The *abc*-problem for Gabor systems

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ABSTRACT. A long standing problem in Gabor theory is to identify time-frequency shifting lattices $a\mathbb{Z} \times b\mathbb{Z}$ and ideal window functions χ_I on intervals I of length c such that $\{e^{-2\pi i n b t}\chi_I(t - ma) : (m, n) \in \mathbb{Z} \times \mathbb{Z}\}$ are Gabor frames for the space of all square-integrable functions on the real line. In this paper, we create a time-domain approach for Gabor frames, introduce novel techniques involving invariant sets of non-contractive and non-measure-preserving transformations on the line, and provide a complete answer to the above *abc*-problem for Gabor systems.

Contents

Preface

A Gabor system generated by a window function ϕ and a rectangular lattice $a\mathbb{Z} \times b\mathbb{Z}$ is given by

$$\mathcal{G}(\phi, a\mathbb{Z} \times b\mathbb{Z}) := \{e^{-2\pi i n b t}\phi(t - ma) : \ (m, n) \in \mathbb{Z} \times \mathbb{Z}\}.$$

Gabor theory could date back to the completeness claim in 1932 by von Neumann and the expansion conjecture in 1946 by Gabor. Gabor theory has close links to Fourier analysis, operator algebra and complex analysis, and it has been applied in a wide range of mathematical and engineering fields.

One of fundamental problems in Gabor theory is to identify window functions ϕ and time-frequency shift lattices $a\mathbb{Z} \times b\mathbb{Z}$ such that $\mathcal{G}(\phi, a\mathbb{Z} \times b\mathbb{Z})$ are Gabor frames for the space $L^2(\mathbb{R})$ of all square-integrable functions on the real line $\mathbb{R}$. Denote by $\mathcal{R}(\phi)$ the set of density parameter pairs (a, b) such that $\mathcal{G}(\phi, a\mathbb{Z} \times b\mathbb{Z})$ is a frame for $L^2(\mathbb{R})$. The range $\mathcal{R}(\phi)$ is an open domain on the plane for window functions ϕ in Feichtinger algebra, but that range is fully known surprisingly only for small numbers of window functions, including the Gaussian window function and totally positive window functions.

The ranges $\mathcal{R}(\phi)$ associated with general window functions ϕ, especially outside Feichtinger algebra, are almost nothing known and Janssen's tie suggests that they could be arbitrarily complicated. Ideal window functions χ_I on intervals I are important examples of such window functions and they have received special attentions. In this paper, we answer that range problem by providing a full classification of triples (a, b, c) for which $\mathcal{G}(\chi_I, a\mathbb{Z} \times b\mathbb{Z})$ generated by the ideal window function χ_I on an interval I of length c is a Gabor frame for $L^2(\mathbb{R})$, i.e., the *abc*-problem for Gabor systems. For an interval I of length c, we show that the range $\mathcal{R}(\chi_I)$ of density parameter pairs (a, b) is neither open nor path-connected, and it is a dense subset of the open region below the equilateral hyperbola $ab = 1$ and on the left of the vertical line $a = c$.

To study the range $\mathcal{R}(\chi_I)$ of density parameter pairs (a, b) associated with ideal window function χ_I, we normalize the interval I to $[0, c)$ and the frequency parameter b to 1. This reduces the *abc*-problem for Gabor systems to finding out all pairs (a, c) of time-spacing and window-size parameters such that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ are Gabor frames.

Denote by $\mathcal{B}^0$ the set of all binary vectors $\mathbf{x} := (\mathbf{x}(\lambda))_{\lambda \in \mathbb{Z}}$ with $\mathbf{x}(0) = 1$ and $\mathbf{x}(\lambda) \in \{0, 1\}$ for all $\lambda \in \mathbb{Z}$, and let $\mathcal{D}_{a,c}$ contain all real numbers t for which there exists a binary solution $\mathbf{x} \in \mathcal{B}^0$ to the following infinite-dimensional linear system

$$\sum_{\lambda \in \mathbb{Z}} \chi_{[0,c)}(t - \mu + \lambda)\mathbf{x}(\lambda) = 2, \quad \mu \in a\mathbb{Z}.$$

We create a time-domain approach to Gabor frames and show that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})$ is a Gabor frame if and only if $\mathcal{D}_{a,c}=\emptyset$.

We do not apply the above empty set characterization of Gabor frames directly, instead we introduce another set $\mathcal{S}_{a,c}$ of real numbers t for which there exists a binary solution $\mathbf{x}\in\mathcal{B}^0$ to another infinite-dimensional linear system

$$\sum_{\lambda\in\mathbb{Z}}\chi_{[0,c)}(t-\mu+\lambda)\mathbf{x}(\lambda)=1,\quad \mu\in a\mathbb{Z}.$$

The set $\mathcal{S}_{a,c}$ is a supset of $\mathcal{D}_{a,c}$ and conversely $\mathcal{D}_{a,c}$ can be obtained from $\mathcal{S}_{a,c}$ by some set operations. Most importantly, $\mathcal{S}_{a,c}$ is a maximal set that is invariant under the transformation $R_{a,c}$ and that has empty intersection with its black hole $[\max(c_0+a-1,0),\min(c_0-a,0)+a)+a\mathbb{Z}$, where

$$R_{a,c}(t):=\begin{cases} t+\lfloor c\rfloor & \text{if } t\in[\min(c_0-a,0),0)+a\mathbb{Z}\\ t+\lfloor c\rfloor+1 & \text{if } t\in[0,\max(c_0+a-1,0))+a\mathbb{Z}\\ t & \text{if } t\in[\max(c_0+a-1,0),\min(c_0-a,0)+a)+a\mathbb{Z}.\end{cases}$$

The piecewise linear transformation $R_{a,c}$ is non-contractive on the whole line and it does not satisfy standard requirement for Hutchinson's remarkable construction of maximal invariant sets. In this paper, we show that Hutchinson's construction works for the maximal invariant set $\mathcal{S}_{a,c}$ of the transformation $R_{a,c}$, and even more surprisingly it requires only finite iterations, i.e.,

$$\mathcal{S}_{a,c}=(R_{a,c})^D(\mathbb{R})\backslash([\max(c_0+a-1,0),\min(c_0-a,0)+a)+a\mathbb{Z})$$

for some nonnegative integer D, whenever it is not an empty set. Therefore complement of the set $\mathcal{S}_{a,c}$ is a periodic set with its restriction on one period consisting of finitely many holes (left-closed right-open intervals). So we may squeeze out those holes on the line and then reconnect their endpoints. This holes-removal surgery yields an isomorphism from the set $\mathcal{S}_{a,c}$ to the line with marks (image of holes). More importantly, restriction of the nonlinear transformation $R_{a,c}$ onto the set $\mathcal{S}_{a,c}$ becomes a linear transformation on a line with marks, and interestingly the set of marks forms a cyclic group for $a\in\mathbb{Q}$.

After exploring deep about locations and sizes of holes, we show that hole-removal surgery is reversible and the set $\mathcal{S}_{a,c}$ can be obtained from the real line by putting marks at appropriate positions and then inserting holes of appropriate sizes at marked positions. The above delicate and complicated augmentation operation leads to parametrization of the set $\mathcal{S}_{a,c}$ via two nonnegative integers for $a\notin\mathbb{Q}$ and via four nonnegative integers for $a\in\mathbb{Q}$. This parametrization yields our complete answer to the *abc*-problem for Gabor systems.

The piecewise linear transformation $R_{a,c}$ is non-measure-preserving on the whole line, but certain ergodic theorem could be established. As it involves fourteen cases (and few more subcases) for full classification of triples (a,b,c) such that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times b\mathbb{Z})$ is a Gabor frame for $L^2(\mathbb{R})$, an algorithm is proposed for that intricate verification. The *abc*-problem for Gabor systems has also close link to the stable recovery problem of rectangular signals f in the shift-invariant space

$$V_2(\chi_{[0,c)},\mathbb{Z}/b):=\Big\{\sum_{\lambda\in\mathbb{Z}/b}d(\lambda)\chi_{[0,c)}(t-\lambda):\ \sum_{\lambda\in\mathbb{Z}/b}|d(\lambda)|^2<\infty\Big\}$$

from their equally-spaced samples $f(t_0+\mu), \mu\in a\mathbb{Z}$, with arbitrary initial sampling position t_0.

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Xin-Rong Dai and Qiyu Sun

CHAPTER 1

Introduction

Let $L^2 := L^2(\mathbb{R})$ be the space of all square-integrable functions on the real line $\mathbb{R}$ with the inner product and norm on L^2 denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|_2$ respectively. A *frame* for L^2 is a collection $\mathcal{F}$ of functions in L^2 satisfying

$$0 < A := \inf_{\|f\|_2=1} \Big(\sum_{\phi \in \mathcal{F}} |\langle f, \phi \rangle|^2 \Big)^{1/2} \le \sup_{\|f\|_2=1} \Big(\sum_{\phi \in \mathcal{F}} |\langle f, \phi \rangle|^2 \Big)^{1/2} =: B < \infty.$$

The constants A and B are known as lower and upper frame bounds of the frame $\mathcal{F}$. Frames for a Hilbert space were introduced in 1952 by Duffin and Schaeffer in the context of nonharmonic Fourier series [**11, 17**], and the notion of frames has been extended to p-frames, Banach frames, g-frames and fusion frames [**3, 9, 10, 45**]. The reader may refer to the textbook by Christensen [**12**] and the survey by Casazza [**7**] for the extensive literature and historical remarks.

The *Gabor system* (also called *Weyl-Heisenberg system*) generated by a window function $\phi \in L^2$ and a rectangular lattice $a\mathbb{Z} \times b\mathbb{Z}$ is defined by

$$\mathcal{G}(\phi, a\mathbb{Z} \times b\mathbb{Z}) := \{ e^{-2\pi inbt} \phi(t - ma) : \ (m,n) \in \mathbb{Z} \times \mathbb{Z} \}; \tag{1.1}$$

and a *Gabor frame* is a Gabor system that forms a frame for L^2, i.e., there exist positive constants A and B such that

$$A\|f\|_2 \le \Big(\sum_{m,n \in \mathbb{Z}} |\langle f, e^{-2\pi inb\cdot} \phi(\cdot - ma) \rangle|^2 \Big)^{1/2} \le B\|f\|_2 \ \text{ for all } f \in L^2.$$

Gabor frames have links to operator algebra and complex analysis, and they have been applied in a wide range of mathematical and engineering field, especially suitable for applications involving time-dependent frequency content [**6, 12, 20, 21, 24, 25, 26, 29**]. The history of Gabor theory could date back to the completeness claim in 1932 by von Neumann on the completeness of the Gabor system $\mathcal{G}(\sqrt[4]{2}\exp(-t^2), \mathbb{Z} \times \mathbb{Z})$ generated by the Gaussian window [**35**, p. 406], and the expansion conjecture in 1946 by Gabor [**19**, Eq. 1.29] on the expansion of the Gabor system $\mathcal{G}(\sqrt[4]{2}\exp(-t^2), \mathbb{Z} \times \mathbb{Z})$ for all square-integrable functions in his fundamental paper. Gabor theory become widely studied after the landmark paper [**16**] in 1986 by Daubechies, Grossmann and Meyer, where they proved that given any positive density parameters a, b satisfying $ab < 1$ there exists a compactly supported smooth function ϕ such that $\mathcal{G}(\phi, a\mathbb{Z} \times b\mathbb{Z})$ is a Gabor frame, see the textbook by Gröchenig [**20**] and the surveys by Janssen [**29**] and Heil [**26**] for more detailed and updated information about Gabor theory and applications.

One of fundamental problems in Gabor theory is to identify window functions and time-frequency shift sets such that the corresponding Gabor systems are Gabor frames. Given a window function $\phi \in L^2$ and a rectangular lattice $a\mathbb{Z} \times b\mathbb{Z}$, a well-known necessary condition for the Gabor system $\mathcal{G}(\phi, a\mathbb{Z} \times b\mathbb{Z})$ to be a Gabor frame, obtained via Banach algebra technique, is that the density parameters a and b satisfy $ab \le 1$ [**5, 13, 28, 33, 37**]. Two other basic necessary conditions for the Gabor system $\mathcal{G}(\phi, a\mathbb{Z} \times b\mathbb{Z})$ to be a Gabor frame are

$$0 < \inf_{t\in\mathbb{R}} \sum_{m\in\mathbb{Z}} |\phi(t - ma)|^2 \le \sup_{t\in\mathbb{R}} \sum_{m\in\mathbb{Z}} |\phi(t - ma)|^2 < \infty, \tag{1.2}$$

and

$$0 < \inf_{\xi\in\mathbb{R}} \sum_{n\in\mathbb{Z}} |\hat{\phi}(\xi - nb)|^2 \le \sup_{\xi\in\mathbb{R}} \sum_{n\in\mathbb{Z}} |\hat{\phi}(\xi - nb)|^2 < \infty \tag{1.3}$$

[**13, 14**]. Here the Fourier transform $\hat{f}$ is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i t\xi} dt$$

for an integrable function f on the real line $\mathbb{R}$, with standard extension to tempered distributions, including square-integrable functions. But the above three necessary conditions on window functions and density parameters are far from providing an answer to the fundamental problem.

Denote by $\mathcal{R}(\phi)$ the range of positive density parameter pairs (a, b) such that the Gabor system $\mathcal{G}(\phi, a\mathbb{Z}\times b\mathbb{Z})$ is a frame for L^2. Then the first necessary condition can be rewritten as

$$\mathcal{R}(\phi) \subset \{(a, b) : ab \le 1\} \tag{1.4}$$

for arbitrary window function ϕ. An important result proved by Feichtinger and Kaiblinger [**18**] states that the range $\mathcal{R}(\phi)$ is an open domain for a window function ϕ in Feichtinger's algebra [**18**], but it is fully characterized unexpectedly only for few families of window functions ϕ [**31, 32, 34, 39, 40**], including recent significant advance made by Gröchenig and Stöckler for a totally positive function of finite type [**22**].

The Gaussian window $\sqrt[4]{2}\exp(-\pi t^2)$ and the "ideal" window χ_I (the characteristic function) on an interval I have received special attention. For the Gaussian window, it is conjectured by Daubechies and Grossmann [**15**] and later proved independently by Lyubarskii [**34**] and by Seip and Wallsten [**39, 40**] via complex analysis technique that the range of positive density parameters a and b is the open domain $\{(a, b) : ab < 1\}$. For the ideal window on an interval I, it is known that $\mathcal{G}(\chi_I, a\mathbb{Z}\times b\mathbb{Z})$ is a Gabor frame if and only if $\mathcal{G}(\chi_{I+d}, a\mathbb{Z}\times b\mathbb{Z})$ is a Gabor frame for every $d \in \mathbb{R}$. Due to the above shift-invariance of Gabor frames, the interval I can be assumed to be left-closed and right-open, and to have zero as its left endpoint, i.e.,

$$I = [0, c) \quad \text{for some} \quad c > 0.$$

Thus the range problem for the ideal window on an interval reduces to the so-called abc**-problem for Gabor systems**: *given a triple (a, b, c) of positive numbers, determine whether $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times b\mathbb{Z})$ is a Gabor frame.*

Applying (1.2) to the ideal window function χ_E on a bounded set E yields the covering property

$$\cup_{m\in\mathbb{Z}}(E + ma) = \mathbb{R}$$

which becomes $a \le c$ for the the ideal window on the interval $[0, c)$ [**8**]. Similarly applying (1.3) for the ideal window on the interval $[0, c)$ shows that bc is not an integer larger than or equal to 2. The above requirements together with (1.4) imply that

$$\mathcal{R}(\chi_{[0,c)}) \subset \{(a,b): \ ab \le 1, a \le c, bc \notin \mathbb{Z}\backslash\{1\}\}.$$

But there is a large gap between the range $\mathcal{R}(\chi_{[0,c)})$ and its supset $\{(a,b): \ ab \le 1, a \le c, bc \notin \mathbb{Z}\backslash\{1\}\}$. In fact the range could be arbitrarily complicated, cf. the famous Janssen's tie [**23, 30**]. In this paper, we introduce a discontinuous periodic transformation, study its two invariant sets, and use them to provide a complete answer to the *abc*-problem for Gabor systems.

Notation: For a real number t, we let $t_+ = \max(t, 0)$, $t_- = \min(t, 0) = t - t_+$, $\lfloor t \rfloor$ be the largest integer not greater than t, $\lceil t \rceil$ the smallest integer not less than t, $\mathrm{sgn}(t)$ be the sign of t, and $\mathbf{t} := (\cdots, t, t, t, \cdots)^T$ be the column vector whose entries take value t. Specially for the window size parameter c, we let $c_0 := c - \lfloor c \rfloor$ be the fractional part of the window size. For a set E, we denote by χ_E the characteristic function on it, by $|E|$ its Lebesgue measure, and by $\#(E)$ its cardinality respectively. We also denote by $\mathbb{Q}$ the set of rational numbers; by $\gcd(s,t)$ the greatest common divisor such that $s/\gcd(s,t), t/\gcd(s,t) \in \mathbb{Z}$ for any given s and t in a lattice $r\mathbb{Z}$ with $r > 0$; by $\mathbf{A}^T$ the transpose of a matrix (vector) $\mathbf{A}$; and by $N(\mathbf{A})$ the null space of a matrix $\mathbf{A}$. In this paper, we also let $\ell^2 := \ell^2(\Lambda)$ be the space of all square-summable vectors $\mathbf{z} := (\mathbf{z}(\lambda))_{\lambda \in \Lambda}$ on a given index set Λ, with standard norm $\|\cdot\|_2 := \|\cdot\|_{\ell^2(\Lambda)}$;

$$\mathcal{B} := \{(\mathbf{x}(\lambda))_{\lambda \in \mathbb{Z}}: \ \mathbf{x}(\lambda) \in \{0,1\} \text{ for all } \lambda \in \mathbb{Z}\}$$

contain all binary column vectors whose components taking values either zero or one; and

$$\mathcal{B}^0 := \{(\mathbf{x}(\lambda))_{\lambda \in \mathbb{Z}} \in \mathcal{B}: \ \mathbf{x}(0) = 1\}$$

be the set of all binary vectors taking value one at the origin.

1.1. Outlines

Given a triple (a,b,c) of positive numbers, one may verify that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times b\mathbb{Z})$ is a Gabor frame if and only if $\mathcal{G}(\chi_{[0,bc)}, (ab)\mathbb{Z} \times \mathbb{Z})$ is. By the above dilation-invariance, we can normalize the frequency-spacing parameter b to 1. Thus the *abc*-problem for Gabor systems reduces to finding out *all pairs* (a,c) *of positive numbers of time-spacing and window-size parameters such that* $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ *are Gabor frames.*

It is known that the Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})$ associated with a pair (a,c) satisfying either $a \ge 1$ or $c \le 1$ is a Gabor frame if and only if $c = 1$ and $0 < a \le 1$, see for instance [**16, 23, 30**] and also Theorem 7.1. So it remains to consider the *abc*-problem for Gabor systems with triples (a,b,c) satisfying

$$0 < a < 1 < c \quad \text{and} \quad b = 1.$$

Define infinite matrices $\mathbf{M}_{a,c}(t), t \in \mathbb{R}$, by

$$\mathbf{M}_{a,c}(t) := \big(\chi_{[0,c)}(t - \mu + \lambda)\big)_{\mu \in a\mathbb{Z}, \lambda \in \mathbb{Z}}, \ t \in \mathbb{R}. \tag{1.5}$$

The infinite matrices $\mathbf{M}_{a,c}(t), t \in \mathbb{R}$, in (1.5) have been used by Ron and Shen in [**38**] to characterize frame property for the Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$. In particular, $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})$ is a Gabor frame if and only if

$$0<\inf_{t\in\mathbb{R}}\inf_{\|\mathbf{z}\|_2=1}\|\mathbf{M}_{a,c}(t)\mathbf{z}\|_2\le\sup_{t\in\mathbb{R}}\sup_{\|\mathbf{z}\|_2=1}\|\mathbf{M}_{a,c}(t)\mathbf{z}\|_2<\infty, \tag{1.6}$$

see also Theorem 2.4. We observe that infinite matrices $\mathbf{M}_{a,c}(t), t\in\mathbb{R}$, in (1.5) are binary, their rows contain $\lfloor c\rfloor+\{0,1\}$ consecutive ones, and they have the following elementary properties about frequency shifts in $\mathbb{Z}$ and time shifts in $a\mathbb{Z}$:

$$\mathbf{M}_{a,c}(t-\lambda')\mathbf{z}=\mathbf{M}_{a,c}(t)\tau_{\lambda'}\mathbf{z}\ \text{ for all } \lambda'\in\mathbb{Z} \tag{1.7}$$

and

$$\mathbf{M}_{a,c}(t-\mu')\mathbf{z}=\tau_{\mu'}(\mathbf{M}_{a,c}(t)\mathbf{z})\ \text{ for all } \mu'\in a\mathbb{Z}, \tag{1.8}$$

where for $\alpha>0$, the shift-operators $\tau_{\nu'}, \nu'\in\alpha\mathbb{Z}$, are defined by

$$\tau_{\nu'}\mathbf{z}:=(\mathbf{z}(\nu+\nu'))_{\nu\in\alpha\mathbb{Z}}\ \text{ for }\ \mathbf{z}:=(\mathbf{z}(\nu))_{\nu\in\alpha\mathbb{Z}}.$$

Using special structures for infinite matrices in (1.5), we establish the equivalence between their uniform stability (1.6) and the non-existence of binary solutions for the infinite-dimensional linear systems

$$\mathbf{M}_{a,c}(t)\mathbf{x}=\mathbf{2},\ t\in\mathbb{R}, \tag{1.9}$$

or equivalently the empty set property for the set $\mathcal{D}_{a,c}$ defined by

$$\mathcal{D}_{a,c}:=\big\{t\in\mathbb{R}:\ \mathbf{M}_{a,c}(t)\mathbf{x}=\mathbf{2} \text{ for some binary vectors } \mathbf{x}\in\mathcal{B}^0\big\}, \tag{1.10}$$

see Theorem 2.1.

Any binary vector $\mathbf{x}\in\mathcal{B}^0$ satisfying $\mathbf{M}_{a,c}(t)\mathbf{x}=\mathbf{2}$ can be written as the sum of two binary vectors $\mathbf{x}_1\in\mathcal{B}^0$ and $\mathbf{x}_2\in\mathcal{B}\backslash\mathcal{B}^0$ such that

$$\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2\quad\text{and}\quad\mathbf{M}_{a,c}(t)\mathbf{x}_1=\mathbf{M}_{a,c}(t)\mathbf{x}_2=\mathbf{1}, \tag{1.11}$$

see Lemma 2.6. The binary vector $\mathbf{x}_1\in\mathcal{B}^0$ in the above decomposition (1.11) is *uniquely* determined by t (see Lemma 3.12), and multiple binary vector solutions $\mathbf{x}$ could exist for the linear system $\mathbf{M}_{a,c}(t)\mathbf{x}=\mathbf{2}, t\in\mathbb{R}$. So we consider binary vector solutions $\mathbf{x}\in\mathcal{B}^0$ to the linear system

$$\mathbf{M}_{a,c}(t)\mathbf{x}=\mathbf{1},\ t\in\mathbb{R}, \tag{1.12}$$

and define

$$\mathcal{S}_{a,c}:=\big\{t\in\mathbb{R}:\ \mathbf{M}_{a,c}(t)\mathbf{x}=\mathbf{1} \text{ for some vector } \mathbf{x}\in\mathcal{B}^0\big\}. \tag{1.13}$$

The sets $\mathcal{D}_{a,c}$ and $\mathcal{S}_{a,c}$ are closely related:

1) they are periodic sets with period a by the time-shift property (1.8);
2) $\mathcal{S}_{a,c}$ is a supset of $\mathcal{D}_{a,c}$ by the decomposition (1.11); and
3) $\mathcal{D}_{a,c}$ can be obtained from $\mathcal{S}_{a,c}$ via some set operations, see Theorem 2.3.

For pairs (a,c) of positive numbers satisfying either $c_0:=c-\lfloor c\rfloor\ge a$ or $c_0\le 1-a$, we can construct the set $\mathcal{D}_{a,c}$ explicitly by applying the above results about the sets $\mathcal{D}_{a,c}$ and $\mathcal{S}_{a,c}$, and then we can determine whether the corresponding Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})$ is a frame, see Theorem 7.2. Thus the *abc*-problem for Gabor systems reduces further to triples (a,b,c) satisfying

$$0<a<1<c,\ 1-a<c_0<a\quad\text{and}\quad b=1.$$

Take $t \in \mathcal{S}_{a,c}$. Let $\mathbf{x}_t \in \mathcal{B}^0$ be the unique solution of the linear system $\mathbf{M}_{a,c}(t)\mathbf{x}_t = \mathbf{1}$, and let $\lambda_{a,c}(t)$ (resp. $\tilde{\lambda}_{a,c}(t)$) be the smallest positive integer (resp. the largest negative integer) such that $\mathbf{x}_t(\lambda_{a,c}(t)) = \mathbf{x}_t(\tilde{\lambda}_{a,c}(t)) = 1$. Then

$$\tau_{\lambda_{a,c}(t)}\mathbf{x}_t, \tau_{\tilde{\lambda}_{a,c}(t)}\mathbf{x}_t \in \mathcal{B}^0$$

and

$$\mathbf{M}_{a,c}(t+\lambda_{a,c}(t))\tau_{\lambda_{a,c}}\mathbf{x}_t = \mathbf{M}_{a,c}(t+\tilde{\lambda}_{a,c}(t))\tau_{\tilde{\lambda}_{a,c}}\mathbf{x}_t = \mathbf{M}_{a,c}(t)\mathbf{x}_t = \mathbf{1}$$

by the frequency-shift property (1.7). This yields two maps on the set $\mathcal{S}_{a,c}$:

$$\mathcal{S}_{a,c} \ni t \longrightarrow t + \lambda_{a,c}(t) \in \mathcal{S}_{a,c} \quad \text{and} \quad \mathcal{S}_{a,c} \ni t \longrightarrow t + \tilde{\lambda}_{a,c}(t) \in \mathcal{S}_{a,c}. \tag{1.14}$$

Our inspection shows that the above two maps can be extended to discontinuous periodic transformations $R_{a,c}$ and $\tilde{R}_{a,c}$ on the line $\mathbb{R}$ respectively, where

$$R_{a,c}(t) := \begin{cases} t + \lfloor c \rfloor & \text{if } t \in [(c_0-a)_-, 0) + a\mathbb{Z} \\ t + \lfloor c \rfloor + 1 & \text{if } t \in [0, (c_0+a-1)_+) + a\mathbb{Z} \\ t & \text{if } t \in [(c_0+a-1)_+, (c_0-a)_- + a) + a\mathbb{Z}, \end{cases} \tag{1.15}$$

and

$$\tilde{R}_{a,c}(t) = \begin{cases} t - \lfloor c \rfloor - 1 & \text{if } t \in [c-(c_0+a-1)_+, c) + a\mathbb{Z} \\ t - \lfloor c \rfloor & \text{if } t \in [c, c-(c_0-a)_-) + a\mathbb{Z} \\ t & \text{if } t \in [c-(c_0-a)_-, c+a-(c_0+a-1)_+) + a\mathbb{Z}, \end{cases} \tag{1.16}$$

see Lemma 3.7. So the set $\mathcal{S}_{a,c}$ is an invariant set under transformations $R_{a,c}$ and $\tilde{R}_{a,c}$ and it has empty intersection with their black holes $[(c_0+a-1)_+, (c_0-a)_- + a) + a\mathbb{Z}$ and $[c-(c_0-a)_-, c+a-(c_0+a-1)_+) + a\mathbb{Z}$. Most importantly, the set $\mathcal{S}_{a,c}$ is *maximal* in the sense that any set E satisfying

$$R_{a,c}(E) = E \quad \text{and} \quad E \cap ([(c_0+a-1)_+, (c_0-a)_- + a) + a\mathbb{Z}) = \emptyset$$

is a subset of $\mathcal{S}_{a,c}$, see Theorem 3.4. Due to the above property, we call $\mathcal{S}_{a,c}$ the *maximal invariant set*.

The piecewise-linear transformations $R_{a,c}$ and $\tilde{R}_{a,c}$ are well-defined as $(c_0 + a - 1)_+ \le (c_0-a)_- + a$. The *black hole* $[(c_0+a-1)_+, (c_0-a)_- + a) + a\mathbb{Z}$ of the transformation $R_{a,c}$ and the black hole $[c-(c_0-a)_-, c+a-(c_0+a-1)_+) + a\mathbb{Z}$ of the transformation $\tilde{R}_{a,c}$ play important role for us to explore the structure of the maximal invariant set $\mathcal{S}_{a,c}$. The following properties for the transformations $R_{a,c}$ and $\tilde{R}_{a,c}$ follow immediately from their definitions (1.15) and (1.16):

1) The transformation $\tilde{R}_{a,c}$ is the left-inverse of the transformation $R_{a,c}$ outside its black hole and vice versa (hence the transformations $R_{a,c}$ and $\tilde{R}_{a,c}$ are one-to-one outside their black holes), i.e.,

$$\begin{cases} \tilde{R}_{a,c}(R_{a,c}(t)) = t \text{ if } t \notin [(c_0+a-1)_+, a+(c_0-a)_-) + a\mathbb{Z}, \\ R_{a,c}(\tilde{R}_{a,c}(t)) = t \text{ if } t \notin [c-(c_0-a)_-, c+a-(c_0+a-1)_+) + a\mathbb{Z}. \end{cases} \tag{1.17}$$

2) The range of the transformation $R_{a,c}$ outside its black hole is the complement of the black hole of the transformation $\tilde{R}_{a,c}$ and vice versa, i.e.,

$$\begin{cases} R_{a,c}\big(\mathbb{R}\backslash([(c_0+a-1)_+, (c_0-a)_- + a) + a\mathbb{Z})\big) \\ \qquad = \mathbb{R}\backslash([c-(c_0-a)_-, c+a-(c_0+a-1)_+) + a\mathbb{Z}), \\ \tilde{R}_{a,c}\big(\mathbb{R}\backslash([c-(c_0-a)_-, c+a-(c_0+a-1)_+) + a\mathbb{Z})\big) \\ \qquad = \mathbb{R}\backslash([(c_0+a-1)_+, (c_0-a)_- + a) + a\mathbb{Z}). \end{cases} \tag{1.18}$$

3) The transformations $R_{a,c}$ and $\tilde{R}_{a,c}$ are measure-preserving outside their black holes, i.e.,

$$\begin{cases} |R_{a,c}(E)| = |E| & \text{if } E \cap ([(c_0+a-1)_+, (c_0-a)_- + a) + a\mathbb{Z}) = \emptyset, \\ |\tilde{R}_{a,c}(E)| = |E| & \text{if } E \cap ([c-(c_0-a)_-, c+a-(c_0+a-1)_+) + a\mathbb{Z}) = \emptyset. \end{cases} \tag{1.19}$$

As the transformation $R_{a,c}$ is *non-contractive* by the above measure-preserving property (1.19), its maximal invariant set $\mathcal{S}_{a,c}$ does not directly follow from the Hutchinson's remarkable construction [**27**]. We observe that invariance of the set $\mathcal{S}_{a,c}$ under the transformation $R_{a,c}$ and its empty intersection with the black hole $[(c_0+a-1)_+, (c_0-a)_- + a) + a\mathbb{Z}$ imply that

$$\mathcal{S}_{a,c} \subset \cap_{n=0}^{\infty} (R_{a,c})^n(\mathbb{R}) \backslash ([(c_0+a-1)_+, (c_0-a)_- + a) + a\mathbb{Z}). \tag{1.20}$$

Surprisingly we show that infinite intersection in the above inclusion can be replaced by *finite* intersection and the inclusion is indeed an equality whenever $\mathcal{S}_{a,c} \neq \emptyset$. This leads to the existence of a nonnegative integer D such that

$$\mathcal{S}_{a,c} = (R_{a,c})^L(\mathbb{R}) \backslash ([(c_0+a-1)_+, (c_0-a)_- + a) + a\mathbb{Z}) \quad \text{for all } L \geq D, \tag{1.21}$$

see Theorem 4.1. Hence the maximal invariant set $\mathcal{S}_{a,c}$ consists of finitely many left-closed and right-open intervals on one period and it is measurable, see Examples 5.1, 6.1 and 6.2 for illustrative examples. Our algorithm to verify frame property for the Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ for given pair (a, c) is based on the observation (1.21), see Appendix A.

In this paper, we prove the finite iteration (1.21) of the maximal invariant set $\mathcal{S}_{a,c}$ from exploring its complement $\mathbb{R} \backslash \mathcal{S}_{a,c}$. We observe that holes $(R_{a,c})^n([c-(c_0-a)_-, c+a-(c_0+a-1)_+)+a\mathbb{Z}), n \geq 0$, obtained from applying the transformation $R_{a,c}$ to the black hole $[c-(c_0-a)_-, c+a-(c_0+a-1)_+)+a\mathbb{Z}$ of the transformation $\tilde{R}_{a,c}$ have empty intersection with the maximal invariant set $\mathcal{S}_{a,c}$, see Proposition 3.6. Furthermore, the black hole $[(c_0+a-1)_+, a+(c_0-a)_-)+a\mathbb{Z}$ of the transformation $R_{a,c}$ and the black hole $[c-(c_0-a)_-, c+a-(c_0+a-1)_+)+a\mathbb{Z}$ of the transformation $\tilde{R}_{a,c}$ are transformable through periodic holes $(R_{a,c})^n([c-(c_0-a)_-, c+a-(c_0+a-1)_+) + a\mathbb{Z}), 0 \leq n \leq D$, in *finite* steps, provided that $\mathcal{S}_{a,c} \neq \emptyset$. Thus

$$\mathcal{S}_{a,c} = \mathbb{R} \backslash \big(\cup_{n=0}^{D} (R_{a,c})^n([c-(c_0-a)_-, c+a-(c_0+a-1)_+) + a\mathbb{Z})\big) \tag{1.22}$$

by its maximal invariance under the transformation $\mathcal{R}_{a,c}$, see Theorem 5.2 for $a \notin \mathbb{Q}$ and Theorems 6.3, 6.4 and 6.5 for $a \in \mathbb{Q}$.

Set $c_1 := \lfloor c \rfloor - \lfloor (\lfloor c \rfloor / a) \rfloor a$. For pairs (a, c) of positive numbers satisfying either $c_1 \geq 2a-1$ or $c_1 = 0$, we can construct the set $\mathcal{S}_{a,c}$ explicitly by applying (1.22), and then we can determine whether the corresponding Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is a frame, see Theorem 7.3. Then it remains to consider the *abc*-problem for Gabor systems with triples (a, b, c) satisfying

$$0 < a < 1 < c, \quad 1-a < c_0 < a, \quad 0 < c_1 < 2a-1 \quad \text{and} \quad b = 1.$$

For the parametrization of the maximal invariant set $\mathcal{S}_{a,c}$, we need some additional properties for the transformation $R_{a,c}$ and perform a topological surgery for the maximal invariant set $\mathcal{S}_{a,c}$. Recall that the maximal invariant set $\mathcal{S}_{a,c}$ has its complement composed by finitely many left-closed right-open intervals, called *holes*, on one period by (1.22). So we may squeeze out those holes on the line and then reconnect their endpoints. The above holes-removal surgery could be described by the map

$$Y_{a,c}(t) := \operatorname{sgn}(t)\big|[t_-, t_+) \cap \mathcal{S}_{a,c}\big|,\ \ t \in \mathbb{R} \tag{1.23}$$

on the line in the sense that it is an isomorphism from the maximal invariant set $\mathcal{S}_{a,c}$ to the *line with marks* (image of the holes). In Figure 1 below, we illustrate the performance of the holes-removal surgery via

$$a\mathbb{T} \ni a\exp(2\pi i t/a) \longmapsto Y_{a,c}(a)\exp\big(-2\pi i Y_{a,c}(t)/Y_{a,c}(a)\big) \in Y_{a,c}(a)\mathbb{T},$$

where $\big(\frac{\pi}{4}, 23-\frac{11\pi}{2}\big)$, $\big(\frac{6}{7}, \frac{23}{7}\big)$, $\big(\frac{13}{17}, \frac{77}{17}\big)$ and $\big(\frac{13}{17}, \frac{75}{17}\big)$ are used as pairs (a, c) in the four subfigures respectively, c.f. Examples 5.1, 6.1 and 6.2. More importantly, after

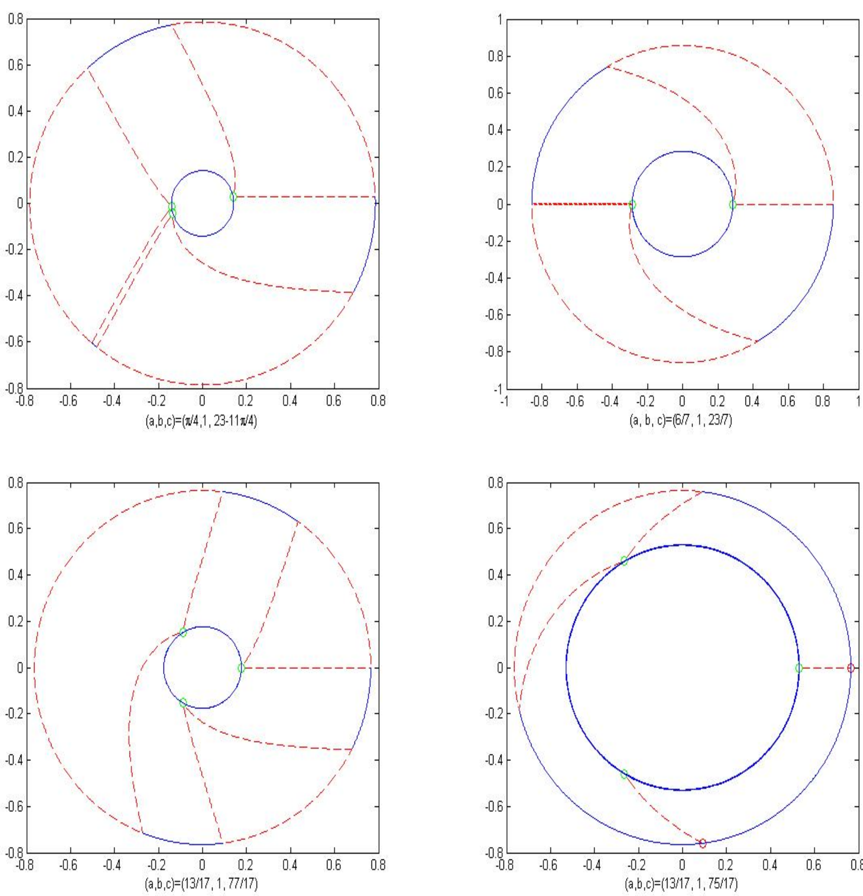


FIGURE 1. The set $a\exp(2\pi i\mathcal{S}_{a,c}/a)$ contains blue arcs in the big circle, while the set $a\exp(2\pi i(\mathbb{R}\backslash\mathcal{S}_{a,c})/a)$ is composed of red dashed arcs in the big circle. The image $Y_{a,c}(a)\exp\big(2\pi i Y_{a,c}(\mathbb{R})/Y_{a,c}(a)\big)$ of the map $Y_{a,c}$ is the small circle, and the set $Y_{a,c}(a)\exp\big(2\pi i\mathcal{K}_{a,c}/Y_{a,c}(a)\big)$ is circled marked, where $\mathcal{K}_{a,c}$ is the set of marks on the line.

performing holes-removal surgery, the restriction of the nonlinear transformation

$R_{a,c}$ onto the maximal invariant set $\mathcal{S}_{a,c}$ becomes a *linear transformation* $S(\theta_{a,c})$ on a line with marks, i.e., the following diagram commutes,

$$\begin{array}{ccc} \mathcal{S}_{a,c} & \xrightarrow{R_{a,c}} & \mathcal{S}_{a,c} \\ {\scriptstyle Y_{a,c}}\downarrow & & \downarrow{\scriptstyle Y_{a,c}} \\ \mathbb{R}/(Y_{a,c}(a)\mathbb{Z}) & \xrightarrow[S(\theta_{a,c})]{} & \mathbb{R}/(Y_{a,c}(a)\mathbb{Z}) \end{array} \tag{1.24}$$

where

$$\theta_{a,c} = Y_{a,c}(\lfloor c \rfloor + 1)$$

and

$$S(\theta_{a,c})(z + Y_{a,c}(a)\mathbb{Z}) = \theta_{a,c} + z + Y_{a,c}(a)\mathbb{Z}, \quad z \in \mathbb{R}/(Y_{a,c}(a)\mathbb{Z}),$$

see Theorem 4.4.

