports are contained in a fixed compact set independent of $l\geq 0$.

Lemma 2.3 *If m is a smooth function with supp $m\subset(-\pi,\pi)$ and f satisfies supp $\hat{f}\subset[-\pi/2,\pi/2]$, then there exists a constant C independent of f such that*

$$\|(m\hat{f})^{\vee}\|_p\leq C\|f\|_p.$$

Proof. By the assumption on m and f, we have

$$m(\xi)\hat{f}(\xi)=\sum_{n\in\mathbb{Z}}m^{\vee}(n)e^{-in\xi}\hat{f}(\xi).$$

Hence

$$(m\hat{f})^{\vee}(x)=\sum_{n\in\mathbb{Z}}m^{\vee}(n)f(x-n)$$

and

$$\|(m\hat{f})^{\vee}\|_p \leq C(\sum_{n\in\mathbb{Z}} |m^{\vee}(n)|^{\min(p,1)})^{1/\min(p,1)}\|f\|_p \leq C\|f\|_p.$$

♠

To state the characterization of Besov spaces and Triebel-Lizorkin spaces by projection operators of appropriate multiresolution in [MS], we need the concept of a multiresolution (see [D]). A *multiresolution* is a family of closed subspaces $\{V_l\}_{l\in\mathbb{Z}}$ of L^2 such that

a) $\cap_{l\in\mathbb{Z}} V_l = \{0\}$ and $\cup_{l\in\mathbb{Z}} V_l$ is dense in L^2;

b) $V_l \subset V_{l+1}, \quad \forall\, l \in \mathbb{Z}$;

c) There exists a function Φ^{Mul} in V_0 such that $\{2^{l/2}\Phi^{Mul}(2^l \cdot - k); k \in \mathbb{Z}\}$ is an orthonormal basis of V_l for all $l \in \mathbb{Z}$.

Here we say that $\{\Phi^{Mul}(\cdot - k), k \in \mathbb{Z}\}$ is *orthonormal* if

$$\int_{\mathbb{R}} \Phi^{Mul}(x)\Phi^{Mul}(x-k)dx = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0. \end{cases}$$

The function Φ^{Mul} in c) is called a *scaling function* of the multiresolution. For $l \geq 0$, denote the orthogonal complement of V_l in V_{l+1} by W_l. Then there exists mother wavelet $\Psi^{Mul} \in W_0$ such that $\{2^{l/2}\Psi^{Mul}(2^l \cdot - k); k \in \mathbb{Z}\}$ is an orthonormal basis of W_l. It is well known that for any $\tau > 0$ there exists a scaling function Φ^{Mul} and a mother wavelet Ψ^{Mul} of a multiresolution such that they are orthonormal, compactly supported and belong to Hölder class C^τ. Here we denote the *Hölder space* with Hölder exponent τ by C^τ. In particular we only need to choose Daubechies' scaling functions and corresponding mother wavelets with their parameter sufficiently large (see [D]).

In this paper, we always use the scaling function Φ^{Mul} and mother wavelet Ψ^{Mul} which are orthonormal, compactly supported and in Hölder class C^τ with τ sufficiently large. For such scaling function Φ^{Mul} and mother wavelet Ψ^{Mul}, define projection operators P_l and $Q_l, l \geq 0$ by

$$P_l f(x) = 2^l \sum_{n\in\mathbb{Z}} \langle f, \Phi^{Mul}(2^l \cdot - n)\rangle \Phi^{Mul}(2^l x - n) \tag{2.5}$$

and

$$Q_l f(x) = 2^l \sum_{n\in\mathbb{Z}} \langle f, \Psi^{Mul}(2^l\cdot -n)\rangle \Psi^{Mul}(2^l x - n), \tag{2.6}$$

where $\langle f, g\rangle = \int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi$ when $\hat{f}(\xi)\overline{\hat{g}(\xi)}$ is integrable.

Lemma 2.4 ([MS]) *Let $0 < p, q < \infty, -\infty < \alpha < +\infty$, and let P_l and $Q_l, l \geq 0$ be as in (2.5) and (2.6). Assume that $\Phi^{Mul}, \Psi^{Mul} \in C^\tau$, $\tau > |\alpha| + 1 + \max(1, 1/p, 1/q)$, and that f is compactly supported distribution. Then $f \in F^\alpha_{p,q}$ if and only if*

$$\|P_0 f\|_p + \|(\sum_{l\geq 0} 2^{lq\alpha}|Q_l f|^q)^{1/q}\|_p < \infty,$$

and $f \in B^\alpha_{p,q}$ if and only if

$$\|P_0 f\|_p + (\sum_{l\geq 0} 2^{lq\alpha}\|Q_l f\|_p^q)^{1/q} < \infty.$$

Lemma 2.5 ([MS]) *Let $0 < p, q < \infty, -\infty < \alpha < +\infty$, and let $Q_l, l \geq 0$ be defined as in Lemma 2.4. Then for the normalized solution ϕ of the refinement equation (1.3), the following statements are equivalent to each other.*

(i) $\phi \in F^\alpha_{p,q}$.

(ii) $\phi \in B^\alpha_{p,q}$.

(iii) $\lim_{l\to\infty} 2^{l\alpha}\|Q_l f\|_p = 0$.

(iv) There exist constants C and $0 < r < 1$ independent of $l \geq 0$ such that

$$2^{l\alpha}\|Q_l f\|_p \leq C r^l, \quad \forall\ l \geq 0.$$

Proof of Theorem 2.1. Let l_0 be the minimal positive integer such that $D^{l_0}\hat{f}_0(0) \neq 0$. Then

$$\hat{f}_0(\xi) = \frac{D^{l_0}\hat{f}_0(0)}{l_0!}\xi^{l_0} + O(\xi^{l_0+1}), \quad \xi \to 0. \tag{2.7}$$

Thus by (1.6) and (2.7), we obtain

$$\begin{aligned}
\|(\hat{\tilde{\Phi}}(\cdot)\hat{f}_k(\cdot))^\vee\|_p &= \Big\|\Big(\hat{\tilde{\Phi}}(\cdot)\hat{\phi}(\cdot)\times\frac{\hat{f}_0(2^{-k}\cdot)}{\hat{\phi}(2^{-k}\cdot)}\Big)^\vee\Big\|_p \\
&= \frac{|D^{l_0}\hat{f}_0(0)|}{l_0!}2^{-l_0k}\|((\cdot)^{l_0}\hat{\tilde{\Phi}}(\cdot)\hat{\phi}(\cdot))^\vee\|_p + O(2^{-(l_0+1)k}), \quad k\to\infty.
\end{aligned}$$

