Behaviour of an Oscillatory Singular Integral
on Weighted Local Hardy Spaces

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Abstract The boundedness on weighted local Hardy spaces $h_w^{1,p}$ of the oscillatory singular integral

$$Tf(x) = \int_{R^n} e^{iQ(x,y)} K(x,y) f(y) dy$$

is considered when $Q(x,y) = P(x-y)$ for some real-valued polynomial P with its degree not less than two. Also a sufficient and necessary condition on polynomial Q on $R^n \times R^n$ such that T maps $h_w^{1,p}$ to weighted integrable function spaces L_w^1 is found.
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1. Introduction and Results

We say that a local integrable function K on $R^n \times R^n \backslash \{(x,x), x \in R^n\}$ is a kernel of Calderon-Zygmund type if $|K(x,y)| \le C|x-y|^{-n}$ and $|\frac{\partial}{\partial x}K(x,y)|+|\frac{\partial}{\partial y}K(x,y)| \le C|x-y|^{-n-1}$ for all $x \neq y$. For a kernel K of Calderon-Zygmund type and a real-valued polynomial Q on $R^n \times R^n$, define an oscillatory singular integral T considered later by

$$Tf(x) = \int_{R^n} e^{iQ(x,y)} K(x,y) f(y) dy. \tag{1}$$

The above oscillatory singular integral T arises in Fourier analysis on lower dimensional variations and has various applications such as Radon transform, Hilbert transform etc. The boundedness of the operator T on various spaces such as unweighted and weighted p-integrable function spaces for $1 < p < \infty$, weak integrable function space w-L^1, unweighted and weighted Hardy spaces are considered in [1], [4]–[8]. Especially they emphasize the connection between the oscillatory singular integral T and the following truncated Calderon-Zygmund operator $\tilde{T}$ defined by

$$\tilde{T}f(x) = \int_{R^n} K(x,y)\phi(|x-y|) f(y) dy, \tag{2}$$

where K is a kernel of Calderon-Zygmund type and ϕ is a fixed nonnegative smooth function satisfying $\phi(t) = 1$ on $[0, \frac{1}{2}]$ and $\phi(t) = 0$ on $[2, \infty)$.

In this paper, we will consider the behaviour of the oscillatory singualr integral T on weighted local Hardy spaces $h_w^{1,p}$. To this end, we introduce some notations and definitions.

We say that w is a Muckenhoupt A_p weight if

$$\frac{1}{|B|}\int_B w(x)dx \Big(\frac{1}{|B|}\int_B w(x)^{-\frac{1}{p-1}} dx\Big)^{p-1} \le C$$

holds for all balls B when $1 < p < \infty$ and

$$Mw(x) \le Cw(x)$$

holds for all $x \in R^n$ when $p = 1$, where constant C independent of the balls B when $1 < p < \infty$ and independent of $x \in R^n$ when $p = 1$. Hereafter M denotes the Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{x \in B} \frac{1}{|B|}\int_B |f(y)| dy$$

as usual where the supremum is taken over all balls B containing x.

Definition 1. Let $1 < p < \infty$. A function a is called an atom of weighted local Hardy spaces $h_w^{1,p}$ if there exists a ball B such that supp$a \subset B$, $\|a\|_{p,w} \le w(B)^{\frac{1}{p-1}}$ and either

(i) $r(B) < 1$ and $\int a(x)dx = 0$

or

(ii) $r(B) \geq 1$.

Hereafter B is called the supporting ball of $h_w^{1,p}$ atom a, we denote $\|f\|_{p,w} = (\int |f(x)|^p w(x)\ dx)^{\frac{1}{p}}$ for $1 \leq p < \infty$, $w(B) = \int_B w(x)dx$ and $r(B)$ denotes the radius of B. Also let $L_w^p = \{f : \|f\|_{p,w} < \infty\}$ be the weighted p-integrable function spaces for $1 \leq p < \infty$. For simplity we use $|B|$ instead of $w(B)$ and L^p instead of L_w^p when $w \equiv 1$.

Definition 2. Let $w \in A_1$ and $1 < p < \infty$. The weighted local Hardy spaces $h_w^{1,p}$ is the set of all tempered distributions f which can be written as

$$f = \sum_{j \in Z} \lambda_j a_j \tag{3}$$

for a family of $h_w^{1,p}$ atoms a_j and a sequences $\{\lambda_j\}$ with $\sum_{j \in Z} |\lambda_j| < \infty$.

Obviously $h_w^{1,p}$ is a Banach spaces for every $1 < p < \infty$ under the norm

$$\|f\|_{h_w^{1,p}} = \inf(\sum_{j \in Z} |\lambda_j|),$$

where the infimum is taken over all possible representation (3) of f. For simplity we use $h^{1,p}$ instead of $h_w^{1,p}$ when $w \equiv 1$. The local Hardy space $h^{1,2}$ was introduced by Goldberg [3] who used the local square function to define it and proved the equivalence with the above definition of $h^{1,2}$. In comparison with the weighted Hardy spaces [11], the only difference between them is that the vanishing moment condition on atoms in $h_w^{1,p}$ is deleted when the radius of its supporting ball B is larger than one. On the other hand, $h_w^{1,p}$ is an subspace of L_w^1, and furthermore a proper subspace of L_w^1 in general.

In Section 2, we will consider the boundedness of osciallatory singular integral T on $h_w^{1,p}$ for Muckenhoupt A_1 weight w when $Q(x,y) = P(x-y)$ for some real-valued polynomial P with $P(0) = 0$ and its degree $\deg(P) \geq 2$. Precisely we have proved the following result:

Theorem 1. *Let $w \in A_1, 1 < p < \infty$ and K be a kernel of Calderon-Zygmund type. Assume $Q(x,y) = P(x-y)$ for some real-valued polynomial P with $P(0) = 0$ and its degree $deg(P) \geq 2$. Then $\tilde{T} - T$, the difference between the corresponding oscillatory singular integral T and the corresponding truncated Calderon-Zygmund operator $\tilde{T}$, is bounded on weighted local Hardy space $h_w^{1,p}$.*

Denote the weighted Hardy space by H_w^1 for $w \in A_1$ [11]. Therefore $H_w^1 \subset h_w^{1,2} \subset L_w^1$. We say that an oscillatory singular integral T is of convolution type if

$$Tf(x) = \int e^{iP(x-y)} \tilde{K}(x-y) f(y) dy$$

for some real-valued polynomial P and a local integrable function $\tilde{K}$ on $R^n\backslash\{0\}$ such that $\tilde{K}(x-y)$ is a kernel of Calderon-Zygmund type. Recall that the conclusion $f\in H^1_w$ and $f, R_jf\in L^1_w$ are equivalent, where R_j $(1\le j\le n)$ denote Riesz transforms as usual. Observe that R_j maps H^1_w to H^1_w for every $w\in A_1$ and $R_jT=TR_j$ when the oscillatory singular integral T is of convolution type. Also observe that $\tilde{T}$ maps H^1_w to L^1_w by the Calderon-Zygmund theory. Therefore T maps H^1_w to H^1_w by Theorem 1 when $\tilde{T}$ is a bounded operator on L^2, $w\in A_1$, $\deg(P)\ge 2$ and the oscillatory singular integral T is of convolution type, which is the case considered by Pan and Hu in [5].

