Convergence and Boundedness of Cascade Algorithm In Besov Spaces and Triebel-Lizorkin Spaces: Part II *

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Abstract

In this part, we shall characterize completely convergence and increment of the cascade algorithm in Besov spaces and Triebel-Lizorkin spaces by joint spectral radius on certain finitely dimensional space, give a new proof of moment conditions for the initial distribution and the refinable distribution in the cascade algorithm, establish close relationship between regularity of the refinable distribution and convergence and boundedness of the cascade algorithm, and apply the characterization to the existence of compactly supported solutions of nonhomogeneous refinement equations. From our results, we see that the initial and the refinable distribution of the cascade algorithm satisfy less moment conditions for the boundedness of the cascade algorithm than for the convergence of the cascade algorithm, and for $0 < p < 1$ than for $p \geq 1$. It is observed that the convergence and boundedness of the cascade algorithm are equivalent to each other under certain

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restriction on the indices of regularity of function space and the rate of convergence of the cascade algorithm, and certain assumptions on the refinable distribution.

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This paper is continuation of [S1]. We shall use the same notations as in Part I. This part is organized as follows. In Section 4, p-norm joint spectral radius on a finitely dimensional space is used to interpret the estimate of $\|P(f_0,H)(\xi)\prod_{j=0}^{k}H(2^j\xi)\|_p^*$ in Theorem 1.1 (Theorems 4.1 and 4.2). From Theorem 4.1, we see that the p-norm joint spectral radius is perfect to characterize convergence of the cascade algorithm. The moment conditions are also discussed in Section 4. In Theorem 4.3, we show that the initial f_0 and the refinable distribution ϕ need to satisfy some moment conditions for the convergence and boundedness of the cascade algorithm. Generally they satisfy less moment conditions for the boundedness of the cascade algorithm than for the convergence of the cascade algorithm, and for $0<p<1$ than for $p\geq 1$. The proof of moment conditions which the initial f_0 and the refinable distribution ϕ satisfy is different with the direct proof in [J1] for L^p and in [JJS] for Sobolev spaces H^l with nonnegative integer l.

In Section 5, we shall discuss the close relationship between regularity of the refinable distribution ϕ and convergence and boundedness of the cascade algorithm. In Theorem 5.1, we give an explicit characterization to regularity of refinable distributions in $X_{p,q}^{\alpha}$ and in $B_{p,\infty}^{\alpha}$ by using some estimate on $\|P(H)(\xi)\prod_{j=0}^{k}H(2^j\xi)\|_p^*$. For appropriate initial f_0, we prove the convergence and boundedness of the cascade algorithm under some assumption on regularity of the refinable distribution ϕ (Theorem 5.2 and Corollary 5.3). Under the stable assumption on ϕ, which means that the integer translates of ϕ are stable, and certain assumption on regularity of ϕ, we show that the necessary conditions on the initial f_0 in Theorems 2.1 and 4.3 are also sufficient for $p\geq 1$, but the assertion above is not true for $0<p<1$ in general (Corollary 5.4 and the remark there). To our surprise, convergence and boundedness of the cascade algorithm are equivalent to each other under certain restriction on the indices α and β and certain assumptions on ϕ (Corollary 5.5). For instance, let $\hat{f}_0(0)=1$ and $\phi, f_0\in H^{\alpha}$. Assume that α

is not a nonnegative integer and the integer translates of ϕ are stable. Then f_k converges to ϕ in H^α if and only if f_k is bounded in H^α. In the proof of Theorem 5.2, Lemma 5.6 plays an important role.

The last section is devoted to discuss the convergence of $f_0 + \sum_{k=1}^{\infty} \gamma^k f_k$ in Besov spaces and Triebel-Lizorkin spaces, and the existence of compactly supported solutions of the following nonhomogeneous refinement equation

$$\phi = \gamma T_c \phi + f_0$$

in Besov spaces and Triebel-Lizorkin spaces. The limit distribution $f_0 + \sum_{k=1}^{\infty} \gamma^k f_k$ is a solution of the nonhomogeneous refinement equation above when it converges in some sense. The nonhomogeneous refinement equation above arises in the construction of wavelets on bounded domain, multiwavelets and biorthogonal wavelets on non-uniform meshes. In almost practical case of nonhomogeneous refinement equation, γ satisfies $|\gamma| \le 1$. This is also our inspiration to consider the rate of increment of the cascade algorithm.

4 Joint Spectral Radius and Moment Conditions

In this section, we shall use p-norm joint spectral radius to characterize convergence and boundedness of the cascade algorithm, and show that the initial distribution f_0 and the refinable distribution ϕ in the cascade algorithm satisfy some moment conditions.

4.1 Joint Spectral Radius

For $N = 0$, we have $H \equiv 1$. Thus the condition about H in Theorem 1.1 is easy to be checked. So in this subsection, we always assume that $N \ge 1$.

Denote the space of all sequences with finite length by $l_0(\mathbb{Z})$. For $d \in l_0(\mathbb{Z})$, define its Fourier series by

$$F(d)(\xi) = \sum_{n \in \mathbb{Z}} d(n) e^{-in\xi}$$

and its quasi-norm by

$$\|d\|_p = \|F(d)\|_p^* = (\sum_{n\in\mathbb{Z}} |d(n)|^p)^{1/p}.$$

For the sequence $c = \{c(n)\}_{n=0}^N$ with $\sum_{n=0}^N c(n) = 2$ and $c_0c_N \neq 0$, define corresponding *subdivision operator* S_c on $l_0(\mathbb{Z})$ by

$$S_c d(n) = \sum_{n'\in\mathbb{Z}} c(n-2n')d(n'). \tag{4.1}$$

Then

$$F(S_c d)(\xi) = 2H(\xi)F(d)(2\xi).$$

For a trigonometrical polynomial $P(\xi)$, define an operator $P(\nabla)$ on $l_0(\mathbb{Z})$ by

$$F(P(\nabla)d)(\xi) = P(\xi)F(d)(\xi), \quad \forall\ d \in l_0(\mathbb{Z}). \tag{4.2}$$

For example, $P(\nabla)$ is the shift operator when $P(\xi) = e^{-ik\xi}$ and the difference operator when $P(\xi) = 1 - e^{-i\xi}$. By computation, we have

$$P(f_0, H)(\xi)\prod_{j=0}^{k-1} H(2^j\xi) = 2^{-k}F(P(f_0, H)(\nabla)S_c^k\delta)(\xi).$$

and

$$\|P(f_0, H)(\xi)\prod_{j=0}^{k-1} H(2^j\xi)\|_p^* = 2^{-k}\|P(f_0, H)(\nabla)S_c^k\delta\|_p$$

where $\delta \in l_0(\mathbb{Z})$ is defined by $\delta(0) = 1$ and $\delta(n) = 0$ for $n \neq 0$.