By definitions of Besov spaces and Triebel-Lizorkin spaces, we have

$$\|(\hat{\tilde{\Phi}}(\cdot)\hat{f}_k(\cdot))^\vee\|_p \le \|f_k\|_{X^\alpha_{p,q}}.$$

Hence

$$2^{-(l_0-\beta)k}\to 0, \quad k\to\infty$$

and $l_0>\beta$ if $\lim_{k\to\infty}A(\beta,f_k,X^\alpha_{p,q})=0$, and $2^{-(l_0-\beta)k}$ is bounded and $l_0\ge\beta$ if $A(\beta,f_k,X^\alpha_{p,q})$ is bounded. This proves (2.1) and (2.2).

To prove that $\phi\in X^{\alpha+\beta}_{p,q}$ under the assumption $\lim_{k\to\infty}A(\beta,f_k,X^\alpha_{p,q})=0$, we introduce auxiliary functions G and G_1 which satisfy

- G and G_1 are Schwartz functions,
- $\hat{G}(\xi),\hat{G}_1(\xi)\ge 0$,
- $\hat{G}$ and $\hat{G}_1$ are supported in $(-4\pi,-\pi]\cup[\pi,4\pi)$,
- $\sum_{l\in\mathbb{Z}}\hat{G}(2^{-l}\xi)=1,\ \forall\ \xi\in\mathbb{R}\backslash\{0\}$,
- $\hat{G}_1(\xi)=1$ on the support of $\hat{G}$.

Then for any $K\in\mathbb{Z}$

$$2^{(\alpha+\beta)l}\|(\hat{G}_1(2^{-l}\xi)\hat{f}_{l+K}(\xi))^\vee\|_p\le A(\beta,f_k,X^\alpha_{p,q})\to 0, \qquad l\to\infty$$

by the assumption. Let $K\ge 1$ be an integer chosen that $|\hat{f}_0(\xi)|$ and $|\hat{\phi}(\xi)|$ are bounded below from zero on $[-2^{-K+2}\pi,-2^{-K}\pi]\cup[2^{-K}\pi,2^{-K+2}\pi]$. The existence of such an integer K follows from the facts that $\hat{\phi}$ and $\hat{f}_0$ are analytic functions and $\hat{\phi}(0)=1$. Set

$$m(\xi)=\hat{G}(\xi)\hat{\phi}(2^{-K}\xi)/\hat{f}_0(2^{-K}\xi).$$

Then m is a Schwartz function with supp $m \subset (-4\pi, -\pi] \cup [\pi, 4\pi)$ and furthermore

$$\hat{G}(2^{-l}\xi)\hat{\phi}(\xi) = m(2^{-l}\xi)\hat{G}_1(2^{-l}\xi)\hat{f}_{l+K}(\xi).$$

By Lemma 2.3, we have

$$2^{l(\alpha+\beta)}\|(\hat{G}(2^{-l}\cdot)\hat{\phi}(\cdot))^\vee\|_p \le C2^{l(\alpha+\beta)}\|(\hat{G}_1(2^{-l}\cdot)\hat{f}_{l+K}(\cdot))^\vee\|_p \to 0, \quad l \to \infty.$$

Let $Q_l, l \ge 0$ be the projection operators in Lemma 2.4. Then by usual estimate, we get

$$2^{l(\alpha+\beta)}\|Q_l\phi\|_p \to 0, \quad l \to \infty$$

and furthermore by Lemma 2.5 there exist constants C and $0 < r < 1$ such that

$$2^{l(\alpha+\beta)}\|Q_l\phi\|_p \le Cr^l.$$

Hence $\phi \in X_{p,q}^{\alpha+\beta}$ and the second assertion is proved.

The assertion $\phi \in X_{p,\infty}^{\alpha+\beta}$ under the assumption that $A(\beta, f_k, X_{p,q}^\alpha)$ is bounded can be proved by the same procedure used above. We omit the detail here.

Now we start to prove that $\phi \in X_{p,q}^{\alpha+\beta}$ under the assumptions that $A(\beta, f_k, X_{p,q}^\alpha)$ is bounded, β is a positive integer and $D^\beta\hat{f}_0(0) \ne 0$. By (1.6) and (2.2), we have

$$2^{\beta k}\hat{f}_k(\xi) = 2^{\beta k}\frac{\hat{\phi}(\xi)\hat{f}_0(2^{-k}\xi)}{\hat{\phi}(2^{-k}\xi)} \to \frac{D^\beta\hat{f}_0(0)}{\beta!}\xi^\beta\hat{\phi}(\xi), \quad k \to \infty.$$

Hence

$$\|(\hat{\tilde{\Phi}}(\cdot)(2^{\beta k}\hat{f}_k(\cdot) - \frac{D^\beta\hat{f}_0(0)}{\beta!}(\cdot)^\beta\hat{\phi}(\cdot)))^\vee\|_p \to 0, \quad k \to \infty$$

and

$$\|(\hat{\tilde{\Psi}}(2^{-l}\cdot)(2^{\beta k}\hat{f}_k(\cdot) - \frac{D^\beta\hat{f}_0(0)}{\beta!}(\cdot)^\beta\hat{\phi}(\cdot)))^\vee\|_p \to 0, \quad k \to \infty$$

for all $l \ge 0$. Thus

$$\|(\hat{\tilde{\Phi}}(\cdot)\hat{\phi}(\cdot))^\vee\|_p + \|(\sum_{l\ge 0}2^{(\alpha+\beta)ql}|(\hat{\tilde{\Psi}}(2^{-l}\cdot)\hat{\phi}(\cdot))^\vee|^q)^{1/q}\|_p$$

