

TIGHT FRAME OVERSAMPLING AND ITS EQUIVALENCE TO SHIFT-INVARIANCE OF AFFINE FRAME OPERATORS

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ABSTRACT. Let $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2 := L^2(-\infty, \infty)$ generate a tight affine frame with dilation factor M , where $2 \leq M \in \mathbf{Z}$, and sampling constant $b = 1$ (for the zeroth scale level). Then for $1 \leq N \in \mathbf{Z}$, $N \times$ oversampling (or oversampling by N) means replacing the sampling constant 1 by $1/N$. The Second Oversampling Theorem asserts that $N \times$ oversampling of the given tight affine frame generated by Ψ preserves a tight affine frame, provided that $N = N_0$ is relatively prime to M (i.e., $\gcd(N_0, M) = 1$). In this paper, we discuss the preservation of tightness in $mN_0 \times$ oversampling, where $1 \leq m|M$ (i.e., $1 \leq m \leq M$ and $\gcd(m, M) = m$). We also show that tight affine frame preservation in $mN_0 \times$ oversampling is equivalent to the property of shift-invariance with respect to $\frac{1}{mN_0}\mathbf{Z}$ of the affine frame operator Q_{0, N_0} defined on the zeroth scale level.

1. INTRODUCTION AND RESULTS

A family $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2 := L^2(-\infty, \infty)$ is said to generate a *tight affine frame*

$$(1) \quad \mathcal{F}_1 = \{\psi_{l;j,k}(x) := M^{j/2}\psi_l(M^jx - k) : j, k \in \mathbf{Z}, l = 1, \dots, L\}$$

of L^2 with dilation factor M where $2 \leq M \in \mathbf{Z}$ (or for simplicity, we say that Ψ is a tight affine frame of L^2), if there exists a positive constant A , called *frame (bound) constant*, such that

$$(2) \quad \sum_{l=1}^L \sum_{j,k \in \mathbf{Z}} |\langle f, \psi_{l;j,k} \rangle|^2 = A \|f\|_2^2 \quad \text{for all } f \in L^2.$$

Here, the standard notation for L^2 -inner product and L^2 -norm is used. In addition, the definition

$$\hat{f}(\omega) := \int_{-\infty}^{\infty} f(x)e^{-ix\omega} dx, \quad f \in L^1,$$

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of the Fourier transform will be used throughout this paper. Also, for any $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2$ that satisfies the property

$$(3) \quad H(\omega) := \sum_{l=1}^L \sum_{k \in \mathbf{Z}} |\widehat{\psi}_l(\cdot + 2k\pi)|^2 \in L^\infty(-\infty, \infty),$$

we consider the affine frame operators $Q_{0,n}$, $1 \leq n \in \mathbf{Z}$, defined by

$$(4) \quad (Q_{0,n}f)(x) := \sum_{l=1}^L \sum_{k \in \mathbf{Z}} \langle f, \psi_l(\cdot - k/n) \rangle \psi_l(x - k/n), \quad f \in L^2,$$

on the zeroth scale level, and denote $Q_0 := Q_{0,1}$. Clearly, we have

$$(5) \quad \|Q_{0,n}f\|_2 \leq n \|H\|_\infty \|f\|_2.$$

The reason for the terminology of ‘‘affine frame operators’’ is that, by introducing the dilation operator

$$(6) \quad D_j f := f(M^j \cdot), \quad j \in \mathbf{Z},$$

$\Psi \subset L^2$ is a tight affine frame of L^2 (in the sense that the family \mathcal{F}_1 in (1) is a tight frame of L^2), if and only if both (3) and, for some constant $A > 0$,

$$(7) \quad \sum_{j \in \mathbf{Z}} D_{-j} Q_0 D_j = AI$$

are satisfied.

For $M = 2$, a complete characterization of tight affine frames (more precisely, orthonormal wavelets) is discussed in [12], and a generalization from $M = 2$ to arbitrary real dilation $a > 1$ is given in [10]. Generalizations to matrix dilation have been studied in [3, 4, 5] for matrices with integer entries, and most recently in [6] for arbitrary real matrices. Of course, all the eigenvalues of the dilation matrices must have magnitudes greater than one. For the univariate setting with dilation factor $a > 1$, where $a^\gamma =: n_a \in \mathbf{Z}$ for some $1 \leq \gamma \in \mathbf{Z}$, and γ being the smallest such integer exponent, the full characterization in [10] reduces to the following.