Define operators $B_\epsilon, \epsilon = 0, 1$ on $l_0(\mathbb{Z})$ by

$$F(B_\epsilon d)(\xi) = H(\frac{\xi}{2})F(d)(\frac{\xi}{2})e^{-i\epsilon\xi/2} + H(\frac{\xi}{2}+\pi)F(d)(\frac{\xi}{2}+\pi)e^{-i\epsilon(\xi/2+\pi)}. \tag{4.3}$$

Recall that $H(\xi) = \frac{1}{2}\sum_{n=0}^N c(n)e^{-in\xi}$. Then

$$B_\epsilon : l_0^{N-1}(\mathbb{Z}) \longrightarrow l_0^{N-1}(\mathbb{Z}), \quad \epsilon = 0, 1,$$

where $l_0^{N-1}(\mathbb{Z})$ denotes the space of all sequence $\{d(n)\} \in l_0(\mathbb{Z})$ with $d(n) = 0$ if $n < 0$ or $n > N-1$.

For any trigonometrical polynomial $R \not\equiv 0$, let $\mathcal{V}(R)$ be the space of all sequences $d \in l_0(\mathbb{Z})$ such that $F(d)(\xi)/R(\xi)$ is still a trigonometrical polynomial. Obviously $\mathcal{V}(R)$ is an ideal of $l_0(\mathbb{Z})$ under convolution, which means $d_1 \in \mathcal{V}(R)$ and $d_2 \in l_0(\mathbb{Z})$ implies $d_1 * d_2 \in \mathcal{V}(R)$. Furthermore

$$B_\epsilon : \mathcal{V}(P(f_0, H)) \longrightarrow \mathcal{V}(P(f_0, H)), \quad \epsilon = 0, 1,$$

and

$$B_\epsilon : \mathcal{V}^{N-1}(P(f_0, H)) \longrightarrow \mathcal{V}^{N-1}(P(f_0, H)), \quad \epsilon = 0, 1,$$

by definitions of $P(f_0, H)$ and $\mathcal{V}(P(f_0, H))$, where

$$\mathcal{V}^{N-1}(P(f_0, H)) = \mathcal{V}(P(f_0, H)) \cap l_0^{N-1}(\mathbb{Z}).$$

For simplicity we identify the space $l_0^{N-1}(\mathbb{Z})$ with Euclidean space $\mathbb{R}^N$, $\mathcal{V}^{N-1}(P(f_0, H))$ with a subspace of $\mathbb{R}^N$, and the restriction of operators B_ϵ on $l_0^{N-1}(\mathbb{Z})$ with $N \times N$ matrices $B_{\epsilon,mat}$. By computation, we have

$$\begin{aligned} \mathcal{V}^{N-1}(P(f_0, H)) \;\; = \;\; & \Big\{ (v_0, \cdots, v_{N-1}) \in \mathbb{R}^N, \quad \sum_{j=0}^{N-1} v_j (j+1)^l e^{-ij\xi_0} = 0, \\ & \forall\; 0 \le l \le \min(\zeta(H, \xi_0), \kappa(f_0, \xi_0)) - 1 \Big\} \end{aligned} \tag{4.4}$$

and

$$B_{\epsilon,mat} = \Big(c(2i - j + \epsilon) \Big)_{0 \le i,j \le N-1}, \quad \epsilon = 0, 1. \tag{4.5}$$

Define $\tilde{N} \times \tilde{N}$ matrices $\tilde{B}_\epsilon, \epsilon = 0, 1$ by

$$\tilde{B}_\epsilon = \Big(\tilde{c}(2i - j + \epsilon) \Big)_{0 \le i,j \le \tilde{N}-1}, \tag{4.6}$$

where $\tilde{N} = N - \deg P(f_0, H)$ and

$$\tilde{H}(\xi) = \frac{1}{2} \sum_{n=0}^{\tilde{N}-1} \tilde{c}(n) e^{-in\xi} = H(\xi) \times \frac{2^{\min(\kappa(f_0,0),\zeta(H,0))} P(f_0, H)(\xi)}{P(f_0, H)(2\xi)}.$$

For operators $T_\epsilon : \mathcal{V} \longrightarrow \mathcal{V}, \epsilon = 0, 1$ on a finitely dimensional space $\mathcal{V}$, define its *p-norm joint spectral radius* $\rho_p(T_0 \mid_{\mathcal{V}}, T_1 \mid_{\mathcal{V}})$ by

$$\rho_p(T_0 \mid_{\mathcal{V}}, T_1 \mid_{\mathcal{V}}) = \inf_{k \ge 1} \Big(\sum_{\epsilon_j \in \{0,1\}, 1 \le j \le k} \| T_{\epsilon_1} T_{\epsilon_2} \cdots T_{\epsilon_k} \mid_{\mathcal{V}} \|^p \Big)^{1/p}. \tag{4.7}$$

It can be check that

$$\rho_p(B_{0,mat}\mid_{\mathcal{V}^{N-1}(P(f_0,H))}, B_{1,mat}\mid_{\mathcal{V}^{N-1}(P(f_0,H))}) = \rho_p(\tilde{B}_0, \tilde{B}_1). \tag{4.8}$$

By standard method on joint spectral radius (see [J1] and [MS]), we have

Theorem 4.1 *Let $B_{\epsilon,mat}, \tilde{B}_\epsilon, \epsilon = 0,1$ and S_c, $P(f_0,H)(\nabla)$, $\mathcal{V}^{N-1}(P(f_0,H))$ and $\rho_p(B_{0,mat}\mid_{\mathcal{V}^{N-1}(P(f_0,H))}, B_{1,mat}\mid_{\mathcal{V}^{N-1}(P(f_0,H))})$, $\rho_p(\tilde{B}_0, \tilde{B}_1)$ be as (4.1)-(4.8). Then the following statements are equivalent to each other.*

(1) $\lim_{k\to\infty} 2^{(\alpha+1-1/p)k}\|P(f_0,H)(\xi)\prod_{j=0}^{k-1} H(2^j\xi)\|_p^* = 0.$

(2) $2^{(\alpha-1/p)k}\|P(f_0,H)(\nabla)S_c^k\delta\|_p$ *tends to zero as k tends to infinity.*

(3) $\sup_{v\in\mathcal{V}^{N-1}(P(f_0,H)),\|v\|_p=1} 2^{(\alpha p-1)k}\sum_{\epsilon_j\in\{0,1\},1\le j\le k}\|B_{\epsilon_1,mat}\cdots B_{\epsilon_k,mat}v\|^p$ *converges to zero.*

(4) $\rho_p(B_{0,mat}\mid_{\mathcal{V}^{N-1}(P(f_0,H))}, B_{1,mat}\mid_{\mathcal{V}^{N-1}(P(f_0,H))}) = \rho_p(\tilde{B}_0, \tilde{B}_1) < 2^{-\alpha+1/p}$.