In Theorem 1, the bound constant of the operator $T-\tilde{T}$ is dependent on the sum of absolute values of the coefficients in P. It is easy to prove that

$$\int_{\lambda^{-1/3}\ge|x|\ge 2}|\int_{|x-y|\ge 2}e^{i\lambda(x-y)^2}\frac{1}{x-y}dy|dx=\frac{1}{3}\ln\lambda^{-1}+O(1)\to+\infty,$$

as $\lambda\to 0$, where $O(1)$ denotes a term bounded by a constant independent of $0<\lambda<1$. Therefore the bound constants of the operators $T-\tilde{T}$ corresponding to $K(x,y)=(x-y)^{-1}$ in (1) and $P(x)=\lambda x^2$ tends to infinity as $\lambda\to 0$. The author believe that the fundamental reason why this phenomenon happens to local Hardy space and does not happen to p-integable spaces is that local Hardy space has not good dilation invariance.

In Section 3, we will consider the behaviour of the oscillatory singular integral T defined by (1) for general polynomial Q on $R^n\times R^n$. First the oscillatory singular integral T defined by

$$Tf(x)=\int\frac{e^{ixy}}{x-y}f(y)dy$$

does not map $h^{1,p}$ to L^1 for all $1<p<\infty$ (see Example 1). Generally the oscillatory factor would damage the vanishing moment on $h^{1,p}_w$ atom which plays an important role. Also the oscillatory factor $Q(x+x_0,y+y_0)$ is completely different from $Q(x,y)$ in the sense of damaging the vanishing moment. These make us to consider the sufficient and necessary condition on polynomials Q on $R^n\times R^n$ under which the corresponding oscillatory singular integral T maps $h^{1,p}_w$ to L^1_w.

Theorem 2. *Let $1<p<\infty$ and $w\in A_1$. Assume that Q is a real-valued polynomial on $R^n\times R^n$ which cannot be written as $R_1(x)+R_2(y)$ for some polynomials R_1 and R_2, and K is a kernel of Calderon-Zygmund type with $|K(x,y)|\ge C|x-y|^{-n}$ for all $0<|x-y|<1$. If the corresponding oscillatory singular integral T defined by (1) is bounded on L^p_w, then the following statements are equivalent to each other:*
1) T maps $h^{1,p}_w$ to L^1_w;
2) $\sum_{1\le j\le\log\left(\min(1,A(x_0))\right)^{-1}}w\left(B(x_0,2^jr)\right)2^{-jn}\min\left(1,B(x_0)\right)\le Cw\left(B(x_0,r)\right)$
holds for all $0<r<1, x_0\in R^n$ and a constant C independent of r and x_0 but dependent of Q, where $A(x_0)=\sum_{\alpha\ne 0,\beta\ne 0}|a_{\alpha\beta}(x_0)|r^{|\alpha|+|\beta|}$, $B(x_0)=\sum_{\beta\ne 0}|a_{0\beta}(x_0)|r^{|\beta|}$, and $a_{\alpha\beta}(x_0)$ be the coefficient of $Q(x+x_0,y+y_0)$, i.e., $Q(x+x_0,y+y_0)=\sum_\alpha\sum_\beta a_{\alpha\beta}(x_0)x^\alpha y^\beta$.

The condition 2) in Theorem 2 seems not very computable. In Section 4, we will give some remarks on condition 2) in Theorem 2 and give a condition on $\tilde{T}$ for which $\tilde{T}$, hence T, is bounded on $h^{1,p}_w$. We prove that

Theorem 2′. *Let $p, w, Q, T, a_{\alpha\beta}(x_0)$ be the same as Theorem 2. Furthermore we assume that the weight w satisfies*

$$C^{-1}w(y) \le w(x) \le Cw(y)$$

for all $|x-y| \le 1, |x| \ge C$ and some constant C. Therefore the following statements are equivalent to each other:

1) T maps $h^{1,p}_w$ to L^1_w;

2) $\sum_{\beta\neq 0} |a_{0\beta}(x_0)|^{\frac{1}{|\beta|}} \le C \sum_{\alpha\neq 0,\beta\neq 0} |a_{\alpha\beta}(x_0)|^{\frac{1}{|\alpha|+|\beta|}}$ holds for all $x_0 \in R^n$ and some constant C independent of x, where $a_{\alpha\beta}(x_0)$ is defined as in Theorem 2.

2. Semi-convolution Type

In this section, we will give the proof of Theorem 1.

Write

$$\begin{aligned}(T-\tilde{T})f(x) =& \int (e^{iP(x-y)} - 1)K(x,y)\phi(|x-y|)f(y)dy \\ &+ \int e^{iP(x-y)} K(x,y)(1-\phi)(|x-y|)f(y)dy \\ =& T_1 f(x) + T_2 f(x).\end{aligned}$$

Observe that the kernel of T_1 satisfies

$$|(e^{iP(x-y)} - 1)K(x,y)\phi(|x-y|)| \le C|x-y|^{1-n}\chi_{|x-y|\le 2}.$$

Therefore the proof of Theorem 1 reduces to

Theorem 3. *Let $1 < p < \infty$ and $w \in A_1$. Assume that a local integrable function K on $R^n \times R^n \backslash \{(x,x); x \in R^n\}$ satisfies $|K(x,y)| \le C|x-y|^{\alpha-n}\chi_{|x-y|\le 2}$ for some constant $C > 0$ and $0 < \alpha < n$. Then the operator T_1 defined by*

$$T_1 f(x) = \int_{R^n} K(x,y)f(y)dy$$

is bounded on $h^{1,p}_w$.

Theorem 4. *Let $1 < p < \infty$ and $w \in A_1$. Assume that P is a non-zero real-valued polynomial with its degree deg$(P) \ge 2$ and K is a kernel of Calderon-Zygmund type with supp$K \cap \{(x,y) : |x-y| \le 1\} = \emptyset$. Then the operator T_2 defined by*

$$T_2 f(x) = \int_{R^n} e^{iP(x-y)} K(x,y)f(y)dy$$

is bounded on $h_w^{1,p}$.

Proof of Theorem 3. Obsverve that T_1 is a linear operator. Hence it sufficies to prove

$$\|T_1 a\|_{h_w^{1,p}} \le C \tag{4}$$

for every $h_w^{1,p}$ atom a and some constant C independent of a. Denote the supporting ball of a by B which has radius $r = r(B)$ and center x_0. Observe that

$$\mathrm{supp} T_1 a \subset B(x_0, r+2).$$

Hereafter $B(z,s)$ denotes the ball with its center $z \in R^n$ and its radius $s > 0$ and tB denotes the ball with the same center as the one of B and radius t times the one of B for $t > 0$. First we know

$$\|T_1 a\|_{p,w} \le C\|Ma\|_{p,w} \le C\|a\|_{p,w} \le Cw(B(x_0,r))^{\frac{1}{p}-1},$$

where M denotes the Hardy-Littlewood maximal operator as usual and the second inequality follows from the L_w^p boundedness of M provided $1 < p < \infty$ and $w \in A_p \subset A_1$. Therefore $C^{-1}T_1 a$ is an $h_w^{1,p}$ atom when $r \ge 1$ and (4) holds when the supporting ball B of a having radius $r \ge 1$. Thus the matter reduces to proving (4) when the supporting ball B of a has its radius $r < 1$. Write

$$\begin{aligned} T_1 a &= (T_1 a)\chi_{2B} + \sum_{k_0 \ge k \ge 2} (T_1 a)\chi_{2^{k+1}B \setminus 2^k B} \\ &= \sum_{1 \le k \le k_0} T_1^k a, \end{aligned}$$

where k_0 is an integer satisfying $2^{k_0} < r + 2 \le 2^{k_0+1}$. Observe that

$$\begin{aligned} w(B(x_0, 2^k r)) \le& (2^k r)^n \mathrm{inf}_{x \in B(x_0,r)} Mw(x) \le C(2^k r)^n \mathrm{inf}_{x \in B(x_0,r)} w(x) \\ \le& C 2^{kn} w(B(x_o, r)) \end{aligned} \tag{5}$$