For irrational time-spacing parameter a, holes $(R_{a,c})^n([c - c_0, c - c_0 + 1) + a\mathbb{Z}), 0 \le n \le D$, in the complement of the maximal invariant set $\mathcal{S}_{a,c}$ have their closure being mutually disjoint, see Theorem 5.2. This gives a one-to-one correspondence between those holes of length $1 - a$ and marks on the line, where marks are obtained by applying the hole removal surgery and conversely holes of same length are inserted at marks by the augmentation operation. From the commutative diagram (1.24) for the transformation $R_{a,c}$ and the above one-to-one correspondence between holes and marks, we conclude that the set of marks are completely determined by the number of marks on one period $[0, Y_{a,c}(a))$ and the position $Y_{a,c}(c-c_0+1)+Y_{a,c}(a)\mathbb{Z}$ and $Y_{a,c}(c_0)+Y_{a,c}(a)\mathbb{Z}$ of two marks associated with black holes $[c - c_0, c - c_0 + 1 - a) + a\mathbb{Z}$ and $[c_0 + a - 1, c_0) + a\mathbb{Z}$ of transformations $\tilde{R}_{a,c}$ and $R_{a,c}$ respectively. Using the above conclusion, we may fully classify the maximal invariant set $\mathcal{S}_{a,c}$ by two parameters d_1 and d_2, the numbers of holes in $[0, c_0 + a - 1)$ and $[c_0, a)$ respectively. This leads to a characterization of nontriviality of the maximal invariant set $\mathcal{S}_{a,c}$, see Theorem 5.5. Also it gives the full classification of pairs (a, c) of positive numbers satisfying $0 < c_1 < 2a - 1$ and $a \notin \mathbb{Q}$ such that the corresponding Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is a frame, see Theorem 7.4.

For rational time-spacing parameter a, one may easily verify that the set $\mathcal{S}_{a,c}$ is finite union of intervals of length $c - \lfloor qc \rfloor/q$ and $(\lfloor qc/ \rfloor + 1)/q - c$ on one period, and it is completely determined by its restriction to the finite set $(\{0, c\} + \mathbb{Z}/q) \cap [0, a)$, where $a = p/q$ for some co-prime integers p and q. In particular,

$$\mathcal{S}_{a,c} = \big(\mathcal{S}_{a,c} \cap \mathbb{Z}/q + [0, c - \lfloor qc \rfloor/q)\big) \cup \big(\mathcal{S}_{a,c} \cap (c + \mathbb{Z}/q) + [0, (\lfloor qc \rfloor + 1)/q - c)\big), \tag{1.25}$$

because infinite matrices $\mathbf{M}_{a,c}(t)$ in (1.5) has the following property for $a \in \mathbb{Q}$:

$$\mathbf{M}_{a,c}(t) = \begin{cases} \mathbf{M}_{a,c}(\lfloor qt \rfloor/q + c - \lfloor qc \rfloor/q) & \text{if } t - \lfloor qt \rfloor/q \ge c - \lfloor qc \rfloor/q \\ \mathbf{M}_{a,c}(\lfloor qt \rfloor/q) & \text{if } 0 \le t - \lfloor qt \rfloor/q < c - \lfloor qc \rfloor/q. \end{cases} \tag{1.26}$$

Furthermore, the maximal invariant set $\mathcal{S}_{a,c}$ has its complement consisting of periodic gaps of two different sizes, see Theorems 6.3, 6.4 and 6.5. Also the transformation $R_{a,c}$ has its restriction on $\mathcal{S}_{a,c}$ being of *finite* order, since there exists a positive integer D such that

$$(R_{a,c})^D(t) - t \in a\mathbb{Z} \quad \text{for all } t \in \mathcal{S}_{a,c}$$

cf. Theorem 4.5. Taking holes-removal surgery described by the map in (1.23) for the maximal invariant set $\mathcal{S}_{a,c}$ leads to a line with marks. Interestingly, it is shown that the set $\mathcal{K}_{a,c}$ of marks forms a cyclic group, see Theorem 6.6 and Corollary 6.7. We observe that the hole-removal surgery is *reversible*, that is, the maximal invariant set $\mathcal{S}_{a,c}$ can be obtained from the real line by putting marks at appropriate positions and then inserting holes of appropriate sizes at marked positions, that augmentation operation is much more delicate and complicated than the holes-removal surgery. Using the above augmentation operation, we characterize nontriviality of the maximal invariant set $\mathcal{S}_{a,c}$ for $a \in \mathbb{Q}$, see Theorem 6.8. Finally using the above characterization and the covering property of the maximal invariant set $\mathcal{S}_{a,c}$ in Theorems 3.2 and 3.3, we provide full classification of pairs (a, c) satisfying $0 < c_1 < 2a - 1$ and $a \in \mathbb{Q}$ such that the corresponding Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is a frame, see Theorem 7.5.

The paper is organized as follows. In Chapter 2, we introduce a new characterization of frame property for the Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ via non-existence of binary solution of infinite-dimensional linear systems (1.9), and we show that the set $\mathcal{D}_{a,c}$ could be obtained from the maximal invariant set $\mathcal{S}_{a,c}$ by some set operations. The main results in that chapter are Theorems 2.1 and 2.3.

In Chapter 3, we consider covering property of the set $\mathcal{S}_{a,c}$ in Theorem 3.2, and show in Theorem 3.4 that the set $\mathcal{S}_{a,c}$ has empty intersection with the black hole $[(c_0 + a - 1)_+, (c_0 - a)_- + a) + a\mathbb{Z}$ of the transformations $R_{a,c}$, and that it is the maximal set that is invariant under the transformation $R_{a,c}$ and has empty intersection with its black hole. The maximal invariance property for the set $\mathcal{S}_{a,c}$ is crucial in our study. Applying the maximal invariance property, we can construct the maximal invariant set $\mathcal{S}_{a,c}$ immediately for pairs (a, c) satisfying either $c_0 \geq a$ or $0 \leq c_0 \leq 1 - a$. Important observations in that chapter also include the dense property of the maximal invariant set $\mathcal{S}_{a,c}$ around the origin, and unique binary solution $\mathbf{x} \in \mathcal{B}^0$ to the linear system $\mathbf{M}_{a,c}(t)\mathbf{x} = \mathbf{1}$ for any given $t \in \mathcal{S}_{a,c}$, see Lemmas 3.10, 3.11 and 3.12.

In Chapter 4, we show in Theorem 4.1 that although the transformation $R_{a,c}$ is not-contractive on the whole, the Hutchinson's remarkable construction [**27**] works for its maximal invariant set $\mathcal{S}_{a,c}$. The surprising observation given in Theorem 4.4 is that the restriction of the piecewise linear transformation $R_{a,c}$ onto the maximal invariant set $\mathcal{S}_{a,c}$ is a shift on the line with marks. In Theorem 4.5 of that chapter, we establish an ergodic theorem for the transformation $R_{a,c}$, even though it is not measure-preserving on the whole line.

In Chapter 5, we study the maximal invariant set $\mathcal{S}_{a,c}$ with $a \notin \mathbb{Q}$. We show that the complement of the maximal invariant set $\mathcal{S}_{a,c}$ consists of left-closed and right-open intervals of same size and it contains a small neighborhood of the origin. After performing the holes-removal surgery described by the isomorphism $Y_{a,c}$ in (1.23), the maximal invariant set $\mathcal{S}_{a,c}$ becomes the real line with marks, and conversely expanding the line with marks by inserting holes $[0, 1-a)$ at every location of marks recovers the maximal invariant set $\mathcal{S}_{a,c}$. Using the above isomorphism $Y_{a,c}$ between the maximal invariant set $\mathcal{S}_{a,c}$ and the real line with marks, we can parameterize the maximal invariant set $\mathcal{S}_{a,c}$ via two nonnegative integer parameters.

In Chapter 6, we study the maximal invariant sets $\mathcal{S}_{a,c}$ with $a \in \mathbb{Q}$. We show that the set $\mathcal{S}_{a,c}$ is the union of mutually disjoint intervals of same size, while its complement may contain holes of two different sizes. We observe that holes in the complement of the set $\mathcal{S}_{a,c}$ have cyclic group structure after performing the holes-removal surgery. Thus the maximal invariant set $\mathcal{S}_{a,c}$ could be obtained from inserting holes of appropriate size at every mark, which forms in a cyclic group. Using the above augmentation operation, we can parameterize the maximal invariant set $\mathcal{S}_{a,c}$ via four nonnegative integer parameters.

In Chapter 7, we provide full classification of all pairs (a, c) of time-spacing and window-size parameters such that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ are Gabor frames. From our classification, we see that the range $\mathcal{R}(\chi_{[0,c)})$ is neither open nor path-connected, but it is a dense subset of the open region $\mathcal{U}_c := \{(a, b) : 0 < a < \max(1/b, c)\}$, cf. [**18**]. Moreover, we confirm a conjecture in [**30**, Section 3.3.5] that $\mathcal{G}(\chi_I, a\mathbb{Z} \times b\mathbb{Z})$ is a Gabor frame if $(a, b) \in \mathcal{U}_c$, the product between a and b is irrational, and c is not a rational combination of a and $1/b$.

In Appendix A, we provide a finite-step algorithm to verify whether the Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times b\mathbb{Z})$ is a Gabor frame for any given triple of (a, b, c) of positive numbers.

In Appendix B, we apply our results on Gabor systems to identify all intervals I and time-sampling spacing lattices $b\mathbb{Z} \times a\mathbb{Z}$ such that signals f in the shift-invariant space

$$V_2(\chi_I, b\mathbb{Z}) := \Big\{ \sum_{\lambda \in b\mathbb{Z}} d(\lambda)\chi_I(t - \lambda) : \sum_{\lambda \in b\mathbb{Z}} |d(\lambda)|^2 < \infty \Big\}$$

can be stably recovered from their equally-spaced samples $f(t_0 + \mu), \mu \in a\mathbb{Z}$, for any initial sampling position $t_0 \in \mathbb{R}$.

CHAPTER 2

Gabor Frames and Infinite Matrices

Infinite matrices $\mathbf{M}_{a,c}(t), t \in \mathbb{R}$, in (1.5) have their rows containing $\lfloor c \rfloor + \{0, 1\}$ consecutive ones. Their rows are obtained by shifting one (or zero) unit of the previous row with possible reduction or expansion by one unit. In the case that the time-spacing parameter a is rational, they also have certain shift-invariance in the sense that their $(\mu + qa)$-th row can be obtained by shifting p-units of the μ-th row, where p and q are coprime integers with $a = p/q$, c.f. [**30**, Eq. 3.3.68]. The above observations could be illustrated from examples below:

$$
\mathbf{M}_{a,c}(0) = \left(\begin{array}{ccccccccccccccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & & & & & & \\
 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & & & & & & & & \\
 & & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & & & & & & & \\
 & & & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & & & & & & \\
 & & & & 0 & 1 & 1 & 1 & 1 & 1 & 0 & & & & & & \\
 & & & & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & & & & & \\
 & & & & & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & & & & \\
 & & & & & & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & & & \\
 & & & & & & & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & & \\
 & & & & & & & & 0 & 1 & 1 & 1 & 1 & 1 & 0 & & \\
 & & & & & & & & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \\
 & & & & & & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)
$$

for the pair $(a, c) = (\pi/4, 23 - 11\pi/2)$ with $a \notin \mathbb{Q}$; and

$$
\mathbf{M}_{a,c}(0) = \left(\begin{array}{ccccccccccccccccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & & & & & & & & \\
 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & & & & & & & & & & & \\
 & & 0 & 1 & 1 & 1 & 1 & 1 & 0 & & & & & & & & & & \\
 & & & 0 & 1 & 1 & 1 & 1 & 1 & 0 & & & & & & & & & \\
 & & & & 0 & 1 & 1 & 1 & 1 & 0 & & & & & & & & & \\
 & & & & & 0 & 1 & 1 & 1 & 1 & 0 & & & & & & & & \\
 & & & & & 0 & 1 & 1 & 1 & 1 & 1 & 0 & & & & & & & \\
 & & & & & & 0 & 1 & 1 & 1 & 1 & 1 & 0 & & & & & & \\
 & & & & & & & 0 & 1 & 1 & 1 & 1 & 0 & & & & & & \\
 & & & & & & & & 0 & 1 & 1 & 1 & 1 & 0 & & & & & \\
 & & & & & & & & 0 & 1 & 1 & 1 & 1 & 1 & 0 & & & & \\
 & & & & & & & & & 0 & 1 & 1 & 1 & 1 & 1 & 0 & & & \\
 & & & & & & & & & & 0 & 1 & 1 & 1 & 1 & 0 & & & \\
 & & & & & & & & & & & 0 & 1 & 1 & 1 & 1 & 0 & & \\
 & & & & & & & & & & & 0 & 1 & 1 & 1 & 1 & 1 & 0 & \\
 & & & & & & & & & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)
$$

for the pair $(a, c) = (13/17, 77/17)$ with $a \in \mathbb{Q}$, cf. Examples 5.1 and 6.1. Those special structures for infinite matrices in (1.5) help us to characterize frame property for the Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ from uniform stability of infinite matrices to non-existence of trinary vectors in their null spaces, and further to non-existence of binary solution of infinite-dimensional linear systems (1.9).

THEOREM 2.1. *Let* $0 < a < 1 < c$, $\mathbf{M}_{a,c}(t), t \in \mathbb{R}$, *be infinite matrices in* (1.5) *and let* $\mathcal{D}_{a,c}$ *be as in* (1.10). *Then the following statements are equivalent.*

(i) $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ *is a Gabor frame.*
(ii) *Infinite matrices* $\mathbf{M}_{a,c}(t), t \in \mathbb{R}$, *have the uniform* ℓ^2*-stability property* (1.6).
(iii) *For every* $t \in \mathbb{R}$, *only zero vector* $\mathbf{0}$ *is contained in the intersection between* $\mathcal{B} - \mathcal{B}$ *and the null space of* $\mathbf{M}_{a,c}(t)$, *i.e.,*

$$N(\mathbf{M}_{a,c}(t)) \cap (\mathcal{B} - \mathcal{B}) = \{\mathbf{0}\} \quad \text{for every } t \in \mathbb{R}.$$

(iv) $\mathcal{D}_{a,c} = \emptyset$.

The implication (iv)$\Longrightarrow$(i) in the above theorem has been implicitly used in [**23, 30**] for their classifications.

The statement (iv) in the above theorem can be rewritten as follows: for any $t \in \mathbb{R}$, there does not exist $\mathbf{x} \in \mathcal{B}^0$ such that $\mathbf{M}_{a,c}(t)\mathbf{x} = \mathbf{2}$. In the next theorem, we show that it suffices to verify nonexistence of binary solutions $\mathbf{x}$ of infinite-dimensional linear systems $\mathbf{M}_{a,c}(t)\mathbf{x} = \mathbf{2}$ for finitely many t, cf. Theorem 3.1 for the set $\mathcal{S}_{a,c}$.

THEOREM 2.2. *Let* $0 < a < 1 < c$, $\mathbf{M}_{a,c}(t), t \in \mathbb{R}$, *be infinite matrices in* (1.5), *and let* $\mathcal{D}_{a,c}$ *be as in* (1.10). *Define*

$$\Theta_{a,c} := \begin{cases} \{0\} & \text{if } a \notin \mathbb{Q}, \\ (\{0, c\} + \gcd(a, 1)\mathbb{Z}) \cap [0, a) & \text{if } a \in \mathbb{Q}. \end{cases}$$

Then $\mathcal{D}_{a,c} = \emptyset$ *if and only if* $\mathcal{D}_{a,c} \cap \Theta_{a,c} = \emptyset$.

The set $\mathcal{D}_{a,c}$ is a periodic set with period a,

$$\mathcal{D}_{a,c} = \mathcal{D}_{a,c} + a\mathbb{Z} \tag{2.1}$$

by the shift property (1.8), and it can be obtained from the maximal invariant set $\mathcal{S}_{a,c}$ in (1.13) by some set operations.

THEOREM 2.3. *Let* $0 < a < 1 < c$. *Then*

$$\begin{aligned} \mathcal{D}_{a,c} \;\; = \;\; & \big(\mathcal{S}_{a,c} \cap (\cup_{\lambda \in [1, \lfloor c \rfloor - 1] \cap \mathbb{Z}} (\mathcal{S}_{a,c} - \lambda))\big) \\ & \cup \big(\mathcal{S}_{a,c} \cap ([0, (c - \lfloor c \rfloor + a - 1)_+) + a\mathbb{Z}) \cap (\mathcal{S}_{a,c} - \lfloor c \rfloor)\big). \end{aligned} \tag{2.2}$$

For pairs (a, c) of positive numbers satisfying either $c_0 := c - \lfloor c \rfloor \geq 1 - a$, or $c_0 \geq a$, or $\lfloor c \rfloor = 1$, we can construct the set $\mathcal{D}_{a,c}$ explicitly. Hence Theorem 2.1 can be used to determine whether Gabor systems $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ corresponding to those pairs are frames, see Theorem 7.2 and the statement (VIII) of Theorem 7.3 for details.

The organization of this chapter is as follows. We recall the characterization of Gabor frames by Ron and Shen [**38**], the equivalence between statements (i) and (ii) of Theorem 2.1, in Section 2.1. To prove the implication (iv)$\Longrightarrow$(i) of Theorem 2.1, we introduce a characterization for $\mathcal{D}_{a,c} = \emptyset$ via uniform boundedness of lengths of consecutive twos in range spaces $\mathbf{M}_{a,c}(t)\mathcal{B}, t \in \mathbb{R}$, in Section 2.2. In Section 2.3, we give a proof of Theorem 2.1. In addition, we provide a frame bound estimate (2.23) via maximal length $Q_{a,c}$ of consecutive twos in range spaces $\mathbf{M}_{a,c}(t)\mathcal{B}, t \in \mathbb{R}$, in Remark 2.8 of that section. We postpone the proofs of Theorems 2.2 and 2.3 to Sections 3.5 and 3.4 of next chapter respectively.

2.1. Gabor frames and uniform stability of infinite matrices

In this section, we recall the equivalence between frame property for the Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ and uniform stability of infinite matrices $\mathbf{M}_{a,c}(t), t \in \mathbb{R}$, in (1.5), i.e., the equivalence of statements (i) and (ii) of Theorem 2.1.

THEOREM 2.4. *Let (a, c) be a pair of positive numbers. Then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is a Gabor frame if and only if $\mathbf{M}_{a,c}(t), t \in \mathbb{R}$, have the uniform ℓ^2-stability property* (1.6)*, i.e., there exist positive constants A and B such that*

$$A\|\mathbf{z}\|_2 \le \|\mathbf{M}_{a,c}(t)\mathbf{z}\|_2 \le B\|\mathbf{z}\|_2 \quad \text{for all } \mathbf{z} \in \ell^2 \text{ and } t \in \mathbb{R}.$$

Furthermore,

$$\inf_{\|f\|_2=1} \Big(\sum_{\phi \in \mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})} |\langle f, \phi\rangle|^2 \Big)^{1/2} = \inf_{t\in\mathbb{R}} \inf_{\|\mathbf{z}\|_2=1} \|\mathbf{M}_{a,c}(t)\mathbf{z}\|_2 \tag{2.3}$$

and

$$\sup_{\|f\|_2=1} \Big(\sum_{\phi \in \mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})} |\langle f, \phi\rangle|^2 \Big)^{1/2} = \sup_{t\in\mathbb{R}} \sup_{\|\mathbf{z}\|_2=1} \|\mathbf{M}_{a,c}(t)\mathbf{z}\|_2. \tag{2.4}$$

The above theorem was proved in [**38**] for arbitrary Gabor system $\mathcal{G}(\phi, a\mathbb{Z} \times \mathbb{Z})$ generated by a window function $\phi \in L^2$. From the equivalence in Theorem 2.4, we see that necessary conditions (1.2), (1.3) and (1.4) for the Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ to be a frame become the nonzero column property for uniform stable matrices $\mathbf{M}_{a,c}(t)$, nonexistence of exponential vectors $(\exp(2\pi i n\xi_0))_{n\in\mathbb{Z}}, \xi_0 \in \mathbb{R}$, in null spaces $N(\mathbf{M}_{a,c}(t))$, and the non-thinness property for infinite matrices $\mathbf{M}_{a,c}(t)$ respectively. The interested reader is referred to [**36, 43**] for various criteria and necessary conditions for the ℓ^2-stability of an infinite matrix.

For the completeness of this paper, we include a short proof of Theorem 2.4.

PROOF. First the sufficiency. Take $t_0 \in [0, 1)$, a sufficiently small positive number $\epsilon \in (0, 1 - t_0)$, and a nonzero vector $\mathbf{z} := (\mathbf{z}(\lambda))_{\lambda\in\mathbb{Z}}$ having finitely many nonzero entries. Define

$$f_{\epsilon,t_0}(t) = \epsilon^{-1/2} \sum_{\lambda\in\mathbb{Z}} \mathbf{z}(\lambda)\chi_{[0,\epsilon)}(t - t_0 - \lambda).$$

Then

$$\begin{aligned} \|f_{\epsilon,t_0}\|_2^2 &= \epsilon^{-1} \int_{\mathbb{R}} \Big| \sum_{\lambda\in\mathbb{Z}} \mathbf{z}(\lambda)\chi_{[0,\epsilon)}(t - t_0 - \lambda)\Big|^2 dt \\ &= \epsilon^{-1} \sum_{\lambda\in\mathbb{Z}} |\mathbf{z}(\lambda)|^2 \int_{\mathbb{R}} \chi_{[0,\epsilon)}(t - t_0 - \lambda) dt = \|\mathbf{z}\|_2^2, \end{aligned}$$

where the second equality holds as $([0,\epsilon)+\lambda))\cap([0,\epsilon)+\lambda'))=\emptyset$ for all distinct integers λ and λ'; and

$$\begin{aligned}
&\sum_{\phi\in\mathcal{G}(\chi_{[0,c)},a\mathbb{Z}\times\mathbb{Z})}|\langle f_{\epsilon,t_0},\phi\rangle|^2\\
= &\sum_{\mu\in a\mathbb{Z}}\sum_{n\in\mathbb{Z}}\Big|\int_0^1\Big(\sum_{\lambda\in\mathbb{Z}}f_{\epsilon,t_0}(t+\lambda)\chi_{[0,c)}(t-\mu+\lambda)\Big)e^{-2\pi int}dt\Big|^2\\
= &\sum_{\mu\in a\mathbb{Z}}\int_0^1\Big|\sum_{\lambda\in\mathbb{Z}}f_{\epsilon,t_0}(t+\lambda)\chi_{[0,c)}(t-\mu+\lambda)\Big|^2dt\\
= &\epsilon^{-1}\sum_{\mu\in a\mathbb{Z}}\int_{t_0}^{t_0+\epsilon}\Big|\sum_{\lambda\in\mathbb{Z}}\mathbf{z}(\lambda)\chi_{[0,c)}(t-\mu+\lambda)\Big|^2dt\\
= &\|\mathbf{M}_{a,c}(t_0)\mathbf{z}\|_2^2,
\end{aligned}$$

where the last equality follows from

$$\sum_{\lambda\in\mathbb{Z}}\mathbf{z}(\lambda)\chi_{[0,c)}(t-\mu+\lambda)=\sum_{\lambda\in\mathbb{Z}}\mathbf{z}(\lambda)\chi_{[0,c)}(t_0-\mu+\lambda)\quad\text{for all } t\in[t_0,t_0+\epsilon]$$

by the assumption that the vector $\mathbf{z}$ has finitely many nonzero entries and $\epsilon>0$ is sufficiently small. Combining the above two equalities with frame property for the Gabor system $\mathcal{G}(\chi_{[0,c)},a\mathbb{Z}\times\mathbb{Z})$, we obtain

$$\begin{aligned}
0 &< \inf_{\|f\|_2=1}\sum_{\phi\in\mathcal{G}(\chi_{[0,c)},a\mathbb{Z}\times\mathbb{Z})}|\langle f,\phi\rangle|^2\\
&\le \inf_{t\in[0,1)}\inf_{\|\mathbf{z}\|_2=1}\|\mathbf{M}_{a,c}(t)\mathbf{z}\|_2^2=\inf_{t\in\mathbb{R}}\inf_{\|\mathbf{z}\|_2=1}\|\mathbf{M}_{a,c}(t)\mathbf{z}\|_2^2\\
&\le \sup_{t\in\mathbb{R}}\sup_{\|\mathbf{z}\|_2=1}\|\mathbf{M}_{a,c}(t)\mathbf{z}\|_2^2=\sup_{t\in[0,1)}\sup_{\|\mathbf{z}\|_2=1}\|\mathbf{M}_{a,c}(t)\mathbf{z}\|_2^2\\
&\le \sup_{\|f\|_2=1}\sum_{\phi\in\mathcal{G}(\chi_{[0,c)},a\mathbb{Z}\times\mathbb{Z})}|\langle f,\phi\rangle|^2<\infty. \qquad (2.5)
\end{aligned}$$

This proves the sufficiency.

Then the necessity. For a compactly supported function $f\in L^2(\mathbb{R})$,

$$\begin{aligned}
\sum_{\phi\in\mathcal{G}(\chi_{[0,c)},a\mathbb{Z}\times\mathbb{Z})}|\langle f,\phi\rangle|^2 &= \sum_{\mu\in a\mathbb{Z}}\int_0^1\Big|\sum_{\lambda\in\mathbb{Z}}\chi_{[0,c)}(t-\mu+\lambda)f(t+\lambda)\Big|^2dt\\
&\ge \int_0^1\Big(\inf_{\|\mathbf{z}\|_2=1}\|\mathbf{M}_{a,c}(t)\mathbf{z}\|_2\Big)^2\times\Big(\sum_{\lambda\in\mathbb{Z}}|f(t+\lambda)|^2\Big)dt\\
&\ge \Big(\inf_{t\in\mathbb{R}}\inf_{\|\mathbf{z}\|_2=1}\|\mathbf{M}_{a,c}(t)\mathbf{z}\|_2\Big)^2\|f\|_2^2
\end{aligned}$$

and similarly

$$\sum_{\phi\in\mathcal{G}(\chi_{[0,c)},a\mathbb{Z}\times\mathbb{Z})}|\langle f,\phi\rangle|^2 \le \Big(\sup_{t\in\mathbb{R}}\sup_{\|\mathbf{z}\|_2=1}\|\mathbf{M}_{a,c}(t)\mathbf{z}\|_2\Big)^2\|f\|_2^2.$$

Combining the above two estimates, we have

$$
\begin{aligned}
0 &< \inf_{t\in\mathbb{R}} \inf_{\|\mathbf{z}\|_2=1} \|\mathbf{M}_{a,c}(t)\mathbf{z}\|_2^2 \\
&\le \inf_{\|f\|_2=1} \sum_{\phi\in\mathcal{G}(\chi_{[0,c)},a\mathbb{Z}\times\mathbb{Z})} |\langle f,\phi\rangle|^2 \le \sup_{\|f\|_2=1} \sum_{\phi\in\mathcal{G}(\chi_{[0,c)},a\mathbb{Z}\times\mathbb{Z})} |\langle f,\phi\rangle|^2 \\
&\le \sup_{t\in\mathbb{R}} \sup_{\|\mathbf{z}\|_2=1} \|\mathbf{M}_{a,c}(t)\mathbf{z}\|_2^2 < \infty. \qquad (2.6)
\end{aligned}
$$

This completes the proof of the necessity.

Finally bound estimates in (2.3) and (2.4) follow immediately from (2.5) and (2.6). □

2.2. Maximal lengths of consecutive twos in range spaces of infinite matrices

For any $t \in \mathbb{R}$ and $\mathbf{x} \in \mathcal{B}$, let

$$
Q_{a,c}(t,\mathbf{x}) := \begin{cases} 0 & \text{if } K(t,\mathbf{x}) = \emptyset \\ \sup\big\{n \in \mathbb{N}\big|\ [\mu,\mu+na) \cap a\mathbb{Z} \\ \qquad\quad \subset K(t,\mathbf{x}) \text{ for some } \mu \in a\mathbb{Z}\big\} & \text{otherwise,} \end{cases}
$$

where

$$
K(t,\mathbf{x}) := \big\{\mu \in a\mathbb{Z}\big|\ \mathbf{M}_{a,c}(t)\mathbf{x}(\mu) = 2\big\}.
$$

Define the maximal length $Q_{a,c}$ of consecutive twos of vectors in range spaces $\mathbf{M}_{a,c}(t)\mathcal{B}, t \in \mathbb{R}$, by

$$
Q_{a,c} := \sup_{t\in\mathbb{R},\mathbf{x}\in\mathcal{B}} Q_{a,c}(t,\mathbf{x}). \qquad (2.7)
$$

Obviously,

$$
Q_{a,c} = +\infty \quad \text{if} \quad \mathcal{D}_{a,c} \neq \emptyset,
$$

because $Q_{a,c}(t,\mathbf{x}_0) = +\infty$ for any $t_0 \in \mathcal{D}_{a,c}$ and $\mathbf{x}_0 \in \mathcal{B}$ satisfying $\mathbf{M}_{a,c}(t_0)\mathbf{x}_0 = \mathbf{2}$. The converse is shown to be true in the next theorem. Hence $\mathcal{D}_{a,c} = \emptyset$ if and only if the maximal length $Q_{a,c}$ of consecutive twos of vectors in range spaces $\mathbf{M}_{a,c}(t)\mathcal{B}, t \in \mathbb{R}$, is finite, cf. Lemma 5.6 for the empty set property for $\mathcal{S}_{a,c}$.

THEOREM 2.5. *Let $0 < a < 1 < c$. Then $\mathcal{D}_{a,c} = \emptyset$ if and only if $Q_{a,c} < +\infty$.*

PROOF. The sufficiency is obvious. Now the necessity. Suppose, on the contrary, that $Q_{a,c} = +\infty$. Then for every $n \ge 1$ there exist $t_n \in \mathbb{R}, \mu_n \in a\mathbb{Z}$ and $\mathbf{x}_n \in \mathcal{B}$ such that

$$
\mathbf{M}_{a,c}(t_n)\mathbf{x}_n(\mu) = 2 \quad \text{for all} \quad \mu_n \le \mu \le \mu_n + 2na.
$$

Applying (1.7) and (1.8) for time-frequency shifts of infinite matrices $\mathbf{M}_{a,c}(t)$, we may assume, without loss of generality, that $t_n \in [0,1)$ and

$$
(\mathbf{M}_{a,c}(t_n)\mathbf{x}_n)(\mu) = 2 \quad \text{for all } \mu \in [-na,na] \cap a\mathbb{Z}, \qquad (2.8)
$$

otherwise replacing t_n by the unique number $t'_n \in [0,1)$ satisfying $t_n - \mu_n - na - t'_n \in \mathbb{Z}$ and $\mathbf{x}_n$ by $\tau_{t'_n - t_n + \mu_n + na}\mathbf{x}_n$. Furthermore, we can assume that $\mathbf{x}_n := (\mathbf{x}_n(\mu))_{\mu\in\mathbb{Z}} \in \mathcal{B}^0, n \ge 1$, satisfy

$$
\mathbf{x}_{n'}(\lambda) = \mathbf{x}_n(\lambda) \quad \text{for all } \lambda \in [-n,n] \cap \mathbb{Z} \text{ and } n' \ge n, \qquad (2.9)
$$

and

$$
\{t_n\}_{n=1}^\infty \ \text{ is a monotone sequence,} \qquad (2.10)
$$

otherwise replacing them by their subsequences satisfying (2.9) and (2.10).

Denote by t_∞ the limit of $\{t_n\}_{n=1}^\infty$ and $\mathbf{x}_\infty$ the limit of $\{\mathbf{x}_n\}_{n=1}^\infty$. Clearly $t_\infty \in [0,1]$ and $\mathbf{x}_\infty \in \mathcal{B}^0$.

If there exists n_0 such that $t_n = t_\infty$ for all $n \ge n_0$, then for any given $\mu \in a\mathbb{Z}$,

$$\big(\mathbf{M}_{a,c}(t_\infty)\mathbf{x}_\infty\big)(\mu) = \big(\mathbf{M}_{a,c}(t_n)\mathbf{x}_n\big)(\mu) = 2$$

for sufficiently large n by (2.9). Thus $\mathbf{M}_{a,c}(t_\infty)\mathbf{x}_\infty = \mathbf{2}$ and $t_\infty \in \mathcal{D}_{a,c}$, which contradicts to the empty-set assumption for $\mathcal{D}_{a,c}$.

If $\{t_n\}_{n=1}^\infty$ is a strictly decreasing sequence, then for any given $\lambda \in \mathbb{Z}$ and $\mu \in a\mathbb{Z}$,

$$\chi_{[0,c)}(t_\infty - \mu + \lambda) = \chi_{[0,c)}(t_n - \mu + \lambda) \tag{2.11}$$

for sufficiently large n. This together with (2.8) and (2.9) implies that

$$\big(\mathbf{M}_{a,c}(t_\infty)\mathbf{x}_\infty\big)(\mu) = 2 \quad \text{for any given } \mu \in a\mathbb{Z},$$

which contradicts to the assumption that $\mathcal{D}_{a,c} = \emptyset$.

If $\{t_n\}_{n=1}^\infty$ is a strictly increasing sequence, then for any given $\lambda \in a\mathbb{Z}$ and $\mu \in a\mathbb{Z}$,

$$\chi_{(0,c]}(t_\infty - \mu + \lambda) = \chi_{[0,c)}(t_n - \mu + \lambda)$$

for sufficiently large n. This together with (2.8) and (2.9) yields that

$$\sum_{\lambda\in\mathbb{Z}} \chi_{(0,c]}(t_\infty - \mu + \lambda)\mathbf{x}_\infty(\lambda) = 2 \quad \text{for all } \mu \in a\mathbb{Z}.$$

Thus $c - t_\infty \in \mathcal{D}_{a,c}$, which is a contradiction. □

2.3. Uniform stability and null spaces of infinite matrices

In this section, we prove Theorem 2.1 by showing (i)⟹(ii)⟹(iii)⟹(iv)⟹(i).

Proof of Theorem 2.1. The implication (i)⟹(ii) has been given by Theorem 2.4.

Next we prove the implication (ii)⟹(iii). Suppose, on the contrary, that there exist $t_0 \in \mathbb{R}$ and a nonzero vector $\mathbf{x} = (\mathbf{x}(\lambda))_{\lambda\in\mathbb{Z}}$ such that

$$\mathbf{M}_{a,c}(t_0)\mathbf{x} = \mathbf{0} \text{ and } \mathbf{x}(\lambda) \in \{-1,0,1\} \text{ for all } \lambda \in \mathbb{Z}. \tag{2.12}$$

Then $\|\mathbf{x}\|_2 = +\infty$ by (2.12) and the assumption (ii). Set

$$\mathbf{x}_N := (\mathbf{x}(\lambda)\chi_{[-N,N]}(\lambda))_{\lambda\in\mathbb{Z}}, \ N \ge 2.$$

Then we obtain from (1.5) and (2.12) that

$$\begin{cases} \lim_{N\to\infty} \|\mathbf{x}_N\|_2 = \infty, \\ \|\mathbf{M}_{a,c}(t_0)\mathbf{x}_N\|_\infty \le \|\mathbf{M}_{a,c}(t_0)\mathbf{1}\|_\infty \le c+1, \text{ and} \\ (\mathbf{M}_{a,c}(t_0)\mathbf{x}_N)(\mu) = 0 \quad \text{for all } \ \mu - t_0 \notin [N-c, N] \cup [-N-c, -N]. \end{cases}$$

Therefore

$$\lim_{N\to\infty} \frac{\|\mathbf{M}_{a,c}(t_0)\mathbf{x}_N\|_2}{\|\mathbf{x}_N\|_2} = 0,$$

which contradicts to the assumption (ii).

Then we establish the implication (iii)⟹(iv). To do so, we need a technical lemma about binary solutions of the infinite-dimensional linear system (1.9).

LEMMA 2.6. *Let* $0 < a < 1 < c, t \in \mathbb{R}$ *and* $\mathbf{x} := (\mathbf{x}(\lambda))_{\lambda\in\mathbb{Z}} \in \mathcal{B}^0$ *satisfy* $\mathbf{M}_{a,c}(t)\mathbf{x} = \mathbf{2}$. *Then there exist binary vectors* $\mathbf{x}_1 \in \mathcal{B}^0$ *and* $\mathbf{x}_2 \in \mathcal{B}\backslash\mathcal{B}^0$ *such that* (1.11) *holds.*

PROOF. Let K be the set of all $\lambda \in \mathbb{Z}$ with $\mathbf{x}(\lambda) = 1$, and write $K = \{\lambda_j : j \in \mathbb{Z}\}$ for a strictly increasing sequence $\{\lambda_j\}_{j=-\infty}^{\infty}$ with $\lambda_0 = 0$. For any $\mu \in a\mathbb{Z}$, it follows from $\mathbf{M}_{a,c}(t_0)\mathbf{x} = \mathbf{2}$ that $K \cap (-t_0 + \mu + [0, c))$ is either $\{\lambda_{2j}, \lambda_{2j+1}\}$ or $\{\lambda_{2j-1}, \lambda_{2j}\}$ for some $j \in \mathbb{Z}$. One may then verify that $\mathbf{x}_l^* := (x_l^*(\lambda))_{\lambda\in\mathbb{Z}}, l = 1, 2$, defined by $\mathbf{x}_l^*(\lambda) = 1$ if $\lambda = \lambda_{2j-l+1}$ for some integer j and $\mathbf{x}_l^*(\lambda) = 0$ otherwise, are binary vectors satisfying (1.11). $\square$

Let us return to the proof of the implication (iii)$\Longrightarrow$(iv). Suppose, on the contrary, that there exist $t_0 \in \mathbb{R}$ and a vector $\mathbf{x} \in \mathcal{B}^0$ such that $\mathbf{M}_{a,c}(t_0)\mathbf{x} = \mathbf{2}$. Let $\mathbf{x}_1, \mathbf{x}_2$ be the binary vectors satisfying (1.11). The existence of such binary vectors follows from Lemma 2.6. Then $\mathbf{z}^* := \mathbf{x}_1 - \mathbf{x}_2$ is a nonzero trinary vector in the null space $N(\mathbf{M}_{a,c}(t_0))$, which contradicts to the assumption (iii).