Theorem 4.2 *Let $B_{\epsilon,mat}, \tilde{B}_\epsilon, \epsilon = 0,1$ and S_c, $P(f_0,H)(\nabla)$, $\mathcal{V}^{N-1}(P(f_0,H))$ and $\rho_p(B_{0,mat}\mid_{\mathcal{V}^{N-1}(P(f_0,H))}, B_{1,mat}\mid_{\mathcal{V}^{N-1}(P(f_0,H))})$, $\rho_p(\tilde{B}_0, \tilde{B}_1)$ be as (4.1)-(4.8). Then the following three statements are equivalent to each other.*

(1) $2^{(\alpha+1-1/p)k}\|P(f_0,H)(\xi)\prod_{j=0}^{k-1} H(2^j\xi)\|_p^*$ *is bounded.*

(2) $2^{(\alpha-1/p)k}\|P(f_0,H)(\nabla)S_c^k\delta\|_p$ *is bounded.*

(3) There exists a constant C independent of $k\ge 1$ such that

$$\sup_{v\in\mathcal{V}^{N-1}(P(f_0,H)),\|v\|_p=1} 2^{(\alpha p-1)k}\sum_{\epsilon_j\in\{0,1\},1\le j\le k}\|B_{\epsilon_1,mat}\cdots B_{\epsilon_k,mat}v\|^p \le C.$$

Furthermore

$$\rho_p(B_{0,mat}\mid_{\mathcal{V}^{N-1}(H(f_0,H))}, B_{1,mat}\mid_{\mathcal{V}^{N-1}(H(f_0,H))}) = \rho_p(\tilde{B}_0, \tilde{B}_1) \le 2^{-\alpha+1/p}$$

if $2^{(\alpha+1-1/p)k}\|P(f_0,H)(\xi)\prod_{j=0}^{k-1} H(2^j\xi)\|_p^*$ *is bounded.*

4.2 Moment Conditions

We say that a compactly supported distribution f satisfies *moment conditions of order* $l, l \geq 0$ if

$$D^l \hat{f}(2n\pi) = 0, \quad \forall\, 0 \leq j \leq l-1, \quad n \in \mathbb{Z}\backslash\{0\}. \tag{4.9}$$

Obviously any compactly supported distribution satisfies moment conditions of order 0 and $l \leq \kappa(f, 0)$.

For $\alpha \in \mathbb{R}$ and $0 < p < \infty$, let $m(\alpha, p)$ be the minimal nonnegative integer strictly larger than $\min(\alpha, \alpha + 1 - 1/p)$ and $\tilde{m}(\alpha, p)$ be the one larger than $\min(\alpha, \alpha + 1 - 1/p)$. Obviously $m(\alpha, p) \geq \tilde{m}(\alpha, p)$, and $m(\alpha, p) = \tilde{m}(\alpha, p)$ only if $\min(\alpha, \alpha + 1 - 1/p)$ is not a nonnegative integer.

Theorem 4.3 *Let p, q, α, β, f_k be as in Theorem 1.1. Then we have*

(1) If $\lim_{k\to\infty} A(\beta, f_k, X_{p,q}^\alpha) = 0$, then f_0 satisfies moment conditions of order $m(\alpha + \beta, p)$.

(2) If $A(\beta, f_k, X_{p,q}^\alpha)$ is bounded, then f_0 satisfies moment conditions of order $\tilde{m}(\alpha + \beta, p)$,

(3) If $\phi \in X_{p,q}^\alpha$, then ϕ satisfies moment conditions of order $m(\alpha, p)$.

(4) If $\phi \in B_{p,\infty}^\alpha$, then ϕ satisfies moment conditions of order $\tilde{m}(\alpha, p)$.

By Theorem 4.3, we obtain the following result which is proved by Jia ([J1]) for $\alpha = 0$ and $1 < p < \infty$, and Jia, Jiang and Lee ([JJL]) for all nonnegative integer α and $p = 2$.

Corollary 4.4 *Let $1 < p < \infty$, $-\infty < \alpha < \infty$, and let f_k be as in (1.2) and $\hat{f}_0(0) \neq 0$. If f_k converges in $F_{p,2}^\alpha$, then f_0 and ϕ satisfy moment conditions of order $m(\alpha, p)$.*

By taking $\beta = 0, 0 \leq \alpha \in \mathbb{Z}$, and $1 < p < \infty, q = 2$ in Theorem 4.3, we see that the initial distribution f_0 and the refinable distribution ϕ satisfy moment conditions of order $\alpha + 1$ when convergence of the cascade algorithm is considered, but the initial distribution f_0 only satisfies moment conditions of order α when we only need to establish boundedness of the cascade algorithm. The assertions in Theorem 4.3 can not be improved in general.

Example 2. Define B-spline B_l of order $l+1$ by convolution of the characteristic function on $[0,1]$ for l times,

$$B_l = \chi_{[0,1]} * \chi_{[0,1]} * \cdots * \chi_{[0,1]} \quad (l \ \ times).$$

Then $B_l, l \geq 1$ are refinable functions, belong to $B^{\alpha}_{p,q}$ for all $\alpha < 1/p + l - 1$ when $0 < p < 1$ and only satisfy moment conditions of order l.

Example 3. Let $f_0(x) = \cos \pi x \chi_{[0,1]}(x)$ and the sequence $\{c(n)\}_{n=0}^{1}$ be defined by $c(0) = c(1) = 1$. Then f_0 only satisfies moment conditions of order 0 and $\phi = \chi_{[0,1]} \in L^\infty$ satisfies the refinement equation $\phi(x) = \phi(2x) + \phi(2x-1)$. On the other hand we have

$$|f_k(x)| = |T_c^k f_0(x)| \leq \sum_{n=0}^{2^k-1} |f_0(2^k x - n)| \leq \chi_{[0,1]}(x).$$

Hence $f_k, k \geq 1$ is bounded in L^p for all $1 < p < \infty$.

Proof of Theorem 4.3 Write

$$H(\xi) = \frac{P(f_0, H)(2\xi)}{2^{\min(\zeta(H,0),\kappa(f_0,0))} P(f_0, H)(\xi)} \tilde{H}(\xi).$$

Then $\tilde{H}(0) = 1$ and there exists a constant C such that

$$C^{-1} \| \prod_{j=0}^{k-1} \tilde{H}(2^j\xi)\|_p^* \leq 2^{k \min(\zeta(H,0),\kappa(f_0,0))} \|P(f_0, H)(\xi) \prod_{j=0}^{k-1} H(2^j\xi)\|_p^* \leq C \| \prod_{j=0}^{k-1} \tilde{H}(2^j\xi)\|_p^*.$$