$$
\begin{aligned}
&\leq \quad C + C\|(\sum_{l\geq 0} 2^{\alpha q l}|(\hat{\hat{\Psi}}(2^{-l}\cdot)(\cdot)^{\beta}\hat{\phi}(\cdot))^{\vee}|^{q})^{1/q}\|_{p} \\
&\leq \quad C + C \lim_{L\to\infty} \|(\sum_{l=0}^{L} 2^{\alpha q l}|(\hat{\hat{\Psi}}(2^{-l}\cdot)(\cdot)^{\beta}\hat{\phi}(\cdot))^{\vee}|^{q})^{1/q}\|_{p} \\
&\leq \quad C + C \lim_{L\to\infty}\lim_{k\to\infty} \|(\sum_{l=0}^{L} 2^{\alpha q l}|(\hat{\hat{\Psi}}(2^{-l}\cdot)2^{\beta k}\hat{f}_k(\cdot))^{\vee}|^{q})^{1/q}\|_{p} \\
&\leq \quad C + \sup_{k\geq 1} 2^{\beta k}\|f_k\|_{F^{\alpha}_{p,q}} \leq C
\end{aligned}
$$

in the case $X^{\alpha}_{p,q} = F^{\alpha}_{p,q}$, and

$$
\begin{aligned}
&\|(\hat{\hat{\Phi}}(\cdot)\hat{\phi}(\cdot))^{\vee}\|_p + (\sum_{l\geq 0} 2^{(\alpha+\beta)ql}\|(\hat{\hat{\Psi}}(2^{-l}\cdot)\hat{\phi}(\cdot))^{\vee}\|_p^q)^{1/q} \\
\leq &\quad C + C \lim_{L\to\infty}\lim_{k\to\infty} (\sum_{l=0}^{L} 2^{\alpha q l}\|(\hat{\hat{\Psi}}(2^{-l}\cdot)2^{\beta k}\hat{f}_k(\cdot))^{\vee}\|_p^q)^{1/q} \\
\leq &\quad C + C\sup_{k\geq 1} 2^{\beta k}\|f_k\|_{B^{\alpha}_{p,q}}
\end{aligned}
$$

in the case $X^{\alpha}_{p,q} = B^{\alpha}_{p,q}$. Hence $\phi \in X^{\alpha+\beta}_{p,q}$ by Lemma 2.4. ♠

2.2 Invariant Ideal and Proof of Theorem 2.2

To prove Theorem 2.2, we need a sampling theorem, stability lemma and result about invariant ideal of trigonometrical polynomials.

Lemma 2.6 *Let $0 < p < \infty$. If f is a compactly supported distribution with its Fourier transform supported in $[-\pi/2, \pi/2]$, then there exists a constant C such that*

$$
C^{-1}\Big(\sum_{n\in\mathbb{Z}} |f(n)|^p\Big)^{1/p} \leq \|f\|_p \leq C\Big(\sum_{n\in\mathbb{Z}} |f(n)|^p\Big)^{1/p}. \tag{2.8}
$$

The lemma above is called sampling theorem (see [T]).

We say that the integer translates of a tempered distribution f is *stable* if $\hat{f}$ is continuous and $N_{\mathbb{R}}(f) = \emptyset$, where

$$
N_{\mathbb{R}}(f) = \{\xi \in \mathbb{R},\ \hat{f}(\xi + 2n\pi) \equiv 0, \quad \forall\ n \in \mathbb{Z}\}. \tag{2.9}
$$

Lemma 2.7 *Let f be a Schwartz function with its Fourier transform being compactly supported. Assume that the integer translates of f are stable. Then there exists a constant C such that*

$$C^{-1}\Big(\sum_{n\in\mathbb{Z}}|d(n)|^p\Big)^{1/p}\le\|\sum_{n\in\mathbb{Z}}d(n)f(x-n)\|_p\le C\Big(\sum_{n\in\mathbb{Z}}|d(n)|^p\Big)^{1/p}. \quad (2.10)$$

The stability lemma above can be proved under weak assumption on f (see [JM] and references therein). For the perfection of this paper, we include a new proof.

Proof. By Lemma 2.6, it suffices to prove that

$$\sum_{n\in\mathbb{Z}}|d(n)|^p\le C\sum_{l=0}^{K_0-1}\sum_{t\in\mathbb{Z}}|\sum_{s\in\mathbb{Z}}d(s)f(\frac{l}{K_0}+t-s)|^p$$

where K_0 is a positive integer such that supp $f\subset[-K_0\pi/2, K_0\pi/2]$. Observe that

$$\sum_{t\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}d(s)f(\frac{l}{K_0}+t-s)e^{-it\xi}=d(\xi)F(f,l)(\xi),$$

where $d(\xi)=\sum_{s\in\mathbb{Z}}d(s)e^{-is\xi}$ and

$$F(f,l)(\xi)=\sum_{s\in\mathbb{Z}}\hat f(\xi+2\pi s)e^{-i(\xi+2\pi s)l/K_0}.$$

Then it suffices to prove that for any $\xi\in[-\pi,\pi]$ there exists $0\le l\le K_0-1$ such that $F(f,l)(\xi)\ne 0$.

Suppose to the contrary that there exists $\xi_0\in[-\pi,\pi]$ such that

$$F(f,l)(\xi_0)=0,\quad \forall\ 0\le l\le K_0-1.$$

Then

$$\sum_{s\in\mathbb{Z}}\hat f(\xi_0+2\pi l+2\pi K_0 s)=0,\quad \forall\ 0\le l\le K_0-1.$$

By the stable assumption on f, there is $s_0\in\mathbb{Z}$ such that $\hat f(\xi_0+2s_0\pi)\ne 0$. Let $0\le l\le K_0-1$ be the unique integer such that $(l-s_0)/K_0$ is an integer. Then by the construction of K_0 we have

$$\sum_{s\in\mathbb{Z}}\hat f(\xi_0+2\pi l+2\pi K_0 s)=\hat f(\xi_0+2s_0\pi)\ne 0,$$

which is a contradiction. ♠

For $-\infty < \gamma < \infty, 0 < p < \infty$ and trigonometrical polynomial H with $H(0) = 1$, let $\mathcal{V}_0(H, \gamma, p)$ be the set of all trigonometrical polynomials $P(\xi)$ such that

$$\lim_{k\to\infty} 2^{\gamma k} \|P(\xi) \prod_{j=0}^{k-1} H(2^j \xi)\|_p^* = 0,$$

and let $\mathcal{V}_1(H, \gamma, p)$ be the set of all trigonometrical polynomials $P(\xi)$ such that $2^{\gamma k}\|P(\xi)\prod_{j=0}^{k-1} H(2^j\xi)\|_p^*$ is bounded. Define operators $B_\epsilon(H), \epsilon = 0, 1$ on the space of trigonometrical polynomials by

$$B_\epsilon(H)P(\xi) = H(\frac{\xi}{2})P(\frac{\xi}{2})e^{-i\epsilon\xi/2} + H(\frac{\xi}{2}+\pi)P(\frac{\xi}{2}+\pi)e^{-i\epsilon(\xi/2+\pi)}, \quad \epsilon = 0, 1. \tag{2.11}$$