Theorem A. *Let $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2$, $b \neq 0$ and $a > 1$ with*

$$(8) \quad \begin{cases} a^\gamma =: n_a \in \mathbf{Z}; \\ a^j \notin \mathbf{Z} \quad \text{for } \gamma > j \in \mathbf{Z} \setminus \{0\}. \end{cases}$$

Then $\{a^{j/2} \psi_l(a^j \cdot - kb) : j, k \in \mathbf{Z}, l = 1, \dots, L\}$ is a tight frame of L^2 with frame constant A , if and only if

$$(9) \quad \sum_{l=1}^L \sum_{j \in \mathbf{Z}} |\widehat{\psi}_l(a^j \omega)|^2 = A \quad \text{a.e. } \omega \in \mathbf{R}$$

and

$$(10) \quad \sum_{l=1}^L \sum_{j=0}^{\infty} \overline{\widehat{\psi}_l(n_a^j \omega)} \widehat{\psi}_l(n_a^j(\omega + 2b^{-1}d\pi)) = 0 \quad \text{a.e. } \omega \in \mathbf{R}$$

for any $d \in \mathbf{Z} \setminus n_a \mathbf{Z}$.

Returning to the special case $a = M \in \mathbf{Z}$ and $b = 1$, let us also recall the following Second Oversampling Theorem established in [7].

Theorem B. *Let $2 \leq M \in \mathbf{Z}$ and assume that $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2$ generates a tight affine frame \mathcal{F}_1 in (1) of L^2 with frame constant $A > 0$. Then for any $2 \leq m \in \mathbf{Z}$ with $\gcd(m, M) = 1$,*

$$(11) \quad \mathcal{F}_m = \{M^{j/2}\psi_l(M^j x - k/m) : j, k \in \mathbf{Z}, l = 1, \dots, L\}$$

is also a tight frame of L^2 , with frame constant mA .

The notion of oversampling affine frames was first introduced in [8], where the result for $M = 2$ was obtained, although [8] has a later publication date than [7], which deals with the theory of affine frames in general. Generalizations to matrix dilation was studied in [9] and, in full generality, in [6]. In addition, oversampling by M^k was discussed in [11] for dilation $M = 2$ for tight affine frames, and arbitrary $2 \leq M \in \mathbf{Z}$ in [13, Chapter 5] for bi-orthogonal wavelets, where $0 < k \in \mathbf{Z}$. A generalization of [11] to matrix dilation was also mentioned in [6]. Observe, however, that since the assumption $\gcd(m, M) = 1$ is violated for $m = M^k, 0 < k \in \mathbf{Z}$, it is necessary to derive additional characterization equations, besides those in (10) for $1 < a = n_a = M \in \mathbf{Z}$ and $b = 1$. For instance, by reducing the matrix consideration in [6] to the scalar setting, a necessary and sufficient condition for tight affine frame preservation in $M^{n_0} \times$ oversampling, with $0 < n_0 \in \mathbf{Z}$, is that

$$(12) \quad \sum_{l=1}^L \sum_{j=0}^{n_0-1} \overline{\widehat{\psi}_l(M^j \omega)} \widehat{\psi}_l(M^j (\omega + 2d\pi)) = 0 \quad \text{a.e. } \omega \in \mathbf{R}$$

for all $d \in \mathbf{Z} \setminus M\mathbf{Z}$.

The objective of this paper is to establish necessary and sufficient conditions for tight affine frame preservation in oversampling by mN_0 , where $\gcd(N_0, M) = 1$, $1 \leq m \leq M$, and $\gcd(m, M) = m$. One of the equivalent conditions, stated in Theorem 2, is that the affine frame operator Q_{0, N_0} in (4), with $n = N_0$, is shift-invariant with respect to $\frac{1}{mN_0}\mathbf{Z}$. To facilitate the statement of our results, we need the notation of the shift operator

$$(13) \quad \tau_y : f \mapsto f(\cdot - y), \quad y \in \mathbf{R}.$$

Theorem 1. *Let $2 \leq M \in \mathbf{Z}$ and assume that $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2$ generates a tight affine frame \mathcal{F}_1 in (1) of L^2 with frame constant $A > 0$. Consider $2 \leq m \leq M$ with $\gcd(m, M) = m$. Then the following statements are equivalent:*

- (i) \mathcal{F}_m in (11) is a tight frame of L^2 ;
- (ii) $\sum_{l=1}^L \overline{\widehat{\psi}_l(\omega)} \widehat{\psi}_l(\omega + 2\pi d) = 0$ a.e. $\omega \in \mathbf{R}$, for all $d \in \mathbf{Z} \setminus m\mathbf{Z}$;
- (iii) Q_0 is shift-invariant with respect to $\frac{1}{m}\mathbf{Z}$, i.e.,

$$(14) \quad \tau_{k/m} Q_0 = Q_0 \tau_{k/m} \quad \text{for all } k \in \mathbf{Z};$$

- (iv) $Q_{0, m} = mQ_0$;

- (v) there exists some 2π -periodic unitary matrix $A(\omega)$ of dimension L such that

$$(15) \quad e^{-i\omega/m} \widehat{\Phi}(\omega) = A(\omega) \widehat{\Phi}(\omega) \quad \text{a.e. } \omega \in \mathbf{R},$$

where $\Phi = [\psi_1 \dots \psi_L]^T$ is the column vector of the functions in Ψ .