For any trigonometrical polynomial $R(\xi)$ with its degree at most 2^k, by Hölder inequality for $p \geq 1$ and $l^p(\mathbb{Z}) \subset l^1(\mathbb{Z})$ for $0 < p < 1$, we obtain

$$|R(0)| \leq C 2^{\max(0,1-1/p)k} \|R(\xi)\|_p^*. \tag{4.10}$$

Thus by $\tilde{H}(0) = 1$ and applying the estimate (4.10) to $\prod_{j=0}^{k-1} \tilde{H}(2^j\xi)$, we get

$$\begin{aligned} 1 &\leq C 2^{\max(0,1-1/p)k} \| \prod_{j=0}^{k-1} \tilde{H}(2^j\xi)\|_p^* \\ &\leq C 2^{k(\min(\zeta(H,0),\kappa(f_0,0)) + \max(0,1-1/p))} \|P(f_0, H)(\xi) \prod_{j=0}^{k-1} H(2^j\xi)\|_p^*. \end{aligned}$$

Hence by Theorem 1.1, we have

$$\min(\zeta(H,0),\kappa(f_0,0)) + \max(0,1-1/p) - \alpha - \beta - 1 + 1/p > 0$$

when $\lim_{k\to\infty} A(\beta, f_k, X^\alpha_{p,q}) = 0$, and

$$\min(\zeta(H,0),\kappa(f_0,0)) + \max(0,1-1/p) - \alpha - \beta - 1 + 1/p \geq 0$$

when $A(\beta, f_k, X^\alpha_{p,q})$ is bounded. This proves the first and second assertion by the definition of $\kappa(f_0, 0)$.

By definitions of Besov spaces and Triebel-Lizorkin spaces,

$$\lim_{k\to\infty} 2^{k\alpha}\|(\hat{\hat{\Psi}}(2^{-k}\cdot)\hat{\phi}(\cdot))^\vee\|_p = 0$$

if $\phi \in X^\alpha_{p,q}$, and $2^{k\alpha}\|(\hat{\hat{\Psi}}(2^{-k}\cdot)\hat{\phi}(\cdot))^\vee\|_p$ is bounded if $\phi \in B^\alpha_{p,\infty}$. Thus by the procedure used in the proof of Theorem 3.1, we get

$$\lim_{k\to\infty} 2^{(\alpha+1-1/p)k}\|P(H)(\xi)\prod_{j=0}^{k-1} H(2^j\xi)\|^*_p = 0$$

if $\phi \in X^\alpha_{p,q}$, and $2^{(\alpha+1-1/p)k}\|P(H)(\xi)\prod_{j=0}^{k-1} H(2^j\xi)\|^*_p$ is bounded if $\phi \in B^\alpha_{p,\infty}$. Hence the third and fourth assertion can be proved by the same procedure used above. ♠

5 Regularity and Cascade Algorithm

In this section, we shall characterize regularity of refinable distributions (Theorem 5.1), and establish some relationship between regularity of the refinable distribution and convergence and boundedness of the cascade algorithm (Theorem 5.2).

5.1 Regularity of Refinable Distribution

For the regularity of refinable distributions, there are a lot of papers on this topics (for instance [CDM], [DL] for Hölder spaces, [J1] and [LW] for L^p and L^p-Lipschitz spaces, [E], [V1], [J2] and [RS] for Sobolev spaces, [MS] and [V2] for Besov spaces and Triebel-Lizorkin spaces). In [MS], regularity of refinable distributions ϕ in $X^{\alpha}_{p,q}$ is characterized by the behavior of $\|R(\xi)\prod_{j=0}^{k-1} H(2^j\xi)\|_p^*$, where R belongs to an ideal of trigonometrical polynomials. In this subsection, we shall show that the ideal in [MS] can be chosen as the minimal ideal of trigonometrical polynomials containing $P(H)(\xi)$. Precisely we have

Theorem 5.1 *Let $0 < p, q < \infty$, $-\infty < \alpha < \infty$ and let ϕ be the normalized solution of the refinement equation (1.3). Then*

(1) $\phi \in X^{\alpha}_{p,q}$ if and only if

$$2^{(\alpha+1-1/p)k} \lim_{k\to\infty} \|P(H)(\xi)\prod_{j=0}^{k-1} H(2^j\xi)\|_p^* = 0. \tag{5.1}$$

(2) $\phi \in B^{\alpha}_{p,\infty}$ if and only if there exists a constant C independent of $k \geq 1$ such that

$$\|P(H)(\xi)\prod_{j=0}^{k-1} H(2^j\xi)\|_p^* \leq C 2^{-(\alpha+1-1/p)k}. \tag{5.2}$$

Proof of Theorem 5.1. By definitions of Besov spaces and Triebel-Lizorkin spaces, we have

$$\lim_{k\to\infty} 2^{(\alpha+1-1/p)k}\|(\hat{\tilde{\Psi}}(2^{-k}\cdot)\hat{\phi}(\cdot))^{\vee}\|_p = 0$$

if $\phi \in X^{\alpha}_{p,q}$, and there exists a constant C independent on $k \geq 1$ such that

$$2^{(\alpha+1-1/p)k}\|(\hat{\tilde{\Psi}}(2^{-k}\cdot)\hat{\phi}(\cdot))^{\vee}\|_p \leq C$$

if $\phi \in B^{\alpha}_{p,\infty}$. Then (5.1) and (5.2) follow from (1.9) and the proof of Theorem 3.1.

Now we prove that $\phi \in X^{\alpha}_{p,q}$ under the assumption (5.1). Write

$$H(\xi) = \frac{P(H)(2\xi)}{2^{\zeta(H,0)}P(H)(\xi)}\tilde{H}(\xi).$$

Then $\tilde{H}(0)=1$, $\tilde{H}(\pi)\neq 0$ and

$$P(H)(\xi)\prod_{j=0}^{k-1}H(2^j\xi)=2^{-k\zeta(H,0)}P(H)(2^k\xi)\prod_{j=0}^{k-1}\tilde{H}(2^j\xi).$$

Furthermore there exists a constant C such that

$$C^{-1}\|\prod_{j=0}^{k-1}\tilde{H}(2^j\xi)\|_p^*\leq\|P(H)(2^k\xi)\prod_{j=0}^{k-1}\tilde{H}(2^j\xi)\|_p^*\leq C\|\prod_{j=0}^{k-1}\tilde{H}(2^j\xi)\|_p^*.$$

Thus $\lim_{k\to\infty}2^{(\alpha-\zeta(H,0)+1-1/p)k}\|\prod_{j=0}^{k-1}\tilde{H}(2^j\xi)\|_p^*=0$ by (5.1) and furthermore by Theorem 4.1 there exist constants C and $0<r<1$ independent of $k\geq 1$ such that

$$\|\prod_{j=0}^{k-1}\tilde{H}(2^j\xi)\|_p^*\leq C2^{-(\alpha-\zeta(H,0)+1-1/p)k}r^k.$$

Let $\phi_{\tilde{H}}$ be the normalized solution of the refinement equation (1.3) with symbol $\tilde{H}$. Then

$$\begin{aligned}\|(\hat{\tilde{\Psi}}(2^{-k}\cdot)\widehat{\phi_{\tilde{H}}}(\cdot))^\vee\|_p &= \|(\hat{\tilde{\Psi}}(2^{-k}\cdot)\widehat{\phi_{\tilde{H}}}(2^{-k}\cdot)\prod_{j=0}^{k-1}\tilde{H}(2^{-j-1}\cdot))^\vee\|_p\\ &\leq C2^{(1-1/p)k}\|\prod_{j=0}^{k-1}\tilde{H}(2^j\xi)\|_p^*\leq C2^{-(\alpha-\zeta(H,0))k}r^k.\end{aligned}$$

Hence $\phi_{\tilde{H}}\in B_{p,q}^{\alpha-\zeta(H,0)}$. Observe that

$$\hat{\phi}(\xi)=C\frac{P(H)(\xi)}{(-i\xi)^{\zeta(H,0)}}\hat{\phi}_{\tilde{H}}(\xi),$$

where C is a constant. Then $\phi\in B_{p,q}^\alpha$ and $\phi\in X_{p,q}^\alpha$ by Lemma 2.5.