Lemma 2.8 *Let $-\infty < \gamma < \infty, 0 < p < \infty$ and let $\mathcal{V}_0(H, \gamma, p)$ and $\mathcal{V}_1(H, \gamma, p)$ be defined as above. Then $\mathcal{V}_0(H, \gamma, p)$ and $\mathcal{V}_1(H, \gamma, p)$ are ideal of the ring of trigonometrical polynomials, and are invariant under operators $B_0(H)$ and $B_1(H)$. Furthermore $P(H) \in \mathcal{V}_0(H, \gamma, p)$ if $\mathcal{V}_0(H, \gamma, p) \neq \{0\}$ and $P(H) \in \mathcal{V}_1(H, \gamma, p)$ if $\mathcal{V}_1(H, \gamma, p) \neq \{0\}$.*

Proof. It is easy to check that $\mathcal{V}_0(H, \gamma, p)$ and $\mathcal{V}_1(H, \gamma, p)$ are ideal of the ring of all trigonometrical polynomials, and invariant under $B_0(H)$ and $B_1(H)$. Hence it remains to prove that $P(H) \in \mathcal{V}_0(H, \gamma, p)$ (resp. $\mathcal{V}_1(H, \gamma, p)$) if $\mathcal{V}_0(H, \gamma, p) \neq \{0\}$ (resp. $\mathcal{V}_1(H, \gamma, p) \neq \{0\}$).

Let P_0 be a nonzero trigonometrical polynomial in $\mathcal{V}_0(H, \gamma, p)$, and let $MC(P_0, P(H))$ be a maximal common factor of P_0 and $P(H)$. Then it suffices to prove that $MC(P_0, P(H))(\xi) \in \mathcal{V}_0(H, \gamma, p)$.

Write

$$P_0(\xi) = MC(P_0, P(H))(\xi)\tilde{P}_0(\xi)$$

and

$$H(\xi) = \frac{MC(P_0, P(H))(2\xi)}{2^{a(P_0,P(H))}MC(P_0, P(H))(\xi)}\tilde{H}(\xi),$$

where $a(P_0, P(H))$ is the maximal integer such that $(e^{-i\xi} - 1)^{a(P_0,P(H))}$ is a factor of $MC(P_0, P(H))(\xi)$. Then $\tilde{P}_0 \not\equiv 0$, $\tilde{H}(0) = 1$ and any common factor between $P(\tilde{H})$ and $\tilde{P}_0$ is a constant. By computation, we have

$$P_0(\xi) \prod_{j=0}^{k-1} H(2^j\xi) = 2^{-ka(P_0,P(H))} \tilde{P}_0(\xi) MC(P_0, P(H))(2^k\xi) \prod_{j=0}^{k-1} \tilde{H}(2^j\xi).$$

Thus $\tilde{P}_0 \in \mathcal{V}_0(\tilde{H}, \gamma - a(P_0, P(H)))$ by the fact that there exists a constant C such that

$$\begin{aligned} C^{-1} \|\tilde{P}_0(\xi) \prod_{j=0}^{k-1} \tilde{H}(2^j\xi)\|_p^* &\leq \|\tilde{P}_0(\xi) MC(P_0, P(H))(2^k\xi) \prod_{j=0}^{k-1} \tilde{H}(2^j\xi)\|_p^* \\ &\leq C\|\tilde{P}_0(\xi) \prod_{j=0}^{k-1} \tilde{H}(2^j\xi)\|_p^*. \end{aligned}$$

Observe that

$$MC(P_0, P(H))(\xi) \prod_{j=0}^{k-1} H(2^j\xi) = 2^{-ka(P_0,P(H))} MC(P_0, P(H))(2^k\xi) \prod_{j=0}^{k-1} \tilde{H}(2^j\xi).$$

Then $MC(P_0, P(H))(\xi) \in \mathcal{V}_0(H, \gamma, p)$ reduces to

$$1 \in \mathcal{V}_0(\tilde{H}, \gamma - a(P_0, P(H)), p). \tag{2.12}$$

Now we start to prove (2.13). First we show that for any $\xi_0 \in \mathbb{R}$ with $\tilde{P}_0(\xi_0) = 0$ there exists $l_0 \geq 1$ and $\epsilon_j \in \{0, 1\}, 1 \leq j \leq l_0$ such that

$$B_{\epsilon_1}(\tilde{H}) B_{\epsilon_2}(\tilde{H}) \cdots B_{\epsilon_{l_0}}(\tilde{H}) \tilde{P}_0(\xi_0) \neq 0.$$

Suppose to the contrary that there exists $\xi_0 \in \mathbb{R}$ such that $\tilde{P}_0(\xi_0) = 0$ and

$$B_{\epsilon_1}(\tilde{H}) B_{\epsilon_2}(\tilde{H}) \cdots B_{\epsilon_l}(\tilde{H}) \tilde{P}_0(\xi_0) = 0, \quad \forall\ \epsilon_j \in \{0, 1\},\ 1 \leq j \leq l. \tag{2.13}$$

Let $\tilde{\phi}$ be the normalized solution of the refinement equation (1.3) with corresponding symbol $\tilde{H}(\xi)$. Then it can be proved that $\xi_0 \notin N_{\mathbb{R}}(\tilde{\phi})$ under the assumption $\tilde{P}_0(\xi_0) = 0$ and any common factor of $\tilde{P}_0$ and $P(\tilde{H})$ is a constant. Furthermore there exists an integer l_0 such that $\xi_0 + 2\pi l_0 \neq 0$ and

$$\hat{\tilde{\phi}}(\xi_0 + 2l_0\pi) \neq 0. \tag{2.14}$$

By computation, we have

$$\tilde{P}_0(\xi)\hat{\tilde{\phi}}(2^{l+1}\xi)/\hat{\tilde{\phi}}(\xi) = \tilde{P}_0(\xi)\prod_{j=0}^{l}\tilde{H}(2^j\xi)$$

$$= \sum_{\epsilon_j\in\{0,1\},1\le j\le l+1} B_{\epsilon_1}(\tilde{H})\cdots B_{\epsilon_{l+1}}(\tilde{H})\tilde{P}_0(2^{l+1}\xi)e^{i\sum_{j=1}^{l+1}\epsilon_j 2^{l-j+1}\xi}. \quad (2.15)$$