The 2π -periodic matrix $A(\omega)$ in (15) of Theorem 1 acts like a bases transform on the shift-invariant space generated by $\psi_l(\cdot - k), l = 1, \dots, L, k \in \mathbf{Z}$. Such an invertible transform of frames was discussed in [1, 13]. In particular, the implication of (v) \implies (i) was given in [13, Theorem 6.1].

In addition, the equivalence of (ii) and (v) was proved in [14] for the special case where $L = 1$, $M = 2$, and that \mathcal{F}_1 is an orthonormal basis of L^2 . But the proof in [14] does not seem to have a simple generalization to the study of frames for $L \geq 1$ and $M \geq 2$. In this regard, our proof of (ii) \implies (v) is fairly technical.

From (iii) of Theorem 1, we see that the range $W_0 := Q_0 L^2$ of the operator Q_0 is shift-invariant over $\frac{1}{m}\mathbf{Z}$. If \mathcal{F}_1 in (1) is an orthonormal basis of L^2 , then as in [14], W_0 is generated by ℓ^2 linear combinations of integer shifts of ψ_1, \dots, ψ_L ; namely,

$$W_0 = \left\{ \sum_{l=1}^L \sum_{j \in \mathbf{Z}} d_l(j) \psi_l(\cdot - j) : \{d_l(j)\}_{j \in \mathbf{Z}} \in \ell^2 \right\},$$

and thus the affine frame operator Q_0 on the zeroth scale level is the projection operator on W_0 , i.e., $Q_0^2 = Q_0$. The interested reader is referred to [2] for a study of closedness of the space generated by ℓ^p linear combinations, $1 \leq p \leq \infty$, of integer shifts in general.

For functions $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2$ and $b \neq 0$, it is easy to check that $\mathcal{F}_{b^{-1}} = \{M^{j/2} \psi_l(M^j \cdot -kb) : j, k \in \mathbf{Z}, l = 1, \dots, L\}$ is a tight frame of L^2 if and only if $\{M^{j/2} \psi_{l,b}(M^j \cdot -k) : j, k \in \mathbf{Z}, l = 1, \dots, L\}$ is a tight frame of L^2 , where $\psi_{l,b} = b^{1/2} \psi_l(b \cdot)$. Let N_0 be a positive integer with $\gcd(N_0, M) = 1$. Then for any tight affine frame $\Psi = \{\psi_1, \dots, \psi_L\}$ of L^2 , it follows from Theorem B that $\{M^{j/2} \psi_{l,N_0^{-1}}(M^j \cdot -k) : j, k \in \mathbf{Z}, l = 1, \dots, L\}$ is also a tight frame of L^2 . Therefore this, together with Theorem 1, give the following extension of Theorem 1.

Theorem 2. *Let $2 \leq M \in \mathbf{Z}$ and assume that $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2$ generates a tight affine frame \mathcal{F}_1 in (1) of L^2 with frame constant $A > 0$. Consider $1 \leq N_0 \in \mathbf{Z}$ with $\gcd(N_0, M) = 1$, and $2 \leq m \leq M$ with $\gcd(m, M) = m$. Then the following statements are equivalent:*

- (i) \mathcal{F}_{mN_0} in (11) is a tight frame of L^2 ;
- (ii) $\sum_{l=1}^L \widehat{\psi}_l(\omega) \widehat{\psi}_l(\omega + 2dN_0\pi) = 0$ a.e. $\omega \in \mathbf{R}$, for all $d \in \mathbf{Z} \setminus m\mathbf{Z}$;
- (iii) the operator Q_{0,N_0} in (4), with $n = N_0$, is shift-invariant with respect to $(mN_0)^{-1}\mathbf{Z}$, i.e.,

$$(16) \quad \tau_{k/(mN_0)} Q_{0,N_0} = Q_{0,N_0} \tau_{k/(mN_0)} \quad \text{for all } k \in \mathbf{Z};$$

- (iv) $Q_{0,mN_0} = mQ_{0,N_0}$;
- (v) there exists some 2π -periodic unitary matrix $A(\omega)$ of dimension L such that

$$(17) \quad e^{-i\omega/(mN_0)} \widehat{\Phi}(\omega) = A(\omega/N_0) \widehat{\Phi}(\omega) \quad \text{a.e. } \omega \in \mathbf{R},$$

where $\widehat{\Phi} = [\psi_1 \ \dots \ \psi_L]^T$ is the column vector of the functions in Ψ .

By applying (7) and (iii) of Theorem 2, we also have the following result for tightness of shifted affine frames.