Similarly we can prove that $2^{(\alpha-\zeta(H,0)+1-1/p)k}\|\prod_{j=0}^{k-1}\tilde{H}(2^j\xi)\|_p^*$ is bounded, $\phi_{\tilde{H}}\in B_{p,\infty}^\alpha$ and hence $\phi\in B_{p,\infty}^\alpha$ under the assumption (5.2). ♠

5.2 Convergence and Boundedness of Cascade Algorithm

For $-\infty < \beta < \infty$, we say that the initial distribution f_0 is *adaptable to convergence of the cascade algorithm with rate* β if

$$\begin{cases} \kappa(f_0, \xi_0) \ge \zeta(H, \xi_0), & \forall\ \xi_0 \in (-\pi, \pi]\backslash\{0\}, \\ \kappa(f_0, 0) > \min(\alpha + \beta, \zeta(H, 0) - 1), \\ D^\gamma \hat{F}_0(0) = 0, & \forall\ 0 \le \gamma \le \beta, \end{cases}$$

and that the initial distribution f_0 is *adaptable to boundedness of the cascade algorithm with rate* β if

$$\begin{cases} \kappa(f_0, \xi_0) \ge \zeta(H, \xi_0), & \forall\ \xi_0 \in (-\pi, \pi]\backslash\{0\}, \\ \kappa(f_0, 0) \ge \min(\alpha + \beta, \zeta(H, 0)), \\ D^\gamma \hat{F}_0(0) = 0, & \forall\ 0 \le \gamma < \beta, \end{cases}$$

where

$$F_0 = \begin{cases} f_0 - \hat{f}_0(0)\phi, & \beta \ge 0, \\ f_0, & \beta < 0. \end{cases}$$

Obviously the adaptability of convergence and boundedness of the cascade algorithm with rate β is equivalent when $\alpha + \beta$ and β are not nonnegative integers. Define

$$F_k = T_c^k F_0, \quad k \ge 1. \tag{5.3}$$

Then

$$F_k = \begin{cases} f_k - \hat{f}_0(0)\phi, & \beta \ge 0, \\ f_k, & \beta < 0. \end{cases}$$

In this subsection, we shall prove the following result about convergence and boundedness of the cascade algorithm under the initial is chosen good enough.

Theorem 5.2 *Let $0 < p, q < \infty$, $-\infty < \alpha, \beta < \infty$, and let ϕ be the normalized solution of the refinement equation (1.3) and $F_k, k \ge 1$ be as in (5.3). Assume that $f_0 \in X_{p,q}^\alpha$ is compactly supported. Then we have*

(1) If $\phi \in X_{p,q}^{\alpha+\beta}$ and f_0 is adaptable to convergence of the cascade algorithm with rate β, then $\lim_{k\to\infty} A(\beta, F_k, X_{p,q}^\alpha) = 0$.

(2) If $\phi \in X^{\alpha+\beta}_{p,q}$ and f_0 is adaptable to boundedness of the cascade algorithm with rate β, then $A(\beta, F_k, X^{\alpha}_{p,q})$ is bounded.

(3) If $\phi \in X^{\alpha+\beta}_{p,\infty}$ and f_0 is adaptable to convergence of the cascade algorithm with rate β, then $A(\beta, F_k, X^{\alpha}_{p,q})$ is bounded.

By Theorems 2.1 and 5.2, we have

Corollary 5.3 *Let p, q, α, ϕ be as in Theorem 5.2 and $F_k, k \geq 1$ be as in (5.3). Assume that $\beta \geq 0$ and f_0 is adaptable to convergence of the cascade algorithm with rate β and satisfies $F_0 \not\equiv 0$. Then $\phi \in X^{\alpha+\beta}_{p,q}$ if and only if* $\lim_{k\to\infty} A(\beta, F_k, X^{\alpha}_{p,q}) = 0$.

By Theorems 2.1, 5.1 and 5.2, and the fact that $\zeta(H, \xi_0) = 0$ holds for all $\xi_0 \in (-\pi, \pi]\backslash\{0\}$ if the integer translates of the normalized solution ϕ are stable, we have

Corollary 5.4 *Let $\beta \geq 0$ and $1 < p < \infty$. Assume that the integer translates of $\phi \in X^{\alpha+\beta}_{p,q}$ are stable and that $f_0 \in X^{\alpha}_{p,q}$ is a compactly supported distribution with $\hat{f}_0(0) = 1$. Then we have*

(1) $2^{\beta k}\|f_k - \phi\|_{X^{\alpha}_{p,q}}$ converges to zero if and only if $D^{\gamma}\hat{f}_0(0) = D^{\gamma}\hat{\phi}(0)$ holds for all $0 \leq \gamma \leq \beta$ and f_0 satisfies moment conditions of order $m(\alpha + \beta, p)$.

(2) $2^{\beta k}\|f_k - \phi\|_{X^{\alpha}_{p,q}}$ is bounded if and only if $D^{\gamma}\hat{f}_0(0) = D^{\gamma}\hat{\phi}(0)$ holds for all $0 \leq \gamma < \beta$ and f_0 satisfies moment conditions of order $\tilde{m}(\alpha + \beta, p)$.

Remark By letting $\beta = 0$, $1 < p < \infty$ and $q = 2$ in Corollary 5.4, we see that under the stable assumption on ϕ and the assumptions $f_0, \phi \in X^{\alpha}_{p,q}$, the necessary conditions in Theorems 2.1 and 4.3 are also sufficient, which is proved by Jia ([J1]) for $\alpha = 0$ and $1 < p < \infty$ and by Jia, Jiang and Lee ([JJL]) for all nonnegative integers α and $p = 2$. In general, the necessary conditions in Theorems 2.1 and 4.3 are not sufficient for $0 < p < 1$.