Finally we prove the implication (iii)$\Longrightarrow$(iv). This is the most technical part of the whole proof. We need the stability inequality (2.13).

LEMMA 2.7. *Let* $0 < a < 1 < c$ *and* $Q_{a,c}$ *be as in* (2.7). *If* $Q_{a,c} < +\infty$, *then*

$$\sum_{0\le\mu\le aQ_{a,c}+a+c+1} |(\mathbf{M}_{a,c}(t)\mathbf{z})(\mu)| \ge \frac{1}{2c}|\mathbf{z}(0)| \tag{2.13}$$

for all $t \in [0, 1)$ *and vectors* $\mathbf{z} = (\mathbf{z}(\lambda))_{\lambda\in\mathbb{Z}}$.

PROOF. For $t \in [0, 1)$, let $\lambda_0 = 0, \mu_0 = \lfloor t/a\rfloor a$ and let $\delta_0 \ge 0$ be the integer in $[c + \mu_0 - t - 1, c + \mu_0 - t)$. If $\delta_0 = 0$, then (2.13) holds as $|(\mathbf{M}_{a,c}(t)\mathbf{z})(\mu_0)| = |\mathbf{z}(0)|$ and $\mu_0 \le t \le aQ_{a,c} + a + c + 1$.

Now we prove (2.13) in the case that $\delta_0 \ge 1$. Take an integer $\lambda^* \in [1, \delta_0]$ with

$$|\mathbf{z}(\lambda^*)| = \max_{1\le\lambda\le\delta_0} |\mathbf{z}(\lambda)|. \tag{2.14}$$

Let us construct a binary vector $\mathbf{x} \in \mathcal{B}^0$ such that $\mathbf{x}(0) = \mathbf{x}(\lambda^*) = 1$, $\mathbf{x}(\lambda) = 0$ for all $\lambda < 0$, and $\mathbf{M}_{a,c}(t)\mathbf{x}$ has maximal length of consecutive twos. To do so, we introduce families of triples $(\lambda_k, \mu_k, \delta_k) \in \mathbb{Z} \times a\mathbb{Z} \times \mathbb{Z}, 0 \le k \le M$, iteratively. For $k = 0$, we let $\lambda_0 = 0, \mu_0 = \lfloor t/a\rfloor a$, and δ_0 be the unique integer in $[c + \mu_0 - t - 1, c + \mu_0 - t)$. Similarly for $k = 1$, we let $\lambda_1 = \lambda^*, \mu_1 = \lfloor(t + \lambda_1)/a\rfloor a$ and δ_1 be the unique integer in $[c + \mu_1 - t - 1, c + \mu_1 - t)$. Inductively suppose that we have defined all triples $(\lambda_m, \mu_m, \delta_m)$ with $m \le k$, we set $M = k$ if $\delta_k \ge c + \mu_k - t + a - 1$, and otherwise we define the triple $(\lambda_{k+1}, \mu_{k+1}, \delta_{k+1})$ by $\lambda_{k+1} = \delta_{k-1} + 1, \mu_{k+1} = \lfloor(t + \lambda_{k+1})/a\rfloor a$ and $\delta_{k+1} \in [c + \mu_{k+1} - t - 1, c + \mu_{k+1} - t) \cap \mathbb{Z}$. By the above construction of triples $(\lambda_k, \mu_k, \delta_k), 0 \le k \le M$,

$$\begin{cases} \lambda_k \in [\mu_k - t, \mu_k - t + a) & \text{if } 0 \le k \le M \\ \lambda_{k+2} \in [c + \mu_k - t, c + \mu_k - t + a) & \text{if } 0 \le k \le M - 2, \end{cases} \tag{2.15}$$

$$[\mu_M - t + c, \mu_M - t + c + a) \cap \mathbb{Z} = \emptyset \quad \text{if } M < \infty, \tag{2.16}$$

and

$$\{\lambda_k\}_{k=0}^{M} \text{ and } \{\mu_k\}_{k=0}^{M} \text{ are strictly increasing sequences.} \tag{2.17}$$

Define $\mathbf{x} := (\mathbf{x}(\lambda))_{\lambda\in\mathbb{Z}}$ by $\mathbf{x}(\lambda) = 1$ if $\lambda = \lambda_k$ for some $0 \le k \le M$, and $\mathbf{x}(\lambda) = 0$ otherwise. Then $\mathbf{x} \in \mathcal{B}^0$ by (2.17), and for $\mu_0 \le \mu \le \mu_{M-1}$,

$$\begin{aligned}(\mathbf{M}_{a,c}(t)\mathbf{x})(\mu) &= \Big(\sum_{0\le k\le M, k \text{ even}} + \sum_{0\le k\le M, k \text{ odd}}\Big)\chi_{[0,c)}(t-\mu+\lambda_k)\\ &= \chi_{[0,\mu_0]}(\mu) + \sum_{2\le k\le M, k \text{ even}} \chi_{(\mu_{k-2},\mu_k]}(\mu)\\ &\quad + \chi_{(t+\lambda_1-c,\mu_1]}(\mu) + \sum_{3\le k\le M, k \text{ odd}} \chi_{(\mu_{k-2},\mu_k]}(\mu) = 2,\end{aligned}$$

where the second equation follows from

$$[\mu_{k-2}+a, \mu_k] \subset (t+\lambda_k - c, t+\lambda_k] \subset (\mu_{k-2}, \mu_k + a), 2 \le k \le M$$

which holds by (2.15). Thus maximal length of consecutive twos for the vector $\mathbf{M}_{a,c}(t)\mathbf{x}$ is at least $(\mu_{M-1} - \mu_0 + a)/a$, which leads to the following estimate:

$$\mu_{M-1} - \mu_0 + a \le aQ_{a,c}. \tag{2.18}$$

By (2.15) and (2.17),

$$\mu_M - \mu_{M-1} \le \mu_M - \mu_{M-2} \le \lambda_M + t - (\lambda_M - c + t - a) \le a + c. \tag{2.19}$$

Combining (2.18) and (2.19) and recalling $\mu_0 \le t < 1$, we have

$$\mu_M \le aQ_{a,c} + c + 1. \tag{2.20}$$

By (2.20), $M < \infty$. Applying (2.15) and (2.16), we obtain

$$|(\mathbf{M}_{a,c}(t)\mathbf{z})(\mu_k)| + |(\mathbf{M}_{a,c}(t)\mathbf{z})(\mu_k + a)| \ge |\mathbf{z}(\lambda_{k+2}) - \mathbf{z}(\lambda_k)| \tag{2.21}$$

for all integers $0 \le k \le M-2$, and

$$|(\mathbf{M}_{a,c}(t)\mathbf{z})(\mu_M)| + |(\mathbf{M}_{a,c}(t)\mathbf{z})(\mu_M + a)| \ge |\mathbf{z}(\lambda_M)|. \tag{2.22}$$

By (2.14), (2.20), (2.21) and (2.22), we get

$$\begin{aligned}&2\sum_{0\le\mu\le aQ_{a,c}+a+c+1} |(\mathbf{M}_{a,c}(t)\mathbf{z})(\mu)|\\ \ge\ & \sum_{k=0}^{M/2} |(\mathbf{M}_{a,c}(t)\mathbf{z})(\mu_{2k})| + |(\mathbf{M}_{a,c}(t)\mathbf{z})(\mu_{2k}+a)|\\ \ge\ & \sum_{k=0}^{M/2-1} |\mathbf{z}(\lambda_{2k+2}) - \mathbf{z}(\lambda_{2k})| + |\mathbf{z}(\lambda_M)| \ge |\mathbf{z}(\lambda_0)| = |\mathbf{z}(0)|\end{aligned}$$

if M is even, and

$$\begin{aligned}&2\delta_0 \sum_{0\le\mu\le aQ_{a,c}+a+c+1} |(\mathbf{M}_{a,c}(t)\mathbf{z})(\mu)|\\ \ge\ & |(\mathbf{M}_{a,c}(t)\mathbf{z})(\mu_0)| + \delta_0 \sum_{k=0}^{(M-1)/2} \big(|(\mathbf{M}_{a,c}(t)\mathbf{z})(\mu_{2k+1})|\\ &+ |(\mathbf{M}_{a,c}(t)\mathbf{z})(\mu_{2k+1}+a)|\big)\\ \ge\ & \Big|\sum_{0\le\lambda\le\delta_0} \mathbf{z}(\lambda)\Big| + \delta_0|\mathbf{z}(\lambda^*)| \ge |\mathbf{z}(0)|\end{aligned}$$

if M is odd. This proves (2.13) in the case that $\delta_0 \ge 1$. □

Let's return the proof of the implication (iv)$\Longrightarrow$(i). Let $Q_{a,c}$ be as in (2.7). Then $Q_{a,c}<\infty$ by Theorem 2.5. For any $f\in L^2$,

$$\begin{aligned}
&\Big(Q_{a,c}+\frac{2a+c+1}{a}\Big)^2\sum_{\phi\in\mathcal{G}(\chi_{[0,c)},a\mathbb{Z}\times\mathbb{Z})}|\langle f,\phi\rangle|^2\\
\geq\ & \Big(Q_{a,c}+\frac{2a+c+1}{a}\Big)\sum_{\mu\in a\mathbb{Z}}\ \sum_{0\le\mu'\le aQ_{a,c}+a+c+1}\ \sum_{n\in\mathbb{Z}}\\
&\quad\Big|\int_0^1\Big(\sum_{\lambda\in\mathbb{Z}}\chi_{[0,c)}(t-\mu'+\lambda)f(t+\mu+\lambda)\Big)e^{-2\pi int}dt\Big|^2\\
=\ & \sum_{\mu\in a\mathbb{Z}}\int_0^1\Big(\big(Q_{a,c}+\frac{2a+c+1}{a}\big)\sum_{0\le\mu'\le aQ_{a,c}+a+c+1}\\
&\quad\Big|\sum_{\lambda\in\mathbb{Z}}\chi_{[0,c)}(t-\mu'+\lambda)f(t+\mu+\lambda)\Big|^2\Big)dt\\
\geq\ & \sum_{\mu\in a\mathbb{Z}}\int_0^1\Big(\sum_{0\le\mu'\le aQ_{a,c}+a+c+1}\Big|\sum_{\lambda\in\mathbb{Z}}\chi_{[0,c)}(t-\mu'+\lambda)f(t+\mu+\lambda)\Big|\Big)^2dt\\
\geq\ & \frac{1}{4c^2}\sum_{\mu\in a\mathbb{Z}}\int_0^1|f(t+\mu)|^2dt\geq\frac{\lfloor 1/a\rfloor}{4c^2}\|f\|_2^2
\end{aligned}$$

where the third inequality follows from Lemma 2.7, and

$$\begin{aligned}
&\sum_{\phi\in\mathcal{G}(\chi_{[0,c)},a\mathbb{Z}\times\mathbb{Z})}|\langle f,\phi\rangle|^2\\
=\ & \sum_{\mu\in a\mathbb{Z}}\int_0^1\Big|\sum_{\lambda\in\mathbb{Z}}\chi_{[0,c)}(t+\lambda)f(t+\lambda+\mu)\Big|^2dt\\
\leq\ & (\lfloor c\rfloor+1)\sum_{\mu\in a\mathbb{Z}}\int_0^1\sum_{\lambda\in\mathbb{Z}}\chi_{[0,c)}(t+\lambda)|f(t+\lambda+\mu)|^2dt\\
\leq\ & (\lfloor c\rfloor+1)(\lfloor c/a\rfloor+1)\|f\|_2^2.
\end{aligned}$$

Hence $\mathcal{G}(\chi_{[0,c)},a\mathbb{Z}\times\mathbb{Z})$ is a Gabor frame. This completes the proof of the implication (iv)$\Longrightarrow$(i) and the proof of Theorem 2.1. $\square$

Remark 2.8. From the argument used to prove the implication (iv)$\Longrightarrow$(i) of Theorem 2.1, we have the following frame bound estimate for the Gabor frame $\mathcal{G}(\chi_{[0,c)},a\mathbb{Z}\times\mathbb{Z})$ via the maximal length $Q_{a,c}$ of consecutive twos in range spaces of infinite matrices $\mathbf{M}_{a,c}(t), t\in\mathbb{R}$:

$$\begin{aligned}
\frac{a^2\lfloor 1/a\rfloor}{4c^2(aQ_{a,c}+2a+c+1)^2}\ &\leq\ \inf_{\|f\|_2=1}\Big(\sum_{\phi\in\mathcal{G}(\chi_{[0,c)},a\mathbb{Z}\times\mathbb{Z})}|\langle f,\phi\rangle|^2\Big)^{1/2}\\
&\leq\ \sup_{\|f\|_2=1}\Big(\sum_{\phi\in\mathcal{G}(\chi_{[0,c)},a\mathbb{Z}\times\mathbb{Z})}|\langle f,\phi\rangle|^2\Big)^{1/2}\\
&\leq\ (\lfloor c\rfloor+1)(\lfloor c/a\rfloor+1). \qquad (2.23)
\end{aligned}$$

CHAPTER 3

Maximal Invariant Sets

The set $\mathcal{D}_{a,c}$ in (1.10) can be used to characterize frame property of the Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})$, and it can be obtained from the set $\mathcal{S}_{a,c}$ in (1.13) by set operations, see Theorems 2.1 and 2.3. In this chapter, we consider various properties of the set $\mathcal{S}_{a,c}$. The advantages to study the set $\mathcal{S}_{a,c}$ instead of the set $\mathcal{D}_{a,c}$ include:

1) For $t\in\mathcal{S}_{a,c}$, there is a **unique** binary solution $\mathbf{x}\in\mathcal{B}^0$ to the linear system $\mathbf{M}_{a,c}(t)\mathbf{x}=\mathbf{1}$, while for $t\in\mathcal{D}_{a,c}$ multiple binary solutions $\mathbf{y}\in\mathcal{B}^0$ could exist for the linear system $\mathbf{M}_{a,c}(t)\mathbf{y}=\mathbf{2}$, see Lemma 3.12.
2) Both $\mathcal{S}_{a,c}$ and $\mathcal{D}_{a,c}$ are invariant under the transformation $R_{a,c}$ and have empty intersection with its black hole, but $\mathcal{S}_{a,c}$ is its **maximal** invariant set, see (3.7) and Theorem 3.4.
3) Both $\mathcal{S}_{a,c}$ and $\mathcal{D}_{a,c}$ can be constructed explicitly by finite steps, but Hutchinson's remarkable construction applies **only** for the set $\mathcal{S}_{a,c}$, see Theorems 2.3, 4.1, 5.2, 6.3, 6.4 and 6.5.
4) The set $\mathcal{S}_{a,c}$ can be fully parameterized, see Theorems 5.5 and 6.8.

The set $\mathcal{S}_{a,c}$ has period a,

$$\mathcal{S}_{a,c}=\mathcal{S}_{a,c}+a\mathbb{Z} \tag{3.1}$$

by the time-shift property (1.8); it is a supset of the set $\mathcal{D}_{a,c}$ in (1.10),

$$\mathcal{D}_{a,c}\subset\mathcal{S}_{a,c} \tag{3.2}$$

by the decomposition (1.11), which is confirmed in Lemma 2.6; and it is not an empty set if and only if it contains some particular points, cf. Theorem 2.2 for the set $\mathcal{D}_{a,c}$.

THEOREM 3.1. *Let $0<a<1<c$ and define*

$$\Omega_{a,c}=\begin{cases}\{0\} & \text{if } a\notin\mathbb{Q},\\ \{0, c-(\lfloor c/\gcd(a,1)\rfloor+1)\gcd(a,1)\} & \text{if } a\in\mathbb{Q}.\end{cases}$$

Then $\mathcal{S}_{a,c}\neq\emptyset$ if and only if $\mathcal{S}_{a,c}\cap\Omega_{a,c}\neq\emptyset$.

The set $\mathcal{S}_{a,c}$ is either an empty set or its $(\lfloor c\rfloor+1)$ copies cover the whole line.

THEOREM 3.2. *Let (a,c) satisfy $0<a<1<c$, and either 1) $a\notin\mathbb{Q}$ or 2) $a\in\mathbb{Q}$ and $c\in\gcd(a,1)\mathbb{Z}$. Assume that $\mathcal{S}_{a,c}\neq\emptyset$. Then*

$$\big(\mathcal{S}_{a,c}\cap([0,(c-\lfloor c\rfloor+a-1)_+)+a\mathbb{Z})+\lfloor c\rfloor\big)\cup\big(\cup_{k=0}^{\lfloor c\rfloor-1}(\mathcal{S}_{a,c}+k)\big)=\mathbb{R}. \tag{3.3}$$

As an application of the covering property in Theorem 3.2, we have that $\mathcal{D}_{a,c}=\emptyset$ if and only if the covering in (3.3) is mutually disjoint, or equivalently the sum of measurement of their restrictions onto one period is a.

THEOREM 3.3. *Let (a, c) satisfy $0 < a < 1 < c$, and either 1) $a \notin \mathbb{Q}$ or 2) $a \in \mathbb{Q}$ and $c \in \gcd(a, 1)\mathbb{Z}$. Assume that $\mathcal{S}_{a,c} \neq \emptyset$. Then $\mathcal{D}_{a,c} = \emptyset$ if and only if*

$$\lfloor c \rfloor |\mathcal{S}_{a,c} \cap [0, a)| + |\mathcal{S}_{a,c} \cap [0, (c - \lfloor c \rfloor + a - 1)_+)| = a. \tag{3.4}$$

The set $\mathcal{S}_{a,c}$ has empty intersection with black holes of transformations $R_{a,c}$ and $\tilde{R}_{a,c}$; and it is a maximal set that is invariant under the transformation $R_{a,c}$ and that has empty intersection with its black hole.

THEOREM 3.4. *Let $0 < a < 1 < c$ and set $c_0 = c - \lfloor c \rfloor$. Then the following statements hold.*

(i) *The set $\mathcal{S}_{a,c}$ has empty intersection with black holes of transformations $R_{a,c}$ and $\tilde{R}_{a,c}$,*

$$\begin{cases} \mathcal{S}_{a,c} \cap ([(c_0 + a - 1)_+, a + (c_0 - a)_-) + a\mathbb{Z}) = \emptyset \\ \mathcal{S}_{a,c} \cap ([c - (c_0 - a)_-, c + a - (c_0 + a - 1)_+) + a\mathbb{Z}) = \emptyset. \end{cases} \tag{3.5}$$

(ii) *The set $\mathcal{S}_{a,c}$ is invariant under transformations $R_{a,c}$ and $\tilde{R}_{a,c}$,*

$$R_{a,c}\mathcal{S}_{a,c} = \mathcal{S}_{a,c} \quad \text{and} \quad \tilde{R}_{a,c}\mathcal{S}_{a,c} = \mathcal{S}_{a,c}. \tag{3.6}$$

(iii) *Any set E satisfying $R_{a,c}E = E$ and having empty intersection with the black hole $[(c_0 + a - 1)_+, a + (c_0 - a)_-) + a\mathbb{Z}$ of the transformation $R_{a,c}$ is contained in $\mathcal{S}_{a,c}$.*

The maximal invariance property for the set $\mathcal{S}_{a,c}$ is **crucial** in our study. So we call the set $\mathcal{S}_{a,c}$ as *maximal invariant set*. We remark that it follows from (1.11) and (3.6) that the set $\mathcal{D}_{a,c}$ in (1.10) is also invariant under transformations $R_{a,c}$ and $\tilde{R}_{a,c}$,

$$R_{a,c}\mathcal{D}_{a,c} = \mathcal{D}_{a,c} \quad \text{and} \quad \tilde{R}_{a,c}\mathcal{D}_{a,c} = \mathcal{D}_{a,c}. \tag{3.7}$$

For some pairs (a, c) of positive numbers, applying the maximal invariance in Theorem 3.4 gives explicit expression for the maximal invariant set $\mathcal{S}_{a,c}$.

THEOREM 3.5. *Let $0 < a < 1 < c$, and set*

$$c_0 = c - \lfloor c \rfloor, \quad c_1 = c - c_0 - \lfloor (c - c_0)/a \rfloor a.$$

Then the following statements hold.

(i) *If $c_0 = 0$, then*

$$\mathcal{S}_{a,c} = \mathbb{R}. \tag{3.8}$$

(ii) *If $c_0 \geq a$ and $c_0 \leq 1 - a$, then*

$$\mathcal{S}_{a,c} = \emptyset. \tag{3.9}$$

(iii) *If $c_0 \geq a$ and $c_0 > 1 - a$, then $\mathcal{S}_{a,c} \neq \emptyset$ if and only if $a \in \mathbb{Q}$ and $c_0 > 1 - \gcd(\lfloor c \rfloor + 1, a)$. Furthermore,*

$$\mathcal{S}_{a,c} = [-\gcd(\lfloor c \rfloor + 1, a), c_0 - 1) + \gcd(\lfloor c \rfloor + 1, a)\mathbb{Z}. \tag{3.10}$$

(iv) *If $0 < c_0 < a$ and $c_0 \leq 1 - a$, then $\mathcal{S}_{a,c} \neq \emptyset$ if and only if $a \in \mathbb{Q}$ and $c_0 < \gcd(\lfloor c \rfloor, a)$. Furthermore,*

$$\mathcal{S}_{a,c} = [c_0, \gcd(\lfloor c \rfloor, a)) + \gcd(\lfloor c \rfloor, a)\mathbb{Z}. \tag{3.11}$$

(v) *If $0 < c_0 < a, a < c_0 < 1 - a$ and $c_1 > 1 - 2a$, then $\mathcal{S}_{a,c} = \emptyset$.*

(vi) *If* $0 < c_0 < a, a < c_0 < 1 - a$ *and* $c_1 = 1 - 2a$*, then*

$$\mathcal{S}_{a,c} = [0, c_0 + a - 1) + a\mathbb{Z}. \tag{3.12}$$

(vii) *If* $0 < c_0 < a, a < c_0 < 1 - a$ *and* $c_1 = 0$*, then*

$$\mathcal{S}_{a,c} = [c_0, a) + a\mathbb{Z}. \tag{3.13}$$

Having the above expression of the set $\mathcal{S}_{a,c}$ (hence the set $\mathcal{D}_{a,c}$ by Theorem 2.3), we can apply Theorem 2.1 to determine whether Gabor systems $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ corresponding to those pairs with either $c_1 \geq 1 - 2a$ or $c_1 = 0$ are frames for L^2, see Theorem 7.3 for details.

This chapter is organized as follows. In Section 3.1, we start from a **pivotal** observation to binary solutions of the infinite-dimensional linear system (1.12) (Lemma 3.7) and a **crucial** characterization of the maximal invariant set $\mathcal{S}_{a,c}$ (Lemma 3.9), and we then use them to prove Theorem 3.4. In Section 3.2, we apply the maximal invariance property and the empty intersection property in Theorem 3.4 to prove Theorem 3.5. In Section 3.3, we study density of the maximal invariant set $\mathcal{S}_{a,c}$ around the origin (see Lemmas 3.10 and 3.11) and use it to prove Theorem 3.1. We use the last two sections to prove Theorems 2.2 and 2.3 of Chapter 2. We postpone the proof of Theorems 3.2 and 3.3 to Section 4.3 of Chapter 4, as we need the property that $\mathcal{S}_{a,c} \cap [0, a)$ is union of finitely many left-closed right-open intervals, which follows from Theorem 4.1.

3.1. Maximality of invariant sets

Let $0 < a < 1 < c$. Define

$$\mathcal{A}_n := (R_{a,c})^n([c - (c_0 - a)_-, c + a - (c_0 + a - 1)_+) + a\mathbb{Z}) \tag{3.14}$$

and

$$\tilde{\mathcal{A}}_n := (\tilde{R}_{a,c})^n([(c_0 + a - 1)_+, a + (c_0 - a)_-) + a\mathbb{Z}), n \geq 0. \tag{3.15}$$

In this section, we prove Theorem 3.4 and the following proposition about holes $\mathcal{A}_n$ and $\tilde{\mathcal{A}}_n, n \geq 0$.

Proposition 3.6. *Let* $0 < a < 1 < c$*, and let* $\mathcal{A}_n$ *and* $\tilde{\mathcal{A}}_n, n \geq 0$*, be as in* (3.14) *and* (3.15) *respectively. Then*

$$\mathcal{A}_n \cap \mathcal{S}_{a,c} = \emptyset \quad \text{and} \quad \tilde{\mathcal{A}}_n \cap \mathcal{S}_{a,c} = \emptyset \quad \text{for all } n \geq 0, \tag{3.16}$$

$$\mathcal{A}_n \cap \mathcal{A}_{n'} \subset [(c_0 + a - 1)_+, a + (c_0 - a)_-) + a\mathbb{Z} \quad \text{for all } n \neq n', \tag{3.17}$$

and

$$\tilde{\mathcal{A}}_n \cap \tilde{\mathcal{A}}_{n'} \subset [c - (c_0 - a)_-, c + a - (c_0 + a - 1)_+) + a\mathbb{Z} \quad \text{for all } n \neq n'. \tag{3.18}$$

To prove them, we need several lemmas about the linear system (1.12), invariant sets of transformations $R_{a,c}$ and $\tilde{R}_{a,c}$, and a characterization for a real number belonging to the set $\mathcal{S}_{a,c}$.

Lemma 3.7. *Let* $0 < a < 1 < c$*. Then for any* $t \in \mathcal{S}_{a,c}$ *and* $\mathbf{x} = (\mathbf{x}(\lambda))_{\lambda \in \mathbb{Z}} \in \mathcal{B}^0$ *satisfying* $\mathbf{M}_{a,c}(t)\mathbf{x} = \mathbf{1}$*,*

$$\mathbf{x}(\lambda) = \begin{cases} 0 & \text{if } \tilde{R}_{a,c}(t) - t < \lambda < R_{a,c}(t) - t \text{ and } \lambda \neq 0, \\ 1 & \text{if } \lambda = R_{a,c}(t) - t, 0, \tilde{R}_{a,c}(t) - t. \end{cases} \tag{3.19}$$

PROOF. By (1.8), we may assume that $t \in [0, a)$. Let λ_1 be the smallest positive integer such that $\mathbf{x}(\lambda_1) = 1$. Then

$$\lambda_1 \geq \lfloor c \rfloor$$

because

$$1 = \chi_{[0,c)}(t) \leq \chi_{[0,c)}(t) + \chi_{[0,c)}(t + \lambda_1) \leq \sum_{\lambda \in \mathbb{Z}} \chi_{[0,c)}(t + \lambda)\mathbf{x}(\lambda) = 1; \tag{3.20}$$

and

$$\lambda_1 \leq \lfloor c \rfloor + 1$$

since otherwise

$$\sum_{\lambda \in \mathbb{Z}} \chi_{[0,c)}(t - a + \lambda)\mathbf{x}(\lambda) = 0.$$

If $\lambda_1 = \lfloor c \rfloor$, then $t \geq c_0$ by (3.20); and if $\lambda_1 = \lfloor c \rfloor + 1$, then $t < c_0 + a - 1$ as

$$1 = \sum_{\lambda \in \mathbb{Z}} \chi_{[0,c)}(t - a + \lambda)\mathbf{x}(\lambda) = \chi_{[0,c)}(t - a + \lfloor c \rfloor + 1).$$

Thus

$$t \notin [(c_0 + a - 1)_+, a + (c_0 - a)_-) \quad \text{and} \quad \lambda_1 = R_{a,c}(t) - t.$$

This implies that

$$\mathcal{S}_{a,c} \cap \big([(c_0 + a - 1)_+, a + (c_0 - a)_-) + a\mathbb{Z}\big) = \emptyset \tag{3.21}$$

and

$$\mathbf{x}(\lambda) = \begin{cases} 0 & \text{if } 0 < \lambda < R_{a,c}(t) - t, \\ 1 & \text{if } \lambda = R_{a,c}(t) - t. \end{cases} \tag{3.22}$$

For the above vector $\mathbf{x} \in \mathcal{B}^0$ satisfying $\mathbf{M}_{a,c}(t)\mathbf{x} = \mathbf{1}$, one may verify that

$$\tilde{\mathbf{M}}_{a,c}(c - t)\tilde{\mathbf{x}} = \mathbf{1},$$

where

$$\tilde{\mathbf{M}}_{a,c}(t) = (\chi_{(0,c]}(t - \mu + \lambda))_{\mu \in a\mathbb{Z}, \lambda \in \mathbb{Z}}, \ t \in \mathbb{R}, \tag{3.23}$$

and $\tilde{\mathbf{x}} = (\mathbf{x}(-\lambda))_{\lambda \in \mathbb{Z}} \in \mathcal{B}^0$. Mimicking the argument used to establish (3.21) and (3.22), we obtain that

$$\mathcal{S}_{a,c} \cap \big([c - a - (c_0 - a)_-, c - (c_0 + a - 1)_+) + a\mathbb{Z}\big) = \emptyset, \tag{3.24}$$

and

$$\tilde{\mathbf{x}}(\lambda) = \begin{cases} 0 & \text{if } 0 < \lambda < t - \tilde{R}_{a,c}(t) \\ 1 & \text{if } \lambda = t - \tilde{R}_{a,c}(t). \end{cases} \tag{3.25}$$

Combining (3.22) and (3.25) proves (3.19). □

Let E have empty intersection with the black hole $[(c_0 + a - 1)_+, a + (c_0 - a)_-) + a\mathbb{Z}$. Then the invariance $R_{a,c}(E) = E$ of the transformation $R_{a,c}$ implies that

$$R_{a,c}(E) \subset E \ \text{ and } \ \tilde{R}_{a,c}(E) \subset E \tag{3.26}$$

by the first equation in (1.17). The converse is true by the second equation in (1.17) if we further assume that E has empty intersection with the black hole $[c - (c_0 - a)_-, c + a - (c_0 + a - 1)_+) + a\mathbb{Z}$ of the transformation $\tilde{R}_{a,c}$. This, together with (1.17), leads to the following characterization of invariant sets of transformations $R_{a,c}$ and $\tilde{R}_{a,c}$.

LEMMA 3.8. *Let* $0 < a < 1 < c$. *If*

$$E\cap([(c_0+a-1)_+, a+(c_0-a)_-)+a\mathbb{Z}) = E\cap([c-(c_0-a)_-, c+a-(c_0+a-1)_+)+a\mathbb{Z}) = \emptyset,$$

then $R_{a,c}(E) = E$ *if and only if* $\tilde{R}_{a,c}(E) = E$ *if and only if* (3.26) *holds.*

The following characterization of the set $\mathcal{S}_{a,c}$ is **important** for us to establish the maximality of the invariant set $\mathcal{S}_{a,c}$.

LEMMA 3.9. *Let* $0 < a < 1 < c$. *Then* $t \notin \mathcal{S}_{a,c}$ *if and only if either* $(R_{a,c})^n(t) \in [(c_0 + a - 1)_+, a + (c_0 - a)_-) + a\mathbb{Z}$ *for some* $n \geq 0$ *or* $(\tilde{R}_{a,c})^m(t) \in [c - a - (c_0 - a)_-, c - (c_0 + a - 1)_+) + a\mathbb{Z}$ *for some* $m \geq 0$.

PROOF. ($\Longleftarrow$) For any $t \in \mathcal{S}_{a,c}$ and $\mathbf{x} \in \mathcal{B}^0$ satisfying $\mathbf{M}_{a,c}(t)\mathbf{x} = \mathbf{1}$, it follows from (1.7) and Lemma 3.7 that

$$\mathbf{M}_{a,c}(R_{a,c}(t))\tau_{R_{a,c}(t)-t}\mathbf{x} = \mathbf{M}_{a,c}(\tilde{R}_{a,c}(t))\tau_{\tilde{R}_{a,c}(t)-t}\mathbf{x} = \mathbf{M}_{a,c}(t)\mathbf{x} = \mathbf{1}.$$

Thus

$$R_{a,c}\mathcal{S}_{a,c} \subset \mathcal{S}_{a,c} \quad \text{and} \quad \tilde{R}_{a,c}\mathcal{S}_{a,c} \subset \mathcal{S}_{a,c}. \tag{3.27}$$

This together with (3.21) and (3.24) proves the sufficiency.

($\Longrightarrow$) Take $t \notin \mathcal{S}_{a,c}$. Suppose, on the contrary, that $(R_{a,c})^n(t) \notin [(c_0 + a - 1)_+, a + (c_0 - a)_-) + a\mathbb{Z}$ for all $n \geq 0$ and $(\tilde{R}_{a,c})^m(t) \notin [c - a - (c_0 - a)_-, c - (c_0 + a - 1)_+) + a\mathbb{Z}$ for all $m \geq 0$. Define

$$t_n = \begin{cases} (R_{a,c})^n(t) & \text{if } n \geq 1 \\ t & \text{if } n = 0 \\ (\tilde{R}_{a,c})^{-n}(t) & \text{if } n \leq -1, \end{cases}$$

and $\lambda_n = t_n - t, n \in \mathbb{Z}$. Then

$$t_{n+m} = (R_{a,c})^m(t_n) \quad \text{for all } n \in \mathbb{Z} \text{ and } 0 \leq m \in \mathbb{Z}$$

and

$$\lambda_n \in \mathbb{Z} \text{ and } \lambda_{n+1} - \lambda_n \in \{\lfloor c \rfloor, \lfloor c \rfloor + 1\} \quad \text{for all } n \in \mathbb{Z} \tag{3.28}$$

by the definitions (1.15) and (1.16) of transformations $R_{a,c}$ and $\tilde{R}_{a,c}$, and the left-inverse properties (1.17) between them. Define $\mathbf{x} := (\mathbf{x}(\lambda))_{\lambda\in\mathbb{Z}} \in \mathcal{B}^0$ by $\mathbf{x}(\lambda) = 1$ if $\lambda = \lambda_n$ for some $n \in \mathbb{Z}$ and $\mathbf{x}(\lambda) = 0$ otherwise, and let $\mu_n \in a\mathbb{Z}$ be so chosen that $\tilde{t}_n := t_n - \mu_n \in [0, a)$. Then $\{\mu_n\}_{n\in\mathbb{Z}}$ is a strictly increasing sequence with

$$\lim_{n\to+\infty} \mu_n = +\infty \text{ and } \lim_{n\to-\infty} \mu_n = -\infty \tag{3.29}$$

by (3.28), and

$$\begin{aligned} &\sum_{\lambda\in\mathbb{Z}} \chi_{[0,c)}(t - \mu_n + \lambda)\mathbf{x}(\lambda) = \sum_{m\in\mathbb{Z}} \chi_{[0,c)}(t - \mu_n + \lambda_m) \\ = &\sum_{m\in\mathbb{Z}} \chi_{[0,c)}(t_m - \mu_n) = \chi_{[0,c)}(t_n - \mu_n) = 1 \quad \text{for all } n \in \mathbb{Z}, \end{aligned} \tag{3.30}$$

where the first equation follows from the definition of the vector $\mathbf{x}$ and the third one holds as

$$t_m - \mu_n \leq t_n - \mu_n - 1 < 0 \quad \text{for all} \quad m < n$$

and

$$t_m - \mu_n \geq (t_{n+1} - t_n) + (t_n - \mu_n) = (\lambda_{n+1} - \lambda_n) + (t_n - \mu_n) \geq c \quad \text{for all} \quad m > n.$$

Similarly for any $\mu \in a\mathbb{Z}$ with $\mu_n < \mu < \mu_{n+1}$,

$$\sum_{\lambda\in\mathbb{Z}} \chi_{[0,c)}(t-\mu+\lambda)\mathbf{x}(\lambda) = \sum_{m\in\mathbb{Z}} \chi_{[0,c)}(t_m-\mu) = 1 \tag{3.31}$$

as

$$t_m - \mu \le t_n - \mu < \mu_n + a - \mu \le 0 \quad \text{for} \quad m \le n,$$
$$0 \le t_{n+1} - \mu_{n+1} < t_m - \mu \le t_{n+1} - \mu_n - a < c \quad \text{for} \quad m = n+1,$$

and

$$t_m - \mu \ge t_{n+2} - \mu_{n+1} + a \ge c \quad \text{for} \quad m \ge n+2.$$

Combining (3.29), (3.30) and (3.31) proves $\mathbf{M}_{a,c}(t)\mathbf{x} = \mathbf{1}$, which contradicts to the assumption $t \notin \mathcal{S}_{a,c}$. □

Now we have all the ingredients to prove Theorem 3.4 and Proposition 3.6.

Proof of Theorem 3.4. (i): The empty-intersection property (3.5) for the set $\mathcal{S}_{a,c}$ has been given in (3.21) and (3.24).

(ii): The invariance (3.6) follows from (3.5), (3.27) and Lemma 3.8.

(iii): Take $t \in E$. Then

$$(R_{a,c})^n(t) \in E \quad \text{for all } n \ge 0 \tag{3.32}$$

by the invariance of the set E. By (1.18) and the invariance $E = R_{a,c}(E)$, we have that

$$E \cap \big([c-a-(c_0-a)_-, c-(c_0+a-1)_+) + a\mathbb{Z}\big) = \emptyset. \tag{3.33}$$

This together with the characterization in Lemma 3.8 implies that $\tilde{R}_{a,c}(E) \subset E$. Hence

$$(\tilde{R}_{a,c})^m(t) \in E \quad \text{for all } m \ge 0. \tag{3.34}$$

Combining (3.32), (3.33) and (3.34) with Lemma 3.9 proves that $t \in \mathcal{S}_{a,c}$. This proves the inclusion $E \subset \mathcal{S}_{a,c}$ and hence maximality of the invariant set $\mathcal{S}_{a,c}$. □

Proof of Proposition 3.6. Suppose, on the contrary, that the first equation in (3.16) does not hold. Then there exists a nonnegative integer m such that

$$(R_{a,c})^m([c-(c_0-a)_-, c+a-(c_0+a-1)_+) + a\mathbb{Z}) \cap \mathcal{S}_{a,c} \ne \emptyset$$

and

$$(R_{a,c})^n([c-(c_0-a)_-, c+a-(c_0+a-1)_+) + a\mathbb{Z}) \cap \mathcal{S}_{a,c} = \emptyset$$

for all $0 \le n < m$. We observe that m is a positive integer by (3.24). Take

$$t \in (R_{a,c})^m([c-(c_0-a)_-, c+a-(c_0+a-1)_+) + a\mathbb{Z}) \cap \mathcal{S}_{a,c}.$$

Then

$$t = R_{a,c}(s)$$

for some $s \in (R_{a,c})^{m-1}([c-(c_0-a)_-, c+a-(c_0+a-1)_+) + a\mathbb{Z})$. If $s \in [(c_0+a-1)_+, a+(c_0-a)_-) + a\mathbb{Z}$, then $t = s$ by (1.15), which contradicts to the assumption on m. If $s \notin [(c_0+a-1)_+, a+(c_0-a)_-) + a\mathbb{Z}$, then

$$s = \tilde{R}_{a,c}(t) \in \mathcal{S}_{a,c}$$

where the equality follows from (1.17) and the inclusion (3.6) in Theorem 3.4. Hence

$$s \in (R_{a,c})^{m-1}([c-(c_0-a)_-, c+a-(c_0+a-1)_+) + a\mathbb{Z}) \cap \mathcal{S}_{a,c},$$

which is a contradiction. This proves the first equality in (3.16).