Thus

$$\tilde{P}_0(2^{-l-1}(\xi_0+2l_0\pi)) = 0, \quad \forall\ l\ge 1.$$

by (2.14)-(2.16), and $\tilde{P}_0\equiv 0$, which is a contradiction. This proves that there exists trigonometrical polynomials $\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_{K_1}\in\mathcal{V}_0(\tilde{H}, \gamma - a(P_0, P(H)), p)$ such that all roots of any nonzero common factor of $\tilde{P}_j, 0\le j\le K_1$ is not on the real line.

Let $\tilde{P}$ be a maximal common factor of $\tilde{P}_j, 0\le j\le K_1$. Then $\tilde{P}\in\mathcal{V}_0(\tilde{H}, \gamma - a(P_0, P(H)), p)$ by the fact that $\mathcal{V}_0(\tilde{H}, \gamma - a(P_0, P(H)), p)$ is an ideal of the ring of trigonometrical polynomials. Observe that Fourier coefficients of $\tilde{P}^{-1}(\xi)$ decay exponentially, in the other words, there exist positive constants C and δ such that

$$|\int_0^{2\pi} e^{in\xi}\tilde{P}^{-1}(\xi)d\xi| \le Ce^{-\delta|n|}.$$

Then

$$2^{(\gamma-a(P_0,P(H)))k}\|\prod_{j=0}^{k-1}\tilde{H}(2^j\xi)\|_p^* \le C2^{(\gamma-a(P_0,P(H)))k}\ \|\tilde{P}(\xi)\prod_{j=0}^{k-1}\tilde{H}(2^j\xi)\|_p^* \to 0$$

as k tends to ∞. Hence $1\in\mathcal{V}_0(\tilde{H}, \gamma - a(P_0, P(H)), p)$ and (2.13) is proved.

By the same procedure as above, we can prove that $P(H)\in\mathcal{V}_1(H,\gamma,p)$ when $\mathcal{V}_1(H,\gamma,p)\ne\{0\}$. ♠

Proof of Theorem 2.2. By the definition of $P(f_0)(\xi)$, there exists compactly supported $\tilde{f}_0\in X_{p,q}^{\alpha-\kappa(f_0,0)}$ such that the integer translates of $\tilde{f}_0$ are stable, $\widehat{\tilde{f}}_0(2l_0\pi)\ne 0$ for some nonzero integer l_0, and

$$\hat{f}_0(\xi) = \frac{P(f_0)(\xi)}{(i\xi)^{\kappa(f_0,0)}}\widehat{\tilde{f}}_0(\xi).$$

Let Ψ be a Schwartz function such that its Fourier transform is compactly supported, $0 \notin$ supp $\hat{\Psi}$ and the integer translates of $(\Psi(\cdot)\widehat{\tilde{f}_0}(\cdot)(\cdot)^{-\kappa(f_0,0)})^\vee$ are stable. Then by (1.6), Lemma 2.7 and definitions of Besov spaces and Triebel-Lizorkin spaces, we have

$$
\begin{aligned}
& 2^{(\alpha+\beta+1-1/p)k}\|P(f_0)(\xi)\prod_{j=0}^{k-1}H(2^j\xi)\|_p^* \\
\leq\; & C2^{(\alpha+\beta+1-1/p)k}\Big\|\Big(\hat{\Psi}(\cdot)\widehat{\tilde{f}_0}(\cdot)(\cdot)^{-\kappa(f_0,0)}P(f_0)(\cdot)\prod_{j=0}^{k-1}H(2^j\cdot)\Big)^\vee\Big\|_p \\
\leq\; & C2^{(\alpha+\beta)k}\|(\hat{\Psi}(2^{-k}\cdot)\hat{f}_k(\cdot))^\vee\|_p \leq CA(\beta, f_k, X_{p,q}^\alpha).
\end{aligned}
$$

Thus $P(f_0) \in \mathcal{V}_0(H, 2^{\alpha+\beta+1-1/p}, p)$ when $\lim_{k\to\infty} A(\beta, f_k, X_{p,q}^\alpha) = 0$, and $P(f_0) \in \mathcal{V}_1(H, 2^{\alpha+\beta+1-1/p}, p)$ when $A(\beta, f_k, X_{p,q}^\alpha)$ is bounded. Hence Theorem 2.2 follows from Lemma 2.8. ♠

3 Sufficient Conditions

In this section, we shall prove some sufficient conditions to the convergence and boundedness of the cascade algorithm (Theorem 3.1), and give the proof of Theorem 1.1.

Theorem 3.1 *Let $p, q, \alpha, \beta, P(f_0, H)$ and f_0 be as in Theorem 1.1. Assume that*

$$D^\gamma \hat{f}_0(0) = 0, \quad \forall\ 0 \leq \gamma \leq \beta.$$

Then we have

(1) If

$$\lim_{k\to\infty} 2^{(\alpha+\beta+1-1/p)k}\|P(f_0, H)(\xi)\prod_{j=0}^{k-1}H(2^j\xi)\|_p^* = 0,$$

then $\lim_{k\to\infty} A(\beta, f_k, X_{p,q}^\alpha) = 0$.

(2) If $2^{(\alpha+\beta+1-1/p)k}\|P(f_0, H)(\xi)\prod_{j=0}^{k-1}H(2^j\xi)\|_p^*$ *is bounded, then* $A(\beta, f_k, X_{p,q}^\alpha)$ *is bounded.*

For a moment, we assume that Theorem 3.1 hold and start to prove Theorem 1.1.