and

$$\int_B w(x)dx \Big(\int_B w^{-\frac{1}{p-1}}(x)dx\Big)^{p-1} \le C|B|^p \tag{6}$$

for every $w \in A_1$. Therefore we have

$$\|T_1^1 a\|_{p,w} \le Cr^\alpha \|Ma\|_{p,w} \le Cw(B)^{\frac{1}{p}-1} r^\alpha$$

and

$$\begin{aligned} \|T_1^k a\|_{p,w} \le& C(2^k r)^{\alpha - n} \|a\|_1 w(2^{k+1}B)^{\frac{1}{p}} \\ \le& C(2^k r)^{\alpha-n} \|a\|_{p,w} \Big(\int_B w^{-\frac{1}{p-1}}(x)dx\Big)^{\frac{p-1}{p}} w(2^{k+2}B)^{\frac{1}{p}} \\ \le& C(2^k r)^\alpha w(2^{k+2}B)^{\frac{1}{p}-1}, \end{aligned}$$

where third inequality follows from (5) and (6). On the other hand, for every $f \in L^p_w$ supported in $B' = B(x_0, s)$ for some $s < 1$, we can write

$$\begin{aligned} f =&(f - c(f)h_{B'}) + c(f)(h_{B'} - h_{2B'}) + \cdots \\ &+ c(f)(h_{2^{k_0}B'} - h_{2^{k_0+1}B'}) + c(f)h_{2^{k_0+1}B'}, \end{aligned}$$

where $c(f) = \int f(x)dx$, k_0 is chosen such that $2^{k_0}s \geq 1 > 2^{k_0-1}s$, $h_{2^kB'} = c_k$ $\chi_{2^{k+1}B'\backslash 2^kB'}$, χ_E denotes the characteristic function of the set E and c_k is chosen such that $\int h_{2^kB'}(x)dx = 1$. Therefore we get

$$\|f\|_{h^{1,p}_w} \leq C\|f\|_{p,w} w(B(x_0,s))^{1-\frac{1}{p}} + C|\int f(x)dx| w(B(x_0,s))s^{-n}\log s^{-1}. \quad (7)$$

Observe that $\mathrm{supp} T^k_1 a \subset B(x_0, 2^{k+2}r)$. Therefore

$$\begin{aligned} \|T_1 a\|_{h^{1,p}_w} \leq& \sum_{k\geq 1, 2^k r\leq 2} \|T^k_1 a\|_{h^{1,p}_w} \\ \leq& C \sum_{k\geq 1, 2^k r\leq 2} (2^k r)^\alpha + \sum_{k\geq 1, 2^k r\leq 2} \|T^k_1 a\|_1 w(B(x_0, 2^{k+2}r))(2^k r)^{-n}\log(2^k r)^{-1} \\ \leq& C + C \sum_{k\geq 1, 2^k r\leq 2} (2^k r)^\alpha (w(2^{k+2}B))^{\frac{1}{p}} \\ &(\int_{2^{k+2}B} w^{-\frac{1}{p-1}}(x)dx)^{\frac{p-1}{p}} (2^k r)^{-n}\log(2^k r)^{-1} \\ \leq& C + C \sum_{k\geq 1, 2^k r\leq 2} (2^k r)^\alpha \log(2^k r)^{-1} \leq C, \end{aligned}$$

and (4) holds true■

To prove theorem 4, we will use the following lemmas.

Lemma 1. *Let Q, K and T_2 be the same as in Theorem 4. Then T_2 is bounded on L^p_w provided $1 < p < \infty$ and $w \in A_p$.*

Proof of Lemma 1. Lemma 1 is proved by Liu and Zhang [6]. For completeness of this paper, we give the sketch of their proof here. Define

$$T^2_j f(x) = \int e^{iP(x-y)} K(x,y)\varphi_j(|x-y|)f(y)dy \quad (8)$$

for $j \geq 1$, where φ_j are smooth functions satisfying $\varphi_j(t) = \varphi(2^{-j}t)(j \geq 1)$ and $\sum_{j\geq 1} \varphi(2^{-j}t) = 1$ on $(1, \infty)$. Therefore we can write

$$T_2 f = \sum_{j\geq 1} T^2_j f.$$

Obviously we have

$$\|T^2_j f\|_{p,w} \leq C\|Mf\|_{p,w} \leq C\|f\|_{p,w} \quad (9)$$

for $1 < p < \infty$ and $w \in A_p$. On the other hand, we have

$$\|T_j^2\|_2 \leq C2^{-\epsilon j}\|f\|_2 \tag{10}$$

for some $\epsilon > 0$ independent of f c.f. [8]. Recall that there exists $p-1 > \delta > 0$ for every $w \in A_p$ and $1 < p < \infty$ such that $w^{1+\delta} \in A_{p-\delta}$ [2]. Therefore by Marcinkiewicz real interpolation theorem [9] between (9) and (10), we get

$$\|T_j^2 f\|_{p,w} \leq C2^{-\epsilon j}\|f\|_{p,w}$$

for some C and ϵ independent of f and $j \geq 1$, and

$$\|T_2 f\|_{p,w} \leq \sum_{j\geq 1}\|T_j^2 f\|_{p,w} \leq C\|f\|_{p,w}.$$

Lemma 1 is proved■

Lemma 2. *Let $Q(x,y) = \sum_\alpha \sum_\beta a_{\alpha\beta}x^\alpha y^\beta$ be a real-valued polynomial. Define*

$$S_k f(x) = \int_B e^{iQ(x,y)} f(y)dy\chi_{2^kB}(x),$$

for $k \geq 1$, where B is a ball with its center zero and its radius $r = r(B)$. Therefore there exist constants C and $\epsilon > 0$ independent of k and f for every $1 < p, q < \infty$ such that

$$\|S_k f\|_p \leq C(1+g(r,k))^{-\epsilon}|2^k r|^{\frac{n}{p}} r^{n(\frac{q-1}{q})}\|f\|_q$$

where we denote $g(r,k) = \sum_{\alpha\neq 0,\beta\neq 0}|a_{\alpha\beta}|(2^k r)^{|\alpha|}r^{|\beta|}$.

Proof of Lemma 2. Obviously we have

$$\|S_k f\|_1 \leq C(2^k r)^n\|f\|_1 \leq C|2^k r|^n r^{n(\frac{q-1}{q})}\|f\|_q \tag{12}$$

and

$$\|S_k f\|_\infty \leq C\|f\|_1 \leq Cr^{n(\frac{q-1}{q})}\|f\|_q. \tag{13}$$

On the other hand, we have

$$\begin{aligned}
&\|S_k f\|_2^2 \\
\leq&(2^k r)^n \int\int f(y)\overline{f(y')}dydy' \int e^{i\left(Q(2^k rx,y)-Q(2^k rx,y')\right)}\psi(x)dx \\
\leq&C(2^k r)^n\|f\|_q^2\Big(\int_{|y|\leq r}\int_{|y'|\leq r} dydy'|\int e^{i\left(Q(2^k rx,y)-Q(2^k rx,y')\right)}\psi(x)dx|^{\frac{q}{q-1}}\Big)^{\frac{q-1}{q}} \\
\leq&C(2^k r)^n r^{\frac{2n(q-1)}{q}}\|f\|_q^2 \\
&\Big(\int_{|y|\leq 1,|y'|\leq 1}\big(1+\sum_{\alpha\neq 0}|\sum_\beta a_{\alpha\beta}r^{|\beta|}(y^\beta - {y'}^\beta)|(2^k r)^{|\alpha|}\big)^{-\epsilon_1}dydy'\Big)^{\frac{q-1}{q}} \\
\leq&C(2^k r)^n r^{\frac{2n(q-1)}{q}}\|f\|_q^2 \\
&\Big(\int_{|y|\leq 1}\big(1+\sum_{\alpha\neq 0}(|\sum_{\beta\neq 0} a_{\alpha\beta}r^{|\beta|}y^\beta| + \sum_{\beta\neq 0} r^{|\beta|}|a_{\alpha\beta}|)(2^k r)^{|\alpha|}\big)^{-\epsilon_2}dy\Big)^{\frac{q-1}{q}} \\
\leq&C(2^k r)^n r^{\frac{2n(q-1)}{q}}(1+g(r,k))^{-\epsilon}\|f\|_q^2,
\end{aligned} \tag{14}$$