The second equality in (3.16) can be established by using similar argument. We leave the detailed arguments to the reader.

Suppose that (3.17) does not hold. Then there exists nonnegative integers n, n' and $y \notin [(c_0+a-1)_, a+(c_0-a)_-)+a\mathbb{Z}$ such that $n > n'$ and $y \in \mathcal{A}_n \cap \mathcal{A}_{n'}$. Thus there exist $z_1, z_2 \in [c-(c_0-a)_-, c+a-(c_0+a-1)_+)+a\mathbb{Z}$ such that

$$y = (R_{a,c})^n(z_1) = (R_{a,c})^{n'}(z_2).$$

Applying (1.17) leads to

$$z_2 = (R_{a,c})^{n-n'}(z_1),$$

which contradicts to the range property (1.18) of the transformation $R_{a,c}$. This completes the proof of the mutually disjoint property (3.17) for holes $\mathcal{A}_n, n \geq 0$.

The mutually disjoint property (3.18) for holes $\tilde{\mathcal{A}}_n, n \geq 0$, can be proved by similar argument. □

3.2. Explicit construction of maximal invariant sets

In this section, we prove Theorem 3.5.

Proof of Theorem 3.5. (i): In this case, the black hole $[(c_0+a-1)_+, a+(c_0-a)_-)+a\mathbb{Z}$ of the transformation $R_{a,c}$ is the empty set. Then the conclusion (3.8) follows from the maximality given in Theorem 3.4.

(ii): In this case, the black hole $[(c_0+a-1)_+, a+(c_0-a)_-)+a\mathbb{Z}$ of the transformation $R_{a,c}$ is the whole line. Hence the empty set property (3.9) holds by the empty intersection property (3.5) in Theorem 3.4.

(iii): ($\Longrightarrow$) Take $t_0 \in \mathcal{S}_{a,c}$. Then

$$(R_{a,c})^n(t_0) = t_0 + n(\lfloor c \rfloor + 1) \in \mathcal{S}_{a,c}, \ n \geq 0$$

by Theorem 3.4 and the definition (1.15) of the transformation $R_{a,c}$. Set

$$\mathcal{E} := \{n(\lfloor c \rfloor + 1) + a\mathbb{Z}, \ n \geq 0\}.$$

Observe that $\mathcal{E}$ is dense in $\mathbb{R}$ if $a \notin \mathbb{Q}$, and $\mathcal{E} = \gcd(\lfloor c \rfloor + 1, a)\mathbb{Z}$ if $a \in \mathbb{Q}$. This observation with $t_0+n(\lfloor c \rfloor+1) \notin [c_0+a-1, a)+a\mathbb{Z}, n \geq 0$, by Theorem 3.4 implies that $a \in \mathbb{Q}$ and $t_0+\gcd(\lfloor c \rfloor+1, a)\mathbb{Z} \in \mathcal{S}_{a,c}$. This together with Theorem 3.4 implies that the length $1-c_0$ of the black hole $[c_0+a-1, a)+a\mathbb{Z}$ of the transformation $R_{a,c}$ on one period must be strictly less than $\gcd(\lfloor c \rfloor+1, a)$, i.e., $1-c_0 < \gcd(\lfloor c \rfloor+1, a)$.

($\Longleftarrow$) Set

$$\mathcal{F} := [-\gcd(\lfloor c \rfloor + 1, a), c_0 - 1) + \gcd(\lfloor c \rfloor + 1, a).$$

Then $\mathcal{F}$ has empty intersection with the black hole $[c_0+a-1, a)+a\mathbb{Z}$ of the transformation $R_{a,c}$ and it is invariant under the transformation $R_{a,c}$, i.e., $R_{a,c}(\mathcal{F}) = \mathcal{F}$. Thus

$$\mathcal{F} \subset \mathcal{S}_{a,c} \tag{3.35}$$

by Theorem 3.4, and hence the sufficiency follows.

Now we prove (3.10). For any $t \notin \mathcal{F}$, we may write $t = t_0 + s$ for some $t_0 \in [c_0-1, 0)$ and $s \in \gcd(\lfloor c \rfloor+1, a)$. One may verify that $(R_{a,c})^n(t) \in [c_0-a, a)+a\mathbb{Z}$, where n is smallest nonnegative integer such that $s + n(\lfloor c \rfloor + 1) \in a\mathbb{Z}$. Thus $\mathcal{S}_{a,c} \subset \mathcal{F}$. This together with (3.35) proves (3.10).

(iv): We may apply the similar argument used in the proof the third statement and (3.11), and leave the details to the reader.

(v) By Theorem 3.4 and Proposition 3.6, it suffices to prove

$$[c_0-a, c_0+a-1)+a\mathbb{Z} \subset \cup_{n=0}^{L}(R_{a,c})^n([c-c_0, c-c_0+1-a)+a\mathbb{Z}), \tag{3.36}$$

where $L=\max(\lfloor (c_0+a-1)/(c_1+1-2a)\rfloor, \lfloor (a-c_0)/(a-c_1)\rfloor)$.

For any $t\in[0, c_0+a-1)$, write $t=l(c_1+1-2a)+t'$ for some $t'\in[0, \min(c_1+1-2a, c_0+a-1))$ and $0\le l\le L$. Then

$$\begin{aligned} t &\in (R_{a,c})^l(t')+a\mathbb{Z}\subset (R_{a,c})^l([0, c_1+1-2a)+a\mathbb{Z})\\ &\subset \cup_{n=0}^{L}(R_{a,b,c})^n([c-c_0, c-c_0+1-a)+a\mathbb{Z}) \end{aligned} \tag{3.37}$$

for all $t\in[0, c_0+a-b)$, where the last inclusion holds as $c_1\le a$.

Similarly for any $s\in[c_0-a, 0)$, let $s=l'(c_1-a)+s'$ for some $s'\in[\max(c_1-a, c_0-a), 0)$ and $0\le l'\le L$. Then

$$\begin{aligned} s &\in (R_{a,c})^{l'}(s')+a\mathbb{Z}\subset (R_{a,c})^{l'}([c_1-a, 0)+a\mathbb{Z})\\ &\subset \cup_{n=0}^{L}(R_{a,b,c})^n([c-c_0, c-c_0+1-a)+a\mathbb{Z}) \end{aligned} \tag{3.38}$$

for all $s\in[c_1-a, 0)$. Combining (3.37) and (3.38) and applying the periodic property (3.1) proves (3.36).

(vi) Mimicking the argument used to prove the statement (v), we can show that

$$\cup_{n=0}^{\infty}(R_{a,c})^n([c-c_0, c-c_0+1-a)+a\mathbb{Z})=[c_0+a-1, a)+a\mathbb{Z}.$$

This together with (1.17) and Theorem 3.4 proves the desired conclusion (3.12).

(vii) The conclusion (3.13) can be obtained by mimicking the argument used to prove the statement (v). □

3.3. Maximal invariant sets around the origin

In this section, we prove Theorem 3.1. To do so, we need two important lemmas about the maximal invariant set $\mathcal{S}_{a,c}$ near the origin for $a\notin\mathbb{Q}$ and $a\in\mathbb{Q}$ respectively.

LEMMA 3.10. *Let $0<a<1<c$ and $a\notin\mathbb{Q}$. If $\mathcal{S}_{a,c}\neq\emptyset$, then*

$$(0,\epsilon)\cap\mathcal{S}_{a,c}\neq\emptyset \tag{3.39}$$

and

$$(-\epsilon,0)\cap\mathcal{S}_{a,c}\neq\emptyset \tag{3.40}$$

for any $\epsilon>0$.

PROOF. For $c_0=0$, the dense properties (3.39) and (3.40) follows from the first conclusion (3.8) of Theorem 3.5. So hereafter we assume that $c_0>0$. In this case, $a+(c_0-a)_- -(c_0+a-1)_+>0$ and the black hole $[(c_0+a-1)_+, a+(c_0-a)_-)+a\mathbb{Z}$ of the transformation $R_{a,c}$ is not an empty set. Take $t_0\in\mathcal{S}_{a,c}$, and let $t_n:=(R_{a,c})^n(t_0)$ and $\tilde t_n:=t_n-\lfloor t_n/a\rfloor a, n\ge 0$. Then

$$\tilde t_n\in\mathcal{S}_{a,c}\cap[0,a)\subset[0,(c_0+a-1)_+)\cup[a+(c_0-a)_-, a) \tag{3.41}$$

by (3.1), (3.5) and (3.6); and

$$\tilde t_n-\tilde t_m\neq 0 \text{ whenever } n\neq m \tag{3.42}$$

by (3.6) and the assumption $a\notin\mathbb{Q}$. Thus without loss of generality, we assume that $\tilde t_n\neq 0$ for all $n\ge 0$, otherwise replacing t_0 by t_{n_0} for a sufficiently large n_0.

Suppose, on the contrary, that (3.39) does not hold. Then there exists $0 < \epsilon < a + (c_0 - a)_- - (c_0 + a - 1)_+$ such that

$$\tilde{t}_n \notin (0, \epsilon) \quad \text{for all } n \geq 1. \tag{3.43}$$

As $\tilde{t}_n, n \geq 0$, lie in the bounded set $(0, a)$, there exist integers $n_1 < n_2$ such that

$$0 < |\tilde{t}_{n_1} - \tilde{t}_{n_2}| < \epsilon$$

by (3.42). Therefore either

$$\tilde{t}_{n_1}, \tilde{t}_{n_2} \in [\epsilon, (c_0 + a - 1)_+)$$

or

$$\tilde{t}_{n_1}, \tilde{t}_{n_2} \in [a + (c_0 - a)_-, a)$$

by (3.41). This implies that

$$t_{n_2+1} - t_{n_1+1} = t_{n_2} - t_{n_1}$$

and

$$\tilde{t}_{n_2+1} - \tilde{t}_{n_1+1} \in \tilde{t}_{n_2} - \tilde{t}_{n_1} + a\mathbb{Z}.$$

Thus either

$$\tilde{t}_{n_2+1} - \tilde{t}_{n_1+1} = \tilde{t}_{n_2} - \tilde{t}_{n_1}$$

or

$$|\tilde{t}_{n_2+1} - \tilde{t}_{n_1+1}| = a - |\tilde{t}_{n_2} - \tilde{t}_{n_1}|.$$

The second case does not happen as in that case either $\tilde{t}_{n_2+1} \in [0, \epsilon)$ or $\tilde{t}_{n_1+1} \in [0, \epsilon)$, which contradicts to (3.43). Thus

$$\tilde{t}_{n_2+k} - \tilde{t}_{n_1+k} = \tilde{t}_{n_2} - \tilde{t}_{n_1} \quad \text{for all} \quad k \geq 1,$$

which implies that $\{\tilde{t}_{n_1+j(n_2-n_1)}\}_{j=0}^{\infty}$ is an arithmetic sequence with common difference $0 \neq \tilde{t}_{n_2} - \tilde{t}_{n_1} \in (-\epsilon, \epsilon)$. This contradicts to $\tilde{t}_n \in (0, a)$ for all $n \geq 0$.

The conclusion (3.40) can be proved by using similar argument. □

To prove Theorem 3.1, we also need the density property that $(-\epsilon, \epsilon) \cap \mathcal{S}_{a,c} \neq \emptyset$ for sufficiently small $\epsilon > 0$, for $a \in \mathbb{Q}$,

LEMMA 3.11. *Let $0 < a < 1 < c$ and $a \in \mathbb{Q}$. If $\mathcal{S}_{a,c} \neq \emptyset$, then there exists a positive number $\epsilon > 0$ such that*

(i) *at least one of two intervals $[0, \epsilon)$ and $(c_0 - a)_- + a + [0, \epsilon)$ is contained in $\mathcal{S}_{a,c}$;*
(ii) *at least one of two intervals $[-\epsilon, 0)$ and $(c_0 + a - 1)_+ + [-\epsilon, 0)$ is contained in $\mathcal{S}_{a,c}$; and*
(iii) *at least one of two intervals $[0, \epsilon)$ and $[-\epsilon, 0)$ is contained in $\mathcal{S}_{a,c}$.*

PROOF. By Theorem 3.5, the statements (i), (ii) and (iii) hold for either $c_0 \leq 1 - a$ or $c_0 \geq a$. So hereafter we assume that $1 - a < c_0 < a$ and write $a = p/q$ for some co-prime integers p and q.

(i) Suppose on the contrary that both $[0, \epsilon)$ and $[c_0, c_0 + \epsilon)$ are not contained in $\mathcal{S}_{a,c}$. Set

$$\epsilon_1 := \begin{cases} \min(c - \lfloor qc \rfloor / q, (\lfloor qc \rfloor + 1)/q - c) & \text{if } c \notin \mathbb{Z}/q \\ 1/q & \text{if } c \in \mathbb{Z}/q. \end{cases} \tag{3.44}$$

Without loss of generality, we assume that $\epsilon \leq \epsilon_1$. Then

$$\mathcal{S}_{a,c} \cap [0, \epsilon) = \mathcal{S}_{a,c} \cap [c_0, c_0 + \epsilon) = \emptyset$$

by (1.25). This together with (3.5) implies that

$$\mathcal{S}_{a,c} \subset ([\epsilon, c_0 + a - 1) \cup [c_0 + \epsilon, a)) + a\mathbb{Z}. \tag{3.45}$$

Thus $\mathcal{S}_{a,c} - \epsilon/2$ has empty intersection with the black hole $[c_0 + a - 1, c_0) + a\mathbb{Z}$ of the transformation $R_{a,c}$, and it is invariant under the transformation $R_{a,c}$ because

$$R_{a,c}(\mathcal{S}_{a,c} - \epsilon/2) = R_{a,c}(\mathcal{S}_{a,c}) - \epsilon/2 = \mathcal{S}_{a,c} - \epsilon/2$$

by (1.15), (3.6) and (3.45). Thus by the maximality of the set $\mathcal{S}_{a,c}$ in Theorem 3.4, we have that

$$\mathcal{S}_{a,c} - \epsilon/2 \subset \mathcal{S}_{a,c},$$

which contradicts to (3.45) and the assumption $\mathcal{S}_{a,c} \neq \emptyset$.

(ii) Suppose on the contrary that both $[-\epsilon, 0)$ and $[c_0 + a - 1 - \epsilon, c_0 + a - 1)$ are not contained in $\mathcal{S}_{a,c}$ for some sufficiently small $\epsilon > 0$. Then

$$[-\epsilon, 0) \cap \mathcal{S}_{a,c} = [c_0 + a - 1 - \epsilon, c_0 + a - 1) \cap \mathcal{S}_{a,c} = \emptyset$$

by (1.25). Following the argument in the proof of the first conclusion, we have that

$$R_{a,c}(\mathcal{S}_{a,c} + \epsilon/2) = \mathcal{S}_{a,c} + \epsilon/2$$

and

$$(\mathcal{S}_{a,c} + \epsilon/2) \cap ([c_0 + a - 1, c_0) + a\mathbb{Z}) = \emptyset.$$

Hence

$$\mathcal{S}_{a,c} + \epsilon/2 \subset \mathcal{S}_{a,c}$$

by Theorem 3.4, which contradicts to the assumption $\mathcal{S}_{a,c} \neq \emptyset$ and $\mathcal{S}_{a,c} \cap [-\epsilon, 0) = \emptyset$.

(iii) Suppose on the contrary that both $[0, \epsilon)$ and $[-\epsilon, 0)$ are not contained in $\mathcal{S}_{a,c}$ for sufficiently small $\epsilon > 0$. Then

$$[0, \epsilon) \cap \mathcal{S}_{a,c} = [-\epsilon, 0) \cap \mathcal{S}_{a,c} = \emptyset \tag{3.46}$$

by (1.25); and

$$[c_0, c_0 + \epsilon) \subset \mathcal{S}_{a,c} \text{ and } [c_0 + a - 1 - \epsilon, c_0 + a - 1) \in \mathcal{S}_{a,c} \tag{3.47}$$

by the first two conclusions of this lemma. We claim that there exists a nonnegative integer $1 \le D \le (2p - q)/(q - p)$ such that

$$(R_{a,c})^D([c - c_0, c - c_0 + 1 - a) + a\mathbb{Z}) \cap ([c_0 + a - 1, c_0) + a\mathbb{Z}) \neq \emptyset. \tag{3.48}$$

Proof of Claim (3.48). Suppose on the contrary that (3.48) does not hold. Then

$$(R_{a,c})^n([c - c_0, c - c_0 + 1 - a) + a\mathbb{Z}) \cap ([c_0 + a - 1, c_0) + a\mathbb{Z}) = \emptyset$$

for all $0 \le n \le (2p - q)/(q - p)$. This together with the one-to-one property of the transformation $R_{a,c}$ out of its black hole and the range property (1.18) implies that $(R_{a,c})^n([c - c_0, c - c_0 + 1 - a) + a\mathbb{Z}), 0 \le n \le (2p - q)/(q - p)$, are mutually disjoint. So

$$\begin{aligned} & |\cup_{n=0}^{(2p-q)/(q-p)} (R_{a,c})^n([c - c_0, c - c_0 + 1 - a) + a\mathbb{Z}) \cap ([0, a) \backslash [c_0 + a - 1, c_0))| \\ = \ & \lfloor p/(q - p) \rfloor (q - p)/q > |[0, a) \backslash [c_0 + a - 1, c_0)| \end{aligned}$$

by (1.19), which is a contradiction. This proves (3.48). □

Now returning to the proof of the third statement of Lemma 3.11. By (3.48), we may assume that the nonnegative integer D in (3.48) is the minimal integer. Hence

$$(R_{a,c})^n([c-c_0, c-c_0+1-a)+a\mathbb{Z}), 0 \le n \le D, \text{ are mutually disjoint.} \tag{3.49}$$

Now let us verify the following claim:

$$(R_{a,c})^n([c-c_0, c-c_0+1-a)+a\mathbb{Z}) = [b_n+a-1, b_n)+a\mathbb{Z} \tag{3.50}$$

for some $b_n \in (0,a] \cap (c+\mathbb{Z}/q), 0 \le n \le D$, and

$$(R_{a,c})^D([c-c_0, c-c_0+1-a)+a\mathbb{Z}) = [c_0+a-1, c_0)+a\mathbb{Z}. \tag{3.51}$$

Proof of Claim (3.50) and (3.51). If $D=0$, then (3.50) and (3.51) follow from (3.5) and (3.47). Now we consider $D \ge 1$. Let $T_0 = [c-c_0, c-c_0+1-a)+a\mathbb{Z}$ and define $T_n, 1 \le n \le D$, inductively by

$$T_n = \begin{cases} R_{a,c}(T_{n-1}) & \text{if } 0 \notin T_{n-1}, \\ R_{a,c}(T_{n-1}) \cup ([c-c_0, c-c_0+1-a)+a\mathbb{Z}) & \text{if } 0 \in T_{n-1}. \end{cases} \tag{3.52}$$

Clearly

$$T_0 = [b_0+a-1, b_0)+a\mathbb{Z}$$

for some $b_0 \in (0,a] \cap (c+\mathbb{Z}/q)$ as the periodic set T_0 has length $1-a$ on one period. Inductively, we assume that

$$T_n = [\tilde{b}_n, b_n)+a\mathbb{Z}$$

for some $\tilde{b}_n, b_n$ with $b_n \in (0,a]$ and $1-a \le b_n-\tilde{b}_n < a, 0 \le n < D$. Here $b_n-\tilde{b}_n < a$ by the assumption $\mathcal{S}_{a,c} \ne \emptyset$ and the emptyset intersection property $T_n \cap \mathcal{S}_{a,c} = \emptyset$ by Proposition 3.6. If $0 \notin T_n$, then either $[\tilde{b}_n, b_n) \subset (0, c_0+a-1)$ or $[\tilde{b}_n, b_n) \subset [c_0, a)$ by (3.49) and (3.50). This implies that

$$\begin{aligned} T_{n+1} &= R_{a,c}(T_n) = [R_{a,c}(\tilde{b}_n), R_{a,c}(\tilde{b}_n)+b_n-\tilde{b}_n)+a\mathbb{Z} \\ &=: [\tilde{b}_{n+1}, b_{n+1})+a\mathbb{Z} \end{aligned} \tag{3.53}$$

for some $\tilde{b}_{n+1}, b_{n+1}$ with $b_{n+1} \in (0,a] \cap (c+\mathbb{Z}/q)$ and $b_{n+1}-\tilde{b}_{n+1} = b_n-\tilde{b}_n$. If $0 \in T_n$, then $\tilde{b}_n \le 0$. Moreover $\tilde{b}_n \ge c_0-a$ and $b_n \le c_0+a-1$, as otherwise T_n has nonempty intersection with the black hole $[c_0+a-1, c_0)+a\mathbb{Z}$ of the transformation $R_{a,c}$, which contradicts to (3.48) and the observation that $T_n \subset \cup_{m=0}^n (R_{a,c})^m([c-c_0, c-c_0+1-a)+a\mathbb{Z})$. Therefore

$$\begin{aligned} T_{n+1} &= R_{a,c}(T_n) \cup ([c-c_0, c-c_0+1-a)+a\mathbb{Z}) \\ &= [\tilde{b}_n+\lfloor c\rfloor, b_n+\lfloor c\rfloor+1-a)+a\mathbb{Z} \\ &=: [\tilde{b}_{n+1}, b_{n+1})+a\mathbb{Z} \end{aligned} \tag{3.54}$$

for some $\tilde{b}_{n+1}, b_{n+1}$ with

$$b_{n+1} \in (0,a] \cap (c+\mathbb{Z}/q) \quad \text{and} \quad b_{n+1}-\tilde{b}_{n+1} = b_n-\tilde{b}_n+1-a.$$

Combining (3.53) and (3.54) proceeds the inductive proof that

$$T_n = [\tilde{b}_n, b_n)+a\mathbb{Z} \tag{3.55}$$

such that $b_n \in (0,a] \cap (c+\mathbb{Z}/q), 0 \le n \le D$ and $b_n-\tilde{b}_n \in [(1-a), a) \cap (1-a)\mathbb{Z}, 0 \le n \le D$ is an increasing sequence. Observe that

$$(R_{a,c})^D([c-c_0, c-c_0+1-a)+a\mathbb{Z}) \subset T_D \subset \cup_{n=0}^D (R_{a,c})^n([c-c_0, c-c_0+1-a)+a\mathbb{Z}).$$

Then T_D has nonempty intersection with the black hole $[c_0+a-1, c_0)+a\mathbb{Z}$ of the transformation $R_{a,c}$ by (3.49). This together with (3.55) implies that either

$$[c_0+a-1-\epsilon_1, c_0+a-1) \subset T_D,$$

or

$$[c_0, c_0+\epsilon_1) \subset T_D$$

or

$$T_D = [c_0+a-1, c_0)+a\mathbb{Z},$$

where ϵ_1 is given in (3.44). Recall that $T_D \cap \mathcal{S}_{a,c} = \emptyset$ by Proposition 3.6. Then both $[c_0+a-1-\epsilon_1, c_0+a-1)$ and $[c_0, c_0+\epsilon_1)$ have empty intersection with T_D by (3.47). Thus

$$T_D = [c_0+a-1, c_0)+a\mathbb{Z}. \tag{3.56}$$

This together with (3.52), (3.53) and (3.54) implies that

$$\tilde{b}_n > 0 \text{ and } b_n - \tilde{b}_n = 1-a \text{ for all } 0 \le n \le D. \tag{3.57}$$

The desired conclusions (3.50) and (3.51) then follow. □

Let us return to the proof of the conclusion (iii). By (1.25), (3.47), (3.49), (3.50), (3.57) and Proposition 3.6, either

$$[b_n+a-1, b_n) \subset [\epsilon_1, c_0+a-b)$$

or

$$[b_n+a-1, b_n) \subset [c_0+\epsilon_1, a),\ 0 \le n < D.$$

This implies that

$$R_{a,c}(b_n+a-1-\epsilon/2)+a\mathbb{Z} = b_{n+1}+a-1-\epsilon/2+a\mathbb{Z} \tag{3.58}$$

for all $0 \le n \le D-1$. By (3.47), (3.50), (3.51), (3.58) and Theorem 3.4, we have that

$$\begin{aligned} & (\tilde{R}_{a,c})^n(c_0+a-1-\epsilon/2)+a\mathbb{Z} \\ = \ & (\tilde{R}_{a,c})^n(b_D+a-1-\epsilon/2)+a\mathbb{Z} \\ = \ & (\tilde{R}_{a,c})^{n-1}(b_{D-1}+a-1-\epsilon/2)+a\mathbb{Z} = \cdots \\ = \ & b_{D-n}+a-1-\epsilon/2+a\mathbb{Z} \subset \mathcal{S}_{a,c}, \quad 0 \le n \le D. \end{aligned}$$

Hence

$$-\epsilon/2+a\mathbb{Z} = \tilde{R}_{a,c}(c-c_0-\epsilon/2)+a\mathbb{Z} = (\tilde{R}_{a,c})^{D+1}(c_0+a-1-\epsilon/2)+a\mathbb{Z} \in \mathcal{S}_{a,c},$$

which contradicts to (3.46). □

We finish this section with the proof of Theorem 3.1.

Proof of Theorem 3.1. The sufficiency is obvious.

Now the necessity for $a \notin \mathbb{Q}$. By Lemma 3.10, there exist $t_n \in \mathbb{R}$ and $\mathbf{x}_n \in \mathcal{B}^0, n \ge 0$, such that $\mathbf{M}_{a,c}(t_n)\mathbf{x}_n = \mathbf{1}$ and $\{t_n\}_{n=1}^\infty$ is a decreasing sequence convergent to zero. Without loss of generality, we may assume that $\mathbf{x}_n$ converges, otherwise replacing it by its subsequence. Therefore

$$\mathbf{M}_{a,c}(0)\mathbf{x} = \lim_{n\to\infty} \mathbf{M}_{a,c}(t_n)\mathbf{x}_n = \mathbf{1},$$

where $\mathbf{x} \in \mathcal{B}^0$ is the limit of $\mathbf{x}_n$ as $n \to \infty$. This proves that $0 \in \mathcal{S}_{a,c}$ and the necessity for $a \notin \mathbb{Q}$.

The necessity for $a \in \mathbb{Q}$ follows directly from (1.25) and Lemma 3.11. $\square$

3.4. Gabor frames and maximal invariant sets

In this section, we shall prove Theorem 2.3.

To prove Theorem 2.3, we need the uniqueness of binary solutions $\mathbf{x} \in \mathcal{B}^0$ to the linear system $\mathbf{M}_{a,c}(t)\mathbf{x} = \mathbf{1}$ for $t \in \mathcal{S}_{a,c}$.

LEMMA 3.12. *Let $0 < a < 1 < c$. Then for any $t \in \mathcal{S}_{a,c}$ there exists a unique vector $\mathbf{x} \in \mathcal{B}^0$ satisfying $\mathbf{M}_{a,c}(t)\mathbf{x} = \mathbf{1}$.*

PROOF. Suppose, on the contrary, that

$$\mathbf{M}_{a,c}(t)\mathbf{x}_0 = \mathbf{M}_{a,c}(t)\mathbf{x}_1 = \mathbf{1}$$

for two distinct vectors $\mathbf{x}_0, \mathbf{x}_1 \in \mathcal{B}^0$. Then there exists $0 \neq \lambda_0 \in \mathbb{Z}$ such that

$$\mathbf{x}_0(\lambda_0) \neq \mathbf{x}_1(\lambda_0)$$

and

$$\mathbf{x}_0(\lambda) = \mathbf{x}_1(\lambda) \quad \text{for all } |\lambda| < |\lambda_0|.$$

Without loss of generality, we assume that $\mathbf{x}_0(\lambda_0) = 1, \mathbf{x}_1(\lambda_0) = 0$ and $\lambda_0 > 0$. Let λ_1 be the largest integer strictly less than λ_0 such that $\mathbf{x}_0(\lambda_1) = \mathbf{x}_1(\lambda_1) = 1$. Thus both $\tau_{\lambda_1}\mathbf{x}_0$ and $\tau_{\lambda_1}\mathbf{x}_1$ belong to $\mathcal{B}^0$,

$$\mathbf{M}_{a,c}(t+\lambda_1)\tau_{\lambda_1}\mathbf{x}_0 = \mathbf{M}_{a,c}(t+\lambda_1)\tau_{\lambda_1}\mathbf{x}_1 = \mathbf{1},$$

and

$$\lambda_0 - \lambda_1 = R_{a,c}(t+\lambda_1) - (t+\lambda_1)$$

by Lemma 3.7. Applying Lemma 3.7 to $\tau_{\lambda_1}\mathbf{x}_1$ leads to $\tau_{\lambda_1}\mathbf{x}_1(\lambda_0 - \lambda_1) = 1$, which contradicts to $\mathbf{x}_1(\lambda_0) = 0$. $\square$

Now we prove Theorem 2.3.

PROOF OF THEOREM 2.3. We use the double inclusion method to prove (2.2). Take $t \in \mathcal{D}_{a,c}$, let $\mathbf{x} \in \mathcal{B}^0$ satisfy $\mathbf{M}_{a,c}(t)\mathbf{x} = \mathbf{2}$. Let K be the set of all $\lambda \in \mathbb{Z}$ with $\mathbf{x}(\lambda) = 1$, and write $K = \{\lambda_j : j \in \mathbb{Z}\}$ for a strictly increasing sequence $\{\lambda_j\}_{j=-\infty}^{\infty}$ with $\lambda_0 = 0$. By Lemma 2.6, the binary vectors $\mathbf{x}_l := (x_l(\lambda))_{\lambda \in \mathbb{Z}}, l = 0, 1$, defined by $\mathbf{x}_l(\lambda) = 1$ if $\lambda = \lambda_{2j-l}$ for some integer j and $\mathbf{x}_l(\lambda) = 0$ otherwise, satisfy

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_1 \quad \text{and} \quad \mathbf{M}_{a,c}(t)\mathbf{x}_0 = \mathbf{M}_{a,c}(t)\mathbf{x}_1 = \mathbf{1}. \tag{3.59}$$

Then either $t \in [0, (c_0 + a - 1)_+) + a\mathbb{Z}$ or $[(c_0 - a)_-, 0) + a\mathbb{Z}$, because

$$\mathcal{D}_{a,c} \cap ([(c_0 + a - 1)_+, a + (c_0 - a)_-) + a\mathbb{Z}) = \emptyset$$

by (3.2) and Theorem 3.4. For the first case that $t \in [0, (c_0 + a - 1)_+) + a\mathbb{Z}$, we have that

$$\lambda_2 = \lfloor c \rfloor + 1$$

by (3.19). Hence λ_1 is an integer in $[1, \lfloor c \rfloor]$ and $t + \lambda_1 \in \mathcal{S}_{a,c}$ as

$$\mathbf{M}_{a,c}(t)\mathbf{x}_1 = \mathbf{1} \quad \text{and} \quad \mathbf{x}_1(\lambda_1) = 1$$

by (3.59). Thus

$$t \in \big(\mathcal{S}_{a,c} \cap ([0, (c_0 + a - 1)_+ + a\mathbb{Z})\big) \cap \big(\cup_{\lambda=1}^{\lfloor c \rfloor} (\mathcal{S}_{a,c} - \lambda)\big) \tag{3.60}$$

for the first case.

Similarly for the second case that $t \in [(c_0-a)_-, 0)+a\mathbb{Z}$, we obtain from (3.19) that

$$\lambda_2 = \lfloor c \rfloor,$$

which together with (3.59) implies that

$$t \in \big(\mathcal{S}_{a,c} \cap ([(c_0-a)_-, 0)+a\mathbb{Z})\big) \cap \big(\cup_{\lambda=1}^{\lfloor c\rfloor -1} (\mathcal{S}_{a,c}-\lambda)\big) \tag{3.61}$$

for the second case. Combining (3.5), (3.60) and (3.61) proves the first inclusion

$$\begin{aligned}\mathcal{D}_{a,c} \quad \subset \quad & \big(\mathcal{S}_{a,c} \cap ([0,(c_0+a-1)_+ + a\mathbb{Z}) \cap (\mathcal{S}_{a,c}-\lfloor c \rfloor)\big) \\ & \cup \big(\mathcal{S}_{a,c} \cap (\cup_{\lambda=1}^{\lfloor c\rfloor -1}(\mathcal{S}_{a,c}-\lambda))\big).\end{aligned} \tag{3.62}$$

Conversely, take

$$t \in \mathcal{S}_{a,c} \cap ([0,(c_0+a-1)_+)+a\mathbb{Z}) \cap (\mathcal{S}_{a,c}-\lambda^*)$$

for some $\lambda^* \in [1, \lfloor c \rfloor] \cap \mathbb{Z}$. Then there exist $\mathbf{x}_0, \mathbf{x}_1 \in \mathcal{B}^0$ such that

$$\mathbf{M}_{a,c}(t)\mathbf{x}_0 = \mathbf{M}_{a,c}(t+\lambda^*)\mathbf{x}_1 = \mathbf{1}. \tag{3.63}$$

Define $\mathbf{x} = \mathbf{x}_0 + \tau_{-\lambda^*}\mathbf{x}_1$. By (1.7) and (3.63),

$$\mathbf{M}_{a,c}(t)\mathbf{x} = \mathbf{M}_{a,c}(t)\mathbf{x}_0 + \mathbf{M}_{a,c}(t+\lambda^*)\mathbf{x}_1 = \mathbf{2}. \tag{3.64}$$

Now let us verify that $\mathbf{x} := (\mathbf{x}(\lambda))_{\lambda\in\mathbb{Z}} \in \mathcal{B}^0$. Observe that $\mathbf{x}(\lambda) \in \{0,1,2\}$ for all $\lambda \in \mathbb{Z}$ and $\mathbf{x}(0) \ge \mathbf{x}_0(0) \ge 1$. Then it suffices to prove that $\mathbf{x}(\lambda) \ne 2$ for all $\lambda \in \mathbb{Z}$. Suppose, on the contrary, that $\mathbf{x}(\lambda_0) = 2$ for some $\lambda_0 \in \mathbb{Z}$. Then

$$\mathbf{x}_0(\lambda_0) = 1 \quad \text{and} \quad \tau_{-\lambda^*}\mathbf{x}_1(\lambda_0) = 1.$$

Hence $\tau_{\lambda_0}\mathbf{x}_0, \tau_{\lambda_0-\lambda^*}\mathbf{x}_1 \in \mathcal{B}^0$ and

$$\mathbf{M}_{a,c}(t+\lambda_0)\tau_{\lambda_0}\mathbf{x}_0 = \mathbf{M}_{a,c}(t+\lambda_0)\tau_{\lambda_0-\lambda^*}\mathbf{x}_1 = \mathbf{1}$$

by (1.7) and (3.63). Thus $\tau_{\lambda_0}\mathbf{x}_0 = \tau_{\lambda_0-\lambda^*}\mathbf{x}_1$ by Lemma 3.12, which is a contradiction because $\tau_{-\lambda^*}\mathbf{x}_1(\lambda^*) = \mathbf{x}_1(0) = 1$ by the assumption that $\mathbf{x}_1 \in \mathcal{B}^0$, and $\mathbf{x}_0(\lambda^*) = 0$ by (3.19) and the assumption that $t \in [0,(c_0+a-1)_+) \cap \mathcal{S}_{a,c}$. Thus $\mathbf{x} \in \mathcal{B}^0$. This together with (3.64) proves that

$$\mathcal{S}_{a,c} \cap ([0,(c_0+a-1)_+)+a\mathbb{Z}) \cap (\mathcal{S}_{a,c}-\lambda^*) \subset \mathcal{D}_{a,c} \tag{3.65}$$

for all positive integers $\lambda^* \in [1, \lfloor c \rfloor - 1] \cap \mathbb{Z}$. Applying similar argument leads to

$$\mathcal{S}_{a,c} \cap ([(c_0-a)_-, 0)+a\mathbb{Z}) \cap (\mathcal{S}_{a,c}-\lambda^*) \subset \mathcal{D}_{a,c} \tag{3.66}$$

for all integers $\lambda^* \in [1, \lfloor c \rfloor - 1]$. The desired equality (2.2) then follows from (3.62), (3.65) and (3.66). □

3.5. Instability of infinite matrices

In this section, we shall prove Theorem 2.2.

Proof of Theorem 2.2. The necessity is obvious. We divide four cases to verify the sufficiency.

Case 1: $c_0 = 0$.

In this case, the sufficiency follows as $\mathcal{D}_{a,c} = \mathcal{S}_{a,c} = \mathbb{R}$ by Theorems 3.5 and 2.3.

Case 2: $a \notin \mathbb{Q}$ and either $c_0 \ge a$ or $0 < c_0 \le 1-a$.

In this case, the sufficiency holds since $\mathcal{D}_{a,c} = \mathcal{S}_{a,c} = \emptyset$ by (3.2) and Theorem 3.5.

Case 3: $a \notin \mathbb{Q}$ and $1 - a < c_0 < a$.

Suppose on the contrary that $\mathcal{D}_{a,c} \neq \emptyset$. Following the argument used in the proof of Lemma 3.10, we can find $t_n \in \mathcal{D}_{a,c} \cap (0, a)$ and $\mathbf{x}_n \in \mathcal{B}^0, n \geq 1$, such that $\{t_n\}_{n=1}^{\infty}$ is a decreasing sequence that converges to zero, $\mathbf{x}_n$ converges to $\mathbf{x}_\infty \in \mathcal{B}^0$, and $\mathbf{M}_{a,c}(t_n)\mathbf{x}_n = \mathbf{2}$. Recall from (2.11) used in the proof of Theorem 2.5 that given any $\lambda \in \mathbb{Z}$ and $\mu \in a\mathbb{Z}$,

$$\chi_{[0,c)}(t_n - \mu + \lambda) = \chi_{[0,c)}(-\mu + \lambda)$$

for sufficiently large n. Thus $\mathbf{M}_{a,c}(0)\mathbf{x}_\infty = 2$ and $0 \in \mathcal{D}_{a,c}$. This leads to the contradiction.