Corollary 1. *Let M, m and N_0 be as in Theorem 2. If both \mathcal{F}_1 and \mathcal{F}_{mN_0} are tight frames of L^2 , then for any integer s , $\{M^{j/2} \psi_l(M^j \cdot -k/N_0 - s/(mN_0)) : j, k \in \mathbf{Z}, l = 1, \dots, L\}$ is also a tight frame of L^2 .*

Note that for the case where $b^{-1} \in \mathbf{Z}$ is a factor of n_a , condition (10) is equivalent to

$$(18) \quad \begin{aligned} & \sum_{l=1}^L \sum_{j=1}^{\infty} \overline{\widehat{\psi}_l(n_a^j \omega)} \widehat{\psi}_l(n_a^j (\omega + 2\pi d)) \\ & + \sum_{l=1}^L \overline{\widehat{\psi}_l(\omega)} \widehat{\psi}_l(\omega + 2\pi d) \delta_{d, b^{-1}\mathbf{Z}} = 0 \quad \text{a.e. } \omega \in \mathbf{R} \end{aligned}$$

for all $d \in \mathbf{Z} \setminus n_a \mathbf{Z}$, where we set $\delta_{d, b^{-1}\mathbf{Z}} = 1$ for $d \in b^{-1}\mathbf{Z}$ and $\delta_{d, b^{-1}\mathbf{Z}} = 0$ otherwise. Then by (18), Theorem A, and Theorem 1, we have the following result about preservation of tightness for oversampling affine frames with dilation a that satisfies (8) and with m being a factor of n_a .

Theorem 3. *Let $a > 1$ satisfy (8), $m|n_a$, and assume that $\{a^{j/2}\psi_l(a^j \cdot -k) : j, k \in \mathbf{Z}, l = 1, \dots, L\}$ is a tight frame of L^2 . Set $\Phi = [\psi_1 \dots \psi_L]^T$. Then the following statements are equivalent:*

- (i) $\{a^{j/2}\psi_l(a^j \cdot -k/m) : j, k \in \mathbf{Z}, l = 1, \dots, L\}$ is a tight frame of L^2 ;
- (ii)

$$\sum_{l=1}^L \overline{\widehat{\psi}_l(\omega)} \widehat{\psi}_l(\omega + 2\pi d) = 0 \quad \text{a.e. } \omega \in \mathbf{R},$$

for all $d \in \mathbf{Z} \setminus m\mathbf{Z}$;

- (iii) Q_0 is shift-invariant with respect to $\frac{1}{m}\mathbf{Z}$;
- (iv) $Q_{0,m} = mQ_0$;
- (v) there exists some 2π -periodic unitary matrix $A(\omega)$ of dimension L such that

$$e^{-i\omega/m} \widehat{\Phi}(\omega) = A(\omega) \widehat{\Phi}(\omega) \quad \text{a.e. } \omega \in \mathbf{R}.$$

For the case $b^{-1} = n_a^\gamma m \in \mathbf{Z}$ with $0 \leq \gamma \in \mathbf{Z}$ and $m|n_a$, it is easy to check that condition (10) is equivalent to

$$(19) \quad \begin{aligned} & \sum_{l=1}^L \sum_{j=\gamma+1}^{\infty} \overline{\widehat{\psi}_l(n_a^j \omega)} \widehat{\psi}_l(n_a^j (\omega + 2\pi d)) \\ & + \sum_{l=1}^L \overline{\widehat{\psi}_l(n_a^\gamma \omega)} \widehat{\psi}_l(n_a^\gamma (\omega + 2\pi d)) \delta_{d, m\mathbf{Z}} = 0 \quad \text{a.e.} \end{aligned}$$

for all $d \in \mathbf{Z} \setminus n_a \mathbf{Z}$. This together with (7) give the following extension of the equivalence of (i) and (iv) in Theorem 1.

Theorem 4. *Let $2 \leq M \in \mathbf{Z}, 0 \leq \gamma \in \mathbf{Z}, 2 \leq m \leq M$ with $\gcd(m, M) = m$, and assume that $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2$ generates a tight affine frame \mathcal{F}_1 in (1) of L^2 . Then $\mathcal{F}_{M^\gamma m}$ in (11) is a tight frame of L^2 if and only if*

$$(20) \quad \begin{aligned} Q_{0, M^\gamma m} &= M^\gamma m D_\gamma Q_0 D_{-\gamma} \\ &+ m \sum_{j=0}^{\gamma-1} M^j (D_j Q_{0, M^{\gamma-j}} D_{-j} - D_{j+1} Q_{0, M^{\gamma-j}} D_{-j-1}). \end{aligned}$$

2. PRELIMINARY RESULTS

Given a measurable set E , we let $M(E, \ell_0)$ be the space of all sequences $\{\alpha_n(\omega)\}_{n=-\infty}^{\infty}$ of measurable functions on E , such that for almost all $\omega \in E$, $\alpha_n(\omega) = 0$ for all but finitely many $n \in \mathbf{Z}$, i.e.,

$$\bigcup_{N \geq 1} \left(\bigcap_{|n| \geq N} \{\omega \in E : \alpha_n(\omega) = 0\} \right) = E.$$

Here, we say that two measurable sets A and B are equal, denoted by $A = B$, if both $A \setminus B$ and $B \setminus A$ have zero Lebesgue measure. For a sequence $X = \{x_n(\omega) \in \mathbf{C}^L\}_{n=-\infty}^{\infty}$ of vector-valued measurable functions on a measurable set E , let

$$S(E, X) := \left\{ \sum_{n=-\infty}^{\infty} \alpha_n(\omega) x_n(\omega) : \{\alpha_n(\omega)\}_{n=-\infty}^{\infty} \in M(E, \ell_0) \right\}.$$

We remark that $S(E, X)$ is well defined since the summation for n in the definition is taken over a finite set for almost all $\omega \in E$.