where ψ is a positive smooth function satisfying $\psi(x) = 1$ on $\{x : |x| \leq 1\}$ and $\psi(x) = 0$ on $\{x : |x| \geq 2\}$, ϵ_1, ϵ_2 and ϵ are sufficient small constants independent of f, r and k, and the third inequality follows the following estimate of of Van de Corput type(see [8] for example),

$$\int_{|y|\leq 1} (1 + |Q(y)|)^{-\epsilon} dy \leq C(1 + \sum_{\alpha} |q_\alpha|)^{\epsilon_1}$$

holds for some constant C, ϵ, ϵ_1 dependent of the degree of Q only, where $Q(y) = \sum_\alpha q_\alpha y^\alpha$.

Therefore Lemma 2 follows from the Marcinkiewicz real interpolation theorem [9] between (12) , (13) and (14)■

Proof of Theorem 4. Recall that T_2 is a linear operator. Therefore it sufficies to prove

$$\|T_2 a\|_{h_w^{1,p}} \leq C \tag{15}$$

for every $h_w^{1,p}$ atom a and some constant C independent of a. We divide two cases to prove (15).

Case 1. The supporting ball B of a has its radius $r = r(B) > 1$.

Write

$$\begin{aligned} T_2 a =& (T_2 a)\chi_{2B} + \sum_{k=1}^{\infty} (T_2 a)\chi_{2^{k+1}B\backslash 2^k B} \\ =& f_0 + \sum_{k=1}^{\infty} f_k. \end{aligned}$$

Recall that $\mathrm{supp} f_0 \subset 2B$, T_2 is boundedon L_w^p for every $1 < p < \infty$ and $w \in A_p$ by Lemma 1. Therefore we get

$$\|f_0\|_{p,w} \leq \|T_2 a\|_{p,w} \leq C\|a\|_{p,w} \leq Cw(B)^{\frac{1}{p}-1}. \tag{16}$$

On the other hand we have

$$\begin{aligned} |f_k(x)| \leq& C2^{-k(n+1)} r^{-n} \|a\|_1 + C2^{-kn} r^{-n} |\int e^{iP(x-y)} a(y) dy| \\ =& I_k(x) + II_k(x) \end{aligned}$$

on $2^{k+1}B\backslash 2^k B$ for $k \geq 1$. For $w \in A_1$, there exist constants $p_0 > 1$ and C for every $q > 0$ such that

$$(|B|^{-1} \int_B w(x)^{p_0} dx)^{\frac{1}{p_0}} \leq C|B|^{-1} \int_B w(x) dx \tag{17}$$

and

$$(|B|^{-1} \int_B w(x)^{-qp_0} dx)^{\frac{1}{p_0}} \leq C|B|^{-1} \int_B w(x)^{-q} dx, \tag{18}$$

by reverse Hölder inequality [2]. Write $P(x-y)=\sum a_{\alpha\beta}x^\alpha y^\beta$. Thus $a_{\alpha\beta}\not\equiv 0$ for $\alpha\neq 0$, $\beta\neq 0$ by our assumption $\deg(P)\geq 2$. Recall that $\mathrm{supp} f_k\subset 2^{k+1}B$ and $\mathrm{supp} a\subset B$. Therefore we get

$$\begin{aligned}\|I_k\|_{p,w}\leq& C2^{-k(n+1)}r^{-n}\|a\|_1 w(2^{k+1}B)^{\frac{1}{p}}\\ \leq& C2^{-k(n+1)}r^{-n}\|a\|_{p,w}(\int_B w(x)^{-\frac{1}{p-1}}dx)^{\frac{1}{p-1}}w(2^{k+1}B)^{\frac{1}{p}}\\ \leq& C2^{-k}w(2^{k+1}B)^{\frac{1}{p}-1}\end{aligned}\tag{19}$$

by (5) and (6), and we also get

$$\begin{aligned}\|II_k\|_{p,w}\leq& C2^{-kn}r^{-n}(\int_{2^{k+1}B}w(x)^{p_0}dx)^{\frac{1}{pp_0}}(\int_{2^{k+1}B}|\int e^{iP(x-y)}a(y)dy|^{\frac{pp_0}{p_0-1}}dx)^{\frac{p_0-1}{pp_0}}\\ \leq& C2^{-kn}r^{-n}w(2^{k+1}B)^{\frac{1}{p}}r^{\frac{n(q-1)}{q}}\|a\|_q(1+g(r,k))^{-\epsilon}\\ \leq& C(1+g(r,k))^{-\epsilon}w(2^{k+1}B)^{\frac{1}{p}-1},\end{aligned}\tag{20}$$

where $g(r,k)=\sum_{\alpha\neq 0,\beta\neq 0}|a_{\alpha\beta}|(2^k r)^{|\alpha|}r^{|\beta|}$, $q=\frac{pp_0}{p+p_0-1}<p$, the second inequality follows from (17) and Lemma 2 (in fact we use $\frac{pp_0}{p_0-1}$ as the p in Lemma 2), and the third one from the Hölder inequality

$$\|a\|_q\leq\|a\|_{p,w}(\int_B w(x)^{-\frac{q}{p-q}}dx)^{\frac{p-q}{pq}}$$

and (18). Recall that $r\geq 1$ and $a_{\alpha\beta}\not\equiv 0$ for $\alpha\neq 0,\beta\neq 0$. Combining (19) and (20), we get

$$\begin{aligned}\|T_2a\|_{h_w^{1,p}}\leq&\|f_0\|_{p,w}w(B)^{1-\frac{1}{p}}+\sum_{k\geq 1}\|f_k\|_{p,w}w(2^{k+1}B)^{1-\frac{1}{p}}\\ \leq& C+C\sum_{k\geq 1}g(r,k)^{-\epsilon}\leq C+C\sum_{k\geq 1}2^{-k\epsilon}\leq C\end{aligned}$$

and (15) holds in Case 1.