Example 2 (continued) Let $0 < p < 1, 0 < q < \infty$ and $\alpha_1 = l - 1 + \delta$, where $\max(0, 1/p - 2) < \delta < \min(1, 1/p - 1)$. Then $m(\alpha_1, p) = l - 1$ and

$B_l \in X^{\alpha_1}_{p,q}$. Let $f_0 \in X^{\alpha_1}_{p,q}$ be any nonzero compactly supported distribution with $\hat{f}_0(0) = 1$ and only satisfies moment conditions of order $l-1$. Then $P(f_0, H)(\xi) = (e^{-i\xi} - 1)^{l-1}$. By computation, there exists a constant C such that

$$\|P(f_0, H)(\xi) \prod_{j=0}^{k-1} H(2^j\xi)\|_p^* = 2^{-lk}\|(e^{-i2^k\xi}-1)^{l-1} \prod_{j=0}^{k-1}(1+e^{-i2^j\xi})\|_p^* \geq C2^{-(l-1/p)k}.$$

Hence f_k does not converge to ϕ in $X^{\alpha_1}_{p,q}$ by Theorem 1.1.

By Corollary 5.4, and the fact that $m(\alpha+\beta, p) = \tilde{m}(\alpha+\beta, p) > \alpha+\beta$ if $p \geq 1$ and $\alpha+\beta$ is not a nonnegative integer, we have

Corollary 5.5 *Let $1 \leq p < \infty$, $\beta \geq 0$ be not a positive integer and $\alpha+\beta$ be not a nonnegative integer. Assume that $f_0 \in X^{\alpha}_{p,q}$ is compactly supported with $\hat{f}_0(0) = 1$ and the integer translates of $\phi \in X^{\alpha+\beta}_{p,q}$ are stable. Then $2^{\beta k}\|f_k - \phi\|_{X^{\alpha}_{p,q}}$ converges to zero if and only if $2^{\beta k}\|f_k - \phi\|_{X^{\alpha}_{p,q}}$ is bounded.*

To prove Theorem 5.2, we need the following lemma.

Lemma 5.6 *Let $R(\xi)$ be a trigonometrical polynomial with $R(0) = 1$. Then we have*

(1) Suppose that $\alpha < 0$ and $\lim_{k\to\infty} 2^{(\alpha-1/p)k}\|\prod_{j=0}^{k-1} R(2^j\xi)\|_p^* = 0$. Then

$$\lim_{k\to\infty} 2^{(\alpha-1/p)k}\|\frac{1-e^{-i2^k\xi}}{1-e^{-i\xi}} \times \prod_{j=0}^{k-1} R(2^j\xi)\|_p^* = 0.$$

(2) Suppose that $\alpha < 0$ and $2^{(\alpha-1/p)k}\|\prod_{j=0}^{k-1} R(2^j\xi)\|_p^*$ is bounded. Then $2^{(\alpha-1/p)k}\|\prod_{j=0}^{k-1} R(2^j\xi) \times (1-e^{-i2^k\xi})/(1-e^{-i\xi})\|_p^*$ is bounded.

(3) Suppose that there exist constants C and $0 < r < 1$ such that

$$2^{-k/p}\|\prod_{j=0}^{k-1} R(2^j\xi)\|_p^* \leq Cr^k.$$

Then $2^{-k/p}\|\prod_{j=0}^{k-1} R(2^j\xi) \times (1-e^{-i2^k\xi})/(1-e^{-i\xi})\|_p^*$ is bounded.

Proof. Let Ψ_l and $\Phi_l, l \ge 0$ be 2π-periodic functions which satisfy the following conditions.

- For any $\gamma \ge 0$ there exists a constant C_γ independent of $l \ge 0$ such that
$$|D^\gamma \Psi_l(\xi)| + |D^\gamma \Phi_l(\xi)| \le C_\gamma 2^{l\gamma},$$
- supp $\Psi_l \subset [-2^{-l+1}\pi, -2^{-l}\pi] \cup [2^{-l}\pi, 2^{-l+1}\pi]$ and supp $\Phi_l \subset [-2^{-l}\pi, 2^{-l}\pi]$,
- $\sum_{l\ge l_0} \Psi_l(\xi) + \Phi_{l_0}(\xi) = 1$ for all $l_0 \ge 0$.

Set $R_k(\xi) = \prod_{j=0}^{k-1} R(2^j\xi)$, $r_k = 2^{(\alpha-1/p)k} \| \prod_{j=1}^{k} R(2^j\xi)\|_p^*$ and $p_* = \min(p,1)$. Then

$$\begin{aligned}
&\Big\| \frac{1-e^{-i2^{k+1}\xi}}{1-e^{-i\xi}} R_k(\xi)\Big\|_p^* \\
\le\ & \Big(\sum_{l=0}^{k} (\|\Psi_l(\xi)\frac{1-e^{-i2^k\xi}}{1-e^{-i\xi}} R_k(\xi)\|_p^*)^{p_*}\Big)^{1/p_*} + \|\Phi_k(\xi)\frac{1-e^{-i2^k\xi}}{1-e^{-i\xi}} R_k(\xi)\|_p^* \\
\le\ & C\Big(\sum_{l=0}^{k} \Big(\Big\|\frac{\Psi_l(\xi)R_k(\xi)}{1-e^{-i\xi}}\Big\|_p^*\Big)^{p_*}\Big)^{1/p_*} + C\|\Phi_k(\xi)\frac{1-e^{-i2^k\xi}}{1-e^{-i\xi}} R_k(\xi)\|_p^*. \qquad (5.4)
\end{aligned}$$

By the construction of Φ_l and $\Psi_l, l \ge 0$, there exists a constant C such that

$$\Big|\int_0^{2\pi} e^{-in\xi}\Phi_k(\xi)\frac{1-e^{-i2^{k+1}\xi}}{1-e^{-i\xi}} R_k(\xi)d\xi\Big| \le C(1+2^{-k}|n|)^{-1-1/p}$$

and

$$\Big|\int_0^{2\pi} e^{-in\xi}\Psi_l(\xi)(1-e^{-i\xi})^{-1} R_l(\xi)d\xi\Big| \le C(1+2^{-l}|n|)^{-1-1/p}, \quad 0 \le l \le k.$$

Hence

$$\|\Phi_k(\xi)\frac{1-e^{-i2^k\xi}}{1-e^{-i\xi}} R_k(\xi)\|_p^* \le C2^{k/p} \qquad (5.5)$$

and

$$\begin{aligned}
&(\|\Psi_l(\xi)(1-e^{-i\xi})^{-1}R_k(\xi)\|_p^*)^p \\
=\ & (\|\Psi_l(\xi)(1-e^{-i\xi})^{-1}R_l(\xi)R_{k-l}(2^{l+1}\xi)\|_p^*)^p
\end{aligned}$$

$$
\begin{aligned}
&\leq C\left\{\begin{array}{ll} \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}(1+2^{-l}|m-2^{l+1}n|)^{-1-1/p}|R_{k-l-1}(n)|^p\times & \\ \quad \left(\sup_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}(1+2^{-l}|m-2^{l+1}n|)^{-1-1/p}\right)^{p-1}, & p\geq 1 \\ \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}(1+2^{-l}|m-2^{l+1}n|)^{-p-1}|R_{k-l-1}(n)|^p, & 0<p\leq 1 \end{array}\right. \\
&\leq C2^{-(\alpha-1/p)kp}2^{\alpha lp}r_{k-l-1}^p, \qquad (5.6)
\end{aligned}
$$

where $\sum_{n\in\mathbb{Z}} R_k(n)e^{-in\xi} = R_k(\xi)$. Hence the assertions follow from (5.4)-(5.6) and the assumptions. ♠

By the procedure used in the proof of Theorem 4.1, we obtain the equivalence of the following two statements for any trigonometrical polynomial R with $R(0)=1$.