For the proof of Theorem 1, we need the following technical lemma.

Lemma 1. *Let $1 \leq L \in \mathbf{Z}$, let E be a measurable set, and let $X = \{x_n(\omega)\}_{n=-\infty}^{\infty}$ be a sequence of vector-valued measurable functions $x_n(\omega) \in \mathbf{C}^L$, $n \in \mathbf{Z}$, on E . Then there exists an L -dimensional square matrix $P(\omega)$ of measurable functions on E such that*

$$(21) \quad P(\omega)v \in S(E, X) \quad \text{for all } v \in \mathbf{C}^L,$$

$$(22) \quad \overline{P(\omega)^T} = P(\omega), \quad P(\omega)^2 = P(\omega) \quad \text{a.e. } \omega \in E,$$

and

$$(23) \quad x_n(\omega) = P(\omega)x_n(\omega) \quad \text{a.e. } \omega \in E \quad \text{for all } n \in \mathbf{Z}.$$

In order to prove Lemma 1, we need the following result.

Lemma 2. *Let $Y = \{y_n(\omega)\}_{n=-\infty}^{\infty}$ and $Z = \{z_n(\omega)\}_{n=-\infty}^{\infty}$ be sequences of vector-valued measurable functions on a measurable set E . If $z_n(\omega) \in S(E, Y)$ for all $n \in \mathbf{Z}$, then $S(E, Z) \subset S(E, Y)$.*

Proof. Write

$$z_n(\omega) = \sum_{k=-\infty}^{\infty} \beta_{n,k}(\omega) y_k(\omega)$$

with $\{\beta_{n,k}(\omega)\}_{k=-\infty}^{\infty} \in M(E, \ell_0)$ for all $n \in \mathbf{Z}$. Then for any sequence $\{\alpha_n(\omega)\}_{n=-\infty}^{\infty}$ of measurable functions in $M(E, \ell_0)$, we have

$$\sum_{n=-\infty}^{\infty} \alpha_n(\omega) z_n(\omega) = \sum_{k=-\infty}^{\infty} \gamma_k(\omega) y_k(\omega)$$

where $\gamma_k(\omega) = \sum_{n=-\infty}^{\infty} \alpha_n(\omega) \beta_{n,k}(\omega)$. Therefore, it suffices to show that

$$(24) \quad \{\gamma_k(\omega)\}_{k=-\infty}^{\infty} \in M(E, \ell_0).$$

For any $0 < C \in \mathbf{R}$, $n \in \mathbf{Z}$, and $1 \leq N, K \in \mathbf{Z}$, set

$$E_{C,K} := \{\omega \in E : |\omega| \leq C \quad \text{and} \quad \gamma_k(\omega) = 0 \quad \text{for all } |k| \geq K\},$$

$$F_{C,N} := \{\omega \in E : |\omega| \leq C \quad \text{and} \quad \alpha_n(\omega) = 0 \quad \text{for all } |n| \geq N\},$$

and

$$G_{C,n,K} := \{\omega \in E : |\omega| \leq C \text{ and } \beta_{n,k}(\omega) = 0 \text{ for all } |k| \geq K\}.$$

It is easy to see that for any $1 \leq K, N \in \mathbf{Z}$ and $0 < C \in \mathbf{R}$,

$$(25) \quad F_{C,N} \cap \left(\bigcap_{|n| \leq N-1} G_{C,n,K} \right) \subset E_{C,K}.$$

By the definition of $M(E, \ell_0)$, the proof of (24) reduces to the existence of an integer $1 \leq K_1 = K_1(\epsilon, C)$ for any pre-assigned positive constants ϵ and C , such that

$$(26) \quad \text{meas}((E \cap \{\omega : |\omega| \leq C\}) \setminus E_{C,K_1}) < \epsilon,$$

where $\text{meas}(A)$ denotes the Lebesgue measure of a measurable set A . Since

$$\{\alpha_n(\omega)\}_{n=-\infty}^{\infty} \in M(E, \ell_0),$$

there exists an integer $N_1 \geq 1$ such that

$$(27) \quad \text{meas}((E \cap \{\omega : |\omega| \leq C\}) \setminus F_{C,N_1}) < \epsilon/3.$$