Case 2. The supporting ball B of a has its radius $r=r(B)<1$.
Write

$$T_2f=\sum_{j\geq 1}T_2^j f$$

as in the proof of Lemma 1. Recall that $\int a(y)dy=0$ by the definition of $h_w^{1,p}$ atom. Therefore we have

$$\begin{aligned}|T_2^j a(x)|\leq&\int\big|K(x,y)\phi(|x-y|)-K(x,x_0)\phi(|x-x_0|)\big||a(y)|dy\\ &+|K(x,x_0)||\phi(|x-x_0|)\int|e^{iP(x-y)}-e^{iP(x-x_0)}||a(y)|dy\\ \leq& C2^{-j(n+1)}r\|a\|_1+C2^{-jn}\sum_\alpha\sum_{\beta\neq 0}|a_{\alpha\beta}|2^{j|\alpha|}r^{|\beta|}\|a\|_1,\end{aligned}\tag{21}$$

where x_0 is the centre of B and

$$\begin{aligned}\|T_2^j a\|_{p,w} \leq& C2^{-j(n+1)}r\|a\|_1 w(B(x_0,2^j))^{\frac{1}{p}}\\ &+ 2^{-jn}\sum_{\alpha}\sum_{\beta\neq 0}|a_{\alpha\beta}|2^{j|\alpha|}r^{|\beta|}\|a\|_1 w(B(x_0,2^j))^{\frac{1}{p}}\\ \leq& C(2^{-j}r+\sum_{\alpha}\sum_{\beta\neq 0}|a_{\alpha\beta}|2^{j|\alpha|}r^{|\beta|})w(B(x_0,2^j))^{\frac{1}{p}-1}.\end{aligned} \tag{22}$$

Observe that

$$\begin{aligned}|T_2^j a(x)| \leq& C2^{-j(n+1)}r\|a\|_1\chi_{|x-x_0|\leq C2^j}(x)\\ &+|K(x,x_0)||\phi(x,x_0)||\int e^{iP(x-x_0-y)}a(y+x_0)dy|\\ \leq& I_1+I_2.\end{aligned}$$

By same argument as in (19) we get

$$\|I_1\|_{p,w}\leq C2^{-j}rw\big(B(x_0,2^j)\big)^{\frac{1}{p}-1}.$$

On the other hand we get

$$\begin{aligned}\|I_2\|_{p,w} \leq& C2^{-jn}(\int_{|x|\leq C2^j}|\int e^{iP(x-x_0-y)}a(y+x_0)dy|^p w(x+x_0)dx)^{\frac{1}{p}}\\ \leq& C2^{-jn}(\int_{|x|\leq C2^j}|\int e^{iP(x-x_0-y)}a(y+x_0)dy|^{\frac{pp_0}{p_0-1}}dx)^{\frac{p_0-1}{pp_0}}\\ &(\int_{|x|\leq C2^j}w(x+x_0)^{p_0}dx)^{\frac{1}{pp_0}}\\ \leq& C(1+\sum_{\alpha\neq 0,\beta\neq 0}|a_{\alpha\beta}|2^{j|\alpha|}r^{|\beta|})^{-\epsilon})w\big(B(x_0,2^j)\big)^{\frac{1}{p}-1},\end{aligned}$$

where the last inequality follows from Lemma 2 and (17). This proves

$$\|T_2^j a\|_{p,w}\leq C\big(2^{-j}r\ +(1+\sum_{\alpha\neq 0,\beta\neq 0}|a_{\alpha\beta}|2^{j|\alpha|}r^{|\beta|})^{-\epsilon}\big)w\big(B(x_0,2^j)\big)^{\frac{1}{p}-1}. \tag{23}$$

Recall that supp$T_2^j a\subset B(x_0,2^{j+1})$. Therefore by (21) and (23) we have

$$\begin{aligned}\|T_2 a\|_{h_w^{1,p}} \leq& C\sum_{j\geq 1}2^{-j}r\\ &C\sum_{j\geq 1}\min\big(\sum_{\alpha}\sum_{\beta\neq 0}|a_{\alpha\beta}|2^{j|\alpha|}r^{|\beta|},(1+\sum_{\alpha\neq 0,\beta\neq 0}|a_{\alpha\beta}|2^{j|\alpha|}r^{|\beta|}\big)^{-\epsilon}).\end{aligned}$$

Let j_0 be the least positive integer such that

$$\sum_{\alpha\neq 0,\beta\neq 0}|a_{\alpha\beta}|2^{j|\alpha|}r^{|\beta|}\geq 1. \tag{24}$$

Then $j_0 \le C\log r^{-1}$. Let (α_0,β_0) be the index satisfying $|a_{\alpha_0\beta_0}|2^{j|\alpha_0|}r^{|\beta_0|} \ge |a_{\alpha\beta}|2^{j|\alpha|}r^{|\beta|}$ for all $\alpha \ne 0, \beta \ne 0$. Therefore we have

$$\begin{aligned}\|T_2 a\|_{h_w^{1,p}} \le & C + C\sum_{1\le j\le j_0}\sum_{\alpha}\sum_{\beta\ne 0}|a_{\alpha\beta}|2^{j|\alpha|}r^{|\beta|} + C\sum_{j\ge j_0}(|a_{\alpha_0\beta_0}|2^{j|\alpha_0|}r^{|\beta_0|})^{-\epsilon}\\ \le & C + C\sum_{\alpha\ne 0,\beta\ne 0}|a_{\alpha\beta}|2^{j_0|\alpha|}r^{|\beta|}\\ & + Cj_0\sum_{\beta\ne 0}|a_{0\beta}|r^{\beta} + C(|a_{\alpha_0\beta_0}|2^{-j_0|\alpha_0|}r^{|\beta_0|})^{-\epsilon}\\ \le & C\end{aligned}$$

and (15) holds in Case 2. Theorem 4 is proved■

3. Non-convolution Type

We begin with an example of polynomial Q on $R^n \times R^n$ and a kernel K of Calderon-Zygmund type in one spatial dimension, for which the corresponding oscillatory singular integral does not map $h^{1,p}$ to L^1 for every $1 < p < \infty$.

Example 1. Let $n = 1$, $K(x-y) = \frac{1}{x-y}$ and $Q(x,y) = xy$. Then the oscillatory singular integral T defined by

$$Tf(x) = \int_R \frac{e^{ixy}}{x-y} f(y)dy$$

does not map $h^{1,p}$ to L^1 for every $1 < p < \infty$.

In particular, for $f_r(y) = r^{-1}e^{i\pi r^{-1}y}\chi_{[\pi r^{-1}-r,\pi r^{-1}+r]}(y)$ $(0 < r < 1/2)$, the $h^{1,p}$ norm $\|f_r\|_{h^{1,p}} \le C$ holds for some constant C independent of $0 < r < 1/2$. On the other hand, we have

$$\begin{aligned}\|Tf_r\|_1 \ge & r^{-1}\int_{2r}^{1}|\int_{-r}^{r}\frac{e^{ixy}}{x-y}dy|dx\\ \ge & r^{-1}\int_{2r}^{1}|\int_{-r}^{r}\frac{1}{x-y}dy|dx - 2r^{-1}\int_{2r}^{1}\int_{-r}^{r}|y|dydx\\ \ge & \int_{2r}^{1}\frac{1}{|x|}dx - 1 = \log(2r)^{-1} - 1 \to \infty \quad (r \to 0).\end{aligned}$$

This show that T does not map $h^{1,p}$ to L^1 boundedly.

Proof of Theorem 2. At first we prove 2)$\Longrightarrow$1). Obviously it suffices to proving

$$\|T_{x_0}a\|_{L^1_{\tau(x_0)w}} \le C \tag{25}$$

for every $h_w^{1,p}$ atoms a with its supporting ball B having center zero and radius $r = r(B)$, where we define

$$T_{x_0} f(x) = \int_{R^n} e^{iQ(x+x_0,y+x_0)} K(x+x_0, y+x_0) f(y) dy$$

and $\tau(x_0)w(\cdot) = w(\cdot + x_0)$. Hereafter the big letter C denotes a constant independent of x_0 and $0 < r < 1$, but would be different at different occurances. We divide two cases to prove (25).