- $\lim_{k\to\infty} 2^{(\alpha-1/p)k}\|\prod_{j=1}^k R(2^j\xi)\|_p^* = 0$.
- There exist constants C and $0<r<1$ such that

$$2^{(\alpha-1/p)k}\|\prod_{j=1}^k R(2^j\xi)\|_p^* \leq Cr^k.$$

Then by using Lemma 5.6 for several times, we have

Corollary 5.7 *Let α_0 be a nonnegative integer and let $R(\xi)$ be a trigonometrical polynomial with $R(0)=1$. Then we have*

(1) Suppose that $\alpha<-\alpha_0$ and $\lim_{k\to\infty} 2^{(\alpha-1/p)k}\|\prod_{j=1}^k R(2^j\xi)\|_p^* = 0$. Then

$$\lim_{k\to\infty} 2^{(\alpha+\alpha_0+1-1/p)k}\|\prod_{j=1}^k (\frac{1+e^{-i2^j\xi}}{2})^{\alpha_0+1}R(2^j\xi)\|_p^* = 0.$$

(2) Suppose that $\alpha<-\alpha_0$ and $2^{(\alpha-1/p)k}\|\prod_{j=1}^k R(2^j\xi)\|_p^*$ is bounded. Then

$$2^{(\alpha+\alpha_0+1-1/p)k}\|\prod_{j=1}^k (\frac{1+e^{-i2^j\xi}}{2})^{\alpha_0+1}R(2^j\xi)\|_p^*$$

is bounded.

(3) Suppose that $\alpha \le -\alpha_0$ *and* $\lim_{k\to\infty} 2^{(\alpha-1/p)k}\|\prod_{j=1}^{k} R(2^j\xi)\|_p^* = 0$*. Then*

$$2^{(\alpha+\alpha_0+1-1/p)k}\|\prod_{j=1}^{k}(\frac{1+e^{-i2^j\xi}}{2})^{\alpha_0+1}R(2^j\xi)\|_p^*$$

is bounded.

Proof of Theorem 5.2 By Theorem 5.1, it suffices to prove the assertion under additional assumption $\kappa(f_0,0) < \zeta(H,0)$. Write

$$H(\xi) = \frac{P(H)(2\xi)}{2^{\zeta(H,0)}P(H)(\xi)}\tilde{H}(\xi).$$

Then

$$P(H)(\xi)\prod_{j=0}^{k-1} H(2^j\xi) = 2^{-k\zeta(H,0)}P(H)(2^k\xi)\prod_{j=0}^{k-1}\tilde{H}(2^j\xi) \tag{5.7}$$

and

$$\begin{aligned} & P(f_0,H)(\xi)\prod_{j=0}^{k-1} H(2^j\xi) \\ = \quad & 2^{-k\kappa(f_0,0)}P(f_0,H)(2^k\xi)\prod_{j=0}^{k-1}(\frac{1+e^{-i2^j\xi}}{2})^{\zeta(H,0)-\kappa(f_0,0)}\tilde{H}(2^j\xi). \end{aligned} \tag{5.8}$$

Hence the assertions follow from (5.7), (5.8), Theorems 4.3 and 5.1, Corollary 5.7 and the assumptions. ♠

6 Application to Nonhomogeneous Refinement Equation

In this section, we shall consider the compactly supported solution of such a nonhomogeneous refinement equation

$$\phi(x) = \gamma\sum_{n=0}^{N} c(n)\phi(2x-n) + f_0(x), \tag{6.1}$$

with $\gamma \neq 0$ and $\sum_{n=0}^{N} c(n) = 2$ in Besov spaces and Triebel-Lizorkin spaces.

The nonhomogeneous refinement equation appears in the construction of wavelets on bounded domain, multiwavelets and biorthogonal wavelets on non-uniform meshes (see [CDD], [CES], [GHM], [Me], [S2] for instance). The existence of compactly supported solutions of the nonhomogeneous refinement equation (6.1) is well-studied (see [DH], [JJS], [StZ] and [S2]).

It is easy to be checked that $f_0 + \sum_{k=1}^{\infty} \gamma^k f_k$ is a solution of the nonhomogeneous refinement equation (6.1) when $\sum_{k=1}^{\infty} \gamma^k f_k$ converges in some sense. In [S2], the author showed the convergence of $f_0 + \sum_{k=1}^{\infty} \gamma^k f_k$ in $L^{\alpha,p} = F^{\alpha}_{p,2}, \alpha \geq 0$ under the assumption that $0 < |\gamma| < 1$ and

$$2^{(\alpha+1-1/p)k} |\gamma|^k \| \prod_{j=0}^{k-1} H(2^j \xi) \|_p^* \leq C r^k$$

holds for some constant C and $0 < r < 1$. A characterization of convergence of $\sum_{k=1}^{\infty} \gamma^k f_k$ in L^p with $1 \leq p < \infty$ is given in [StZ]. In this section, we first characterize the convergence of $\sum_{k=1}^{\infty} \gamma^k f_k$ in Besov spaces and Triebel-Lizorkin spaces by using the characterization of convergence of the cascade algorithm.

Theorem 6.1 *Let $0 < p, q < \infty$, $-\infty < \alpha < \infty$, $\gamma \neq 0$ and $f_0 \in X^{\alpha}_{p,q}$ be compactly supported and $\rho_p(\tilde{B}_0, \tilde{B}_1)$ be as in Theorem 4.1. Then the following three statements are equivalent to each other.*

(1) $\sum_{k=1}^{\infty} \gamma^k f_k$ *converges in* $X^{\alpha}_{p,q}$*,*

(2) $\lim_{k\to\infty} 2^{(\alpha+1-1/p)k} |\gamma|^k \| P(f_0, H)(\xi) \prod_{j=0}^{k-1} H(2^j \xi) \|_p^* = 0$ *and* $D^l \hat{f}_0(0) = 0$ *holds for all* $0 \leq l \leq \ln_2 |\gamma|$.

(3) $\rho_p(\tilde{B}_0, \tilde{B}_1) < 2^{-\alpha+1/p} |\gamma|^{-1}$ *and* $D^l \hat{f}_0(0) = 0$ *holds for all* $0 \leq l \leq \ln_2 |\gamma|$.