Proof of Theorem 1.1. Obviously the first and second assertion follows from Theorems 2.1, 2.2 and 3.1. Now we prove the third assertion. Set

$$\tilde{f}_0 = f_0 - \frac{D^\beta \hat{f}_0(0)}{\beta!} D^\beta \phi.$$

Then $\tilde{f}_0 \in X^\alpha_{p,q}$,

$$\widehat{\tilde{f}_0}(\xi) = \hat{f}_0(\xi) - \frac{D^\beta \hat{f}_0(0)}{\beta!} \xi^\beta \hat{\phi}(\xi)$$

and $D^\gamma \widehat{\tilde{f}_0}(0) = 0, \quad 0 \leq \gamma \leq \beta$. Define $\tilde{f}_k = T_c \tilde{f}_{k-1} = T_c^k \tilde{f}_0,\ k \geq 1$ Then

$$\tilde{f}_k = f_k - 2^{-\beta k} \frac{D^\beta \hat{f}_0(0)}{\beta!} D^\beta \phi$$

and

$$A(\beta, f_k, X^\alpha_{p,q}) \leq CA(\beta, \tilde{f}_k, X^\alpha_{p,q}) + C\|\phi\|_{X^{\alpha+\beta}_{p,q}}.$$

Hence by Theorem 3.1 it suffices to prove

$$P(\tilde{f}_0, H)(\xi) = P(f_0, H)(\xi) \tag{3.1}$$

under the assumption $\tilde{f}_0 \not\equiv 0$. Observe that for any integer $\kappa \geq 0$ and $\xi_0 \in \mathbb{R}$,

$$D^\gamma \widehat{\tilde{f}_0}(\xi_0) = D^\gamma \widehat{\phi}(\xi_0) = 0, \quad \forall\ 0 \leq \gamma \leq \kappa$$

holds if and only if

$$D^\gamma \widehat{f_0}(\xi_0) = D^\gamma \widehat{\phi}(\xi_0) = 0, \quad \forall\ 0 \leq \gamma \leq \kappa$$

is true. Then (3.1) follows from (1.9) and (1.10). ♠

To prove Theorem 3.1, we need some lemmas.

Lemma 3.2 *Let p, q, α be as in Theorem 1.1 and let Q_0, P_0 be defined as in Lemma 2.4 with Φ^{mul} and $\Psi^{mul} \in C^\tau$ for some sufficiently large τ. Assume that $f_0 \in X^\alpha_{p,q}$ satisfies*

$$D^\gamma \hat{f}_0(0) = 0, \quad \forall\ 0 \leq \gamma \leq l_0 - 1$$

for some nonnegative integer l_0. Then there exists a constant C independent of $k \geq 1$ such that

$$\|P_0(f_k)\|_p + \|Q_0(f_k)\|_p \leq C2^{-kl_0}, \quad k \geq 1.$$

Observe that $f_k, k \geq 1$ are supported in some compact set independent of k. Then Lemma 3.2 follows from the estimate below.

Lemma 3.3 *Let $f_k, k \geq 1$ be as (1.2). Assume that $f_0 \in X^\alpha_{p,q}$ satisfies*

$$D^\gamma \hat{f}_0(0) = 0, \quad \forall\ 0 \leq k \leq l_0 - 1$$

for some nonnegative integer l_0. Then there exists a constant C independent of $k \geq 0$ such that

$$|\hat{f}_k(\xi)| \leq C2^{-kl_0}(1 + |\xi|)^C. \tag{3.2}$$

Proof. First we prove that there exists a constant C independent of $\xi \in \mathbb{R}$ such that

$$|\hat{f}_0(\xi)| \leq C|\xi|^{l_0}(1 + |\xi|)^{-\alpha - l_0 - \min(0,1-1/p)}. \tag{3.3}$$

For a moment, we assume that (3.3) hold and start to prove the estimate (3.2). Obviously we have

$$|\prod_{j=1}^k H(2^{-j}\xi)| \leq C \min(2^{kB}, (1 + |\xi|)^B), \tag{3.4}$$

where $B = \ln_2 \sup_{\xi\in\mathbb{R}} |H(\xi)|$. Thus by (1.6), (3.3) and (3.4), we obtain

$$\begin{aligned}
|\hat{f}_k(\xi)| &= |\prod_{j=1}^k H(2^{-j}\xi)\hat{f}_0(2^{-k}\xi)| \\
&\leq C2^{-kl_0}|\xi|^{l_0}(1 + 2^{-k}|\xi|)^{-\alpha-l_0-\min(0,1-1/p)} \min(2^{kB}, (1 + |\xi|)^B) \\
&\leq \begin{cases} C2^{-kl_0}(1 + |\xi|)^{l_0+B}, & |\xi| \leq 2^k \\ C2^{k(\alpha+\min(0,1-1/p)+B)}|\xi|^{-\alpha-\min(0,1-1/p)}, & |\xi| \geq 2^k \end{cases} \\
&\leq C2^{-kl_0}(1 + |\xi|)^{l_0+B+|\alpha+\min(0,1-1/p)|}.
\end{aligned}$$

This proves (3.2) if (3.3) holds. Now we divide two cases $p \geq 1$ and $0 < p < 1$ to prove (3.3).

Case 1. $p \geq 1$.

By Lemma 2.4 and the fact that $Q_l f_0, l \geq 0$ are supported in a compact set independent of $l \geq 0$, we obtain $f_0 \in B^{\alpha}_{1,\infty}$. Thus

$$\|\hat{\tilde{\Phi}}(\cdot)\hat{f}_0(\cdot)\|_\infty \leq \|(\hat{\tilde{\Phi}}(\cdot)\hat{f}_0(\cdot))^\vee\|_1 \leq \|f_0\|_{B^{\alpha}_{1,\infty}}$$

and

$$\|\hat{\tilde{\Psi}}(2^{-l}\cdot)\hat{f}_0(\cdot)\|_\infty \leq \|(\hat{\tilde{\Phi}}(2^{-l}\cdot)\hat{f}_0(\cdot))^\vee\|_1 \leq C2^{-l\alpha}\|f_0\|_{B^{\alpha}_{1,\infty}}, \quad l \geq 0.$$

Thus

$$|\hat{f}_0(\xi)| \leq C(1+|\xi|)^{-\alpha}.$$

Hence the estimate (3.3) for $p \geq 1$ follows from the assumption

$$D^\gamma \hat{f}_0(0) = 0, \quad \forall\ 0 \leq k \leq l_0 - 1$$

and the fact that $\hat{f}_0$ is an analytic function.

Case 2. $p < 1$.