Case 1. $r = r(B) \geq 1$
As in the proof of Theorem 4, write

$$T_{x_0} a = f_0 + \sum_{k=1}^{\infty} f_k.$$

Therefore we have

$$\begin{aligned}\|f_0\|_{1,\tau(x_0)w} &\leq \|T_{x_0}a\|_{p,\tau(x_0)w} \big(\tau(x_0)w\big)(2B)^{\frac{p-1}{p}} \leq C\\ \|f_k\|_{1,\tau(x_0)w} &\leq \|f_k\|_{p,\tau(x_0)w} \big(\tau(x_0)w\big)(2^{k+1}B)^{\frac{p-1}{p}}\\ &\leq C\big(2^{-k} + (1+g_{x_0}(r,k)\big)^{-\epsilon}),\end{aligned}$$

where $g_{x_0}(r,k) = \sum_{\alpha\neq 0,\beta\neq 0} |a_{\alpha\beta}(x_0)|(2^k r)^{|\alpha|} r^{|\beta|}$ and constants $C, \epsilon > 0$ are independent of k and a. We say index $\gamma = (\gamma_1, .., \gamma_n) \geq \delta = (\delta_1, .., \delta_n)$ if $\gamma_i \geq \delta_i$ for all $1 \leq i \leq n$. Observe that $a_{\alpha\beta}(x_0) = a_{\alpha\beta}(0)$ for all index pairs (α, β) and $x_0 \in R^n$ for which there does not exist index pairs (γ, δ) such that $a_{\gamma\delta}(0) \neq 0$, $(\gamma, \delta) \neq (\alpha, \beta), \gamma \geq \alpha$ and $\delta \geq \beta$. Therefore $g_{x_0}(r,k) \geq C2^{ks}$ holds for some constants C and s independent of x_0 and k provided $r \geq 1$. This shows that

$$\|T_{x_0}a\|_{1,w} \leq \sum_{k\geq 0} \|f_k\|_{1,w} \leq C + C\sum_{k\geq 1} 2^{-ks\epsilon} \leq C$$

and (25) holds in Case 1.

Case 2. $r = r(B) < 1$.
Write

$$T_{x_0} a = f_0 + \sum_{k=1}^{\infty} f_k \tag{26}$$

as in Case 1. As in Case 2 in the proof of Theorem 4, we have

$$\|f_0\|_{1,\tau(x_0)w} \leq C,$$

and

$$\begin{aligned}\|f_k\|_{1,\tau(x_0)w} \leq& \|f\|_{p,\tau(x_o)w} (\tau(x_0)w)(2^{k+1}B)^{\frac{p-1}{p}}\\ \leq& C2^{-k} + C\min\Big(g_{x_0}(r,k) + \sum_{\beta\neq 0} |a_{0\beta}(x_0)| r^{|\beta|}, (1+g_{x_0}(r,k))^{-\varepsilon}\Big)\\ &2^{-kn}\big(\tau(x_0)w\big)(2^kB)\big(\tau(x_0)w\big)(B)^{-1},\end{aligned}$$

where $g_{x_0}(r,k) = \sum_{\alpha\neq 0,\beta\neq 0} |a_{\alpha\beta}(x_0)|(2^k r)^{|\alpha|} r^{|\beta|}$ as in Case 1. Define the first positive integer k such that $g_{x_0}(r,k) \geq 1$ by k_0 if it exists and define $k_0 = 0$ otherwise. Observe that

$$2^k g_{x_0}(r,0) \leq g_{x_0}(r,k) \leq 2^{Nk} g_{x_0}(r,0)$$

for some positive integer N. Thus we get

$$C_1 \log\big(\min(1, g_{x_0}(r,0))\big)^{-1} \leq k_0 \leq C_2 \log\big(\min(1, g_{x_0}(r,0))\big)^{-1} \tag{27}$$

for some constants $C_2 \geq C_1 > 0$ independent of x_0 and $r < 1$. Therefore

$$\begin{aligned}\|T_{x_0} a\|_{1,\tau(x_0)_w} \leq & C + C \sum_{k_0 \geq k} g_{x_0}(r,k)^{-\epsilon} + C \sum_{1\leq k\leq k_0} g_{x_0}(r,k) \\ & + C \sum_{1\leq k\leq k_0} 2^{-kn}\big(\tau(x_0)w\big)(2^k B)\big(\tau(x_0)w\big)(B)^{-1} \\ & \min(1, \sum_{\beta\neq 0} |a_{0\beta}(x_0)| r^{|\beta|}) \\ \leq & C + C\min(1, \sum_{\beta\neq 0} |a_{0\beta}(x_0)| r^{|\beta|}) \\ & \sum_{1\leq k\leq \log\left(\min(1, g_{x_0}(r,0))\right)^{-1}} 2^{-kn}(\tau(x_0)w)(2^k B)(\tau(x_0)w)(B)^{-1} \\ \leq & C < \infty,\end{aligned}$$

where the first inequality follows from (5) and the second one from our assumption 2),

$$\sum_{k\leq j\leq 2k} 2^{-jn}\big(\tau(x_0)w\big)(2^j B) \leq C \sum_{1\leq j\leq k} 2^{-jn}\big(\tau(x_0)w\big)(2^j B), \tag{28}$$

(5) and (7). Thus (25) holds in Case 2.

Secondly we prove 1)$\Longrightarrow$2). Let a be an $h^{1,p}_w$ atom with its supporting B having radius $r = r(B) < 1$ and center zero. Write

$$T_{x_0} a = f_0 + \sum_{k=1}^{\infty} f_k$$

where f_k are defined as in (26). Observe that

$$\begin{aligned}|f_k(x)| \geq & |K(x+x_0, x_0)|\Big|\int e^{i\sum_{\beta\neq 0} a_{0\beta}(x_0) y^\beta} a(y)dy\Big| \\ & - |K(x+x_0,x_0)| \int \sum_{\alpha\neq 0,\beta\neq 0} |a_{\alpha\beta}(x_0)|(2^k r)^{|\alpha|} r^{|\beta|} |a(y)|dy \\ & - \int |K(x+x_0,x_0) - K(x+x_0, y+x_0)||a(y)|dy\end{aligned} \tag{29}$$

on $2^{k+1}B\backslash 2^kB$ for $k \geq 1$. Also we know from (27) that $g_{x_0}(r,0) \geq Cr^N$ and $2^kr \leq 1$ for all $k \leq \epsilon_1 \log\big(\min(1, g_{x_0}(r,0))\big)^{-1}$, where C, N and $0 < \epsilon_1 < 1$ are constants independent of x_0 and $r < 1$. Recall that (28), (29) and $|K(x+x_0,x_0)| \geq C|x|^{-n}$ for all $|x| < 1$ and $x_0 \in R^n$ by our assumption. Therefore we get

$$
\begin{aligned}
&|\int e^{i\sum_{\beta\neq 0} a_{0\beta}(x_0)y^\beta} a(y)dy| \sum_{1\leq k\leq \log\left(\min\left(1, g_{x_0}(r,0)\right)\right)^{-1}} r^{-n}2^{-kn}(\tau(x_0)w)(2^kB)\\
\leq & C\int_{R^n} |T_{x_0}a(x)|(\tau(x_0)w)(x)dx\\
&+ C \sum_{1\leq k\leq \epsilon_1\log\left(\min\left(1, g_{x_0}(r,0)\right)\right)^{-1}} \big(2^{-k} + \sum_{\alpha\neq 0,\beta\neq 0} |a_{\alpha\beta}(x_0)|(2^kr)^{|\alpha|}r^{|\beta|}\big)\\
\leq & \|T_{x_0}a\|_{1,\tau(x_0)w} + C \leq C,
\end{aligned}
$$

where the first inequality follows from (5) , and the last one from our assumption 1). Therefore the matter reduces to

$$
\begin{aligned}
&\min\big(1, \sum_{\beta\neq 0} |a_{0\beta}(x_0)|r^{|\beta|}\big)\big(\tau(x_0)w\big)(B)^{-1}r^n\\
&\leq C\sup_a |\int e^{i\sum_{\beta\neq 0} a_{0\beta}(x_0)y^\beta} a(y)dy|,
\end{aligned}
\tag{30}
$$

where the supremum on a is taken over all function a satisfying supp$a \subset B(0,r)$, $\int a(y)dy = 0$ and $\|a\|_{p,\tau(x_0)w} \leq \big(\tau(x_0)w\big)\big(B(0,r)\big)^{\frac{1}{p}-1}$. Observe that

$$
\|a\|_{p,\tau(x_0)w} \leq \big(\tau(x_0)w\big)\big(B(0,r)\big)^{\frac{1}{p}-1}
$$

provided $\|a\|_\infty \leq \big(\tau(x_0)w\big)\big(B(0,r)\big)^{-1}$. Denote

$$
\mathcal{R}'_N = \big\{R(y) = \sum_{\beta\neq 0} a_\beta y^\beta,\ R \text{ is real-valued polynomial, deg}R \leq N\big\}.
$$