Proof. The equivalence of the second statement and the third one in Theorem 6.1 follows from Theorem 4.1.

From (1) it follows that $\lim_{k\to\infty} |\gamma|^k \|f_k\|_{X^{\alpha}_{p,q}} = 0$. Hence (2) holds by Theorem 1.1. This proves that (1) implies (2).

Conversely, there exists $0 < r < 1$ such that $\rho_p(\tilde{B}_0, \tilde{B}_1) < 2^{-\alpha+1/p} r|\gamma|^{-1}$. Then

$$|\gamma|^k \|P(f_0, H)(\xi) \prod_{j=0}^{k-1} H(2^j\xi)\|_p^* \le C 2^{-(\alpha+1-1/p)k} r^k$$

holds for some constant C independent of $k \ge 1$. Therefore by Theorem 1.1 and the assumption, $A(\ln_2(r|\gamma|), f_k, X_{p,q}^\alpha)$ is bounded. Thus $|\gamma|^k \|f_k\|_{X_{p,q}^\alpha} \le Cr^k$ and $\sum_{k=1}^\infty \gamma^k f_k$ converges in $X_{p,q}^\alpha$. ♠

Now we give the existence of compactly supported solution of the nonhomogeneous refinement equation (6.1) in Besov spaces and Triebel-Lizorkin spaces.

Theorem 6.2 *Let $\alpha, p, q, \gamma, f_0$ be as in Theorem 6.1, and let ϕ be the normalized solution of the refinement equation (1.3). Suppose that*

$$D^{l_0}(\hat{f}_0 \hat{\phi}^{-1})(0) = 0 \tag{6.2}$$

if $\gamma > 0$ and $l_0 = \ln_2 \gamma$ is a nonnegative integer. Assume that

$$\lim_{k\to\infty} 2^{(\alpha+1-1/p)k} |\gamma|^k \|P(f_0, H)(\xi) \prod_{j=0}^{k-1} H(2^j\xi)\|_p^* = 0.$$

Then there is a solution of the nonhomogeneous refinement equation (6.1) in $X_{p,q}^\alpha$.

Proof. Let $g \in X_{p,q}^\alpha$ be a compactly supported distribution chosen later. Set

$$\tilde{f}_0 = f_0 + \gamma T_c g - g.$$

Then

$$\sum_{k=1}^\infty \gamma^k T_c^k \tilde{f}_0 + f_0 + \gamma T_c g$$

is a solution of the nonhomogeneous refinement equation (6.1) in $X_{p,q}^\alpha$ if it converges in $X_{p,q}^\alpha$. By Theorem 6.1, it suffices to find such a function g such that $P(f_0, H)$ is a factor of $P(\tilde{f}_0, H)$ and satisfies

$$D^l \widehat{\tilde{f}_0}(0) = 0, \quad 0 \le l \le \ln_2 |\gamma|. \tag{6.3}$$

Obviously it suffices to choose $g=0$ when $|\gamma|<1$. Now we start to choose the function g when $|\gamma|\ge 1$. Write

$$\hat g(\xi)=\frac{P(f_0,H)(\xi)}{(i\xi)^{\min(\kappa(f_0,0),\zeta(H,0))}}\hat{\tilde g}(\xi), \tag{6.4}$$

where $\tilde g\in X_{p,q}^{\alpha-\min(\kappa(f_0,0),\zeta(H,0))}$ is chosen that

$$D^l\widehat{\tilde f_0}(0)=0,\quad \forall\ 0\le l\le \ln_2|\gamma|.$$

Then $P(f_0,H)$ is a factor of $P(\tilde f_0,H)$ by (6.4), (1.10) and

$$\widehat{\tilde f_0}(\xi)=\hat f_0(\xi)+\gamma H(\frac{\xi}{2})\hat g(\frac{\xi}{2})-\hat g(\xi).$$

Hence it remains to prove the existence of $\tilde g$ such that (6.3) hold. By computation, we can write (6.3) as

$$(1-2^{-l}\gamma)D^l\hat g(0)=D^l\hat f_0(0)+2^{-l}\gamma\sum_{s=0}^{l-1}\begin{pmatrix} l\\ s\end{pmatrix}D^{l-s}H(0)D^s\hat g(0),\quad 0\le l\le \ln_2|\gamma|. \tag{6.5}$$

Obviously there is unique solution of the linear equation (6.5) if $\gamma\le -1$ or $\gamma\ge 1$ and $\ln_2\gamma$ is not an integer. If $\ln_2\gamma$ is an integer, then there is unique solution of the following linear equation

$$(1-2^{-l}\gamma)D^l\hat g(0)=D^l\hat f_0(0)+2^{-l}\gamma\sum_{s=0}^{l-1}\begin{pmatrix} l\\ s\end{pmatrix}D^{l-s}H(0)D^s\hat g(0),\quad 0\le l\le \ln_2|\gamma|-1. \tag{6.6}$$

By (6.2), the right hand side of (6.5) equals zero for $l=\ln_2\gamma$ (see Theorem 2.4 in [S2]). So we obtain a solution of (6.5) by setting $D^l\hat g(0), 0\le l\le \ln_2|\gamma|-1$ in (6.6) and $D^l\hat g(0)=0$ for $l=\ln_2\gamma$ when $\ln_2\gamma$ is an integer. Therefore for any $|\gamma|\ge 1$ we can find a solution $D^l\hat g(0), 0\le l\le \ln_2|\gamma|$ of the linear equation (6.5). By (6.4), we see that $D^l\hat{\tilde g}(0), 0\le l\le \ln_2|\gamma|$ is uniquely determined by $D^l\hat g(0), 0\le l\le \ln_2|\gamma|$. On the other hand for any numbers $a_l, 0\le l\le \ln_2|\gamma|$ there exists a compactly supported distribution in $\tilde g\in X_{p,q}^{\alpha-\min(\kappa(f_0,0),\zeta(H,0))}$ such that $D^l\hat{\tilde g}(0)=a_l, 0\le l\le \ln_2|\gamma|$. This proves the existence of $\tilde g$ such that (6.3) holds. ♠

Remark The condition (6.2) in Theorem 6.2 is the necessary condition in [S2] such that there exists a compactly supported solution of the nonhomogeneous refinement equation. The relationship between regularity of the solution of the nonhomogeneous refinement equation (6.1) and convergence of the cascade algorithm seems different to the corresponding one of a refinement equation. For example for any Schwartz function g with compact support, g satisfies the following nonhomogeneous refinement equation,

$$g(x) = g(2x) + f_0(x)$$

where $f_0(x) = g(x) - g(2x)$. It can be proved that that $f_k(x) = g(2^k x) - g(2^{k+1}x)$ and $\sum_{k=1}^{\infty} f_k$ converges to $g(2x)$ in $X^{\alpha}_{p,q}$ only when $\alpha < 1/p$.

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