Case 4: $a \in \mathbb{Q}$ and $c_0 > 0$.

Write $a = p/q$ for some co-prime integers p and q. By (1.26), we obtain

$$\mathcal{D}_{a,c} = \big(\mathcal{D}_{a,c} \cap \mathbb{Z}/q + [0, c - \lfloor qc \rfloor/q)\big) \cup \big(\mathcal{D}_{a,c} \cap (c + \mathbb{Z}/q) + [0, (\lfloor qc \rfloor + 1)/q - c)\big).$$

Thus $\mathcal{D}_{a,c} \neq \emptyset$ if and only if $\mathcal{D}_{a,c} \cap (\{0, c\} + \mathbb{Z}/q) \neq \emptyset$. This together with the periodicity $\mathcal{D}_{a,c} = \mathcal{D}_{a,c} + a\mathbb{Z}$ proves the sufficiency. $\square$

CHAPTER 4

Piecewise Linear Transformations

The piecewise linear transformations $R_{a,c}$ and $\tilde{R}_{a,c}$ are non-contractive on the real line. They do not satisfy standard requirements for Hutchinson's remarkable construction of their maximal invariant sets [**27**]. Define

$$\mathcal{E}_m := (R_{a,c})^m(\mathbb{R})\backslash([(c_0+a-1)_+, (c_0-a)_- + a) + a\mathbb{Z}),\ m \geq 0. \tag{4.1}$$

By the invariance property (3.6) in Theorem 3.4, we obtain the following inclusion by applying the transformation $R_{a,c}$ iteratively,

$$\mathcal{S}_{a,c} \subset \cap_{m=0}^{\infty} \mathcal{E}_m. \tag{4.2}$$

In this chapter, we first show that infinite intersection in the above inclusion can be replaced by finite intersection and the inclusion is indeed an equality.

THEOREM 4.1. *Let $0 < a < 1 < c$ and $\mathcal{E}_m, m \geq 0$, be as in* (4.1)*. Then the following statements hold.*

(i) *If $a \in \mathbb{Q}$, then*

$$\mathcal{S}_{a,c} = \mathcal{E}_{a/\gcd(a,1)}. \tag{4.3}$$

(ii) *If $a \notin \mathbb{Q}$ and $\mathcal{S}_{a,c} \neq \emptyset$, then*

$$\mathcal{S}_{a,c} = \mathcal{E}_{\lfloor a/(1-a) \rfloor}. \tag{4.4}$$

Combining (4.2) and Theorems 3.4 and 4.1 leads to the following characterization whether the maximal invariant set $\mathcal{S}_{a,c}$ is an empty set, cf. Theorem 5.5.

COROLLARY 4.2. *Let $0 < a < 1 < c$ and $a \notin \mathbb{Q}$. Then $\mathcal{S}_{a,c} \neq \emptyset$ if and only if $\mathcal{E}_{\lfloor a/(1-a) \rfloor}$ is a nonempty set invariant under the transformation $R_{a,c}$.*

By Theorem 4.1, we have the following topological property for the maximal invariant set $\mathcal{S}_{a,c}$.

COROLLARY 4.3. *Let $0 < a < 1 < c$. Then complement of the maximal invariant set $\mathcal{S}_{a,c}$ consists of finitely many left-closed right-open intervals on one period.*

In next theorem, we show that the restriction of the transformation $R_{a,c}$ onto its maximal invariant set $\mathcal{S}_{a,c}$ is a linear isomorphism on the line with marks, i.e., the commutative diagram (1.24) holds.

THEOREM 4.4. *Let $0 < a < 1 < c$. Assume that $\mathcal{S}_{a,c} \neq \emptyset$. Then under the isomorphism $Y_{a,c}$ from $\mathcal{S}_{a,c}$ to the line with marks, the restriction of the piecewise linear transformation $R_{a,c}$ onto the maximal invariant set $\mathcal{S}_{a,c}$ becomes a shift on the line with marks; i.e.,*

$$Y_{a,c}(R_{a,c}(t) + a\mathbb{Z}) = Y_{a,c}(t) + Y_{a,c}(\lfloor c \rfloor + 1) + Y_{a,c}(a)\mathbb{Z} \quad \text{for all } t \in \mathcal{S}_{a,c}. \tag{4.5}$$

Recall that the piecewise linear transformation $R_{a,c}$ is not measure-preserving on the whole line, but it has measure-preserving property on the maximal invariant set $\mathcal{S}_{a,c}$ by (1.19). In this chapter, we also establish an ergodic theorem for the transformation $R_{a,c}$. The reader may refer to [**48**] for ergodic theory of various dynamic systems.

THEOREM 4.5. *Let $0 < a < 1 < c$. Then for all continuous periodic functions f with period a, the limit*

$$F(t) := \lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} f((R_{a,c})^k(t))}{n} \tag{4.6}$$

exists for any $t \in \mathbb{R}$. Moreover

$$F(t) = \begin{cases} \frac{1}{|\mathcal{S}_{a,c}\cap[0,a)|}\int_{\mathcal{S}_{a,c}\cap[0,a)} f(s)ds & \text{if } t \in \mathcal{S}_{a,c} \text{ and } a \notin \mathbb{Q} \\ \frac{1}{D+1}\sum_{k=0}^{D} f((R_{a,c})^k(t)) & \text{if } t \in \mathcal{S}_{a,c} \text{ and } a \in \mathbb{Q} \\ f(t_0) & \text{if } t \notin \mathcal{S}_{a,c}, \end{cases} \tag{4.7}$$

where $D \ge 0$ is a nonnegative integer and $t_0 \in [(c_0 + a - 1)_+, (c_0 - a)_- + a) + a\mathbb{Z}$ is the limit of $(R_{a,c})^n(t)$ as $n \to \infty$ for $t \notin \mathcal{S}_{a,c}$.

Applying the above theorem, we conclude that $\mathcal{S}_{a,c} = \emptyset$ if and only if

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} f((R_{a,c})^k(t))}{n} = 0, \ t \in \mathbb{R}$$

for all periodic functions f vanishing on the black hole $[(c_0 + a - 1)_+, (c_0 - a)_- + a) + a\mathbb{Z}$ of the transformation $R_{a,c}$, cf. Theorems 5.5 and 6.8 and Corollary 4.2.

This chapter is organized as follows. In Section 4.1, we show that Hutchison's remarkable construction works for the maximal invariant set $\mathcal{S}_{a,c}$ of the transformation $R_{a,c}$ and prove the first conclusion in Theorem 4.1. In Section 4.2, we discuss the restriction of the transformation $R_{a,c}$ on its maximal invariant set and establish Theorem 4.4. In Section 4.3, we consider covering properties of the maximal invariant set $\mathcal{S}_{a,c}$ and prove Theorems 3.2 and 3.3 in Chapter 3. The proofs of Theorem 4.5 and the second conclusion of Theorem 4.1 will be given in Section 5.1 and 5.3 of next chapter respectively, as we need additional information about complement of the maximal invariant set $\mathcal{S}_{a,c}$ with $a \notin \mathbb{Q}$ in Theorem 5.2.

4.1. Hutchinson's construction of maximal invariant sets

In this section, we prove the first conclusion of Theorem 4.1 and postpone the proof of the second conclusion to Section 5.1.

PROOF OF THEOREM 4.1. (i) For $c_0 = 0$, the conclusion (4.3) is obvious because in this case $\mathcal{S}_{a,c} = \mathbb{R}$ by Theorem 3.5, the black hole $[(c_0 + a - 1)_+, (c_0 - a)_- + a) + a\mathbb{Z}$ is an empty set, and $\mathcal{E}_L = \mathbb{R}$ for all $L \ge 0$. So it remains to consider the case that $c_0 > 0$.

Write $a = p/q$ for some co-prime integers p and q. Then by (4.2) and Theorem 3.4, it suffices to prove that the set $\mathcal{E}_p$ is invariant under the transformation $R_{a,c}$. Take $t \in \mathcal{E}_p$. Then there exists s such that

$$t = (R_{a,c})^p(s)$$

and

$$t_n := (R_{a,c})^n(s) \notin [(c_0 + a - 1)_+, (c_0 - a)_- + a) + a\mathbb{Z}, \ 0 \le n \le p. \tag{4.8}$$

As $t_n - t \in \{0, 1, \dots, p-1\}/q + a\mathbb{Z}, 0 \le n \le p$, there exist two distinct integers n_1 and n_2 such that

$$t_{n_1} - t_{n_2} \in a\mathbb{Z} \quad \text{and} \quad 0 \le n_1 < n_2 \le p. \tag{4.9}$$

By (1.17), (4.8) and (4.9), we have that

$$(R_{a,c})^{n_2-n_1}(s) - s \in a\mathbb{Z}.$$

Let $s_1 = R_{a,c}(s)$ and $s_2 = (R_{a,c})^{n_2-n_1-1}(s)$. Then

$$R_{a,c}(t) = (R_{a,c})^p(s_1) \quad \text{and} \quad \tilde{R}_{a,c}(t) = (R_{a,c})^p(s_2)$$

with

$$(R_{a,c})^n(s_1), (R_{a,c})^n(s_2) \in \{t_0, \dots, t_{n_2-n_1-1}\} + a\mathbb{Z} \subset [(c_0-a)_-, (c_0+a-1)_+) + a\mathbb{Z}$$

for all $0 \le n \le p$. This proves that $R_{a,c}(t), \tilde{R}_{a,c}(t) \in \mathcal{E}_p$ for any $t \in \mathcal{E}$, and hence invariance of the set $\mathcal{E}_p$ under the transformation $R_{a,c}$ follows. □

4.2. Piecewise linear transformations onto maximal invariant sets

In this section, we prove Theorem 4.4.

Proof of Theorem 4.4. Recall that the maximal invariant set $\mathcal{S}_{a,c}$ is a measurable periodic set by (1.25) and Corollary 4.3. Then the map $Y_{a,c}$ is well-defined and its restriction on $\mathcal{S}_{a,c}$ is periodic by (1.23),

$$Y_{a,c}(t+a) = Y_{a,c}(t) + Y_{a,c}(a) \quad \text{for all } t \in \mathcal{S}_{a,c}. \tag{4.10}$$

Hence it remains to verify (4.5) for $t \in [0, a) \cap \mathcal{S}_{a,c} = ([0, (c_0+a-1)_+) \cup [(c_0 - a)_- + a, a)) \cap \mathcal{S}_{a,c}$, where the last equality follows from (3.5).

For $t \in [0, (c_0+a-1)_+) \cap \mathcal{S}_{a,c}$, we obtain from (1.15), (1.19), (1.23) and (3.6) that

$$\begin{aligned} Y_{a,c}(R_{a,c}(t)) &= |[0, R_{a,c}(t)) \cap \mathcal{S}_{a,c}| \\ &= Y_{a,c}(R_{a,c}(0)) + |[R_{a,c}(0), R_{a,c}(t)) \cap \mathcal{S}_{a,c}| \\ &= Y_{a,c}(R_{a,c}(0)) + |R_{a,c}([0,t) \cap \mathcal{S}_{a,c})| \\ &= Y_{a,c}(R_{a,c}(0)) + Y_{a,c}(t). \end{aligned} \tag{4.11}$$

Similarly for $t \in [(c_0-a)_- + a, a) \cap \mathcal{S}_{a,c}$, we get $c_0 < a$ and

$$\begin{aligned} Y_{a,c}(R_{a,c}(t)) &= |[R_{a,c}(c_0), R_{a,c}(t)) \cap \mathcal{S}_{a,c}| + Y_{a,c}(R_{a,c}(c_0)) \\ &= |R_{a,c}([c_0, t) \cap \mathcal{S}_{a,c})| + |[0, c_0 + \lfloor c \rfloor + a) \cap \mathcal{S}_{a,c}| \\ &\quad -|[c_0 + \lfloor c \rfloor, c_0 + \lfloor c \rfloor + a) \cap \mathcal{S}_{a,c}| \\ &= |[c_0, t) \cap \mathcal{S}_{a,c}| + |[0, \lfloor c \rfloor + 1) \cap \mathcal{S}_{a,c}| \\ &\quad +|R_{a,c}([0, c_0+a-1)) \cap \mathcal{S}_{a,c}| - Y_{a,c}(a) \\ &= Y_{a,c}(t) + Y_{a,c}(R_{a,c}(0)) - Y_{a,c}(a). \end{aligned} \tag{4.12}$$

Combining (4.11) and (4.12) proves (4.5) for $t \in [0, a) \cap \mathcal{S}_{a,c}$, and hence all $t \in \mathcal{S}_{a,c}$ by (4.10). □

4.3. Gabor frames and covering of maximal invariant sets

In this section, we prove Theorems 3.2 and 3.3 in Chapter 3.

Proof of Theorem 3.2. Set

$$A_\lambda := \mathcal{S}_{a,c} \cap [0, (c_0 + a - 1)_+) + \lambda + a\mathbb{Z}$$

and

$$B_\lambda := \mathcal{S}_{a,c} \cap [(c_0 - a)_- + a, a) + \lambda + a\mathbb{Z}, \quad \lambda \in \mathbb{Z}.$$

We divide the proof into two cases.

Case 1: $a \notin \mathbb{Q}$.

Take $t_0 \in \mathcal{S}_{a,c}$. Then $(R_{a,c})^n(t_0) \in \mathcal{S}_{a,c}$ by Theorem 3.4. Write

$$(R_{a,c})^n(t_0) = t_0 + k_n,$$

with $k_n \in \mathbb{Z}, n \geq 0$, are defined inductively by $k_0 = 0$ and

$$k_{n+1} - k_n = \begin{cases} \lfloor c \rfloor + 1 & \text{if } t_0 + k_n \in [0, (c_0 + a - 1)+) + a\mathbb{Z} \\ \lfloor c \rfloor & \text{if } t_0 + k_n \in [(c_0 - a)_- + a, a) + a\mathbb{Z}. \end{cases} \tag{4.13}$$

Then for any nonnegative integer l,

$$\begin{aligned} t_0 + l &= t_0 + k_n + (l - k_n) \\ &\in \big(\cup_{\lambda_2=0}^{\lfloor c \rfloor - 1} B_{\lambda_2} \cup \big(\cup_{\lambda_1=0}^{\lfloor c \rfloor} A_{\lambda_1}\big) \end{aligned} \tag{4.14}$$

by (4.13), where k_n is so chosen that $k_n \leq l < k_{n+1}$. Therefore

$$\begin{aligned} &\big\{t_0 + l - \lfloor (t_0 + l)/a \rfloor a \big|\ 0 \leq l \in \mathbb{Z}\big\} \\ \subset\ &\big(\cup_{\lambda_2=0}^{\lfloor c \rfloor - 1} B_{\lambda_2} \cap [0, a)\big) \cup \big(\cup_{\lambda_1=0}^{\lfloor c \rfloor} A_{\lambda_1} \cap [0, a)\big) \end{aligned}$$

by (3.1) and (4.14). Observe that the left hand side of the above inclusion is a dense subset of $[0, a)$ by the assumption $a \notin \mathbb{Q}$, while its right hand side is the union of finitely many intervals that are right-open and left-closed by Corollary 4.3. Thus

$$[0, a) = \big(\cup_{k=0}^{\lfloor c \rfloor - 1} (\mathcal{S}_{a,c} + k) \cap [0, a)\big) \cup \big((\mathcal{S}_{a,c} \cap [0, (c_0 + a - 1)_+) + \lfloor c \rfloor) \cap [0, a)\big)$$

and the conclusion (3.3) follows.

Case 2: $a \in \mathbb{Q}$ *and* $c \in \gcd(a, 1)\mathbb{Z}$

Write $a = p/q$ for some coprime integers p and q. Take $t_0 \in \mathcal{S}_{a,c} \cap \gcd(a, 1)\mathbb{Z}$. The existence of such a point t_0 follows from (1.25) and the assumption that $\mathcal{S}_{a,c} \neq \emptyset$. Following the argument in (4.15), we have that

$$t_0 + l \in \big(\cup_{\lambda_1=0}^{\lfloor c \rfloor} A_{\lambda_1}\big) \cup \big(\cup_{\lambda_2=0}^{\lfloor c \rfloor - 1} B_{\lambda_2}\big) \tag{4.15}$$

for all $0 \leq l \in \mathbb{Z}$. Observe that $\{t_0, t_0+1, \ldots, t_0+a/\gcd(a, 1)-1\}+a\mathbb{Z} = \gcd(a, 1)\mathbb{Z}$. The above observation together with (4.15) implies that

$$\gcd(a, 1)\mathbb{Z} \subset \big(\cup_{\lambda_1=0}^{\lfloor c \rfloor} A_{\lambda_1}\big) \cup \big(\cup_{\lambda_2=0}^{\lfloor c \rfloor - 1} B_{\lambda_2}\big). \tag{4.16}$$

Combining (1.25) and (4.16) proves the desired covering property (3.3). □

We finish this section with the proof of Theorem 3.3.

Proof of Theorem 3.3. ($\Longrightarrow$) By Theorem 2.3 and the assumption that $\mathcal{D}_{a,c}=\emptyset$, we then have that

$$t+\lambda\notin \mathcal{S}_{a,c}\quad \text{for all } t\in \mathcal{S}_{a,c}\cap[0,(c_0+a-1)_+) \text{ and } \lambda\in[1,\lfloor c\rfloor]\cap\mathbb{Z};$$

and

$$t+\lambda\notin \mathcal{S}_{a,c}\quad \text{for all } t\in \mathcal{S}_{a,c}\cap[(c_0-a)_-+a,a) \text{ and } \lambda\in[1,\lfloor c\rfloor-1]\cap\mathbb{Z}.$$

Therefore the sets $\mathcal{S}_{a,c}\cap[0,(c_0+a-1)_+)+\lambda_1+a\mathbb{Z}, \lambda_1\in[0,\lfloor c\rfloor]\cap\mathbb{Z}$, and $\mathcal{S}_{a,c}\cap[(c_0-a)_-+a,a)+\lambda_2+a\mathbb{Z}, \lambda_2\in[0,\lfloor c\rfloor-1]\cap\mathbb{Z}$, are mutually disjoint. This together with the covering property in Theorem 3.2 and the periodic property (2.1) for the set $\mathcal{S}_{a,c}$ implies that

$$\begin{aligned}
a &= \sum_{\lambda_1\in[0,\lfloor c\rfloor]\cap\mathbb{Z}} |(\mathcal{S}_{a,c}\cap[0,(c_0+a-1)_+)+\lambda_1+a\mathbb{Z})\cap[0,a)|\\
&\quad+\sum_{\lambda_2\in[0,\lfloor c\rfloor-1]\cap\mathbb{Z}} |(\mathcal{S}_{a,c}\cap[(c_0-a)_-+a,a)+\lambda_2+a\mathbb{Z})\cap[0,a)|\\
&= \sum_{\lambda_1\in[0,\lfloor c\rfloor]\cap\mathbb{Z}} |(\mathcal{S}_{a,c}\cap[0,(c_0+a-1)_+)+\lambda_1+a\mathbb{Z})\cap[\lambda_1,a+\lambda_1)|\\
&\quad+\sum_{\lambda_2\in[0,\lfloor c\rfloor-1]\cap\mathbb{Z}} |(\mathcal{S}_{a,c}\cap[(c_0-a)_-+a,a)+\lambda_2+a\mathbb{Z})\cap[\lambda_2,a+\lambda_2)|\\
&= (\lfloor c\rfloor+1)|\mathcal{S}_{a,c}\cap[0,(c_0+a-1)_+)|+\lfloor c\rfloor|\mathcal{S}_{a,c}\cap[(c_0-a)_-+a,a)|,
\end{aligned}$$

which proves (3.4).

($\Longleftarrow$) Set

$$A_\lambda=(\mathcal{S}_{a,c}\cap[0,(c_0+a-1)_+)+\lambda+a\mathbb{Z})\cap[0,a)$$

and

$$B_\lambda=(\mathcal{S}_{a,c}\cap[(c_0-a)_-+a,a)+\lambda+a\mathbb{Z})\cap[0,a), \lambda\in\mathbb{Z}.$$

By Theorem 3.2, the sets $A_{\lambda_1}, \lambda_1\in[0,\lfloor c\rfloor]\cap\mathbb{Z}$ and $B_{\lambda_2}, \lambda_2\in[0,\lfloor c\rfloor-1]\cap\mathbb{Z}$ form a covering for the interval $[0,a)$. This together with the assumption (3.4) and the periodic property (2.1) for the set $\mathcal{S}_{a,c}$ implies that

$$\begin{aligned}
a &= (\lfloor c\rfloor+1)|\mathcal{S}_{a,c}\cap[0,(c_0+a-1)_+)|+\lfloor c\rfloor|\mathcal{S}_{a,c}\cap[c_0,a)|\\
&= \sum_{\lambda_1\in[0,\lfloor c\rfloor]\cap\mathbb{Z}}|A_{\lambda_1}|+\sum_{\lambda_2\in[0,\lfloor c\rfloor-1]\cap\mathbb{Z}}|B_{\lambda_2}|\\
&\ge \big|\big(\cup_{\lambda_1\in[0,\lfloor c\rfloor]\cap\mathbb{Z}}A_{\lambda_1}\big)\cup\big(\cup_{\lambda_2\in[0,\lfloor c\rfloor-1]\cap\mathbb{Z}}B_{\lambda_2}\big)\big|=a.
\end{aligned}$$

Thus the intersection of any two of those sets $A_{\lambda_1}, \lambda_1\in[0,\lfloor c\rfloor]\cap\mathbb{Z}$ and $B_{\lambda_2}, \lambda_2\in[0,\lfloor c\rfloor-1]\cap\mathbb{Z}$, has zero Lebesgue measure. Hence they have empty intersection as those sets are finite union of intervals that are left-closed and right-open by (1.25). This together with Theorem 2.3 proves that $\mathcal{D}_{a,c}=\emptyset$. $\square$

CHAPTER 5

Maximal Invariant Sets with Irrational Time Shifts

Let $c_0 := c - \lfloor c \rfloor$ be the fractional part of window parameter c. For either $c_0 \le 1-a$ or $c_0 \ge a$, the maximal invariant set $\mathcal{S}_{a,c}$ has been explicitly constructed, see Theorem 3.5. In this chapter, we consider the maximal invariant set $\mathcal{S}_{a,c}$ with

$$0 < a < 1 < c, \quad 1-a < c_0 < a \quad \text{and} \quad a \notin \mathbb{Q}. \tag{5.1}$$

Before exploring further, let us have an illustrative example.

EXAMPLE 5.1. Take $a = \pi/4 \approx 0.7854$, and $c = 23 - 11\pi/2 \approx 5.7212$. The black holes of the corresponding transformations $R_{a,c}$ and $\tilde{R}_{a,c}$ are $[17-21\pi/4, 18-11\pi/2)+\pi\mathbb{Z}/4$ and $[5-3\pi/2, 6-7\pi/4)+\pi\mathbb{Z}/4$ respectively, which can be transformed back and forth via the hole $[11-7\pi/2, 12-15\pi/4)+\pi\mathbb{Z}/4$; i.e.,

$$\begin{cases} R_{a,c}([5-3\pi/2, 6-7\pi/4)+\pi\mathbb{Z}/4) = [11-7\pi/2, 12-15\pi/4)+\pi\mathbb{Z}/4 \\ (R_{a,c})^2([5-3\pi/2, 6-7\pi/4)+\pi\mathbb{Z}/4) = [17-21\pi/4, 18-11\pi/2)+\pi\mathbb{Z}/4, \end{cases}$$

and

$$\begin{cases} \tilde{R}_{a,c}([17-21\pi/4, 18-11\pi/2)+\pi\mathbb{Z}/4) = [11-7\pi/2, 12-15\pi/4)+\pi\mathbb{Z}/4 \\ (\tilde{R}_{a,c})^2([17-21\pi/4, 18-11\pi/2)+\pi\mathbb{Z}/4) = [5-3\pi/2, 6-7\pi/4)+\pi\mathbb{Z}/4. \end{cases}$$

Therefore for the pair $(a,c) = (\pi/4, 23-11\pi/2)$,

$$\begin{aligned} \mathcal{S}_{a,c} &= [18-23\pi/4, 11-7\pi/2) \cup [12-15\pi/4, 5-3\pi/2) \\ &\quad \cup [6-7\pi/4, 17-21\pi/4) + \pi\mathbb{Z}/4 \\ &\approx [-0.0642, 0.0044) \cup [0.2190, 0.2876) \cup [0.5022, 0.5066) + 0.7864\mathbb{Z} \end{aligned}$$

by Theorem 3.4, which consists of intervals of different lengths on one period and contains a small neighborhood of the lattice $\pi\mathbb{Z}/4$, cf. Figure 1.

For arbitrary $a \notin \mathbb{Q}$, the black hole $[c_0+a-1, c_0)+a\mathbb{Z}$ of the transformation $R_{a,c}$ and the black hole $[c-c_0, c-c_0+1-a)+a\mathbb{Z}$ of the transformation $\tilde{R}_{a,c}$ are inter-transformable through mutually disjoint periodic holes

$$(R_{a,c})^n([c-c_0, c-c_0+1-a)+a\mathbb{Z}) = (\tilde{R}_{a,c})^{D-n}([c_0+a-1, c_0)+a\mathbb{Z}), 0 \le n \le D,$$

in finite steps, provided that $\mathcal{S}_{a,c} \ne \emptyset$, where $D \le \lfloor a/(1-a) \rfloor - 1$ is a nonnegative integer. This together with the maximal invariance property in Theorem 3.4 leads to the following conclusion for the set $\mathcal{S}_{a,c}$, cf. Example 5.1.

THEOREM 5.2. *Let (a,c) satisfy (5.1). Assume that $\mathcal{S}_{a,c} \ne \emptyset$. Then there exists a nonnegative integer $D \le \lfloor a/(1-a) \rfloor - 1$ such that $\mathcal{A}_n := (R_{a,c})^n([c-c_0, c-c_0+1-a)+a\mathbb{Z}), 0 \le n \le D$, satisfy the following properties:*

$$\mathcal{A}_n = (R_{a,c})^n(c-c_0) + [0, 1-a) + a\mathbb{Z}, \quad 0 \le n \le D; \tag{5.2}$$

$$\text{closure of } \mathcal{A}_n, \ 0 \le n \le D, \ \text{are mutually disjoint}; \tag{5.3}$$

$$\mathcal{A}_D = [c_0 + a - 1, c_0) + a\mathbb{Z}; \tag{5.4}$$

and

$$\mathbb{R}\backslash\mathcal{S}_{a,c} = \cup_{n=0}^{D}\mathcal{A}_n. \tag{5.5}$$

For $a \notin \mathbb{Q}$, it follows from Lemma 3.10 and Theorems 3.5 and 5.2 that the maximal invariant set $\mathcal{S}_{a,c}$ consists of finitely many left-closed and right-open intervals on one period (hence it is measurable) and it contains a small neighborhood of the origin.

COROLLARY 5.3. *Let $0 < a < 1 < c$ and $a \notin \mathbb{Q}$. Assume that $\mathcal{S}_{a,c} \neq \emptyset$. Then the following statements hold.*

(i) *The set $\mathcal{S}_{a,c}$ contains finitely many left-closed and right-open intervals on one period and it contains a small neighborhood of the lattice $a\mathbb{Z}$.*
(ii) *The complement of the set $\mathcal{S}_{a,c}$ contains finitely many left-closed right-open intervals of length $1-a$ on one period whose closure are mutually disjoint.*

Combining Theorems 4.4 and 5.2 , we have the following result about the marks $\mathcal{K}_{a,c}$.

COROLLARY 5.4. *Let (a,c) satisfy* (5.1). *Assume that $\mathcal{S}_{a,c} \neq \emptyset$. Then the set $\mathcal{K}_{a,c}$ of marks is given by*

$$\mathcal{K}_{a,c} = \{nY_{a,c}(\lfloor c\rfloor + 1), 1 \le n \le D+1\} + Y_{a,c}(a)\mathbb{Z}, \tag{5.6}$$

where D is the smallest nonnegative integer satisfying (5.4).

After performing the holes-removal surgery, the maximal invariant set $\mathcal{S}_{a,c}$ becomes the real line with marks in $\mathcal{K}_{a,c}$. This suggests that for the case that $a \notin \mathbb{Q}$ we can expand the line with marks by inserting holes $[0, 1-a)$ at every location of marks to recover the maximal invariant set $\mathcal{S}_{a,c}$ by Theorem 5.2. Using the equivalence between the application of the piecewise linear transformation $R_{a,c}$ on the set $\mathcal{S}_{a,c}$ and a rotation on the circle with marks given in Theorem 4.4, we can characterize the non-triviality of the maximal invariant set $\mathcal{S}_{a,c}$ via two nonnegative integer parameters d_1 and d_2 for the case that $a \notin \mathbb{Q}$.

THEOREM 5.5. *Let (a,c) be a pair of positive numbers satisfying $\lfloor c\rfloor \ge 2, 0 < c_1 := c - c_0 - \lfloor (c-c_0)/a\rfloor a < 2a-1$ and* (5.1). *Then $\mathcal{S}_{a,c} \neq \emptyset$ if and only if there exist nonnegative integers d_1 and d_2 such that*

$$(d_1 + d_2 + 1)c_1 - c_0 + (d_1+1)(1-a) \in a\mathbb{Z}, \tag{5.7}$$

$$(d_1+1)(1-a) < c_0 < 1 - (d_2+1)(1-a), \tag{5.8}$$

and

$$\#E_{a,c} = d_1, \tag{5.9}$$

where

$$m = \frac{(d_1 + d_2 + 1)c_1 - c_0 + (d_1+1)(1-a)}{a}$$

and

$$\begin{aligned} E_{a,c} \quad = \quad & \big\{n \in [1, d_1+d_2+1] \big|\; n(c_1 - m(1-a)) \\ & \in [0, c_0 - (d_1+1)(1-a)) + (a - (d_1+d_2+1)(1-a))\mathbb{Z}\big\}. \end{aligned} \tag{5.10}$$

The nonnegative integers d_1 and d_2 in Theorem 5.5 satisfy $(d_1 + d_2 + 1) < a/(1-a)$ by (5.8), and they are uniquely determined by the pair (a, c) of positive numbers by (5.7) and the assumptions that $\lfloor c \rfloor \geq 2$ and $a \notin \mathbb{Q}$. We also notice that the nonnegative integer parameters d_1 and d_2 in Theorem 5.5 are indeed the numbers of holes contained in $[0, c_0 + a - 1)$ and $[c_0, a)$ respectively, and the set of marks is given by

$$\mathcal{K}_{a,c} = \big\{n(c_1 - m(1-a))\big\}_{n=1}^{d_1+d_2+1} + (a - (d_1 + d_2 + 1)(1-a))\mathbb{Z}.$$

For pairs (a, c) satisfying $\lfloor c \rfloor \geq 2, 0 < c_1 < 2a - 1$ and (5.1), we can apply Theorem 5.5 to determine whether the corresponding Gabor systems $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is a frame, see Theorem 7.4 for details.

This chapter is organized as follows. In Section 5.1, we prove Theorem 5.2, Corollary 5.4 and the second conclusion of Theorem 4.1. In Section 5.2, we parameterize the maximal invariant set $\mathcal{S}_{a,c}$ and establish Theorem 5.5. In Section 5.3, we consider the ergodic theory associated with the transformation $R_{a,c}$ and prove Theorem 4.5 of Chapter 4.

5.1. Maximal invariant sets with irrational time shifts

In this section, we prove Theorem 5.2, Corollary 5.4 and the second conclusion of Theorem 4.1.

Proof of Theorem 5.2. By Proposition 3.6, $\mathcal{A}_n, n \geq 0$, have empty intersection with the maximal invariant set $\mathcal{S}_{a,c}$,

$$\mathcal{A}_n \cap \mathcal{S}_{a,c} = \emptyset, \tag{5.11}$$

and they have the following mutual intersection property:

$$\mathcal{A}_m \cap \mathcal{A}_n \subset [c_0 + a - 1, c_0) + a\mathbb{Z} \quad \text{whenever} \quad m \neq n. \tag{5.12}$$

Let D be the smallest nonnegative integer such that

$$\mathcal{A}_D \cap \big([c_0 + a - 1, c_0) + a\mathbb{Z}\big) \neq \emptyset \tag{5.13}$$

if it exists, and set $D = \infty$ if $\mathcal{A}_n \cap ([c_0 + a - 1, c_0) + a\mathbb{Z}) = \emptyset$ for all $n \geq 0$. By (1.19) and the definition (5.13) of the integer D, $\mathcal{A}_n \cap [0, a), 0 \leq n < D$, have the same Lebesgue measure $1 - a$. On the other hand, $\mathcal{A}_n \cap [0, a), 0 \leq n < D$, are mutually disjoint sets contained in $[0, a)\backslash[c_0 + a - 1, c_0)$ by (5.12) and (5.13). Therefore

$$D \leq \lfloor a/(1-a) \rfloor - 1.$$

By (5.11), (5.13) and Lemma 3.10, we can prove immediately that $\mathcal{A}_n \cap [0, a), 0 \leq n < D$, are intervals of length $1 - a$ contained either in $[0, c_0 + 1 - a)$ or in $[c_0, a)$ (and hence (5.2) follows) by induction on $n \geq 0$.

Now we prove (5.3). Suppose, on the contrary, there exist $0 \leq n \neq m \leq D$ such that

$$(R_{a,c})^n(c - c_0) + 1 - a \in (R_{a,c})^m(c - c_0) + a\mathbb{Z} \tag{5.14}$$

by (5.2) and (5.12). This together with the assumption $a \notin \mathbb{Q}$ and the definition of the transformation $R_{a,c}$ implies that

$$\lfloor c \rfloor = 1, \; m = n + 1, \; \text{and} \; (R_{a,c})^n(c - c_0) \in [c_0, a) + a\mathbb{Z}. \tag{5.15}$$

Applying (5.14) and (5.15) repeatedly gives that

$$\begin{aligned}\cup_{k=n}^{n+L}\mathcal{A}_k &= ((R_{a,c})^n(c-c_0)+[0,1-a)+a\mathbb{Z})\\ &\quad\cup((R_{a,c})^n(c-c_0)+1-a+[0,1-a)+a\mathbb{Z})\\ &\quad\cup\cdots\cup((R_{a,c})^n(c-c_0)+L(1-a)+[0,1-a)+a\mathbb{Z})\\ &= (R_{a,c})^n(c-c_0)+[0,(L+1)(1-a))+a\mathbb{Z},\end{aligned}$$

where $L \geq 0$ is largest nonnegative integer such that

$$(R_{a,c})^n(c-c_0)+L(1-a)+[0,1-a)\subset[c_0,a)+a\mathbb{Z}.$$

This contradicts to the density property in Lemma 3.10 as

$$(-\epsilon,0)\subset(R_{a,c})^n(c-c_0)+[0,(L+1)(1-a))+a\mathbb{Z}$$

for sufficiently small $\epsilon > 0$.

By (5.2) and (5.13), we may write

$$\mathcal{A}_D\cap[0,a)=[u_D,u_D+1-a)$$

for some u_D satisfying

$$c_0+2a-2<u_D<c_0. \tag{5.16}$$

Then the proof of (5.4) reduces to showing

$$u_D=c_0+a-1. \tag{5.17}$$

Suppose that

$$c_0+2a-2<u_D<c_0+a-1.$$

Take $t\in\mathcal{S}_{a,c}\cap(0,\epsilon)$ with sufficiently small $\epsilon>0$, where the existence follows from Lemma 3.10. Then

$$(R_{a,c})^{D+1}(t)\in[u_D+1-a,u_D+1-a+\epsilon)+a\mathbb{Z}\subset[c_0+a-1,c_0)+a\mathbb{Z},$$

which implies that $t\notin\mathcal{S}_{a,c}$ by Lemma 3.9. On the other hand,

$$(R_{a,c})^{D+1}(t)\in\mathcal{S}_{a,c}$$

by (3.6). This leads to a contradiction. Thus

$$u_D\notin(c_0+2a-2,c_0+a-1). \tag{5.18}$$

Similarly we can prove that

$$u_D\notin(c_0+a-1,c_0). \tag{5.19}$$

Combining (5.16), (5.18) and (5.19) proves (5.17) and hence (5.4).

Define

$$\mathcal{T}_{a,c}=\mathbb{R}\backslash\big(\cup_{n=0}^{D}(R_{a,c})^n([c-c_0,c-c_0+1-a)+a\mathbb{Z})\big).$$

The proof of (5.5) reduces to showing

$$\mathcal{S}_{a,c}=\mathcal{T}_{a,c}. \tag{5.20}$$

By Proposition 3.6, we obtain that

$$\mathcal{S}_{a,c}\subset\mathcal{T}_{a,c}.$$

Therefore it remains to prove

$$\mathcal{T}_{a,c}\subset\mathcal{S}_{a,c},$$

which in turn reduces to establishing

$$R_{a,c}(\mathcal{T}_{a,c})=\mathcal{T}_{a,c} \tag{5.21}$$

by Theorem 3.4 and the observation that $\mathcal{T}_{a,c}$ has empty intersection with the black hole $[c_0+a-1, c_0)+a\mathbb{Z}$. We first make the following claim:

Claim 1: $R_{a,c}(\mathcal{T}_{a,c}) \subset \mathcal{T}_{a,c}$.

PROOF. Take $t \in \mathcal{T}_{a,c}$. Therefore it suffices to prove that

$$R_{a,c}(t) \notin \cup_{n=1}^{D} (R_{a,c})^n([c-c_0, c-c_0+1-a)+a\mathbb{Z})$$

by (1.18). Suppose on the contrary that

$$R_{a,c}(t) \in (R_{a,c})^n([c-c_0, c-c_0+1-a)+a\mathbb{Z})$$

for some $1 \le n \le D$. Recall that

$$(R_{a,c})^D([c-c_0, c-c_0+1-a)+a\mathbb{Z}) = [c_0+a-1, c_0)+a\mathbb{Z}.$$

Then

$$t \in (R_{a,c})^{n-1}([c-c_0, c-c_0+1-a)+a\mathbb{Z})$$

by (1.17) and the fact that $t \notin [c_0+a-1, c_0)+a\mathbb{Z}$, which is a contradiction. □

Using similar argument, we have the following claim:

Claim 2: $\tilde{R}_{a,c}(\mathcal{T}_{a,c}) \subset \mathcal{T}_{a,c}$.