Therefore the matter reduces to

Lemma 3. *Let $\mathcal{R}'_N$ be defined as above. Thus*

$$
\sup_a |\int e^{iR(y)}a(y)dy| \geq C min(1, \|R\|) \tag{31}
$$

holds for all $R \in \mathcal{R}'_N$ and a constant C independent of R, where the supremum on a is taken over all function a satisfying supp$a \subset B(0,1)$, $\int a(y)dy = 0$ and $\|a\|_\infty \leq 1$, and we define $\|R\| = \sum_{\beta\neq 0}|a_\beta|$ for all $R \in \mathcal{R}'_N$.

Proof of Lemma 3. Denote

$$
\|R\|_* = \sup_a |\int R(y)a(y)dy|
$$

for all $R \in \mathcal{R}'_N$, where the supremum on a is taken over the same region as in (31). Obviously $\|R\|_* \geq 0, \|CR\|_* = |C|\|R\|_*$ and $\|R_1 + R_2\|_* \leq \|R_1\|_* + \|R_2\|_*$ for all $R, R_1, R_2 \in \mathcal{R}'_N$ and real number C. Furthermore $\|R\|_* = 0$ implies $\int R(y)a(y)dy = 0$ for all bounded functions a satisfying supp$a \subset B(0,1)$ and $\int a(y)dy = 0$. Therefore

$$\frac{1}{|B(0,1)|}\int_{B(0,1)} |R(y)|^2 dy = (\frac{1}{|B(0,1)|}\int_{B(0,1)} R(y)dy)^2,$$

where $|B(0,1)|$ denotes the Lebesgue measure of $B(0,1)$, and R must be a constant. Recall that $R \in \mathcal{R}'_N$. Thus $R = 0$, and $\|R\|_* = 0$ implies $R = 0$. Hence we prove that $\|\cdot\|_*$ is a norm on $\mathcal{R}'_N$. By the equivalence of two norms on finite dimensions spaces, we get $\|R\|_* \geq C_1\|R\|$ for all $R \in \mathcal{R}'_N$ and some constant C_1.

Observe that

$$|e^{iR(y)} - 1 - iR(y)| \leq \|R\|^2$$

for all $|y| \leq 1$ by Taylor formula. Hence we get

$$\begin{aligned}\sup_a |\int e^{iR(y)}a(y)dy| &\geq \|R\|_* - \|R\|^2 \\ &\geq C_1\|R\| - \|R\|^2 \geq \frac{C_1}{2}\|R\|,\end{aligned}$$

when $\|R\|$ is chosen sufficient small.

As in the procedure to prove $\|R\|_* = 0$ holds only for $R = 0$, we get

$$\sup_a |\int e^{iR(y)}a(y)dy| = 0$$

holds only for $R = 0$, where $R \in \mathcal{R}'_N$ and the supremum on a is taken over all bounded functions a satisfying supp$a \subset B(0,1)$ and $\int a(y)dy = 0$. Observe that $\int e^{iR(y)}a(y)dy$ is continuous on $R \in \mathcal{R}'_N$ for all bounded functions a. Therefore the matter reduces to proving that (31) holds for all $R \in \mathcal{R}'_N$ when $\|R\|$ is large enough.

Define

$$R_{B(0,1)} = \frac{1}{|B(0,1)|}\int_{B(0,1)} e^{iR(y)}dy.$$

Therefore by estimates of Van de Corput type [8], we get

$$\|R_{B(0,1)}\| \leq C\|R\|^{-\epsilon}$$

holds for all $R \in \mathcal{R}'_N$, where constants C and $\epsilon > 0$ is independent of $R \in \mathcal{R}'_N$. Observe that

$$\int_{B(0,1)} \big(e^{-iR(y)} - \bar{R}_{B(0,1)}\chi_{B(0,1)}(y)\big)dy = 0$$

and

$$|e^{-iR(y)} - \bar{R}_{B(0,1)}\chi_{B(0.1)}(y)| \leq 2.$$

Therefore we get

$$\begin{aligned}&\sup_a \Big|\int e^{iR(y)}a(y)dy\Big|\\ \geq&\frac{1}{2}\int |e^{iR(y)}-R_{B(0,1)}\chi_{B(0,1)}(y)|^2dy\\ \geq&\frac{1}{2}|B(0,1)|-C\|R\|^{-2\epsilon}\geq \frac{1}{4}|B(0,1)|,\end{aligned}$$

provided that $\|R\|$ chosen large enough. Lemma 3 and hence Theorem 2 is proved■

Example 1. (revised) Let $Q(x,y)=xy$. Then $a_{01}(x_0)=x_0, a_{11}(x_0)=1, a_{10}(x_0)=x_0, a_{00}(x_0)=x_0^2$ and $a_{\alpha\beta}(x_0)=0$ otherwise. Now the condition 2) in Theorem 2 becomes

$$\min(1,|x_0|r)\int_{x_0-1}^{x_0+1} w(x)\big(1+\frac{|x-x_0|}{r}\big)^{-1}dx\leq C\int_{x_0-r}^{x_0+r} w(x)dx$$

for all $x_0\in R^n, 0<r<1$. The authors believe that a weight $w\in A_1$ satisfying the the above condition does not exist.

4. Remarks

Observe that $2^kr\leq 1$ when $k\leq \varepsilon_1\log\big(\min(1,\sum_{\alpha\neq 0,\beta\neq 0}|a_{\alpha\beta}(x_0)|r^{|\alpha|+|\beta|})\big)^{-1}$, where $\varepsilon_1>0$ is a constant independent of x_0 and $r<1$. Therefore condition 2) in Theorem 2 is equivalent to

$$\min\big(1,\sum_{\beta\neq 0}|a_{0\beta}(x_0)|r^{|\beta|}\big)\log\big(\min\big(1,\sum_{\alpha\neq 0,\beta\neq 0}|a_{\alpha\beta}(x_0)|r^{|\alpha|+|\beta|}\big)\big)^{-1}\leq C \tag{32}$$

holds for all $x_0\in R^n$ and $0<r<1$, provided $w\in A_1$ and

$$C^{-1}w(y)\leq w(x)\leq Cw(y) \tag{33}$$

holds for all $|x-y|\leq 1, |x|\geq C$ and a constant C.

Example 2. $w(x)=|x|^\alpha, -n<\alpha\leq 0$ satisfies (33).