By Lemma 2.6, $l^p \subset l^1$ for $0 < p \leq 1$ and definitions Besov spaces and Triebel-Lizorkin spaces, we have

$$\|(\hat{\tilde{\Phi}}(\cdot)\hat{f}_0(\cdot))^\vee\|_p \leq C$$

and

$$\|(\hat{\tilde{\Psi}}(2^{-l}\cdot)\hat{f}_0(\cdot))\|_p \leq C2^{-l\alpha}.$$

Therefore by Lemma 2.6, we get

$$\begin{aligned}
\|\hat{\tilde{\Phi}}(\cdot)\hat{f}_0(\cdot)\|_\infty &\leq \|(\hat{\tilde{\Phi}}(\cdot)\hat{f}_0(\cdot))^\vee\|_1 \leq C\sum_{n\in\mathbb{Z}}|(\hat{\tilde{\Phi}}(\cdot)\hat{f}_0(\cdot))^\vee(n/8)| \\
&\leq C(\sum_{n\in\mathbb{Z}}|(\hat{\tilde{\Phi}}(\cdot)\hat{f}_0(\cdot))^\vee(n/8)|^p)^{1/p} \leq C\|(\hat{\tilde{\Phi}}(\cdot)\hat{f}_0(\cdot))^\vee\|_p \leq C
\end{aligned}$$

and

$$\begin{aligned}
\|\hat{\tilde{\Psi}}(2^{-l}\cdot)\hat{f}_0(\cdot)\|_\infty &\leq \|(\hat{\tilde{\Psi}}(2^{-l}\cdot)\hat{f}_0(\cdot))^\vee\|_1 \\
&\leq C2^{-l}\sum_{n\in\mathbb{Z}}|(\hat{\tilde{\Psi}}(2^{-l}\cdot)\hat{f}_0(\cdot))^\vee(2^{-l-3}n)| \\
&\leq C2^{-l(1-1/p)}(2^{-l}\sum_{n\in\mathbb{Z}}|(\hat{\tilde{\Psi}}(2^{-l}\cdot)\hat{f}_0(\cdot))^\vee(2^{-l-3}n)|^p)^{1/p} \\
&\leq C2^{-l(1-1/p)}\|(\hat{\tilde{\Psi}}(2^{-l}\cdot)\hat{f}_0(\cdot))^\vee\|_p \leq C2^{-l(1-1/p+\alpha)},
\end{aligned}$$

where $l \geq 0$. Thus
$$|\hat{f}_0(\xi)| \leq C(1+|\xi|)^{-\alpha-1+1/p}$$
and (3.3) follows from the assumption on f_0. This completes the proof of the estimate (3.3) and Lemma 3.3. ♠

Proof of Theorem 3.1. Write
$$H(\xi) = \frac{P(f_0,H)(2\xi)}{2^{\min(\zeta(H,0),\kappa(f_0,0))} P(f_0,H)(\xi)} \tilde{H}(\xi). \tag{3.5}$$

Then $\tilde{H}(0) = 1$ and
$$2^{-k\min(\zeta(H,0),\kappa(f_0,0))} P(f_0,H)(2^k\xi) \prod_{j=0}^{k-1} \tilde{H}(2^j\xi) = P(f_0,H)(\xi) \prod_{j=0}^{k-1} H(2^j\xi). \tag{3.6}$$

Furthermore there exists a constant C such that
$$\begin{aligned} C^{-1}\| \prod_{j=0}^{k-1} \tilde{H}(2^j\xi)\|_p^* &\leq 2^{k\min(\zeta(H,0),\kappa(f_0,0))} \|P(f_0,H)(\xi) \prod_{j=0}^{k-1} H(2^j\xi)\|_p^* \\ &\leq C\| \prod_{j=0}^{k-1} \tilde{H}(2^j\xi)\|_p^*. \end{aligned} \tag{3.7}$$

Define an operator $\tilde{T}$ associated with the trigonometrical polynomial $\tilde{H}$ by
$$(\widehat{\tilde{T}f})(\xi) = \tilde{H}(\frac{\xi}{2})\hat{f}(\frac{\xi}{2}). \tag{3.8}$$

Observe that $\tilde{T}^k f, k \geq 1$ is supported in a compact set independent of $k \geq 1$ when f is compactly supported. Then
$$\|\tilde{T}^k f\|_p \leq C r_k 2^{-(\alpha'+\beta)k} \|f\|_p \tag{3.9}$$
holds for all $f \in L^p$ supported in a fixed compact set, where $\alpha' = \alpha - \min(\zeta(H,0),\kappa(f_0,0))$ and $r_k = 2^{(\alpha'+\beta+1-1/p)k} \| \prod_{j=0}^{k-1} \tilde{H}(2^j\xi)\|_p^*$.

By the definition of $P(f_0,H)$, there exists compactly supported distribution $\tilde{f}_0$ such that
$$\hat{f}_0(\xi) = \frac{P(f_0,H)(\xi)}{(-i\xi)^{\min(\zeta(H,0),\kappa(f_0,0))}} \widehat{\tilde{f}}_0(\xi).$$

Define the cascade algorithm $\tilde{f}_k, k \ge 1$ by $\tilde{f}_k = \tilde{T}\tilde{f}_{k-1},\ k \ge 1$. Then

$$\hat{f}_k(\xi) = \prod_{j=1}^{k} H(2^{-j}\xi)\hat{f}_0(2^{-k}\xi) = \frac{P(f_0, H)(\xi)}{(-i\xi)^{\min(\zeta(H,0),\kappa(f_0,0))}}\widehat{\tilde{f}}_k(\xi), \quad k \ge 0$$

and $D^{\min(\zeta(H,0),\kappa(f_0,0))} f_k$ is finitely linear combination of integer translates of $\tilde{f}_k$. Thus

$$\|\tilde{f}_k\|_{X^{\alpha'}_{p,q}} \approx \|D^{\min(\zeta(H,0),\kappa(f_0,0))} f_k\|_{X^{\alpha'}_{p,q}}. \tag{3.10}$$