The invariance (5.21) of the set $\mathcal{T}_{a,c}$ under the transformation $R_{a,c}$ follows from the above two claims and Lemma 3.8. This completes the proof of the equation (5.20) (and hence (5.5)). □

PROOF OF COROLLARY 5.4. By Theorem 5.2, $(R_{a,c})^n(c-c_0+1-a)+[a-1, 0]+a\mathbb{Z}, 0 \le n \le D$, have their closures being mutually disjoint, and

$$\mathbb{R}\backslash\mathcal{S}_{a,c} = \cup_{n=0}^{D}((R_{a,c})^n(c-c_0+1-a)+[a-1,0)+a\mathbb{Z}).$$

Therefore $(R_{a,c})^n(c-c_0+1-a) \in \mathcal{S}_{a,c}$ for all $0 \le n \le D$, and

$$\begin{aligned}\mathcal{K}_{a,c} &= \cup_{n=0}^{D}\{Y_{a,c}((R_{a,c})^n(c-c_0+1-a))+Y_{a,c}(a)\mathbb{Z}\}\\ &= \cup_{n=0}^{D}\{(n+1)Y_{a,c}(c-c_0+1-a)+Y_{a,c}(a)\mathbb{Z}\}\\ &= \cup_{m=1}^{D+1}\{mY_{a,c}(\lfloor c\rfloor+1)+Y_{a,c}(a)\mathbb{Z}\}\end{aligned}$$

where the second equality follows from (4.10) and Theorem 4.4. This proves (5.6). □

Finally we prove the second conclusion of Theorem 4.1 of Chapter 4.

PROOF OF THEOREM 4.1. (ii) For $c_0 = 0$, the conclusion (4.4) holds because $\mathcal{S}_{a,c} = \mathcal{E}_L = \mathbb{R}$ for all $L \ge 0$ in this case. So we may assume that $c_0 > 0$ from now on.

By $a \notin \mathbb{Q}$, Theorem 3.5 and the assumptions $\mathcal{S}_{a,c} \ne \emptyset$, we have that $1-a < c_0 < a$. By (4.2) and Theorem 3.4, it suffices to prove that

$$\mathcal{E}_{\lfloor a/(1-a)\rfloor} \subset \mathcal{S}_{a,c}.$$

Suppose, on the contrary, that there exists $t \in \mathcal{E}_{\lfloor a/(1-a)\rfloor}$ and $t \notin \mathcal{S}_{a,c}$. Then

$$t = (R_{a,c})^{\lfloor a/(1-a)\rfloor}(s)$$

for some $s \in \mathbb{R}\backslash([c_0+a-1, c_0)+a\mathbb{Z})$; and

$$t = (R_{a,c})^n(s_0)$$

for some nonnegative integer $n \le \lfloor a/(1-a)\rfloor - 1$, and $s_0 \in [c-c_0, c-c_0+1-a)+a\mathbb{Z}$ by Theorem 5.2. As $R_{a,c}$ is one-to-one outside its black holes by (1.17), we then have that $s_0 = (R_{a,c})^{\lfloor a/(1-a)\rfloor - n}(s)$, which contradicts to the range property (1.18) of the transformation $R_{a,c}$. $\square$

5.2. Nontriviality of maximal invariant sets with irrational time shifts

In this section, we prove Theorem 5.5. The necessity follows essentially from Theorem 5.2. For the sufficiency, we perform the augmentation operation by inserting hole $[0, 1-a)$ from the line with marks at $\{n(c_1 - m(1-a))\}_{n=1}^{d_1+d_2+1} + (a-(d_1+d_2+1)(1-a))\mathbb{Z}$ and then show that the set obtained through the augmentation operation is invariant under the transformation $R_{a,c}$.

Proof of Theorem 5.5. ($\Longrightarrow$) Assume that $\mathcal{S}_{a,c} \neq \emptyset$. Let $D \le \lfloor a/(1-a)\rfloor - 1$ be the nonnegative integer in Theorem 5.2 such that

$$[a_n, a_n + 1 - a) := ((R_{a,c})^n([c-c_0, c-c_0+1-a) + a\mathbb{Z})) \cap [0, a), 0 \le n \le D, \tag{5.22}$$

are mutually disjoint;

$$\mathcal{S}_{a,c} = \mathbb{R}\backslash\big(\cup_{n=0}^{D}([a_n, a_n + 1 - a) + a\mathbb{Z})\big); \tag{5.23}$$

$$(R_{a,c})^D([c-c_0, c-c_0+1-a) + a\mathbb{Z}) = [c_0 + a - 1, c_0) + a\mathbb{Z}; \tag{5.24}$$

and

$$a_n - (R_{a,c})^n(c - c_0) \in a\mathbb{Z},\ \ 0 \le n \le D. \tag{5.25}$$

Applying (5.25) with $n = D$ and using (5.22) and (5.24), we obtain

$$c_0 + a - 1 - \big(c - c_0 + d_1(\lfloor c\rfloor + 1) + d_2\lfloor c\rfloor\big) \in a\mathbb{Z}, \tag{5.26}$$

where d_1, d_2 are the numbers of the indices $n \in [0, D-1]$ such that $[a_n, a_n + 1 - a)$ is contained in $[0, c_0 + a - 1)$ and in $[c_0, a)$ respectively. Then the desired inclusion (5.7) follows from (5.26).

Observe that

$$D = d_1 + d_2 \tag{5.27}$$

because it follows from (5.22) and (5.24) that

$$[a_n, a_n + 1 - a) \subset [0, a)$$

and

$$[a_n, a_n + 1 - a) \cap [c_0 + a - 1, c_0) = [a_n, a_n + 1 - a) \cap [a_D, a_D + 1 - a) = \emptyset$$

for every $0 \le n \le D - 1$. Recall that there are d_1 (resp. d_2) mutually disjoint holes of length $1 - a$ contained in $[0, c_0 + a - 1)$ (resp. $[c_0, a)$) by (5.22), (5.27) and the definition of integer parameters d_1 and d_2; and that $(-\epsilon, \epsilon) \subset \mathcal{S}_{a,c}$ for sufficiently small $\epsilon > 0$ by Corollary 5.3. Therefore

$$d_1(1-a) < c_0 + a - 1 = |[0, c_0 + a - 1)|$$

and

$$d_2(1-a) < a - c_0 = |[c_0, a)|,$$

which proves (5.8).

Let $\theta_{a,c} := Y_{a,c}(c - c_0 + 1)$ be as in Theorem 4.4, and set $\tilde{\theta}_{a,c} = Y_{a,c}(c_1)$. Then

$$Y_{a,c}(a) = a - (d_1 + d_2 + 1)(1-a) \tag{5.28}$$

by (5.22) and (5.23); and

$$\tilde{\theta}_{a,c} \in Y_{a,c}(c-c_0)+Y_{a,c}(a)\mathbb{Z} = Y_{a,c}(c-c_0+1-a)+Y_{a,c}(a)\mathbb{Z} = \theta_{a,c}+Y_{a,c}(a)\mathbb{Z} \tag{5.29}$$

by (4.10) and the fact that the black hole $[c-c_0, c-c_0+1-a)$ of the transformation $\tilde{R}_{a,c}$ having empty intersection with the set $\mathcal{S}_{a,c}$ (see (3.2) in Theorem 3.2). Combining (5.22), (5.23), (5.28), (5.29) with Theorem 4.4, we conclude that the marks are located at

$$n\tilde{\theta}_{a,c} + (a-(d_1+d_2+1)(1-a))\mathbb{Z}, 1 \le n \le d_1+d_2+1,$$

with the first mark $\tilde{\theta}_{a,c}+a\mathbb{Z}$ and the last mark $(d_1+d_2+1)\tilde{\theta}_{a,c}+a\mathbb{Z}$ being obtained from the holes $[c-c_0, c-c_0+1-a)+a\mathbb{Z}$ and $[c_0+a-1, c_0)+a\mathbb{Z}$ respectively. As there are d_1 holes contained in $[0, c_0+a-1)$, we have that

$$Y_{a,c}(c_0+a-1) = c_0+a-1-d_1(1-a)$$

by the definition of the map $Y_{a,c}$. Thus

$$c_0-(d_1+1)(1-a)-(d_1+d_2+1)\tilde{\theta}_{a,c} \in (a-(d_1+d_2+1)(1-a))\mathbb{Z}. \tag{5.30}$$

Let m be the number of holes $[a_n, a_n+1-a), 0 \le n \le d_1+d_2$, contained in $[0, c_1)$. By the definition $\tilde{\theta}_{a,c} = Y_{a,c}(c_1)$ and the one-to-one correspondence between holes and marks,

$$m = \#\big\{1 \le n \le d_1+d_2+1 \big| n\tilde{\theta}_{a,c} \in [0, \tilde{\theta}_{a,c}) + (a-(d_1+d_2+1)(1-a))\mathbb{Z}\big\}.$$

This, together with the observation that the black hole $[c_1, c_1+1-a)$ of the transformation $\tilde{R}_{a,c}$ has empty intersection with the set $\mathcal{S}_{a,c}$, implies that

$$\tilde{\theta}_{a,c} = c_1 - m(1-a). \tag{5.31}$$

Let $\tilde{m}$ be another integer such that

$$(d_1+d_2+1)\tilde{\theta}_{a,c} \in \tilde{m}(a-(d_1+d_2+1)(1-a)) + [0, a-(d_1+d_2+1)(1-a)).$$

We want to prove that

$$\tilde{m} = m. \tag{5.32}$$

For any $1 \le l \le \tilde{m}$, there exists one and only one $1 \le n_l \le d_1+d_2+1$ such that

$$n_l\tilde{\theta}_{a,c} \in l(a-(d_1+d_2+1)(1-a)) + [0, \tilde{\theta}_{a,c}),$$

which implies that $\tilde{m} \le m$. Now we prove that $m \le \tilde{m}$. Suppose on the contrary that $m > \tilde{m}$. Then there exists an integer $1 \le n \le d_1+d_2+1$ such that

$$n\tilde{\theta}_{a,c} \in [0, \tilde{\theta}_{a,c}) + (a-(d_1+d_2+1)(1-a))(\mathbb{Z}\backslash\{1, \dots, \tilde{m}\}).$$

This implies that

$$n\tilde{\theta}_{a,c} \ge (\tilde{m}+1)(a-(d_1+d_2+1)(1-a)),$$

which is a contradiction as

$$\tilde{\theta}_{a,c} \le n\tilde{\theta}_{a,c} \le (d_1+d_2+1)\tilde{\theta}_{a,c} < (\tilde{m}+1)(a-(d_1+d_2+1)(1-a))$$

by the definition of the integer $\tilde{m}$, and hence (5.32) is established.

From (5.30), (5.31) and (5.32), it follows that

$$\begin{aligned} & (d_1+d_2+1)(c_1-m(1-a)) = (d_1+d_2+1)\tilde{\theta}_{a,c} \\ = \; & c_0-(d_1+1)(1-a)+m(a-(d_1+d_2+1)(1-a)), \end{aligned}$$

which implies that

$$ma = (d_1 + d_2 + 1)c_1 - c_0 + (d_1 + 1)(1 - a). \tag{5.33}$$

Then the condition (5.9) follows from (5.30), (5.31) and (5.33), and the definition of the integer d_1.

($\Longleftarrow$) Let d_1 and d_2 be nonnegative integers in (5.7) and (5.8), and $c_1 = c - c_0 - \lfloor (c - c_0)/a \rfloor a$. Then

$$0 < c_1 < a < 1$$

by $a \notin \mathbb{Q}$; and

$$-a < -c_0 + 1 - a < (d_1 + d_2 + 1)c_1 - c_0 + (d_1 + 1)(1 - a) < (d_1 + d_2 + 1)c_1 < (d_1 + d_2 + 1)a$$

by (5.7) and (5.8). Also from (5.7) and (5.8), we see that

$$(d_1 + d_2 + 1)c_1 - c_0 + (d_1 + 1)(1 - a) \in (d_1 + d_2 + 1)\lfloor c \rfloor - c + \lfloor c \rfloor + (d_1 + 1) + a\mathbb{Z} = a\mathbb{Z}.$$

Thus

$$(d_1 + d_2 + 1)c_1 - c_0 + (d_1 + 1)(1 - a) = ma \tag{5.34}$$

for some integer $0 \le m \le d_1 + d_2$. Set

$$\tilde{\theta}_{a,c} = c_1 - m(1 - a). \tag{5.35}$$

Then

$$(d_1 + d_2 + 1)\tilde{\theta}_{a,c} = c_0 - (d_1 + 1)(1 - a) + m(a - (d_1 + d_2 + 1)(1 - a)) \tag{5.36}$$

by (5.34). This together with $0 \le m \le d_1 + d_2$ and $0 < c_0 - (d_1 + 1)(1 - a) < a - (d_1 + d_2 + 1)(1 - a)$ implies that

$$\tilde{\theta}_{a,c} \in (0, a - (d_1 + d_2 + 1)(1 - a)). \tag{5.37}$$

We next claim that

$$(n - n')\tilde{\theta}_{a,c} \notin (a - (d_1 + d_2 + 1)(1 - a))\mathbb{Z} \tag{5.38}$$

for all $0 \le n \ne n' \le d_1 + d_2 + 1$.

Proof of Claim (5.38). For $n - n' = \pm(d_1 + d_2 + 1)$, the conclusion (5.38) follows from (5.8) and (5.36). Then it remain to prove (5.38) for $1 \le |n - n'| \le d_1 + d_2$. Suppose on the contrary that (5.38) are not true. Then

$$k\tilde{\theta}_{a,c} = l(a - (d_1 + d_2 + 1)(1 - a))$$

for some integers $l \in \mathbb{Z}$ and $k \in [1, d_1 + d_2] \cap \mathbb{Z}$. Then

$$k(m - \lfloor c \rfloor) = l(d_1 + d_2 + 1) \quad \text{and} \quad k(m - \lfloor (\lfloor c \rfloor / a) \rfloor) = l(d_1 + d_2 + 2)$$

by the assumption $a \notin \mathbb{Q}$. Thus

$$l = k(\lfloor (\lfloor c \rfloor / a) \rfloor - \lfloor c \rfloor),$$

which is a contradiction as $1 \le l < k$ by (5.37). $\square$

Denote

$$\mathcal{K}_{a,c} := \{n\tilde{\theta}_{a,c}\}_{n=1}^{d_1+d_2+1} + (a-(d_1+d_2+1)(1-a))\mathbb{Z}$$

and rewrite $\mathcal{K}_{a,c}$ as $\{z_n\}_{n=1}^{d_1+d_2+1} + (a-(d_1+d_2+1)(1-a))\mathbb{Z}$ for some increasing sequence

$$0 < z_1 < z_2 < \ldots < z_{d_1+d_2+1} < a-(d_1+d_2+1)(1-a).$$

The existence of such a positive strictly increasing sequence $\{z_n\}_{n=1}^{d_1+d_2+1}$ follows from (5.38). Given any $\delta \in (0, c_0-(d_1+1)(1-a))$ (respectively $\delta \in (c_0-(d_1+1)(1-a), a-(d_1+d_2+1)(1-a))$), it follows from (5.36) and (5.37) that for any integer $k \in [0, m]$ (resp. $k \in [0, m-1]$) there is one and only one integer $n \in [1, d_1+d_2+1]$ such that

$$n\tilde{\theta}_{a,c} \in k(a-(d_1+d_2+1)(1-a)) + [\delta, \delta+\tilde{\theta}_{a,c})$$

and for $k \in \mathbb{Z}\backslash[0, m]$ (resp. $l \in \mathbb{Z}\backslash[0, m-1]$) there is no integer $n \notin [1, d_1+d_2+1]$ such that

$$n\tilde{\theta}_{a,c} \in k(a-(d_1+d_2+1)(1-a)) + [\delta, \delta+\tilde{\theta}_{a,c}).$$

The above observations together with (5.37) and (5.38) imply that

$$\#\big([\delta, \delta+\tilde{\theta}_{a,c}) \cap \big(\{z_l\}_{l=1}^{d_1+d_2+1} + \{0, a-(d_1+d_2+1)(1-a)\}\big) = m+1 \tag{5.39}$$

for $\delta \in (0, c_0-(d_1+1)(1-a))$, and

$$\#\big([\delta, \delta+\tilde{\theta}_{a,c}) \cap \big(\{z_l\}_{l=1}^{d_1+d_2+1} + \{0, a-(d_1+d_2+1)(1-a)\}\big) = m \tag{5.40}$$

for $\delta \in (c_0-(d_1+1)(1-a), a-(d_1+d_2+1)(1-a))$.

Now let us expand marks located at $\{z_l\}_{l=1}^{d_1+d_2+1} + (a-(d_1+d_2+1)(1-a))\mathbb{Z}$ to holes of length $1-a$ located at $\{y_l\}_{l=1}^{d_1+d_2+1} + a\mathbb{Z}$ on the real line by

$$y_l = z_l + (l-1)(1-a),\ 1 \le l \le d_1+d_2+1. \tag{5.41}$$

Clearly

$$0 < y_1 < y_2 < \ldots < y_{d_1+d_2+1} < a.$$

Now let claim that

$$(R_{a,c})^n(c-c_0) + a\mathbb{Z} = y_{l(n)} + a\mathbb{Z} \quad \text{for all } 0 \le n \le d_1+d_2, \tag{5.42}$$

by induction on $0 \le n \le d_1+d_2$, where $l(n) \in [1, d_1+d_2+1]$ is the unique integer such that

$$z_{l(n)} \in (n+1)\tilde{\theta}_{a,c} + (a-(d_1+d_2+1)(1-a))\mathbb{Z}.$$

Proof of Claim (5.42). Applying (5.39) gives

$$[\delta, \delta+\tilde{\theta}_{a,c}) \cap \big(\{z_l\}_{l=1}^{d_1+d_2+1} + \{0, a-(d_1+d_2+1)(1-a)\}\big) = \{z_1, \ldots, z_{m+1}\},$$

where $\delta > 0$ is so chosen that

$$\delta < z_1 \quad \text{and} \quad (\tilde{\theta}_{a,c}, \tilde{\theta}_{a,c}+\delta) \cap \{z_1, \ldots, z_{d_1+d_2+1}\} = \emptyset.$$

Thus we obtain that

$$z_{m+1} = \tilde{\theta}_{a,c} \tag{5.43}$$

which together with (5.35) implies that

$$y_{l(0)} = y_{m+1} = z_{m+1} + m(1-a) = \tilde{\theta}_{a,c} + m(1-a) = c_1. \tag{5.44}$$

Combining (5.9) and (5.36) gives

$$z_{d_1+1} = c_0 + a - 1 - d_1(1-a) = (d_1+d_2+1)\tilde{\theta}_{a,c} - m(a-(d_1+d_2+1)(1-a)). \tag{5.45}$$

Thus

$$y_{l(d_1+d_2)} = y_{d_1+1} = z_{d_1+1} + d_1(1-a) = c_0 + a - 1, \tag{5.46}$$

Having the above information about $y_{l(0)}$ and $y_{l(d_1+d_2)}$, we now prove (5.42) by induction on n. Clearly the conclusion (5.42) for $n = 0$ follows from (5.44). Inductively we assume that (5.42) holds for $n = k \le d_1 + d_2 - 1$. Then

$$z_{l(k)} \ne c_0 - (d_1+1)(1-a)$$

by (5.36), (5.38) and the observation that $l(k) \ne d_1 + d_2 + 1$. If $z_{l(k)} < c_0 - (d_1 + 1)(1-a)$, then

$$y_{l(k)} < c_0 + a - 1$$

by (5.46) and

$$\begin{aligned}(R_{a,c})^{k+1}(c-c_0) &= R_{a,c}((R_{a,c})^k(c-c_0)) \in R_{a,c}(y_{l(k)}) + a\mathbb{Z}\\ &= y_{l(k)} + \lfloor c \rfloor + 1 + a\mathbb{Z}\\ &= z_{l(k)} + \tilde{\theta}_{a,c} + (m + l(k))(1-a) + a\mathbb{Z}. \qquad (5.47)\end{aligned}$$

Note that either

$$z_{l(k+1)} = z_{l(k)} + \tilde{\theta}_{a,c}$$

or

$$z_{l(k+1)} = z_{k(l)} + \tilde{\theta}_{a,c} - (a - (d_1+d_2+1)(1-a)).$$

For the first case,

$$l(k+1) = l(k) + m + 1$$

because

$$[z_{l(k)}, z_{l(k)} + \tilde{\theta}_{a,c}) \cap \left(\{z_k\}_{k=1}^{d_1+d_2+1} + \{0, a-(d_1+d_2+1)(1-a)\}\right) = m+1$$

by (5.39) and hence

$$\begin{aligned}(R_{a,c})^{k+1}(c-c_0) &\in z_{l(k)} + \tilde{\theta}_{a,c} + (m + l(k))(1-a) + a\mathbb{Z}\\ &= z_{l(k+1)} + (l(k+1)-1)(1-a) + a\mathbb{Z} = y_{l(k+1)} + a\mathbb{Z}. \qquad (5.48)\end{aligned}$$

Similarly for the second case,

$$l(k+1) = l(k) + m + 1 - (d_1 + d_2 + 1)$$

since

$$\begin{aligned}&\#\big([0, z_{l(k+1)}) + (a - (d_1+d_2+1)(1-a))\\ &\qquad \cap\big(\{z_k\}_{k=1}^{d_1+d_2+1} + \{0, a-(d_1+d_2+1)(1-a)\}\big)\big)\\ =\ &\#\big(\big([0, z_{l(k)}) \cap \{z_k\}_{k=1}^{d_1+d_2+1}\big) \cup \big([z_{l(k)}, z_{l(k)} + \tilde{\theta}_{a,c}) \cap \{z_k\}_{k=1}^{d_1+d_2+1}\big)\big)\\ =\ &l(k) - 1 + m + 1 = m + l(k)\end{aligned}$$

by (5.39). Thus

$$\begin{aligned}(R_{a,c})^{k+1}(c-c_0) &\in z_{l(k)} + \tilde{\theta}_{a,c} + (m + l(k))(1-a) + a\mathbb{Z}\\ &= z_{l(k+1)} + (a - (d_1+d_2+1)(1-a))\\ &\quad + (l(k+1) + (d_1+d_2+1) - 1)(1-a) + a\mathbb{Z}\\ &= y_{l(k+1)} + a\mathbb{Z}. \qquad (5.49)\end{aligned}$$

This shows that the inductive conclusion holds when $z_{l(k)} < c_0 - (d_1+1)(1-a)$. Similarly we can show that the inductive conclusion (5.42) holds when $z_{l(k)} > c_0 - (d_1+1)(1-a)$. □

We continue our proof of the sufficiency. From (5.42), we see that for any $0 \le n \le d_1 + d_2 - 1$,

$$(R_{a,c})^n([c - c_0, c - c_0 + 1 - a)) + a\mathbb{Z} = [y_{l(n)}, y_{l(n)} + 1 - a) + a\mathbb{Z}$$

is contained either in $[0, c_0 + a - 1) + a\mathbb{Z}$ or $[c_0, a) + a\mathbb{Z}$, and

$$(R_{a,c})^D([c - c_0, c - c_0 + 1 - a)) + a\mathbb{Z} = [c_0 + a - 1, c_0) + a\mathbb{Z}$$

by (5.42). Therefore $\mathcal{S}_{a,c}$ is the complement of $\cup_{n=0}^{d_1+d_2}([y_{l(n)}, y_{l(n)} + 1 - a) + a\mathbb{Z})$, which implies that its restriction on $[0, a)$ has measure $a - (d_1 + d_2 + 1))1 - a) > 0$ and hence it is not an empty set. □

5.3. Ergodicity of piecewise linear transformations

In this section, we prove Theorem 4.5 of Chapter 4. Define

$$\tilde{Q}_{a,c} := \sup_{t\in\mathbb{R}} \sup_{\mathbf{x}\in\mathcal{B}} \tilde{Q}_{a,c}(t, \mathbf{x}),$$

where

$$\tilde{K}(t, \mathbf{x}) = \{\mu \in a\mathbb{Z} : \mathbf{M}_{a,c}(t)\mathbf{x}(\mu) = 1\}.$$

Let

$$\tilde{Q}_{a,c}(t, \mathbf{x}) = \begin{cases} 0 & \text{if } \tilde{K}(t, \mathbf{x}) = \emptyset \\ \sup\{n \in N \mid [\mu, \mu + na) \subset \tilde{K}(t, \mathbf{x}) & \\ \qquad \text{for some } \mu \in a\mathbb{Z}\} & \text{otherwise,} \end{cases}$$

cf. the index $Q_{a,c}$ in (2.7). Following the argument used in the proof of Theorem 2.5, we have the following result.

LEMMA 5.6. *Let $0 < a < 1 < c$. Then*

$$\mathcal{S}_{a,c} = \emptyset \text{ if and only if } \tilde{Q}_{a,c} < \infty.$$

To prove Theorem 4.5, we need another technical lemma, cf. Lemma 3.9.

LEMMA 5.7. *Let $0 < a < 1 < c$. Then then exists a nonnegative integer L such that*

$$(R_{a,c})^L(t) \in [(c_0 + a - 1)_+, (c_0 - a)_- + a) + a\mathbb{Z} \quad \text{for all } t \notin \mathcal{S}_{a,c}. \tag{5.50}$$

PROOF. For $a \in \mathbb{Q}$, the conclusion (5.50) with $L = a/\gcd(a, 1)$ follows from Theorem 3.4 and the first conclusion of Theorem 4.1.

For $a \notin \mathbb{Q}$ and $\mathcal{S}_{a,c} \neq \emptyset$, the conclusion (5.50) with $L = \lfloor a/(1 - a) \rfloor$ holds by Theorem 3.4 and the second conclusion of Theorem 4.1.

Now it remains to prove (5.50) under the assumption that $\mathcal{S}_{a,c} = \emptyset$. Suppose, on the contrary, there exists $t \in \mathbb{R}$ such that

$$(R_{a,c})^L(t) \notin [(c_0 + a - 1)_+, (c_0 - a)_- + a) + a\mathbb{Z} \tag{5.51}$$

for all $L \ge 0$. Define $\mathbf{x} = (\mathbf{x}_t(\lambda))_{\lambda\in\mathbb{Z}}$ by $\mathbf{x}_t(\lambda) = 1$ if $\lambda = (R_{a,c})^L(t) - t$ for some nonnegative integer L, and $\mathbf{x}_t(\lambda) = 0$ otherwise. Then $\mathbf{x}_t \in \mathcal{B}$ as $0 \neq (R_{a,c})^L(t) - t \in \mathbb{Z}$ for all $L \ge 1$. Following the argument in Lemma 3.9, we have that

$$\mathbf{M}_{a,c}(t)\mathbf{x}_t(\mu) = 1 \quad \text{for all } \ 0 \le \mu \in a\mathbb{Z}, \tag{5.52}$$

which implies that $\tilde{Q}_{a,c} = +\infty$. This is a contradiction by Lemma 5.6. □

Now we prove Theorem 4.5.

Proof of Theorem 4.5. We divide three cases to verify (4.6) and (4.7).

Case 1: $t \notin \mathcal{S}_{a,c}$.

In this case, there exists $L \geq 0$ by Lemma 5.7 such that

$$(R_{a,c})^n(t) = (R_{a,c})^L(t) \in [(c_0 + a - 1)_+, (c_0 - a)_- + a) + a\mathbb{Z}$$

for all $n \geq L$. Thus (4.6) and (4.7) follow.

Case 2: $a \in \mathbb{Q}$ and $t \in \mathcal{S}_{a,c}$.

In this case, $t_n = (R_{a,c})^n(t) \in \mathcal{S}_{a,c}$ for all $n \geq 0$. Following the argument used in the proof of Theorem 4.1, there exists a nonnegative integer D such that $t_{D+1} - t \in a\mathbb{Z}$, which in turn implies that

$$(R_{a,c})^{n+D+1}(t) - (R_{a,c})^n(t) \in a\mathbb{Z} \tag{5.53}$$

for all $n \geq 0$. This together with the periodicity of the function f proves (4.6) and (4.7).

Case 3: $a \notin \mathbb{Q}$ and $t \in \mathcal{S}_{a,c}$.

If $c_0 = 0$, then $\mathcal{S}_{a,c} = \mathbb{R}$ by Theorem 3.5; and $R_{a,c}(t) = \lfloor c \rfloor$ for all $t \in \mathbb{R}$. Thus (4.7) follows from ergodic theorem for irrational rotation [**48**].

Now we consider $c_0 > 0$. In this case, we further assume that $1 - a < c_0 < a$ by Theorem 3.5. Define g on the real line by

$$g(Y_{a,c}(t)) = f(t),\ t \in \mathcal{S}_{a,c}, \tag{5.54}$$

where $Y_{a,c}$ is given in (1.23). The function g is well-defined as $Y_{a,c}$ is an isomorphism between the maximal invariant set $\mathcal{S}_{a,c}$ to the line with marks. Furthermore it follows the periodic of the function f that g is piecewise continuous and satisfies

$$g(u + Y_{a,c}(a)) = g(u),\ u \in \mathbb{R}. \tag{5.55}$$

By Theorem 4.4, we then have that

$$\frac{\sum_{k=0}^{n-1} f((R_{a,c})^k(t))}{n} = \frac{\sum_{k=0}^{n-1} g(Y_{a,c}(t) + kY_{a,c}(\lfloor c \rfloor + 1))}{n}. \tag{5.56}$$

Then by (5.54), (5.55), (5.56) and the ergodic theorem for irrational rotation [**48**], it remains to prove that

$$\frac{Y_{a,c}(\lfloor c \rfloor + 1)}{Y_{a,c}(a)} \notin \mathbb{Q}. \tag{5.57}$$

Suppose, on the contrary, that $\frac{Y_{a,c}(\lfloor c \rfloor+1)}{Y_{a,c}(a)} \in \mathbb{Q}$. Let r, s be co-prime nonnegative integers with

$$\frac{Y_{a,c}(\lfloor c \rfloor + 1)}{Y_{a,c}(a)} = \frac{r}{s}. \tag{5.58}$$

Denote the number of holes in the interval $[0, \lfloor c \rfloor+1)$ and $[0, a)$ by M, N respectively. Then it follows from (1.23), (5.58) and Theorem 5.2 that

$$s(\lfloor c \rfloor + 1 - M(1 - a)) = r(a - N(1 - a)),$$

which together with $a \notin \mathbb{Q}$ implies

$$s(\lfloor c \rfloor + 1 - M) = -rN \quad \text{and} \quad sM = r(N + 1). \tag{5.59}$$

Thus

$$r = s(\lfloor c \rfloor + 1).$$

Substituting the above equality into (5.58) gives

$$Y_{a,c}(\lfloor c \rfloor + 1) \in Y_{a,c}(a)\mathbb{Z},$$

which is a contradiction as $\lfloor c \rfloor + 1 = R_{a,c}(0) \in \mathcal{S}_{a,c}$ and $\lfloor c \rfloor + 1 \notin a\mathbb{Z}$ by $a \notin \mathbb{Q}$. This proves (5.57) and completes the proof of (4.6) and (4.7) for Case 3. □

CHAPTER 6

Maximal Invariant Sets with Rational Time Shifts

In this chapter, we study maximal invariant sets $\mathcal{S}_{a,c}$ for pairs (a,c) satisfying

$$0 < a < 1 < c, \quad 1-a < c_0 < a \quad \text{and} \quad a \in \mathbb{Q}. \tag{6.1}$$

Before doing that, let us have some illustrative examples.

Example 6.1. For the pair $(a,c) = (13/17, 77/17)$, black holes of the corresponding transformations $R_{a,c}$ and $\tilde{R}_{a,c}$ are $[5/17, 9/17)+13\mathbb{Z}/17$ and $[3/17, 7/17)+13\mathbb{Z}/17$ respectively. Applying the transformation $R_{a,c}$ to the black hole of the transformation $\tilde{R}_{a,c}$, we obtain that

$$\begin{cases} R_{a,c}([3,7)/17+13\mathbb{Z}/17) = ([5,7)\cup[10,12))/17+13\mathbb{Z}/17, \\ (R_{a,c})^2([3,7)/17+13\mathbb{Z}/17) = ([5,7)\cup[0,2))/17+13\mathbb{Z}/17, \\ (R_{a,c})^3([3,7)/17+13\mathbb{Z}/17) = [5,9)/17+13\mathbb{Z}/17. \end{cases}$$

Thus

$$\begin{aligned} \mathcal{S}_{a,c} &= ([2,3)\cup[9,10)\cup[12,13))/17+13\mathbb{Z}/17 \\ &\approx [0.1176, 0.1764)\cup[0.5294, 0.5882)\cup[0.7059, 0.7647)+0.7647\mathbb{Z} \end{aligned}$$

consists of intervals of same length $1/17$ on the period $[0, 13/17)$ and contains small left neighborhood of the lattice $13\mathbb{Z}/17$. On the other hand, its complement $\mathbb{R}\backslash\mathcal{S}_{a,c} = ([0,2)\cup[3,9)\cup[10,12))/17+13\mathbb{Z}/17$ contains one big gap of size $6/17$, two small gaps of size $2/17$ on the period $[0,13/17)$, and a small gap attached to the right-hand side of the lattice $13\mathbb{Z}/17$.

For the pair $(a,c) = (13/17, 73/17)$, the maximal invariant set $\mathcal{S}_{a,c} = ([0,1)\cup[7,8)\cup[10,11))/17+13\mathbb{Z}/17$ contains a small right neighborhood of the lattice $13\mathbb{Z}/17$, while its complement $\mathbb{R}\backslash\mathcal{S}_{a,c} = ([1,7)\cup[8,10)\cup[11,13))/17+13\mathbb{Z}/17$ contains a small gap attached to the left-hand side of the lattice $13/17$.

For the pair $(a,c) = (13/17, 75/17)$, the maximal invariant set

$$\begin{aligned} \mathcal{S}_{a,c} &= ([0,3)\cup[7,10)\cup[10,13))/17+13\mathbb{Z}/17 \\ &= [0, 0.1765)\cup[0.4118, 0.5882)\cup[0.5882, 0.7647)+0.7647\mathbb{Z} \end{aligned}$$

consists of intervals of "same" length $3/17$ and contains small left and right neighborhoods of the lattice $13\mathbb{Z}/17$. On the other hand, its complement $\mathbb{R}\backslash\mathcal{S}_{a,c} = [3,7)/17+13\mathbb{Z}/17$ contains one big gap of size $4/17$ and two small gaps of size "zero" at $\{0, 10/17\}$ on the period $[0, 13/17)$, c.f. Figure 1.

Example 6.2. For the pair $(a,c) = (6/7, 23/7+\delta)$ with $-1/14 < \delta < 1/14$, black holes of the corresponding transformations $R_{a,c}$ and $\tilde{R}_{a,c}$ are $[1/7+\delta, 2/7+$

$\delta) + 6\mathbb{Z}/7$ and $[3/7, 4/7) + 6\mathbb{Z}/7$ respectively. Observe that

$$\begin{cases} R_{a,c}([3/7,4/7)+6\mathbb{Z}/7) = [0,1/7)+6\mathbb{Z}/7 \\ (R_{a,c})^2([3/7,4/7)+6\mathbb{Z}/7) = ([4/7,5/7+\delta)\cup[1/7+\delta,1/7))+6\mathbb{Z}/7 \\ (R_{a,c})^3([3/7,4/7)+6\mathbb{Z}/7) = [1/7+\delta,2/7+\delta)+6\mathbb{Z}/7 \end{cases}$$

if $\delta < 0$, and

$$\begin{cases} R_{a,c}([3/7,4/7)+6\mathbb{Z}/7) = [0,1/7)+6\mathbb{Z}/7 \\ (R_{a,c})^2([3/7,4/7)+6\mathbb{Z}/7) = [4/7,5/7)+6\mathbb{Z}/7 \\ (R_{a,c})^3([3,4)/7+6\mathbb{Z}/7) = [1/7,2/7)+6\mathbb{Z}/7 \\ (R_{a,c})^4([3,4)/7+6\mathbb{Z}/7) = (([5/7,5/7+\delta)\cup[1/7+\delta,2/7))+6\mathbb{Z}/7 \\ (R_{a,c})^5([3,4)/7+6\mathbb{Z}/7) = [1/7+\delta,2/7+\delta)+6\mathbb{Z}/7, \end{cases}$$

if $\delta \ge 0$. Therefore for the pair $(a,c) = (6/7, 23/7+\delta)$

$$\mathcal{S}_{a,c} \quad = \quad [2/7+\delta,3/7)\cup[5/7+\delta,6/7)+6\mathbb{Z}/7$$

consists of intervals of length $1/7-\delta$, while its complement $\mathbb{R}\backslash\mathcal{S}_{a,c} = ([0,2/7+\delta)\cup[3/7,5/7+\delta))+6\mathbb{Z}/7$ contains a small left neighborhood of the lattice $6\mathbb{Z}/7$, c.f. (1.25) and Figure 1.

For arbitrary $a \in \mathbb{Q}$, as shown in Lemma 3.11, the set $\mathcal{S}_{a,c}$ contains at least one of two intervals $[0,\epsilon)$ and $[-\epsilon,0)$ whenever it is not an empty set, where $\epsilon > 0$ is sufficiently small. For the case that the set $\mathcal{S}_{a,c}$ contains a small neighborhood of the origin, the restriction of its complement $\mathbb{R}\backslash\mathcal{S}_{a,c}$ on one period consists of finitely many left-closed right-open intervals of length $1-a$, cf. Theorem 5.2 and Example 6.2 with $(a,c) = (13/17, 75/17)$.

Theorem 6.3. *Let (a,c) satisfy* (6.1). *Define*

$$\mathcal{A}_n := (R_{a,c})^n([c-c_0, c-c_0+1-a)+a\mathbb{Z}),\ \ n \ge 0.$$

Let N be the smallest nonnegative integers such that

$$\mathcal{A}_N \cap \big([c_0+a-1,c_0)+a\mathbb{Z}\big) \neq \emptyset. \tag{6.2}$$

Assume that $[-\epsilon,\epsilon) \subset \mathcal{S}_{a,c}$ for some sufficiently small $\epsilon > 0$. Then the following statements hold for gaps $\mathcal{A}_n, 0 \le n \le N$.

(i) *Their restrictions on one period are intervals of length $1-a$,*

$$\mathcal{A}_n = (R_{a,c})^n(c-c_0+1-a)+[a-1,0)+a\mathbb{Z},\ \ 0\le n\le N. \tag{6.3}$$

(ii) *The last gap $\mathcal{A}_N$ concoides with the black hole $[c_0+a-1,c_0)+a\mathbb{Z}$ of the transformation $R_{a,c}$,*

$$\mathcal{A}_N = [c_0+a-1,c_0)+a\mathbb{Z}. \tag{6.4}$$

(iii) *Their closures $\bar{\mathcal{A}}_n, 0\le n\le N$, are mutually disjoint,*

$$\bar{\mathcal{A}}_n \cap \bar{\mathcal{A}}_{n'} = \emptyset \quad \text{for all} \quad 0 \le n \neq n' \le N. \tag{6.5}$$

(iv) *Their union is same as the complement of the maximal invariant set $\mathcal{S}_{a,c}$,*

$$\mathbb{R}\backslash\mathcal{S}_{a,c} = \cup_{n=0}^N \mathcal{A}_n. \tag{6.6}$$

For the case that only one of two intervals $[0,\epsilon)$ and $[-\epsilon,0)$ is contained in the set $\mathcal{S}_{a,c}$, the restriction of its complement $\mathbb{R}\backslash\mathcal{S}_{a,c}$ on the period $[0,a)$ consists of finitely many gaps of two different sizes, cf. Examples 6.1 and 6.2.