Observe that

$$\log\big(\min\big(1,\sum_{\alpha\neq 0,\beta\neq 0}|a_{\alpha\beta}(x_0)|r^{|\alpha|+|\beta|}\big)\big)^{-1}$$

is equivalent to

$$\log\big(\min\big(1,\sum_{\alpha\neq 0,\beta\neq 0}|a_{\alpha\beta}(x_0)|^{\frac{1}{|\alpha|+|\beta|}}r\big)\big)^{-1}. \tag{34}$$

Therefore (32) is equivalent to

$$\min(1,|a_{0\beta}(x_0)|r^{|\beta|})\log\big(\min(1,g(x_0)^{|\beta|}r^{|\beta|})\big)^{-1}\leq C \tag{35}$$

for all $\beta \neq 0$, where $g(x_0) = \sum_{\alpha\neq 0,\beta\neq 0} |a_{\alpha\beta}(x_0)|^{\frac{1}{|\alpha|+|\beta|}}$. Recall that $a_{\alpha\beta}(x_0) = a_{\alpha\beta}(0) \neq 0$ for all index pair (α, β) for which there does not exist index (γ, δ) satisfying $a_{\gamma\delta}(0) \neq 0, (\gamma, \delta) \neq (\alpha, \beta), \gamma \geq \alpha$ and $\delta \geq \beta$. Thus

$$g(x_0) \geq C_1 \tag{36}$$

for some constant C_1 independent of $x_0 \in R^n$. Now we can prove

$$|a_{0\beta}(x_0)|^{\frac{1}{|\beta|}} \leq Cg(x_0).$$

Conversely there exists a sequence $x_k \in R^n$ $(k \geq 1)$ such that

$$|a_{0\beta}(x_k)|^{\frac{1}{|\beta|}} \geq kg(x_k) \tag{37}$$

Recall $g(x_k) \geq C_1$ by (36). Hence $|a_{0\beta}(x_k)|^{\frac{1}{|\beta|}} \geq kC_1 > 1$ when k is large enough. Let $r_k = |a_{0\beta}(x_k)|^{-\frac{1}{|\beta|}} < 1$. Then $g(x_k)r_k \leq k^{-1}$ and

$$\min\bigl(1, |a_{0\beta}(_k)|r_k^{|\beta|}\bigr)\log\bigl(\min\bigl(1, g(x_k)^{|\beta|}r_k^{|\beta|}\bigr)\bigr)^{-1} \geq |\beta|\log k,$$

which contradicts to (35). Therefore we prove

Theorem 5. *Let $w \in A_1$ satisfy (33). Then condition 2) in Theorem 2 is equivalent to*

$$\sum_{\beta\neq 0} |a_{0\beta}(x)|^{\frac{1}{|\beta|}} \leq C \sum_{\alpha\neq 0,\beta\neq 0} |a_{\alpha\beta}(x_0)|^{\frac{1}{|\alpha|+|\beta|}}$$

holds for all $x \in R^n$ and some constant C independent of x.

Combining with Theorem 2 and 5, we get Thereom $2'$.

Example 3. Let $Q(x, y) = (x - y)^2 y$. Then $a_{02}(x_0) = x_0, a_{03}(x_0) = 1, a_{10}(x_0) = x_0, a_{11}(x_0) = -2x_0, a_{12}(x_0) = -2, a_{21}(x_0) = 1$ and $a_{\alpha\beta}(x_0) = 0$ otherwise. Furthermore we have

$$\sum_{\alpha\neq 0,\beta\neq 0} |a_{\alpha\beta}(x_0)|^{\frac{1}{|\alpha|+|\beta|}} = 1 + 2^{\frac{1}{3}} + |2x_0|^{\frac{1}{2}},$$

and

$$\sum_{\beta\neq 0} |a_{0\beta}(x_0)|^{\frac{1}{|\beta|}} = 1 + |x_0|^{\frac{1}{2}}.$$

This shows the condition 2) in Theorem $2'$ holds for $Q(x, y) = (x - y)^2 y$ in one spatial dimension.

Now we give a condition for which $\tilde{T}$, hence T, is bounded on $h_w^{1,p}$.

Theorem 6. *Let $w \in A_1$ and $1 < p < \infty$. Assume that $\tilde{T}$ is bounded on L_w^p and furthermore*

$$|\int_{R^n} \tilde{T}a(x)dx| \leq Cw(B(x_0, r))^{-1}r^n(\log r^{-1})^{-1} \tag{38}$$

holds for all $h_w^{1,p}$ atom a with its supporting ball $B(x_0, r)$ having radius $r < 1$. Then $\tilde{T}$ is bounded on $h_w^{1,p}$.

Proof of Theorem 6. Let a be a $h_w^{1,p}$ atom and $B(x_0, r)$ be its supporting ball with radius r and center x_0. Observe that supp$\tilde{T} \subset B(x_0, r+2)$. Therefore

$$\|\tilde{T}a\|_{h_w^{1,p}} \leq C\|\tilde{T}a\|_{p,w} w(B(x_0, 2r))^{1-\frac{1}{p}} \leq C$$

when $r > 1$. Hence the matter reduces to $r(B) < 1$.

Let $h_k = c_k \chi_{B(x_0, 2^{k+1}r)\setminus B(x_0, 2^k r)}(x)$ and $d_k = \int_{R^n} (\tilde{T}a)(x)\chi_{R^n\setminus B(x_0, 2^k r)}(x)dx$, where $c_k = \int \chi_{B(x_0, 2^{k+1}r)\setminus B(x_0, 2^k r)}(x)dx$. Write

$$\begin{aligned}\tilde{T}a =&(\tilde{T}a)\chi_{B(x_0,2r)} + d_1 h_1 \\ &+ \sum_{1\leq k\leq k_0} ((\tilde{T}a)\chi_{B(x_0,2^{k+1}r)\setminus B(x_0,2^k r)} - d_k h_k + d_{k+1}h_{k+1}), \\ =&\tilde{f}_0 + \sum_{1\leq k\leq k_0} \tilde{f}_k\end{aligned}$$

where k_0 is an integer such that $2^{k_0} r \leq 4 \leq 2^{k_0+1}r$. Obviously $\int_{R^n} \tilde{f}_k(x)dx = 0$, supp$\tilde{f}_k \subset B(x_0, 2^{k+1}r)$ and

$$\begin{aligned}\|\tilde{f}_k\|_{p,w} \leq& Cr^{-n}2^{-k}\|a\|_1 w(B(x_0, 2^{k+1}r))^{\frac{1}{p}} \\ \leq& C2^{-k} w(B(x_0, 2^{k+1}))^{\frac{1}{p}-1}\end{aligned}$$

for all $k \geq 1$. On the other hand, we have

$$\text{supp}\tilde{f}_0 \subset B(x_0, 2r), \|\tilde{f}_0\|_{p,w} \leq w(B(x_0, 2r))^{\frac{1}{p}-1}$$

and

$$\begin{aligned}|\int \tilde{f}_0(x)dx| =& |\int (\tilde{T}a)(x)\chi_{B(x_0,2r)}(x)dx + d_1| \\ =& |\int (\tilde{T}a)(x)dx| \leq Cw(B(x_0, r))^{-1}(\text{log}r^{-1})^{-1}.\end{aligned}$$

Therefore we get $\|f_0\|_{h_w^{1,p}} \leq C$ by (7) and

$$\|\tilde{T}a\|_{h_w^{1,p}} \leq \sum_{k\geq 0} \|\tilde{f}_k\|_{h_w^{1,p}} \leq C + C\sum_{k\geq 1} 2^{-k} \leq C.$$

This proved Theorem 6■

Remark. Let K be a kernel of Calderon-Zygmund type. Define

$$T^* f(x) = \int_{R^n} K(x, y)f(y)dy$$

Observe that

$$\int_{R^n} |\int K(x, y)(1-\phi)(|x-y|)a(y)dy|dx \leq Cr\|a\|_1$$

provided $\int_{R^n} a(x)dx = 0$. Hence $\int T^*a(x)dx = 0$ implies (38) and $\tilde{T}$ satisfies (38) when T^* is bounded on weighted Hardy space H_w^1.

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