From the assumption $\{\beta_{n,k}(\omega)\}_{k=-\infty}^{\infty} \in M(E, \ell_0)$ for any $n \in \mathbf{Z}$, we may find an integer $K_1 \geq 1$ so that

$$(28) \quad \text{meas}((E \cap \{\omega : |\omega| \leq C\}) \setminus G_{C,n,K_1}) < \epsilon/(3N_1)$$

for all $n \in \mathbf{Z}$ with $|n| \leq N_1 - 1$. Hence, (26) follows from (25), (27), and (28). \square

Proof of Lemma 1. Set $x_{0,n}(\omega) = x_n(\omega)$, $n \in \mathbf{Z}$, and $X_0 = X$, and define inductively, $e_l(\omega)$ and $X_l = \{x_{l,n}(\omega)\}_{n=-\infty}^{\infty}$, $1 \leq l \leq L$, by

$$(29) \quad e_l(\omega) = \begin{cases} x_{l-1, I(X_{l-1})(\omega)}(\omega) / |x_{l-1, I(X_{l-1})(\omega)}(\omega)| & \text{if } \omega \in E \setminus E_{l-1}, \\ 0 & \text{if } \omega \in E_{l-1}, \end{cases}$$

and

$$(30) \quad x_{l,n}(\omega) = x_{l-1,n}(\omega) - e_l(\omega) \overline{e_l(\omega)}^T x_{l-1,n}(\omega), \quad n \in \mathbf{Z},$$

where

$$(31) \quad E_{l-1} = \{\omega \in E : x_{l-1,n}(\omega) = 0 \text{ for all } n \in \mathbf{Z}\},$$

and where $I(X_{l-1})(\omega)$ is an integer-valued measurable function of $\omega \in E$, so chosen that

$$(32) \quad x_{l-1, I(X_{l-1})(\omega)}(\omega) \neq 0 \text{ for any } \omega \in E \setminus E_{l-1}.$$

For instance, given a sequence $Y = \{y_n(\omega)\}_{n=-\infty}^{\infty}$ of measurable functions, for any $\omega \in \bigcup_{n=-\infty}^{\infty} \text{supp } y_n$, we may choose $I(Y)(\omega)$ to be the smallest integer n such that $y_n(\omega) \neq 0$ and $|y_{n'}(\omega)| = 0$ for all $n' \in \mathbf{Z}$ with $|n'| < |n|$.

Now by (29), (30), and (31), we see that for any $1 \leq l \leq L$ and $n \in \mathbf{Z}$, $e_l(\omega)$ and $x_{l,n}(\omega)$ are measurable functions on E , $e_l(\omega), x_{l,n}(\omega) \in S(E, X_{l-1})$, and

$\overline{x_{l,n}(\omega)}^T e_l(\omega) = 0$. This together with Lemma 2 leads to

$$(33) \quad S(E, X_L) \subset S(E, X_{L-1}) \subset \dots \subset S(E, X_1) \subset S(E, X),$$

and for all $1 \leq l \leq L$,

$$(34) \quad e_l(\omega) \in S(E, X)$$

and

$$(35) \quad \overline{y(\omega)}^T e_l(\omega) = 0 \quad \text{for all } y(\omega) \in S(E, X_l).$$

By (29), (32), (33), and (35), we obtain

$$(36) \quad \overline{e_l(\omega)}^T e_{l'}(\omega) = \begin{cases} \chi_{E \setminus E_{l-1}}(\omega) & \text{if } l = l', \\ 0 & \text{if } l \neq l'. \end{cases}$$

Hence from (34) and (36), we see that the choice of

$$P(\omega) = \sum_{l=1}^L e_l(\omega) \overline{e_l(\omega)}^T$$

gives (21) and (22). To verify (23), we set

$$E_L = \{\omega \in E : x_{L,n}(\omega) = 0 \quad \text{for all } n \in \mathbf{Z}\},$$

and observe that this set contains the sets introduced in (31), namely,

$$E_0 \subset E_1 \subset \dots \subset E_{L-1} \subset E_L.$$

Therefore by (36), we have

$$(37) \quad \overline{Q(\omega)}^T Q(\omega) = I_L \quad \text{a.e. } \omega \in E \setminus E_L,$$

where the L -dimensional square matrix $Q(\omega)$ of measurable functions is defined by $Q(\omega) = [e_1(\omega) \dots e_L(\omega)]$. It is easy to see that $P(\omega) = Q(\omega) \overline{Q(\omega)}^T$ which, together with (37), leads to

$$(38) \quad P(\omega) = I_L \quad \text{a.e. } \omega \in E \setminus E_L.$$

By (30) and (36), we obtain

$$(39) \quad \begin{aligned} x_{L,n}(\omega) &= x_n(\omega) - \sum_{l=1}^L e_l(\omega) \overline{e_l(\omega)}^T x_{l-1,n}(\omega) \\ &= x_n(\omega) - P(\omega) x_n(\omega) \quad \text{a.e. } \omega \in E. \end{aligned}$$

Thus, combining (38) and (39), we have

$$x_{L,n}(\omega) \equiv 0 \quad \text{a.e. } \omega \in E \setminus E_L \quad \text{for all } n \in \mathbf{Z}.$$

This, together with (39) and the definition of the set E_L , imply (23). \square

3. PROOF OF THE MAIN RESULTS

We only give the proof of Theorem 1, since the other three theorems and Corollary 1 follow accordingly. We divide the proof of Theorem 1 into the following steps: (i) \implies (ii) \implies (iii) \implies (iv) \implies (i), and (v) \implies (ii) \implies (v). The proof of (ii) \implies (v) is the most technical part in our proof, and will be dealt with last.