THEOREM 6.4. *Let (a, c) satisfy (6.1). Assume that*

$$[-\epsilon, 0) \subset \mathcal{S}_{a,c} \quad \text{and} \quad [0, \epsilon) \subset \mathbb{R}\backslash\mathcal{S}_{a,c}$$

for some positive $\epsilon > 0$. Define

$$\delta = \sup\{\epsilon \in (0, c_0 + a - 1],\ [0, \epsilon) \subset \mathbb{R}\backslash\mathcal{S}_{a,c}\}, \tag{6.7}$$

$$\mathcal{A}_n := (R_{a,c})^n([c - c_0, c - c_0 + 1 - a + \delta) + a\mathbb{Z}),\ n \geq 0,$$

and

$$\mathcal{B}_m := (R_{a,c})^m\big([c_0 + a - 1 - \delta, c_0 + a - 1) + a\mathbb{Z}\big),\ m \geq 0.$$

Then there are nonnegative integers N, D with

$$N \leq D \leq 1/\gcd(a, 1) - 1, \tag{6.8}$$

such that periodic gaps $\mathcal{A}_n, 0 \leq n \leq N$, and $\mathcal{B}_m, 0 \leq m \leq D - N$, have the following properties:

(i) *Big gaps $\mathcal{A}_n, 0 \leq n \leq N$, have their restrictions on one period being intervals of length $1 - a + \delta$,*

$$\mathcal{A}_n = (R_{a,c})^n(c - c_0 + 1 - a + \delta) + [a - 1 - \delta, 0) + a\mathbb{Z},\ 0 \leq n \leq N. \tag{6.9}$$

(ii) *The last big gap $\mathcal{A}_N$ concoides with the gap $[c_0 + a - 1 - \delta, c_0) + a\mathbb{Z}$ containing the black hole of the transformation $R_{a,c}$,*

$$\mathcal{A}_N = [c_0 + a - 1 - \delta, c_0) + a\mathbb{Z}. \tag{6.10}$$

(iii) *Gaps $\mathcal{A}_n, 0 \leq n \leq N$, and $\mathcal{B}_m, 1 \leq m \leq D - N$, has their closures being mutually disjoint,*

$$\begin{cases} \bar{\mathcal{A}}_n \cap \bar{\mathcal{A}}_{n'} = \emptyset \text{ for all } 0 \leq n \neq n' \leq N, \\ \bar{\mathcal{A}}_n \cap \bar{\mathcal{B}}_m = \emptyset \text{ for all } 0 \leq n \leq N \text{ and } 1 \leq m \leq D - N, \\ \bar{\mathcal{B}}_m \cap \bar{\mathcal{B}}_{m'} = \emptyset \text{ for all } 1 \leq m, m' \leq D - N. \end{cases} \tag{6.11}$$

(iv) *Small gaps $\mathcal{B}_m, 1 \leq m \leq D - N$, have their restrictions on $[0, a)$ being intervals of length δ,*

$$\mathcal{B}_m = (R_{a,c})^m(c_0) + [-\delta, 0) + a\mathbb{Z},\ 1 \leq m \leq D - N. \tag{6.12}$$

(v) *The last small gap $\mathcal{B}_{D-N}$ is $[0, \delta) + a\mathbb{Z}$,*

$$\mathcal{B}_{D-N} = [0, \delta) + a\mathbb{Z}. \tag{6.13}$$

(vi) *The union of big gaps $\mathcal{A}_n, 0 \leq n \leq N$, and small gaps $\mathcal{B}_m, 1 \leq m \leq D-N$, is the complement of the maximal invariant set $\mathcal{S}_{a,c}$,*

$$\mathbb{R}\backslash\mathcal{S}_{a,c} = \big(\cup_{n=0}^{N} \mathcal{A}_n\big) \cup \big(\cup_{m=1}^{D-N} \mathcal{B}_m\big). \tag{6.14}$$

THEOREM 6.5. *Let (a, c) satisfy (6.1). Assume that*

$$[0, \epsilon) \subset \mathcal{S}_{a,c} \quad \text{and} \quad [-\epsilon, 0) \subset \mathbb{R}\backslash\mathcal{S}_{a,c}$$

for some positive $\epsilon > 0$. Define

$$\delta' = \inf\{-\epsilon \in [c_0 - a, 0),\ [-\epsilon, 0) \subset \mathbb{R}\backslash\mathcal{S}_{a,c}\}, \tag{6.15}$$

$$\mathcal{A}_n := (R_{a,c})^n([c - c_0 + \delta', c - c_0 + 1 - a) + a\mathbb{Z}),\ n \geq 0,$$

and

$$\mathcal{B}_m := (R_{a,c})^m\big([c_0, c_0 - \delta') + a\mathbb{Z}\big),\ m \geq 0.$$

Then there are nonnegative integers N and D with $N \leq D \leq 1/\gcd(a, 1) - 1$, such that periodic gaps $\mathcal{A}_n, 0 \leq n \leq N$, and $\mathcal{B}_m, 0 \leq m \leq D - N$, have the following properties:

(i) $\mathcal{A}_n = (R_{a,c})^n(c - c_0 + 1 - a) + [a - 1 + \delta', 0) + a\mathbb{Z},\ 0 \le n \le N$;
(ii) $\mathcal{A}_N = [c_0 + a - 1, c_0 - \delta') + a\mathbb{Z}$;
(iii) $\mathcal{A}_n, 0 \le n \le N$, *and* $\mathcal{B}_m, 1 \le m \le D - N$, *have their closures being mutually disjoint;*
(iv) $\mathcal{B}_m = (R_{a,c})^m(c_0 - \delta') + [\delta', 0) + a\mathbb{Z},\ 1 \le m \le D - N$;
(v) $\mathcal{B}_{D-N} = [\delta', 0) + a\mathbb{Z}$; *and*
(vi) $\mathbb{R}\backslash\mathcal{S}_{a,c} = \big(\cup_{n=0}^{N} \mathcal{A}_n\big) \cup \big(\cup_{m=1}^{D-N} \mathcal{B}_m\big)$.

For $a \in \mathbb{Q}$, the black hole $[c_0 + a - 1, c_0) + a\mathbb{Z}$ of the transformation $R_{a,c}$ and the black hole $[c - c_0, c - c_0 + 1 - a) + a\mathbb{Z}$ of the transformation $\tilde{R}_{a,c}$ are inter-transformable in the sense that

$$[c - c_0, c - c_0 + 1 - a) + a\mathbb{Z} \xmapsto{R_{a,c}} \cdots \xmapsto{R_{a,c}} [c_0 + a - 1, c_0) + a\mathbb{Z}$$

and

$$[c_0 + a - 1, c_0) + a\mathbb{Z} \xmapsto{\tilde{R}_{a,c}} \cdots \xmapsto{\tilde{R}_{a,c}} [c - c_0, c - c_0 + 1 - a) + a\mathbb{Z},$$

i.e., there exists a nonnegative integer L such that

$$(R_{a,c})^L([c - c_0, c - c_0 + 1 - a) + a\mathbb{Z}) = [c_0 + a - 1, c_0) + a\mathbb{Z}$$

and

$$(\tilde{R}_{a,c})^L([c_0 + a - 1, c_0) + a\mathbb{Z}) = [c - c_0, c - c_0 + 1 - a) + a\mathbb{Z},$$

see (6.62), (6.63) and (6.64). But gaps $(R_{a,c})^n([c - c_0, c - c_0 + 1 - a) + a\mathbb{Z}), 0 < n < L$, and $(\tilde{R}_{a,c})^n([c_0 + a - 1, c_0) + a\mathbb{Z}), 0 < n < L$, to connect black holes of transformations $R_{a,c}$ and $\tilde{R}_{a,c}$ could have overlaps, see Examples 6.1 and 6.2 and compare Theorem 5.2.

For $a \notin \mathbb{Q}$, the complement $\mathbb{R}\backslash\mathcal{S}_{a,c}$ of the maximal invariant set $\mathcal{S}_{a,c}$ is composed of gaps of length $1 - a$ by Theorem 5.2, while for $a \in \mathbb{Q}$ it may contain gaps of two different sizes by Theorems 6.3, 6.4 and 6.5. For $a \notin \mathbb{Q}$, the maximal invariant set $\mathcal{S}_{a,c}$ is the union of intervals of different size (see Example 5.1), while we show in the next theorem that for $a \in \mathbb{Q}$, it is union of intervals of same size.

THEOREM 6.6. *Let* (a, c) *satisfy* (6.1). *Assume that* $\mathcal{S}_{a,c} \neq \emptyset$. *Let*

$$\mathcal{G}_{a,c} := \{(R_{a,c})^n(c - c_0 + 1 - a + \delta)\}_{n=0}^{D} + a\mathbb{Z} \tag{6.16}$$

where

$$\delta = \inf\{\epsilon \ge 0,\ c - c_0 + 1 - a + \epsilon \in \mathcal{S}_{a,c}\}$$

and D *is the smallest nonnegative integer such that*

$$(R_{a,c})^{D+1}(c - c_0 + 1 - a + \delta) - (c - c_0 + 1 - a + \delta) \in a\mathbb{Z}. \tag{6.17}$$

Then the maximal invariant set $\mathcal{S}_{a,c}$ *is the union of mutually disjoint intervals of same size,*

$$\mathcal{S}_{a,c} = \mathcal{G}_{a,c} + [0, h), \tag{6.18}$$

and

$$(\alpha + [0, h)) \cap (\beta + [0, h)) = \emptyset \text{ for all distinct } \alpha, \beta \in \mathcal{G}_{a,c}, \tag{6.19}$$

where $h > 0$.

By Theorems 4.4 and 6.6, we obtain the following cyclic group structure for the set $\mathcal{K}_{a,c}$ of marks on the line, i.e., images of gaps in the complement of the set $\mathcal{S}_{a,c}$ under the isomorphism $Y_{a,c}$ in (1.23).

COROLLARY 6.7. *Let (a, c) satisfy (6.1). Assume that $\mathcal{S}_{a,c} \neq \emptyset$. Then the following statements hold.*

(i) *If $\mathcal{S}_{a,c}$ contains small neighborhood of the origin, then*

$$\mathcal{K}_{a,c} = \{(n+1)Y_{a,c}(c - c_0 + 1 - a)\}_{n=0}^{N} + Y_{a,c}(a)\mathbb{Z}, \tag{6.20}$$

where N is the nonnegative integer in Theorem 6.3.

(ii) *If either $[0, \epsilon)$ or $[-\epsilon, 0)$ is contained in $\mathbb{R}\backslash\mathcal{S}_{a,c}$ for sufficiently small $\epsilon > 0$, then the set $\mathcal{K}_{a,c}$ of marks on the line form a finite cyclic group generated by $Y_{a,c}(c - c_0 + 1 - a) + Y_{a,c}(a)\mathbb{Z}$. Moreover,*

$$\mathcal{K}_{a,c} = \gcd\big(Y_{a,c}(c - c_0 + 1 - a), Y_{a,c}(a)\big)\mathbb{Z} = \frac{Y_{a,c}(a)}{D+1}\mathbb{Z}, \tag{6.21}$$

where D is the nonnegative integer in Theorems 6.4 and 6.5.

After performing the holes-removal surgery, the maximal invariant set $\mathcal{S}_{a,c}$ becomes the real line with marks, and the set of marks form a cyclic group. This suggests that for the case that $a \in \mathbb{Q}$ we can start from a cyclic group, put marks on the real line using elements in that group, and then expand the line with marks by inserting holes of appropriate size at every location of marks to recover the maximal invariant set $\mathcal{S}_{a,c}$. Using the above augmentation operation, we can characterize the non-triviality of the maximal invariant set $\mathcal{S}_{a,c}$ via four nonnegative integer parameters $d_i, 1 \leq i \leq 4$, for $a \in \mathbb{Q}$.

THEOREM 6.8. *Let (a, c) satisfy (6.1) and*

$$0 < c_1 := \lfloor c \rfloor - \lfloor (\lfloor c \rfloor / a) \rfloor a < 2a - 1, \lfloor c \rfloor \geq 2 \text{ and } c \in \gcd(a, 1)\mathbb{Z}. \tag{6.22}$$

Then $\mathcal{S}_{a,c} \neq \emptyset$ if and only if the pair (a, c) of positive numbers is one of the following three types:

1) $c_0 < \gcd(c_1, a)$.
2) $1 - c_0 < \gcd(c_1 + 1, a)$.
3) *There exist nonnegative integers d_1, d_2, d_3, d_4 such that*

$$0 < B_d := a - (d_1 + d_2 + 1)(1 - a) \in (D+1)\gcd(a, 1)\mathbb{Z}; \tag{6.23}$$

$$(D+1)c_1 + (d_1 + d_3 + 1)(1 - a) \in a\mathbb{Z}; \tag{6.24}$$

$$(d_1 + d_2 + 1)((D+1)c_1 + (d_1 + d_3 + 1)(1 - a)) - (d_1 + d_3 + 1)a \in (D+1)a\mathbb{Z}; \tag{6.25}$$

$$c_0 = (d_1 + 1)(1 - a) + (d_1 + d_3 + 1)B_d/(D+1) + \gamma \tag{6.26}$$

for some $\gamma \in (-\min(B_d/(D+1), a - c_0), \min(B_d/(D+1), c_0 + 1 - a))$;

$$\gcd((D+1)c_1 + (d_1 + d_3 + 1)(1 - a), (D+1)a) = a; \tag{6.27}$$

and

$$\#E_{a,c}^d = d_1, \tag{6.28}$$

where $D = d_1 + d_2 + d_3 + d_4 + 1$ and

$$\begin{aligned} E_{a,c}^d \;=\; & \big\{n \in [1, d_1 + d_2 + 1] \big|\; n((D+1)c_1 + (d_1 + d_3 + 1)(1 - a)) \\ & \in (0, (d_1 + d_3 + 1)a) + (D+1)a\mathbb{Z}\big\}. \end{aligned} \tag{6.29}$$

In Theorem 6.8, we insert a gap of large size at the origin for the first two cases, while a gap of small size is inserted at the origin for the third case. For the first two cases, no gaps of small size have been inserted at any location of marks and the size of gaps inserted is always c_0 for the first case and $1-c_0$ for the second case. For the third case, the nonnegative integer parameters d_1, d_2 are indeed the numbers of gaps of size $1-a+|\gamma|$ inserted in $[0, c_0+a-1)$ and $[c_0, a)$ respectively, and the nonnegative integer parameters d_3, d_4 are the numbers of gaps of size $|\gamma|$ inserted in $[0, c_0+a-1)$ and $[c_0, a)$, excluding the one inserted at the origin, respectively.

Recall that for $c \notin \gcd(a,1)\mathbb{Z}$, $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})$ is a Gabor frame if and only if both $\mathcal{G}(\chi_{[0,\lfloor c/\gcd(a,1)\rfloor \gcd(a,1))}, a\mathbb{Z}\times\mathbb{Z})$ and $\mathcal{G}(\chi_{[0,\lfloor c/\gcd(a,1)+1\rfloor \gcd(a,1))}, a\mathbb{Z}\times\mathbb{Z})$ are Gabor frames [**30**, Section 3.3.6.1]. Then for pairs (a,c) satisfying $\lfloor c\rfloor \ge 2, 0<c_1<2a-1$ and (5.1), we can apply Theorem 6.8 and the above observation to determine whether the corresponding Gabor systems $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})$ is a frame for L^2, see Theorem 7.5 for details.

This chapter is organized as follows. In Section 6.1, we consider maximal invariant sets $\mathcal{S}_{a,c}$ containing a small neighborhood of the origin and prove Theorem 6.3. In Section 6.2, we discuss maximal invariant sets $\mathcal{S}_{a,c}$ containing a small half neighborhood of the origin and prove Theorems 6.4 and 6.5. In Section 6.3, we consider the group structure of the set of marks and prove Theorem 6.6 and Corollary 6.7. Finally in Section 6.4, we parameterize maximal invariant sets $\mathcal{S}_{a,c}$ and prove Theorem 6.8.

6.1. Maximal invariant sets with rational time shifts I

In this section, we prove Theorem 6.3.

Proof of Theorem 6.3. We follow the arguments used in the proof of Theorem 5.2. Write $a=p/q$ for some co-prime integers p and q. By Proposition 3.6, the holes $\mathcal{A}_n, n\ge 0$, have the following properties:

$$\mathcal{A}_n \cap \mathcal{S}_{a,c} = \emptyset,\ n\ge 0, \tag{6.30}$$

and

$$\mathcal{A}_m \cap \mathcal{A}_n \subset [c_0+a-1, c_0) + a\mathbb{Z} \ \text{ whenever } \ m\ne n. \tag{6.31}$$

Following the argument used in the proof of Theorem 5.2, a nonnegative integer N satisfying (6.2) exists and satisfies

$$N \le a/(1-a) - 1.$$

From (6.2) and (6.31) it follows that

$$\mathcal{A}_n \cap \big([c_0+a-1, c_0)+a\mathbb{Z}\big) = \emptyset \ \text{ for all } 0\le n<N. \tag{6.32}$$

By (6.30), (6.32) and Theorem 3.4, the proof of (6.3), (6.4) and (6.5) reduces to showing that

$$\mathcal{A}_n = [b_n+a-1, b_n) + a\mathbb{Z} \tag{6.33}$$

for some $b_n \in (0,a]$ with

$$b_n - (R_{a,c})^n(c-c_0+1-a) \in a\mathbb{Z}, \tag{6.34}$$

and

$$[b_n+a-1-\epsilon_0, b_n+a-1)\cup[b_n, b_n+\epsilon_0) + a\mathbb{Z} \subset \mathcal{S}_{a,c}, \quad 0\le n\le N \tag{6.35}$$

for some sufficiently small $\epsilon_0 > 0$.

PROOF OF (6.33), (6.34) AND (6.35). Observe that

$$[c - c_0 - \epsilon_0, c - c_0) \subset \mathcal{S}_{a,c} \text{ and } [c - c_0 + 1 - a, c - c_0 + 1 - a + \epsilon_0) \subset \mathcal{S}_{a,c}$$

because for sufficiently small $\epsilon_0 > 0$,

$$[c - c_0 - \epsilon_0, c - c_0) = R_{a,c}[-\epsilon_0, 0) \subset R_{a,c}\mathcal{S}_{a,c} = \mathcal{S}_{a,c}$$

and

$$[c - c_0 + 1 - a, c - c_0 + 1 - a + \epsilon_0) = R_{a,c}([0, \epsilon_0) + a) \subset \mathcal{S}_{a,c}$$

where the last inclusion holds as $\mathcal{S}_{a,c}$ is the maximal invariant set under the transformation $R_{a,c}$ by Theorem 3.4. Then the conclusions (6.33), (6.34) and (6.35) holds for $n = 0$. Hence the proof is finished if $N = 0$ and we may assume that $N \geq 1$ from now on. Inductively we assume that the conclusions (6.33), (6.34) and (6.35) hold for all $n \leq k \leq N-1$. By the inductive hypothesis and Proposition 3.6,

$$[b_k + a - 1, b_k) \subset [\epsilon_0, c_0 + a - 1 - \epsilon_0) \cup [c_0 + \epsilon_0, a - \epsilon_0), \tag{6.36}$$

which implies that (6.33) for $n = k + 1$. Applying (6.36) again and using the conclusion (6.33) for $n = k+1$, we have that

$$[b_{k+1} + a - 1 - \epsilon_0, b_{k+1} + a - 1) + a\mathbb{Z} = R_{a,c}([b_k + a - 1 - \epsilon_0, b_k + a - 1) + a\mathbb{Z})$$

and

$$[b_{k+1}, b_{k+1} + \epsilon_0) + a\mathbb{Z} = R_{a,c}([b_k, b_k + \epsilon_0) + a\mathbb{Z}).$$

The above equalities together with the inductive hypothesis and the invariance of the set $\mathcal{S}_{a,c}$ given in Theorem 3.4 prove (6.34) and (6.35) for $n = k + 1$. This completes the inductive proof (hence (6.3), (6.4) and (6.5) are proved). □

We can follow the argument used in the proof of Theorem 5.2 to prove (6.6), and then leave the details to the reader. □

6.2. Maximal invariant sets with rational time shifts II

In this section, we will prove Theorems 6.4 and 6.5.

PROOF OF THEOREM 6.4. Let $\delta \in (0, c_0 + a - 1]$ be as in (6.7). As $[0, \delta)$ is the maximal interval contained in the complement of the set $\mathcal{S}_{a,c}$, there exists sufficiently small $\epsilon_0 > 0$ such that

$$[\delta, \delta + \epsilon_0) \subset \mathcal{S}_{a,c} \quad \text{provided that} \quad \delta < c_0 + a - 1. \tag{6.37}$$

By Lemma 3.11 and the assumption about the set $\mathcal{S}_{a,c}$ around the neighborhood of the origin, we have that

$$[-\epsilon_0, 0) \subset \mathcal{S}_{a,c} \quad \text{and} \quad [c_0, c_0 + \epsilon_0) \subset \mathcal{S}_{a,c} \tag{6.38}$$

for some sufficiently small $\epsilon_0 > 0$. Therefore

$$[c - c_0 - \epsilon_0, c - c_0) = R_{a,c}[-\epsilon_0, 0) \subset R_{a,c}\mathcal{S}_{a,c} = \mathcal{S}_{a,c} \tag{6.39}$$

by (1.15), (6.38) and Theorem 3.4;

$$\begin{aligned} & [c - c_0 + 1 - a + \delta, c - c_0 + 1 - a + \delta + \epsilon_0) \\ = & \begin{cases} R_{a,c}[\delta, \delta + \epsilon_0) - a & \text{if } 0 < \delta < c_0 + a - 1 \\ R_{a,c}[c_0, c_0 + \epsilon_0) & \text{if } \delta = c_0 + a - 1 \end{cases} \\ \subset & \ R_{a,c}\mathcal{S}_{a,c} = \mathcal{S}_{a,c} \end{aligned} \tag{6.40}$$

by (1.15), (6.37), (6.38) and Theorem 3.4; and

$$
\begin{aligned}
& [c-c_0, c-c_0+1-a+\delta)\cap \mathcal{S}_{a,c}\\
= & \big(R_{a,c}[-a,\delta-a)\cup[c-c_0,c-c_0+1-a)\big)\cap\mathcal{S}_{a,c}\\
= & R_{a,c}([-a,\delta-a)\cap\mathcal{S}_{a,c})=\emptyset \qquad (6.41)
\end{aligned}
$$

by (1.15), Theorem 3.4 and the assumption $[0,\delta)\subset[0,c_0+a-1)$. Thus $[c-c_0, c-c_0+1-a+\delta)$ is a gap (i.e., a left-closed right-open interval with empty intersection with $\mathcal{S}_{a,c}$) with length $1-a+\delta$ and boundary intervals of length ϵ_0 at each side in the maximal invariant set $\mathcal{S}_{a,c}$.

Let N be the smallest nonnegative integer such that

$$(R_{a,c})^N([c-c_0,c-c_0+1-a+\delta)+a\mathbb{Z})\cap([c_0+a-1,c_0)+a\mathbb{Z})\neq\emptyset \qquad (6.42)$$

if it exists and $N=+\infty$ otherwise.

At first we verify (6.9) *and* (6.10) *about big gaps* $\mathcal{A}_n, 0\le n\le N$. For $N=0$, the conclusions (6.9) and (6.10) follows from (6.39), (6.40) and (6.41). So we may assume that $1\le N\le\infty$. Thus

$$\mathcal{A}_n\cap([c_0+a-1,c_0)+a\mathbb{Z})=\emptyset,\ 0\le n<N. \qquad (6.43)$$

We claim that $\mathcal{A}_n, 0\le n<N$, have the following properties for $0\le n<N$:

$$\mathcal{A}_n=[b_n+a-1-\delta,b_n)+a\mathbb{Z} \qquad (6.44)$$

for some $b_n\in(0,a]$ satisfying

$$[b_n+a-1-\delta-\epsilon_0,b_n+\epsilon_0)\subset[0,c_0+a-b)\cup[c_0,a), \qquad (6.45)$$

$$([b_n+a-1-\delta,b_n)+a\mathbb{Z})\cap\mathcal{S}_{a,c}=\emptyset, \qquad (6.46)$$

and

$$[b_n+a-1-\delta-\epsilon_0,b_n+a-1-\delta)\cup[b_n,b_n+\epsilon_0)+a\mathbb{Z}\subset\mathcal{S}_{a,c}. \qquad (6.47)$$

Proof of (6.44), (6.45), (6.46) and (6.47). For $n=0$, write

$$
\begin{aligned}
(R_{a,c})^n([c-c_0,c-c_0+1-a+\delta)+a\mathbb{Z}) &= [c-c_0,c-c_0+1-a+\delta)+a\mathbb{Z}\\
&= [b_0+a-1-\delta,b_0)+a\mathbb{Z}
\end{aligned}
$$

with $b_0\in(0,a]$. Then the conclusions (6.44), (6.45), (6.46) and (6.47) for $n=0$ follow from (6.39), (6.40), (6.41), (6.43) and the empty intersection property in Theorem 3.4. Inductively we assume that the conclusions (6.44), (6.45), (6.46) and (6.47) hold for all $0\le n\le k<N-1$. Then for $n=k+1$,

$$
\begin{aligned}
& (R_{a,c})^n([c-c_0,c-c_0+1-a+\delta)+a\mathbb{Z})\\
= & R_{a,c}[b_k+a-1-\delta,b_k)+a\mathbb{Z} \quad (\text{by (6.44) with } n=k)\\
= & [R_{a,c}(b_k+a-1-\delta),R_{a,c}(b_k+a-1-\delta)+1-a+\delta)+a\mathbb{Z}\\
& \qquad (\text{by (6.45) with } n=k)\\
=: & [b_{k+1}+a-1-\delta,b_{k+1})+a\mathbb{Z}
\end{aligned}
$$

for some $b_{k+1}\in(0,a]$,

$$
\begin{aligned}
& ([b_{k+1}+a-1-\delta,b_{k+1})+a\mathbb{Z})\cap\mathcal{S}_{a,c}\\
= & R_{a,c}\big(([b_k+a-1-\delta,b_k)+a\mathbb{Z})\cap\mathcal{S}_{a,c}\big)\\
& \qquad (\text{by (1.17), (6.43), and (6.45) for } n=k)\\
= & \emptyset,
\end{aligned}
$$

$$\begin{aligned}
& [b_{k+1}+a-1-\delta-\epsilon_0, b_{k+1}+a-1-\delta)+a\mathbb{Z} \\
= & [R_{a,c}(b_k+a-1-\delta)-\epsilon_0, R_{a,c}(b_k+a-1-\delta))+a\mathbb{Z} \\
= & R_{a,c}[b_k+a-1-\delta-\epsilon_0, b_k+a-1-\delta)+a\mathbb{Z} \\
& \qquad \text{(by (6.46) with } n=k) \\
= & (R_{a,c})^{k+1}[c-c_0-\epsilon_0, c-c_0)+a\mathbb{Z}\subset \mathcal{S}_{a,c} \quad \text{(by (6.39))}
\end{aligned}$$

and similarly

$$\begin{aligned}
& [b_{k+1}, b_{k+1}+\epsilon_0)+a\mathbb{Z} \\
= & (R_{a,c})^{k+1}[c-c_0+1-a+\delta, c-c_0+1-a+\delta+\epsilon_0)+a\mathbb{Z}\subset \mathcal{S}_{a,c}
\end{aligned}$$

by (6.40), (6.44), (6.45) and (6.47) for $n=k$, the definition (1.15) of the transformation $R_{a,c}$, and the invariance property (3.6) in Theorem 3.4. Those together with (6.43) completes the inductive proof of (6.44), (6.45), (6.46), and (6.47). □

By (6.43) and Proposition 3.6, $(R_{a,c})^n([c-c_0, c-c_0+1-a)+a\mathbb{Z}), 0\le n\le N$, are mutually disjoint. This together with (6.44), (6.45), (6.46), (6.47) and the inclusion $(R_{a,c})^n([c-c_0, c-c_0+1-a)+a\mathbb{Z})\subset \mathcal{A}_n, 0\le n\le N$, shows that $\mathcal{A}_n, 0\le n<N$, are mutually disjoint, i.e.,

$$\mathcal{A}_n\cap\mathcal{A}_{n'}=\emptyset \quad \text{for all} \quad 0\le n\ne n'<N. \tag{6.48}$$

This together with (6.44) implies that

$$1\le N\le 1/\gcd(a,1)-1. \tag{6.49}$$

Applying (6.44) and (6.47), we obtain (6.9) immediately by inductive proof. Next we can show that (6.10) holds by (6.38), (6.42), (6.44) and (6.45), because $(R_{a,c})^N[c-c_0, c-c_0+1-a+\delta)$ is a periodic set with its restriction on one period being an interval of length $1-a+\delta$ by using (6.44) and (6.45) with $n=N-1$, and its right neighborhood $(R_{a,c})^N([c-c_0+1-a+\delta, c-c_0+1-a+\delta+\epsilon_0)+a\mathbb{Z})$ is contained in $\mathcal{S}_{a,c}$ by (6.47), while $[c_0, c_0+\epsilon_0)\subset\mathcal{S}_{a,c}$ by (6.38) and $\mathcal{S}_{a,c}$ has empty intersection with the black hole $[c_0+a-1, c_0)+a\mathbb{Z}$.

Let D be the minimal nonnegative integer such that

$$(R_{a,c})^{D-N}([c_0+a-1-\delta, c_0+a-1)+a\mathbb{Z})\cap([0,\delta)+a\mathbb{Z})\ne\emptyset \tag{6.50}$$

if it exists and $D=+\infty$ otherwise.

Secondly we divide three cases to verify (6.12) *and* (6.13) *for small gaps* $\mathcal{B}_m, 0\le m\le D-N$.

Case 1: $D=N$.

In this case, the conclusions (6.12) and (6.13) hold as

$$[-\epsilon_0, 0)\cup[c_0-a+1-\delta-\epsilon_0, c_0+a-1-\delta)\subset\mathcal{S}_{a,c}$$

for sufficiently small $\epsilon_0>0$ by (6.38) and (6.47); and $[0,\delta)\subset\mathbb{R}\backslash\mathcal{S}_{a,c}$ by the assumption.

Case 2: $D=N+1$.

In this case,

$$R_{a,c}[c_0+a-1-\delta, c_0+a-1)+a\mathbb{Z}=[\tilde{b}_1-\delta, \tilde{b}_1)+a\mathbb{Z} \tag{6.51}$$

for some $\tilde b_1 \in (0, a]$ with $\tilde b_1 - \delta - R_{a,c}(c_0 + a - 1 - \delta) \in a\mathbb{Z}$, since $[c_0 + a - 1 - \delta, c_0 + a - 1) \subset [0, c_0 + a - 1)$. Recall that

$$[c_0 + a - 1 - \delta - \epsilon_0, c_0 + a - b - \delta) \cup [c_0, c_0 + \epsilon_0) \subset \mathcal{S}_{a,c}$$

by (6.38) and (6.47). We then obtain from (6.51) and Theorem 3.4 that

$$[\tilde b_1 - \delta - \epsilon_0, \tilde b_1 - \delta) \subset \mathcal{S}_{a,c} \text{ and } [\tilde b_1, \tilde b_1 + \epsilon_0) \subset \mathcal{S}_{a,c}. \tag{6.52}$$

Therefore $[\tilde b_1 - \delta, \tilde b_1)$ is a gap of length δ with boundary intervals of length ϵ_0 at each side in the set $\mathcal{S}_{a,c}$. Thus

$$[\tilde b_1 - \delta, \tilde b_1) \cap ([c_0 + a - 1, c_0) + a\mathbb{Z}) = \emptyset$$

as the gap containing $[c_0 + a - 1, c_0)$ is $(R_{a,c})^N[c - c_0, c - c_0 + 1 - a + \delta) + a\mathbb{Z}$ which has length $1 - a + \delta$ and boundary intervals of length ϵ_0 at each side in $\mathcal{S}_{a,c}$. By the definition of the nonnegative integer D,

$$([\tilde b_1 - \delta, \tilde b_1) + a\mathbb{Z}) \cap ([0, \delta) + a\mathbb{Z}) \neq \emptyset.$$

This together with (6.38), (6.52) and $[0, \delta) \in \mathbb{R}\backslash\mathcal{S}_{a,c}$ implies that $\tilde b_1 = \delta$ and

$$R_{a,c}[c_0 + a - b - \delta, c_0 + a - b) + a\mathbb{Z} = [0, \delta) + a\mathbb{Z}. \tag{6.53}$$

The conclusion (6.12) and (6.13) for $D = N + 1$ follow from (6.51) and (6.53).

Case 3: $N + 2 \le D \le +\infty$.

In this case, following the arguments used to prove (6.44), (6.45), (6.46) and (6.47), and the conclusions (6.12) and (6.13) for $D = N + 1$, we may inductively show that

$$\mathcal{B}_m = [\tilde b_m - \delta, \tilde b_m) + a\mathbb{Z},\ 1 \le m < D - N, \tag{6.54}$$

for some $\tilde b_m \in (0, a]$ with

$$[\tilde b_m - \delta - \epsilon_0, \tilde b_m + \epsilon_0) \subset [0, c_0 + a - 1) \cup [c_0, a), \tag{6.55}$$

$$\mathcal{B}_m \cap \mathcal{S}_{a,c} = \emptyset, 1 \le m < D - N, \tag{6.56}$$

and

$$[\tilde b_m - \delta - \epsilon_0, \tilde b_m - \delta) \cup [\tilde b_m, \tilde b_m + \epsilon_0) \subset \mathcal{S}_{a,c}. \tag{6.57}$$

Using (6.44), (6.47), (6.50), (6.54) and (6.57), we obtain that

$$\mathcal{B}_m \cap ([c_0 + a - 1, c_0) + a\mathbb{Z}) = \emptyset \ \text{ for all } \ 0 \le m < D - N. \tag{6.58}$$

Next we prove that

$$\mathcal{B}_m \cap \mathcal{B}_{m'} = \emptyset \text{ for all } 0 \le m \neq m' < D - N. \tag{6.59}$$

Suppose, on the contrary, that $\mathcal{B}_{m_1} \cap \mathcal{B}_{m_2} \neq \emptyset$ for some $0 \le m_1 < m_2 < D - N$. Then

$$\mathcal{B}_{m_1} = \mathcal{B}_{m_2}$$

by (6.54), (6.56), and (6.57). This, together with (6.43), (6.58) and the one-to-one correspondence of the transformation $R_{a,c}$ on the complement of $[c_0+a-1, c_0)+a\mathbb{Z}$, leads to

$$(R_{a,c})^{m_2-m_1}([c-c_0, c-c_0+\min(\delta, 1-a))+a\mathbb{Z}) = [c-c_0, c-c_0+\min(\delta, 1-a))+a\mathbb{Z},$$

which contradicts to the range property (1.18). This proves (6.48).

Combining (6.54) and (6.48), we conclude that the restriction of $\mathcal{B}_m, 0 \le m < D-N$, on one period $[0,a)$ are mutually disjoint interval of length δ. This implies that

$$D < +\infty.$$

Applying (6.54) with $m = D-N-1$, and recalling the definition of the integer $D < \infty$ we obtain that

$$\mathcal{B}_{D-N} + a\mathbb{Z} = [0,\delta) + a\mathbb{Z} \tag{6.60}$$

and

$$[-\epsilon_0, 0) \cup [\delta, \delta+\epsilon_0) \in \mathcal{S}_{a,c}. \tag{6.61}$$

Therefore the conclusions (6.12) and (6.13) for $N+2 \le D \le \infty$ follow from (6.54), (6.60) and (6.61).

In the third place we prove the mutually disjointness property (6.11) *for periodic gaps* $\mathcal{A}_n, 0 \le n \le N$, *and* $\mathcal{B}_m, 1 \le m \le D-N$. Observe that the big gaps $\mathcal{A}_n, 0 \le n \le N$, have their restrictions on one period being intervals of length $1-a+\delta$ by (6.10), (6.38), (6.40), (6.44), (6.45), (6.46), (6.47) and (6.48). Then their mutually disjointness follows from (6.42) and (6.48).

Observe that small gaps $\mathcal{B}_m, 1 \le m \le D-N$, have their restrictions on one period being intervals of length δ by (6.54), (6.55), (6.57), (6.60) and (6.61). Recall that big gaps $\mathcal{A}_n, 0 \le n \le N$, have their restrictions on one period being intervals of length $1-a+\delta$. Therefore small gaps $\mathcal{B}_m, 1 \le m \le D-N$, and big gaps $\mathcal{A}_n, 0 \le n \le N$, have their closure being disjoint.

Recall that small gaps $\mathcal{B}_m, 1 \le n \le D-N$, have their restrictions on one period being intervals of length δ. This together with (6.59) and (6.60) proves the mutual disjointness of the closure of small gaps $\mathcal{B}_m, 1 \le n \le D-N$.

Next we establish the upper bound estimate (6.8) *for* D. For $D = N$, the upper bound estimate (6.8) has been given in (6.49). For $D \ge N+1$, we observe that

$$(R_{a,c})^n([c-c_0, c-c_0+\min(\delta, 1-a)) + a\mathbb{Z}) \subset \begin{cases} \mathcal{A}_n & \text{if } 0 \le n \le N \\ \mathcal{B}_{n-N} & \text{if } N+1 \le n \le D, \end{cases}$$

which implies that $(R_{a,c})^n([c-c_0, c-c_0+\min(\delta,1-a)) + a\mathbb{Z}), 0 \le n \le D$, are mutually disjoint. Also observe that $(R_{a,c})^n([c-c_0, c-c_0+\min(\delta,1-a))+a\mathbb{Z}), 0 \le n \le D$, have their restriction on one period being intervals with left endpoints in $\gcd(a,1)\mathbb{Z}$. Therefore $D+1 \le 1/\gcd(a,1)$ and (6.8) follows.