Hereafter $A \approx B$ means that there exists an absolute constant C such that $C^{-1}A \le B \le CA$. On the other hand

$$\|f_k\|_{X^{\alpha}_{p,q}} \approx \|(\hat{\tilde{\Phi}}(8\cdot)\hat{f}_k(\cdot))^\vee\|_p + \|D^{\min(\zeta(H,0),\kappa(f_0,0))} f_k\|_{X^{\alpha'}_{p,q}} \tag{3.11}$$

and

$$\|(\hat{\tilde{\Phi}}(8\cdot)\hat{f}_k(\cdot))^\vee\|_p \approx \|(\hat{\tilde{\Phi}}(8\cdot)\hat{\tilde{f}}_k(\cdot))^\vee\|_p. \tag{3.12}$$

Hence it follows from (3.10)-(3.12) that

$$\|f_k\|_{X^{\alpha}_{p,q}} \approx \|\tilde{f}_k\|_{X^{\alpha'}_{p,q}}. \tag{3.13}$$

Let $Q_l, l \ge 0$ be as in Lemma 2.4. Then by computation we obtain

$$Q_l(\tilde{f}_k) = \tilde{T}^l(Q_0(\tilde{f}_{k-l})), \quad l \le k \tag{3.14}$$

and

$$Q_l(\tilde{f}_k) = \tilde{T}^k(Q_{l-k}(f_0)), \quad l \ge k. \tag{3.15}$$

Thus in the case $X^{\alpha}_{p,q} = F^{\alpha}_{p,q}$ by (3.9), (3.13)-(3.15) and Lemmas 2.4 and 3.2, we get

$$\begin{aligned}
&\|f_k\|_{F^{\alpha}_{p,q}} \le C\|\tilde{f}_k\|_{F^{\alpha'}_{p,q}} \le C\|P_0(\tilde{f}_k)\|_p + C\|(\sum_{l\ge 0} 2^{l\alpha' q}|Q_l(\tilde{f}_k)|^q)^{1/q}\|_p \\
\le\ & C\|P_0(\tilde{f}_k)\|_p + C(\sum_{l=0}^{k-1} 2^{l\alpha' q}2^{\delta(k-l)q}\|Q_l(\tilde{f}_k)\|_p^q)^{1/q} + C\|(\sum_{l\ge k} 2^{l\alpha' q}|Q_l(\tilde{f}_k)|^q)^{1/q}\|_p \\
\le\ & C2^{-kl_0} + C\Big(\sum_{l=0}^{k-1} r_l^q 2^{\delta(k-l)q}2^{-\beta l q}\|Q_0(\tilde{f}_{k-l})\|_p^q\Big)^{1/q} + Cr_k 2^{-k\beta}\|\tilde{f}_0\|_{F^{\alpha'}_{p,q}} \\
\le\ & C2^{-kl_0} + C2^{-\beta k}\Big(\sum_{l=0}^{k} r_l^q 2^{-(l_0-\beta-\delta)(k-l)q}\Big)^{1/q} + Cr_k 2^{-\beta k},
\end{aligned}$$

where l_0 is the minimal nonnegative integer strictly larger than β, and $\delta > 0$ is chosen that $l_0 - \beta - \delta > 0$. Similarly we have

$$\|f_k\|_{B^\alpha_{p,q}} \le 2^{-kl_0} + C2^{-\beta k}\Big(\sum_{l=0}^{k} r_l^q 2^{-(l_0-\beta-\delta)(k-l)q}\Big)^{1/q} + Cr_k 2^{-\beta k}$$

in the case $X^\alpha_{p,q} = B^\alpha_{p,q}$. It is easy to check that $\sum_{l=0}^{k-1} r_l^q 2^{-(l_0-\beta-\delta)(k-l)q}$ tends to zero too as r_k is, and is also bounded as r_k is. This completes the proof of Theorem 3.1. ♠

References

[CDM] A. Cavaretta, W. Dahmen and C. A. Micchelli, Stationary subdivision, *Memoir Amer. Math. Soc.*, 453(1991), 1-186.

[CS] A. Cohen and Q. Sun, An arithmetic characterization of conjugate quadrature filters associated to orthonormal wavelet bases, *SIAM J. Math. Anal.*, 24(1993), 1355-1360.

[D] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, PA 1992.

Vol. 216, Amer. Math. Soc., 1998.

[GL] T. N. T. Goodman and S. L. Lee, Convergence of cascade algorithms, In *Mathematical Methods for Curves and Surfaces II*, M. Dahlen, T. Lyche and L. L. Schumaker eds., Vanderbilt University Press, Nashville, 1998, pp. 191-212.

[GMW] T. N. T. Goodman, C. A. Micchelli and J. Ward, Spectral radius formulas for subdivision operators, In *Recent Advances in Wavelet Analysis*, L. L. Schumaker and G. Webb eds., Academic Press, New York, 1995, pp. 335-360.

[J1] R. Q. Jia, Subdivision schemes in L_p spaces, *Adv. Comput. Math.*, 3(1995), 309-341.

[JJL] R. Q. Jia, Q. Jiang and S. L. Lee, Convergence of cascade algorithm in Sobolev spaces and integrals of wavelets, Preprint 1998.

[JM] R. Q. Jia and C. A. Micchelli, Using the refinement equation for the construction of pre-wavelets II: powers of two, In *Curve and Surfaces*, P. J. Laurent, A. Le Méhauté and L. L. Schumaker eds., Academic Press, New York 1991, pp. 209-246.

[JW] R. Q. Jia and J.-Z. Wang, Stability and linear independence associated with wavelet decompositions, *Proc. Amer. Math. Soc.*, 117(1993), 1115-1124.

[LLS] W. Lawton, S. L. Lee and Z. Shen, Convergence of multidimensional cascade algorithms, *Numer. Math.*, 78(1998), 427-438.

[MS] B. Ma and Q. Sun, Compactly supported refinable distribution in Triebel-Lizorkin spaces and Besov spaces, *J. Fourier Anal. Appl.*, 5(1999), 87-104.

[R] A. Ron, Characterization of linear independence and stability of the shifts of a univariate refinable functions in terms of its mask, *CMS Technical Report 93-3*, University of Wisconsin, Madison.

[Sh] Z. Shen, Refinable function vectors, *SIAM J. Math. Anal.*, 29(1998), 239-250.

[St] G. Strang, Eigenvalues of $(\downarrow 2)H$ and convergence of cascade algorithm, *IEEE Trans. SP.*, 44(1996), 233-238.

[T] H. Triebel, *Theory of Function Spaces,* Monographs in Math. No: 78, Birkhäuser, 1983.