Set $\mathcal{G} = \{\psi_l(\cdot - n/m) : 1 \leq l \leq L, 0 \leq n \leq m-1\}$. Then for \mathcal{F}_m in (11), we have

$$(40) \quad \mathcal{F}_m = \{M^{j/2}g(M^j \cdot -k) : g \in \mathcal{G}, j, k \in \mathbf{Z}\}.$$

By direct computation, we obtain

$$\sum_{g \in \mathcal{G}} \sum_{j \in \mathbf{Z}} |\widehat{g}(M^j \omega)|^2 = m \sum_{l=1}^L \sum_{j \in \mathbf{Z}} |\widehat{\psi}_l(M^j \omega)|^2,$$

and for all integers d ,

$$(41) \quad \begin{aligned} & \sum_{g \in \mathcal{G}} \sum_{j=0}^{\infty} \overline{\widehat{g}(M^j \omega)} \widehat{g}(M^j(\omega + 2\pi d)) \\ &= \sum_{n=0}^{m-1} \sum_{l=1}^L \sum_{j=0}^{\infty} \overline{\widehat{\psi}_l(M^j \omega)} \widehat{\psi}_l(M^j(\omega + 2\pi d)) e^{i2\pi d n M^j / m} \\ &= m \sum_{l=1}^L \sum_{j=1}^{\infty} \overline{\widehat{\psi}_l(M^j \omega)} \widehat{\psi}_l(M^j(\omega + 2\pi d)) + m \sum_{l=1}^L \overline{\widehat{\psi}_l(\omega)} \widehat{\psi}_l(\omega + 2\pi d) \delta_{d, m\mathbf{Z}}. \end{aligned}$$

Therefore, by (40), (41), the assumption (i), and Theorem A, we have

$$(42) \quad \sum_{l=1}^L \sum_{j=1}^{\infty} \overline{\widehat{\psi}_l(M^j \omega)} \widehat{\psi}_l(M^j(\omega + 2\pi d)) + \sum_{l=1}^L \overline{\widehat{\psi}_l(\omega)} \widehat{\psi}_l(\omega + 2\pi d) \delta_{d, m\mathbf{Z}} = 0$$

for any $d \in \mathbf{Z} \setminus M\mathbf{Z}$. Similarly by Theorem A, (40) and (41) with $m = 1$, and the assumption that \mathcal{F}_1 is a tight frame of L^2 , we also have

$$(43) \quad \sum_{l=1}^L \sum_{j=0}^{\infty} \overline{\widehat{\psi}_l(M^j \omega)} \widehat{\psi}_l(M^j(\omega + 2\pi d)) = 0$$

for any $d \in \mathbf{Z} \setminus M\mathbf{Z}$. Hence (ii) follows from (42) and (43).

Taking the Fourier transform on both sides of (4) leads to

$$(Q_0 f)^\wedge(\omega) = \sum_{l=1}^L \sum_{k \in \mathbf{Z}} \widehat{f}(\omega + 2k\pi) \overline{\widehat{\psi}_l(\omega + 2k\pi)} \widehat{\psi}_l(\omega), \quad j \geq 0.$$

Hence, for any $f \in L^2$ and $d \in \mathbf{Z}$,

$$\begin{aligned} & (\tau_{d/m} Q_0 \tau_{-d/m} f - Q_0 f)^\wedge(\omega) \\ &= \sum_{l=1}^L \sum_{k \in \mathbf{Z}} \widehat{f}(\omega + 2k\pi) \overline{\widehat{\psi}_l(\omega + 2k\pi)} \widehat{\psi}_l(\omega) (e^{2idk\pi/m} - 1). \end{aligned}$$

This, together with assumption (ii), lead to the shift-invariance of the operator Q_0 with respect to $\frac{1}{m}\mathbf{Z}$, and this establishes (ii) \implies (iii).

To prove (iii) \implies (iv), note that

$$(44) \quad Q_{0,m} = \sum_{k=0}^{m-1} \tau_{k/m} Q_0 \tau_{-k/m}$$

from its definition (4). Hence (iv) follows from (14) and (44).

To prove (iv) \implies (i), note that since \mathcal{F}_1 is a tight frame, the function H in (3) is bounded. Therefore by (7), \mathcal{F}_m in (11) is a tight frame if and only if

$$(45) \quad \sum_{j \in \mathbf{Z}} D_{-j} Q_{0,m} D_j = AI$$

for some positive constant A . Hence (i) follows from (45), the tight frame assumption on \mathcal{F}_1 , and the hypothesis $Q_{0,m} = mQ_0$.

Let (v) be satisfied. Then we have

$$e^{i(\omega+2\pi d)/m} \widehat{\Phi}(\omega + 2d\pi) = A(\omega) \widehat{\Phi}(\omega + 2d\pi)$$

for any integer d . Thus,

$$\begin{aligned} e^{2di\pi/m} \overline{\widehat{\Phi}(\omega)^T} \widehat{\Phi}(\omega + 2\pi d) &= \overline{\widehat{\Phi}(\omega)^T} \overline{A(\omega)^T} A(\omega) \widehat{\Phi}(\omega + 2\pi d) \\ &= \overline{\widehat{\Phi}(\omega)^T} \widehat{\Phi}(\omega + 2\pi d), \end{aligned}$$

which implies (ii).

Finally, we come to the proof of (ii) \implies (v). Let $P(\omega)$ be the L -dimensional square matrix of measurable functions on $[-m\pi, m\pi)$ in Lemma 1, with $E = [-m\pi, m\pi)$ and $X = \{\widehat{\Phi}(\omega + 2mn\pi)\}_{n=-\infty}^{\infty}$. For notational convenience, we denote the $2m\pi$ -periodization of $P(\omega)$ again by $P(\omega)$. Then by Lemma 1, we have

$$(46) \quad P(\omega)v \in S(\mathbf{R}, \mathcal{G}_m) \quad \text{for all } v \in \mathbf{C}^L,$$

$$(47) \quad P(\omega + 2m\pi) = P(\omega), \quad \overline{P(\omega)^T} = P(\omega), \quad P(\omega)^2 = P(\omega) \quad \text{a.e. } \omega \in \mathbf{R},$$

and

$$(48) \quad \widehat{\Phi}(\omega) = P(\omega) \widehat{\Phi}(\omega) \quad \text{a.e. } \omega \in \mathbf{R},$$

where $\mathcal{G}_m = \{\widehat{\Phi}(\omega + 2mn\pi)\}_{n=-\infty}^{\infty}$. By (46) and assumption (ii), we obtain

$$(49) \quad \overline{P(\omega)^T} P(\omega + 2d\pi) = 0 \quad \text{for all } d \in \mathbf{Z} \setminus m\mathbf{Z}.$$

Therefore, the function

$$A(\omega) := I_L + \sum_{d=0}^{m-1} (e^{i(\omega+2\pi d)/m} - 1) P(\omega + 2\pi d)$$

is an L -dimensional square matrix of measurable functions that satisfies $A(\omega + 2\pi) = A(\omega)$ by (47), and

$$\begin{aligned}
& \overline{A(\omega)^T} A(\omega) \\
= & I_L + \sum_{d=0}^{m-1} (e^{i(\omega+2\pi d)/m} - 1)P(\omega + 2\pi d) + \sum_{d=0}^{m-1} (e^{-i(\omega+2\pi d)/m} - 1)\overline{P(\omega + 2\pi d)^T} \\
& + \left(\sum_{d=0}^{m-1} (e^{-i(\omega+2\pi d)/m} - 1)\overline{P(\omega + 2\pi d)^T} \right) \\
& \times \left(\sum_{d=0}^{m-1} (e^{i(\omega+2\pi d)/m} - 1)P(\omega + 2\pi d) \right) \\
= & I_L + \sum_{d=0}^{m-1} (e^{i(\omega+2\pi d)/m} - 1)P(\omega + 2\pi d) + \sum_{d=0}^{m-1} (e^{-i(\omega+2\pi d)/m} - 1)P(\omega + 2\pi d) \\
& + \sum_{d=0}^{m-1} (e^{-i(\omega+2\pi d)/m} - 1) \times (e^{i(\omega+2\pi d)/m} - 1)P(\omega + 2\pi d) \\
= & I_L,
\end{aligned}$$

where we have used (47) and (49) to obtain the second equality. Moreover, by (47), (48), and (49), we get

$$\begin{aligned}
A(\omega)\widehat{\Phi}(\omega) &= \widehat{\Phi}(\omega) + \sum_{d=0}^{m-1} (e^{i(\omega+2\pi d)/m} - 1)P(\omega + 2\pi d)\widehat{\Phi}(\omega) \\
&= \widehat{\Phi}(\omega) + \sum_{d=0}^{m-1} (e^{i(\omega+2\pi d)/m} - 1)\overline{P(\omega + 2\pi d)^T}P(\omega)\widehat{\Phi}(\omega) \\
&= e^{i\omega/m}\widehat{\Phi}(\omega).
\end{aligned}$$

This completes the proof of (v), and hence Theorem 1. \square

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