Finally we prove (6.14). Write $\delta = l_0(1-a) + \tilde{\delta}$ for some $0 \le l_0 \in \mathbb{Z}$ and $\tilde{\delta} \in (0, 1-a]$. From (6.9)–(6.13), we obtain that

$$\begin{aligned} &(R_{a,c})^n([c-c_0, c-c_0+1-a) + a\mathbb{Z}) \\ &= \begin{cases} b_{n-\tilde{l}(D+1)} + \tilde{l}(1-a) - \delta + [a-1, 0) + a\mathbb{Z} \\ \qquad \text{if } 0 \le n - \tilde{l}(D+1) \le N \text{ for some } 0 \le \tilde{l} \le l_0, \\ \tilde{b}_{n-\tilde{l}(D+1)-N} + \tilde{l}(1-a) - \delta + [a-1, 0) + a\mathbb{Z} \\ \qquad \text{if } N+1 \le n - \tilde{l}(D+1) \le D \text{ for some } 0 \le \tilde{l} \le l-1, \\ ((\tilde{b}_{n-l(D+1)-N} + [-\tilde{\delta}, 0)) \cup [c_0+a-1, c_0-\tilde{\delta})) + a\mathbb{Z} \\ \qquad \text{if } N+1 \le n - l(D+1) \le D, \\ ((b_{n-(l+1)(D+1)} + [-\tilde{\delta}, 0)) \cup [c_0+a-1, c_0-\tilde{\delta})) + a\mathbb{Z} \\ \qquad \text{if } 0 \le n - (l+1)(D+1) \le N, \end{cases} \end{aligned} \tag{6.62}$$

where

$$b_n = (R_{a,c})^n(c - c_0 + 1 - a + \delta), 0 \le n \le N,$$

and

$$\tilde{b}_m = (R_{a,c})^m(c_0), 1 \le m \le D - N.$$

Therefore

$$\begin{aligned}(\cup_{n=0}^{N}\mathcal{A}_n) \cup (\cup_{m=1}^{D-N}\mathcal{B}_m) &= \cup_{n=0}^{D}(R_{a,c})^n([c - c_0, c - c_0 + 1 - a + \delta) + a\mathbb{Z}) \\ &= \cup_{n=0}^{(l_0+1)(D+1)+N}(R_{a,c})^n[c - c_0, c - c_0 + 1 - a) + a\mathbb{Z}, \qquad (6.63)\end{aligned}$$

and

$$(R_{a,c})^{(l_0+1)(D+1)+N}[c - c_0, c - c_0 + 1 - a) + a\mathbb{Z} = [c_0 + a - 1, c_0) + a\mathbb{Z}. \qquad (6.64)$$

Hence the union of the gaps $\mathcal{A}_n, 0 \le n \le N$, and $\mathcal{B}_m, 1 \le m \le D - N$, is invariant under the transformation $R_{a,c}$ and contains the black holes of the transformations $R_{a,c}$ and $\tilde{R}_{a,c}$. It also indicates that any points not in that union will not be in that union under the transformation $R_{a,c}$. Thus $\mathbb{R}\backslash((\cup_{n=0}^{N}\mathcal{A}_n) \cup (\cup_{m=1}^{D-N}\mathcal{B}_m))$ is invariant under the transformation $R_{a,c}$. This together with Theorem 3.4 proves (6.14). □

We finish this section with the proof of Theorem 6.5.

Proof of Theorem 6.5. We follow the argument used in the proof of Theorem 6.4 with δ, N, D replaced by δ' in (6.15), the smallest nonnegative integer such that

$$(R_{a,c})^N([c - c_0 + \delta', c - c_0 + 1 - a) + a\mathbb{Z}) \cap ([c_0 + a - 1, c_0) + a\mathbb{Z}) \ne \emptyset,$$

and the minimal nonnegative integer such that

$$(R_{a,c})^{D-N}([c_0, c_0 - \delta') + a\mathbb{Z}) \cap ([\delta', 0) + a\mathbb{Z}) \ne \emptyset$$

respectively. We leave the detailed arguments to the reader. □

6.3. Cyclic group structure of maximal invariant sets

In this section, we prove Theorem 6.6 and Corollary 6.7.

Proof of Theorem 6.6. We divide into three cases to establish (6.18) and (6.19).

Case 1: The maximal invariant set $\mathcal{S}_{a,c}$ contains a small neighborhood of the origin.

In this case, $\delta = 0$ as

$$(R_{a,c})^n(c - c_0 + 1 - a) + [0, \epsilon) \in \mathcal{S}_{a,c}, \ n \ge 0, \qquad (6.65)$$

by (6.3), (6.6), Theorem 3.4 and the assumption on the neighborhood of the origin. Observe that

$$(R_{a,c})^n(c - c_0 + 1 - a) \in \text{gad}(a, 1)\mathbb{Z} \quad \text{for all} \quad n \ge 0$$

by $a \in \mathbb{Q}$. Then there exist two distinct integers $m, n \ge 0$ such that

$$(R_{a,c})^m(c - c_0 + 1 - a) - (R_{a,c})^n(c - c_0 + 1 - a) \in a\mathbb{Z}.$$

This together with (1.17), (6.65) and Theorem 3.4 implies that

$$(R_{a,c})^{|m-n|}(c - c_0 + 1 - a) - (c - c_0 + 1 - a) \in a\mathbb{Z}.$$

Then the existence of a nonnegative integer D satisfying (6.17) follows and the set $\mathcal{G}_{a,c}$ in (6.16) is well-defined. Furthermore,

$$(R_{a,c})^n(c-c_0+1-a)-(R_{a,c})^m(c-c_0+1-a)\notin a\mathbb{Z}\quad\text{for all } 0\le n\ne m\le D,$$

and

$$N\le D\le 1/\gcd(a,1)-1 \tag{6.66}$$

by (6.3) and (6.5), where N is given in Theorem 3.4.

By (6.6) and (6.65), the restriction of the maximal invariant set $\mathcal{S}_{a,c}$ on $[0,a)$ is finitely union of finitely many left-closed right-open intervals. More precisely, $\mathcal{S}_{a,c}\cap[0,a)$ is union of mutually disjoint intervals $[b_k,b_k+h_k), 0\le k\le D$,

$$\mathcal{S}_{a,c}\cap[0,a)=\cup_{k=0}^{D}[b_k,b_k+h_k) \tag{6.67}$$

and

$$[b_k,b_k+h_k)\cap[b_{k'},b_{k'}+h_{k'})=\emptyset \text{ if } k\ne k', \tag{6.68}$$

where

$$b_k\in((R_{a,c})^k(c-c_0+1-a)+a\mathbb{Z})\cap[0,a)$$

and b_k+h_k is chosen so that either

$$b_k+h_k\in\cup_{n=N+1}^{D}(R_{a,c})^n(c-c_0+1-a)+a\mathbb{Z} \tag{6.69}$$

or

$$b_k+h_k+[0,1-a)\subset\cup_{n=0}^{N}\mathcal{A}_n. \tag{6.70}$$

Therefore the proof of (6.18) and (6.19) reduces to showing that

$$h_n=h_0,\ 1\le n\le D. \tag{6.71}$$

Suppose on the contrary that $h_m\ne h_0$ for some $1\le m\le D$. Then either $h_m>h_0$ or $h_m<h_0$. For the case that $h_m>h_0$,

$$[b_m,b_m+h_0]\subset[b_m,b_m+h_m)\subset\mathcal{S}_{a,c}. \tag{6.72}$$

Observe that

$$(\tilde R_{a,c})^m([b_m,b_m+h_0]+a\mathbb{Z})=[b_0,b_0+h_0]+a\mathbb{Z}\subset\mathcal{S}_{a,c} \tag{6.73}$$

by (6.72) and Theorem 3.4. This together with (6.69) and (6.70) with $k=0$ implies that the existence of a unique $N\le m'\le D$ such that

$$b_0+h_0\in(R_{a,c})^{m'}(c-c_0+1-a)+a\mathbb{Z}.$$

Applying $(R_{a,c})^m$ leads to

$$b_m+h_0\in(R_{a,c})^m(b_0+h_0)+a\mathbb{Z}=(R_{a,c})^{m''}(c-c_0+1-a)+a\mathbb{Z},$$

where $0\le m''\le D$ is the unique integer with $m+m'-m''\in D\mathbb{Z}$. Hence $[b_m,b_m+h_m)\cap[b_{m''},b_{m''}+h_{m''})$ has nonempty intersection, which contradicts to (6.68). Therefore

$$h_n\le h_0\quad\text{for all}\quad 1\le n\le D.$$

Using similarly argument, we can prove that

$$h_n\ge h_0\quad\text{for all}\quad 1\le n\le D.$$

Hence (6.71) is established, and (6.18) and (6.19) are proved.

We remark that

$$h=\frac{a-(N+1)(1-a)}{D+1}$$

because

$$|[0,a)\backslash\mathcal{S}_{a,c}| = (N+1)(1-a)$$

by (6.3), (6.5), and (6.6); and

$$|\mathcal{S}_{a,c}\cap[0,a)| = \sum_{n=0}^{D} h_n = (D+1)h_0$$

by (6.67), (6.68) and (6.71).

Case 2: The maximal invariant set $\mathcal{S}_{a,c}$ and its complement $\mathbb{R}\backslash\mathcal{S}_{a,c}$ contain a small left and right neighborhood of the origin respectively.

In this case, δ is the same as the one in (6.7) and D satisfying (6.17) is the same as the one in Theorem 6.4. Thus $\mathcal{G}_{a,c}$ is well-defined. Let N be as Theorem 6.4. Set

$$\mathcal{A}_n = (R_{a,c})^n([c-c_0, c-c_0+1-a+\delta)+a\mathbb{Z}), n\ge 0$$

and

$$\mathcal{B}_m = (R_{a,c})^m([c_0+a-1-\delta, c_0+a-1)+a\mathbb{Z}), m\ge 0.$$

Recall that the mutually disjoint big gaps $\mathcal{A}_n, 0\le n\le N$, and small gaps $\mathcal{B}_m, 1\le m\le D-N$, have neighborhood of length ϵ_0 at each side are contained in the maximal invariant set $\mathcal{S}_{a,c}$. Therefore

$$\mathcal{S}_{a,c} = \big(\cup_{n=0}^{N}([b_n, b_n+h_n)+a\mathbb{Z})\big)\cup\big(\cup_{m=1}^{D-N}([\tilde b_m, \tilde b_m+\tilde h_m)+a\mathbb{Z})\big),$$

where $h_n, 0\le n\le N$, and $\tilde h_m, 1\le m\le D-N$, are so chosen that

$$[b_n+h_n, b_n+h_n+\epsilon_0)+a\mathbb{Z}\subset\mathbb{R}\backslash\mathcal{S}_{a,c}, 0\le n\le N$$

and

$$[\tilde b_m+\tilde h_m, \tilde b_m+\tilde h_m+\epsilon_0)+a\mathbb{Z}\subset\mathbb{R}\backslash\mathcal{S}_{a,c}, 1\le m\le D-N$$

for sufficiently small $\epsilon_0>0$. As $[0,\delta)+a\mathbb{Z}$ and $[c_0+a-1, c_0)+a\mathbb{Z}$ are contained in the union of the mutually disjoint gaps, each of the intervals $[b_n, b_n+h_n)+a\mathbb{Z}, 0\le n\le N$, and $[\tilde b_m, \tilde b_m+\tilde h_m)+a\mathbb{Z}, 1\le m\le D-N$, is contained either in $[0, c_0+a-1)+a\mathbb{Z}$ or in $[c_0, a)+a\mathbb{Z}$, and its boundary interval of length ϵ_0 at each side is not contained in the set $\mathcal{S}_{a,c}$. Recall that

$$b_n-(R_{a,c})^n(c-c_0+b-a+\delta)\in a\mathbb{Z}, 0\le n\le N,$$

and

$$\tilde b_m-(R_{a,c})^{m+N}(c-c_0+1-a+\delta)\in a\mathbb{Z}, 1\le m\le D-N$$

by Theorem 6.4. Hence the interval $[b_n, b_n+h_n)+a\mathbb{Z} = (R_{a,c})^n[b_0, b_0+h_0)+a\mathbb{Z}$ and $[\tilde b_m, \tilde b_m+\tilde h_m)+a\mathbb{Z} = (R_{a,c})^{m+N_1}[b_0, b_0+h_0)+a\mathbb{Z}$. This together with the measure-preserving property (1.19) implies that the length of intervals contained in the set $\mathcal{S}_{a,c}$ are the same, i.e.,

$$h_n = \tilde h_m = h \text{ for all } 0\le n\le N \text{ and } 1\le m\le D-N$$

where $h>0$. This completes the proof of (6.18) and (6.19) in Case 2. We remark that

$$h = \frac{a-(N+1)(1-a)}{D+1}-\delta$$

because the measure of the gaps contained in $[0,a)$ is equal to $(N+1)(1-a+\delta)+(D-N)\delta$, while the measure of the intervals contained in $\mathcal{S}_{a,c}\cap[0,a)$ is $(D+1)h$.

Case 3: The maximal invariant set $\mathcal{S}_{a,c}$ and its complement $\mathbb{R}\backslash\mathcal{S}_{a,c}$ contain a small right and left neighborhood of the origin respectively.

In this case, $\delta = 0$ and D satisfying (6.17) is the same as the one in Theorem 6.5. Following the argument used in Case 2, with applying Theorem 6.4 replaced by using Theorem 6.5, we can prove (6.18) and (6.19) in Case 3. Also we have

$$h = \frac{a-(N+1)(1-a)}{D+1} + \delta'$$

where N, δ' are given in Theorem 6.5. □

Proof of Corollary 6.7. (i): In this case, it follows from Theorem 6.3 that the complement of the maximal invariant set $\mathcal{S}_{a,c}$ is the union of gaps $(R_{a,c})^n(c-c_0+1-a)+[a-1,0), 0 \le n \le N$, which have their closure being mutually disjoint. Therefore

$$\begin{aligned}\mathcal{K}_{a,c} &= \{Y_{a,c}((R_{a,c})^n(c-c_0+1-a)),\ 0\le n\le N\} + Y_{a,c}(a)\mathbb{Z}\\ &= \{(n+1)Y_{a,c}(c-c_0+1-a)\}_{n=0}^N + a\mathbb{Z},\end{aligned}$$

where the last equality follows from Theorem 4.4, and hence (6.20) is proved.

(ii): We divide our proof into two cases.

Case 1: The maximal invariant set $\mathcal{S}_{a,c}$ and its complement $\mathbb{R}\backslash\mathcal{S}_{a,c}$ contain a small left and right neighborhood of the origin respectively.

Let D and δ be as in Theorem 6.4. Then

$$\mathcal{K}_{a,c} = \{Y_{a,c}((R_{a,c})^n(c-c_0+1-a+\delta)),\ 0\le n\le D\} + Y_{a,c}(a)\mathbb{Z}$$

by Theorem 6.4. This, together with Theorem 4.4 and the fact that $[0,\delta)$ is contained in $\mathbb{R}\backslash\mathcal{S}_{a,c}$, implies that

$$\begin{aligned}\mathcal{K}_{a,c} &= Y_{a,c}(R_{a,c}(\delta)-a) + \{nY_{a,c}(c-c_0+1-a)|0\le n\le D\} + Y_{a,c}(a)\mathbb{Z}\\ &= \{nY_{a,c}(c-c_0+1-a)|\ 1\le n\le D+1\} + Y_{a,c}(a)\mathbb{Z}.\end{aligned} \tag{6.74}$$

On the other hand,

$$(R_{a,c})^n(c-c_0+1-a+\delta)\notin c-c_0+1-a+\delta+a\mathbb{Z}$$

for all $1\le n\le D$ and

$$(R_{a,c})^{D+1}(c-c_0+1-a+\delta)\in c-c_0+1-a+\delta+a\mathbb{Z}$$

by Theorem 6.4. This together with Theorem 4.4 leads to

$$nY_{a,c}(c-c_0+1-a)\notin Y_{a,c}(a)\mathbb{Z}, 1\le n\le D. \tag{6.75}$$

and

$$(D+1)Y_{a,c}(c-c_0+1-a)\in Y_{a,c}(a)\mathbb{Z} \tag{6.76}$$

Combining (6.74), (6.75) and (6.76) proves (6.21).

Case 2: The maximal invariant set $\mathcal{S}_{a,c}$ and its complement $\mathbb{R}\backslash\mathcal{S}_{a,c}$ contain a small right and left neighborhood of the origin respectively.

To prove (6.21), we can follow the argument used in the proof of Case 1 with δ and Theorem 6.4 replaced by 0 and Theorem 6.5 respectively. We omit the details here. □

6.4. Nontriviality of maximal invariant sets with rational time shifts

In this section, we prove Theorem 6.8. The necessity of Theorem 6.8 follows essentially from Theorems 6.3, 6.4, 6.5 and 6.6. We examine five cases to verify the sufficiency. For the case 1) $c_0 < \gcd(c_1, a)$, we show that $[c_0, \gcd(c_1, a)) + \gcd(c_1, a)\mathbb{Z}$ is an invariant set under the transformation $R_{a,c}$ and it has empty intersection with black holes of transformations $R_{a,c}$ and $\tilde{R}_{a,c}$. This together with the maximality of the set $\mathcal{S}_{a,c}$ implies that $\mathcal{S}_{a,c} \neq \emptyset$. Similarly for the case 2) $1 - c_0 < \gcd(a, c_1 + 1)$, we verify that $[0, \gcd(a, c_1 + 1) - 1 + c_0) + \gcd(a, c_1 + 1)\mathbb{Z}$ is invariant under the transformation $R_{a,c}$ and it has empty intersection with black holes of transformations $R_{a,c}$ and $\tilde{R}_{a,c}$. For the case 3), we start from putting marks at $h\mathbb{Z}$ and insert gaps of size $1 - a + |\gamma|$ at marks located at $lmh + NhZ, 1 \le l \le d_1 + d_2 + 1$, and $|\gamma|$ at other marked locations, where $D = d_1 + d_2 + d_3 + d_4 + 1$, $h = (a - (d_1 + d_2 + 1)(1 - a))/(D + 1) - |\gamma|$ and $m = ((D + 1)c_1 + (d_1 + d_3 + 1)(1 - a))/a$. We then show that the gaps just inserted form a set that is invariant under the transformation $R_{a,c}$ and that contains black holes of the transformations $\tilde{R}_{a,c}$ and $R_{a,c}$.

Proof of Theorem 6.8. ($\Longrightarrow$) By Lemma 3.11, there exists a sufficiently small $\epsilon > 0$ such that one of the following three cases holds:

(i) $[-\epsilon, \epsilon) \subset \mathcal{S}_{a,c}$; or
(ii) $[-\epsilon, 0) \subset \mathcal{S}_{a,c}$ and $[0, \epsilon) \subset \mathbb{R}\backslash\mathcal{S}_{a,c}$; and
(iii) $[0, \epsilon) \subset \mathcal{S}_{a,c}$ and $[-\epsilon, 0) \subset \mathbb{R}\backslash\mathcal{S}_{a,c}$.

Define

$$\gamma = \begin{cases} \delta & \text{if } [-\epsilon, 0) \subset \mathcal{S}_{a,c} \text{ and } [0, \epsilon) \subset \mathbb{R}\backslash\mathcal{S}_{a,c} \\ 0 & \text{if } [-\epsilon, \epsilon) \subset \mathcal{S}_{a,c} \\ \delta' & \text{if } [0, \epsilon) \subset \mathcal{S}_{a,c} \text{ and } [-\epsilon, 0) \subset \mathbb{R}\backslash\mathcal{S}_{a,c}, \end{cases}$$

where δ, δ' are given in Theorems 6.4 and 6.5 respectively. Then

$$c_0 - a \le \gamma \le c_0 + a - 1.$$

Now we divide the proof of necessity into five cases.

Case 1: $\gamma = c_0 + a - 1$.
Let D, N be as in Theorem 6.4. Then

$$D = N$$

and $(R_{a,c})^n([c_1, c_1 + c_0) + a\mathbb{Z}), 0 \le n \le N$, are mutually disjoint gap with

$$(R_{a,c})^N([c_1, c_1 + c_0) + a\mathbb{Z}) = [0, c_0) + a\mathbb{Z}$$

by Theorem 6.4 and the assumption on γ. Thus

$$N \ge 1$$

by the assumption $c_1 > 0$. Observe that

$$(R_{a,c})^n([c_1, c_1 + c_0) + a\mathbb{Z}) = [c_1, c_1 + c_0) + n(c_1 - a) + a\mathbb{Z}, \ 0 \le n \le N$$

because $-a < c_1 - a < 0$ and

$$(R_{a,c})^n([c_1, c_1 + c_0) + a\mathbb{Z}) \subset [c_0, a) + a\mathbb{Z} \quad \text{for all} \quad 0 \le n \le N - 1.$$

Replacing n by N in the above equality and recalling that $(R_{a,c})^N([c_1, c_1+c_0)+a\mathbb{Z}) = [0, c_0) + a\mathbb{Z}$ gives

$$c_1 + N(c_1 - a) = ka \tag{6.77}$$

for some integer k. Thus

$$N + 1 \in a\mathbb{Z}/\gcd(c_1, a).$$

This together with mutual disjointness of gaps $[c_1, c_1 + c_0) + n(c_1 - a) + a\mathbb{Z}, 0 \le n \le N$, implies that

$$N + 1 = a/\gcd(c_1, a),$$

as otherwise $a/\gcd(c_1, a) \le N$ and

$$([c_1, c_1 + c_0) + a\mathbb{Z}) \cap ([c_1, c_1 + c_0) + n(c_1 - a) + a\mathbb{Z}) = [c_1, c_1 + c_0) + a\mathbb{Z} \ne \emptyset$$

for $n = a/\gcd(c_1, a) \le N$. Thus

$$\cup_{n=0}^{N}(n(c_1 - a) + a\mathbb{Z}) = \cup_{n=0}^{a/\gcd(c_1,a)-1}(n(c_1 - a) + a\mathbb{Z}) = \gcd(c_1, a)\mathbb{Z}.$$

Therefore the mutual disjointness of the gaps $[c_1, c_1+c_0)+n(c_1-a)+a\mathbb{Z}, 0 \le n \le N$, becomes

$$c_0 \le \gcd(c_1, a).$$

Observe that

$$\cup_{n=0}^{N}[c_1, c_1+c_0)+n(c_1-a)+a\mathbb{Z} = \cup_{n=0}^{a/\gcd(c_1,a)-1}([0, \gcd(c_1, a))+n\gcd(c_1, a)+a\mathbb{Z}) = \mathbb{R}$$

if $c_0 = \gcd(c_1, a)$, which contradicts to $\mathcal{S}_{a,c} \ne \emptyset$. This proves the desired first condition $c_0 < \gcd(c_1, a)$ in the theorem.

Case 2: $0 < \gamma < c_0 + a - 1$.
Let D, N be as in Theorem 6.4. Then

$$N \ge 0 \quad \text{and} \quad D \ge N + 1$$

by Theorem 6.4 and the assumption on γ. Denote by d_1, d_2 the number of big gaps

$$\mathcal{A}_n := (R_{a,c})^n(c - c_0 + [0, 1 - a + \gamma)) + a\mathbb{Z},\ 0 \le n \le N - 1,$$

of length $1-a+\gamma$ contained in $[0, c_0+a-1-\gamma)+a\mathbb{Z}$ and in $[c_0, a)+a\mathbb{Z}$ respectively, and similarly denote by d_3 and d_4 the number of small gaps

$$\mathcal{B}_m := (R_{a,c})^m([c_0 - \gamma, c_0)) + a\mathbb{Z},\ 1 \le m \le D - N,$$

of length γ contained in $[\gamma, c_0+a-1-\gamma)+a\mathbb{Z}$ and in $[c_0, a)+a\mathbb{Z}$ respectively. Now let us verify (6.23)–(6.28) for the above nonnegative integer parameters d_1, d_2, d_3 and d_4.

Proof of (6.23). By Theorem 6.4, the big gaps $\mathcal{A}_n, 0 \le n \le N - 1$, and the small gaps $\mathcal{B}_m, 1 \le m \le D - N - 1$, are either contained in $[\gamma, c_0 + a - 1 - \gamma) + a\mathbb{Z}$ or $[c_0, a) + a\mathbb{Z}$. Hence

$$N = d_1 + d_2 \tag{6.78}$$

and

$$D - N - 1 = d_3 + d_4. \tag{6.79}$$

Combining (6.10), (6.13), (6.18), (6.19), (6.78) and (6.79), we obtain that there are $(d_1 + d_2 + 1)$ gaps of length $1 - a + \gamma$ and $(d_3 + d_4 + 1)$ gaps of length γ, and $D + 1 := d_1 + d_2 + d_3 + d_4 + 2$ intervals of length h on one period $[0, a)$. Therefore

$$0 < B_d := a - (d_1 + d_2 + 1)(1 - a) = (D + 1)(h + \gamma) \in (D + 1)\gcd(a, 1)\mathbb{Z}. \tag{6.80}$$

This proves (6.23) and

$$0<\gamma<\frac{B_d}{D+1}.$$

Proof of (6.24). By (6.10), (6.13) and the definition of nonnegative integers $d_i, 1\le i\le 4$, we obtain that

$$c-c_0+1-a+\delta+d_1(\lfloor c\rfloor+1)+d_2\lfloor c\rfloor\in c_0+a\mathbb{Z}, \tag{6.81}$$

and

$$c_0+\lfloor c\rfloor-\delta+d_3(\lfloor c\rfloor+1)+d_4\lfloor c\rfloor\in a\mathbb{Z}. \tag{6.82}$$

Adding (6.81) and (6.82) leads to

$$c+(d_1+d_3+1)(\lfloor c\rfloor+1)+(d_2+d_4)\lfloor c\rfloor\in c_0+a\mathbb{Z}.$$

Then $(D+1)c_1+(d_1+d_3+1)(1-a)\in a\mathbb{Z}$ and (6.24) is true.

Proof of (6.26). By Theorem 6.4 and the definition of the integers d_1 and d_3, the interval $[0, c_0+a-1-\gamma)$ is covered by d_1 gaps of length $1-a+\gamma$, d_3+1 gaps of length γ, and d_1+d_3+1 intervals of length h. This together with (6.80) leads to

$$\begin{aligned} c_0+a-1-\gamma &= d_1(1-a+\gamma)+(d_3+1)\gamma+(d_1+d_3+1)h\\ &= d_1(1-a)+(d_1+d_3+1)B_d/(D+1). \end{aligned} \tag{6.83}$$

This proves (6.26).

Proof of (6.25). Substituting the expression (6.83) into (6.81), we obtain that

$$\begin{aligned} a\mathbb{Z} &\ni c-c_0+1-a+\delta-c_0+d_1(\lfloor c\rfloor+1)b+d_2\lfloor c\rfloor-d_1a\\ &= d_1(\lfloor c\rfloor+1)+d_2\lfloor c\rfloor+\lfloor c\rfloor+1-(d_1+1)a\\ &\quad -(d_1+1)(1-a)-(d_1+d_3+1)B_d/(D+1)\\ &= (d_1+d_2+1)\lfloor c\rfloor-(d_1+d_3+1)B_d/(D+1). \end{aligned}$$

Multiplying $D+1$ at both sides of the above equation leads to the desired inclusion (6.25).

Proof of (6.27). Define

$$m=\frac{(D+1)c_1+(d_1+d_3+1)(1-a))|}{a}. \tag{6.84}$$

Then m is a positive integer in $[1, D]$ by (6.24) and the observation that

$$\begin{aligned} 0 &< (D+1)c_1+(d_1+d_3+1)(1-a)\\ &< (D+1)(2a-1)+(d_1+d_3+1)(1-a)\le(D+1)a. \end{aligned}$$

Let $Y_{a,c}$ be as in (1.23) and let m_1 be the nonnegative integer in $[0, D]$ such that $Y_{a,c}(c_1+1-a)\in m_1h+Y_{a,c}(a)\mathbb{Z}$. We claim the following:

$$m_1=m. \tag{6.85}$$

Recall that

$$(R_{a,c})^N([c_1, c_1+1-a+\gamma)+a\mathbb{Z})=[c_0+a-1-\gamma, c_0)+a\mathbb{Z}$$

by Theorem 6.4, and that there are d_1+d_3+1 gaps in the interval $[0, c_0+a-1-\gamma)$. This together with Theorem 4.4 that

$$(d_1+d_2+1)m_1h-(d_1+d_3+1)h\in Y_{a,c}(a)\mathbb{Z}=(D+1)h\mathbb{Z}. \tag{6.86}$$

Then the number of gaps of length $1-a+\gamma$ contained $[0,c_1)$ is

$$\frac{(d_1+d_2+1)m_1h-(d_1+d_3+1)h}{Y_{a,c}(a)}=\frac{(d_1+d_2+1)m_1-(d_1+d_3+1)}{D+1}.$$

This implies that there are m_1 gaps contained in $[0,c_1)$ with $((d_1+d_2+1)m_1-(d_1+d_3+1))/(D+1)$ of them are gaps of length $1-a+\gamma$. Hence

$$\begin{aligned} c_1 &= m_1h+\big(m_1-\frac{(d_1+d_2+1)m_1-(d_1+d_3+1)}{D+1}\big)\gamma \\ &\quad +\frac{(d_1+d_2+1)m_1-(d_1+d_3+1)}{D+1}(1-a+\gamma) \\ &= \frac{m_1a-(d_1+d_3+1)(1-a)}{D+1}. \end{aligned}$$

This together with (6.84) proves (6.85).

We return to the proof of (6.27). The above claim (6.85), together with (6.78), (6.79), and the cyclic group property (6.21) for the set $\mathcal{K}_{a,c}$ of marks, proves (6.27).

Proof of (6.28). Applying (6.84) and (6.85), and recalling that $[c_1,c_1+1-a)$ is a gap in the complement of the maximal invariant set $\mathcal{S}_{a,c}$, we have that

$$Y_{a,c}(c_1)-mh\in(D+1)h\mathbb{Z}.$$

Then $n\in E^d_{a,c}$ if and only if $(R_{a,c})^n[c_1,c_1+1-a+\delta)$ is a big gap contained in $[0,c_0+a-1)+a\mathbb{Z}$. This implies that the cardinality of the set $E^d_{a,c}$ is equal to d_1 from the definition of the nonnegative integer d_1.

This completes the proof of the necessity for the case that $\gamma\in(0,c_0+a-1)$.

Case 3: $\gamma=0$.

Let N and D be as in Theorems 6.3 and 6.6 respectively. Then

$$N\ge 0\quad\text{and}\quad D\ge N+1.$$

Denote by d_1,d_2 the numbers of gaps $(R_{a,c})^n(c-c_0+[0,1-a)), 0\le n\le N-1$, of length $1-a$ contained in $[0,c_0+a-1)+a\mathbb{Z}$ and in $[c_0,a)+a\mathbb{Z}$ respectively, and similarly denote by d_3 and d_4 the numbers of $(R_{a,c})^m(c_0), 1\le m\le D-N$, contained in $(0,c_0+a-1)+a\mathbb{Z}$ or $[c_0,a)+a\mathbb{Z}$ respectively. Then we may follow the argument for Case 2 line by line and establish (6.23)–(6.28) with the above nonnegative integer parameters d_1,d_2,d_3 and d_4.

Case 4: $\gamma\in(c_0-a,0)$.

Let N,D be as in Theorem 6.5. By Theorem 6.5, $N\ge 0$ and $D\ge N+1$. Denote by d_1,d_2 the numbers of big gaps $(R_{a,c})^n(c-c_0+[\gamma,1-a)), 0\le n\le N-1$, of length $1-a-\gamma$ contained in $[0,c_0+a-1)+a\mathbb{Z}$ and in $[c_0-\gamma,a+\gamma)+a\mathbb{Z}$ respectively, and similarly denote by d_3 and d_4 the numbers of small gaps $(R_{a,c})^m([c_0,c_0-\gamma)), 1\le m\le D-N$, of length $-\gamma$ contained in $[0,c_0+a-1)+a\mathbb{Z}$ and in $[c_0-\gamma,a+\gamma)+a\mathbb{Z}$ respectively. We may follow the argument for the second case and prove the desired properties (6.23)–(6.28) with the above nonnegative integers d_1,d_2,d_3 and d_4.

Case 5: $\gamma=c_0-a$.

We follow the argument used in Case 1. Let D,N be as in Theorem 6.5. Then $D=N$ by the assumption on γ, and $(R_{a,c})^n([c_1+c_0-a,c_1+1-a)+a\mathbb{Z}), 0\le n\le N$, are mutually disjoint gap with $(R_{a,c})^N([c_1+c_0-a,c_1+1-a)+a\mathbb{Z})=[c_0+a-1,a)+a\mathbb{Z}$ by Theorem 6.5. Thus $N\ge 1$ as $c_1<2a-1$. Observe that

$$(R_{a,c})^n([c_1+c_0-a,c_1+1-a)+a\mathbb{Z})=[c_1+c_0-a,c_1+1-a)+n(c_1+1-a)+a\mathbb{Z}$$

for all $0 \le n \le N$, because $0 < c_1 + 1 - a < a$ and

$$(R_{a,c})^n([c_1 + c_0 - a, c_1 + 1 - a) + a\mathbb{Z}) \subset [0, c_0 + a - 1) + a\mathbb{Z},\ 0 \le n \le N - 1.$$

Replacing n by N in the above equality, recalling that $(R_{a,c})^N([c_1 + c_0 - a, c_1 + 1 - a) + a\mathbb{Z}) = [c_0 + a - 1, a) + a\mathbb{Z}$ and applying mutual disjointness of $[c_1 + c_0 - a, c_1 + 1 - a) + n(c_1 + 1 - a) + a\mathbb{Z}, 0 \le n \le N$, we obtain

$$N + 1 = a/\gcd(c_1 + 1, a),$$

Hence the desired second condition $1 - c_0 < \gcd(c_1 + 1, a)$ follows from the assumption $\mathcal{S}_{a,c} \neq \emptyset$ and the mutual disjointness of the gaps $(R_{a,c})^n([c_1 + c_0 - a, c_1 + 1 - a) + a\mathbb{Z}), 0 \le n \le N$.

($\Longleftarrow$) We examine five cases to prove the sufficiency.

Case 1: $c_0 < \gcd(c_1, a)$.

Let $D = a/\gcd(c_1, a) - 1$ and define

$$T = \big(\cup_{n=0}^{D} [c_0, \gcd(c_1, a)) + n(a - c_1)\big) + a\mathbb{Z}.$$

Then

$$\begin{aligned} T &= \big(\cup_{i=0}^{D} [c_0, \gcd(c_1, a)) + i\gcd(c_1, a)\big) + a\mathbb{Z} \\ &= [c_0, \gcd(c_1, a)) + \gcd(c_1, a)\mathbb{Z}, \end{aligned} \tag{6.87}$$

and T has empty intersection with black holes of the transformations $R_{a,c}$ and $\tilde{R}_{a,c}$, since

$$T \cap [0, c_0) = T \cap [c_1, c_0 + c_1) = \emptyset,$$

and for any $t \in T$,

$$R_{a,c}(t) = t + c_1 \in [c_0, \gcd(c_1, a)) + c_1 + \gcd(c_1, a)\mathbb{Z} = T.$$

Therefore $T \subset \mathcal{S}_{a,c}$ (in fact $T = \mathcal{S}_{a,c}$) as $\mathcal{S}_{a,c}$ is the maximal invariant set that has empty intersection with the black hole of the transformation $R_{a,c}$ by Theorem 3.4. Thus $\mathcal{S}_{a,c}$ is not an empty set as the restriction of the set T on $[0, a)$ consists of $a/\gcd(c_1, a)$ intervals of length $\gcd(c_1, a) - c_0 > 0$.

Case 2: $1 - c_0 < \gcd(a, c_1 + 1)$.

Let $D = a/\gcd(a, c_1 + 1) - 1$ and define

$$\begin{aligned} T' &= \big(\cup_{i=0}^{D} [0, \gcd(a, c_1 + 1) - 1 + c_0) + i(c_1 + 1 - a)\big) + a\mathbb{Z} \\ &= [0, \gcd(a, c_1 + 1) - 1 + c_0) + \gcd(a, c_1 + 1)\mathbb{Z}. \end{aligned}$$

We may follow the argument used in Case 1 to show that T' has empty intersection with black holes of the transformations $R_{a,c}$ and $\tilde{R}_{a,c}$, and it is invariant under the transformation $R_{a,c}$. Then $\mathcal{S}_{a,c} \supset T'$ is not an empty set.

Case 3: There exist nonnegative integers d_1, d_2, d_3, d_4 and $\gamma \in (0, \min(B_d/(D+1), c_0 + a - 1))$ satisfying (6.23)–(6.28).

In this case, we set

$$h = \frac{a - (d_1 + d_2 + 1)(1 - a)}{D + 1} - \gamma > 0,$$

$$m = \frac{(D + 1)c_1 + (d_1 + d_3 + 1)(1 - a)}{a},$$

and

$$\tilde{m} = \frac{(d_1 + d_2 + 1)m - (d_1 + d_3 + 1)}{D + 1}.$$

Then

$$0 < h \in \gcd(a, 1)\mathbb{Z}$$

by (6.23) and (6.26); m is a positive integer no larger than D, i.e.,

$$m \in \mathbb{Z} \cap [1, D]$$

as

$$0 < (D+1)c_1/a \leq m < ((D+1)(2a-1) + (d_1+d_3+1)(1-a))/a < D+1;$$

and $\tilde{m}$ is a nonnegative integer no larger than m,

$$\tilde{m} \in [0, m] \cap \mathbb{Z} \tag{6.88}$$

by (6.25). Moreover,

$$\begin{aligned} & m\frac{a-(d_1+d_2+1)(1-a)}{D+1} + \tilde{m}(1-a) \\ = \ & \frac{m}{D+1}a - \frac{d_1+d_3+1}{D+1}(1-a) = c_1. \end{aligned} \tag{6.89}$$

In order to expand the real line $\mathbb{R}$ with marks at $h\mathbb{Z}$ to create an invariant set under the transformation $R_{a,c}$, we insert gaps $[0, 1-a+\gamma)$ located at $lmh + (D+1)h\mathbb{Z}, 1 \leq l \leq d_1 + d_2 + 1$, and gaps $[0, \delta)$ otherwise. Recall from (6.27) that $(l - l')mh \notin (D+1)h\mathbb{Z}$ for all $1 \leq l \neq l' \leq D+1$. Therefore we have inserted d_1+d_2+1 gaps $[0, 1-a+\gamma)$ and d_3+d_4+1 gaps $[0, \gamma)$ on the interval $[0, (D+1)h)$. Thus after performing the above expansion, the interval $[0, (D+1)h)$ with marks on $[0, (D+1)h) \cap h\mathbb{Z}$ becomes the interval

$$[0, (D+1)h + (d_1+d_2+1)(1-a+\gamma) + (d_3+d_4+1)\gamma) = [0, a)$$

with gaps $[y_i, y_i + h_i), 0 \leq i \leq D$, where $0 = y_0 \leq y_1 \leq \ldots \leq y_D$ and $h_i \in \{1-a+\gamma, \gamma\}, 0 \leq i \leq D$. Now we want to prove that

$$y_m = c_1. \tag{6.90}$$

For that purpose, we need the following claim:

Claim 6.9. *For $s \in [0, D] \cap \mathbb{Z}$, the cardinality of the set $\{l \in [1, d_1+d_2+1] \cap \mathbb{Z} |\ lmh \in sh + [0, mh) + (D+1)h\mathbb{Z}\}$ is equal to $\tilde{m}+1$ if $1 \leq s \leq d_1+d_3+1$, and $\tilde{m}$ otherwise.*

Proof. For any $i \in \mathbb{Z}$, let $k_i = \lfloor((D+1)i + m + s - 1)/m\rfloor$ the unique integer such that $k_i mh \in [sh, sh+mh) + i(D+1)h$. Therefore $1 \leq k_i \leq d_1+d_2+1$ if and only if $m \leq i(D+1) + m + s - 1 \leq (d_1+d_2+1)m + m - 1$ if and only if $1 - s \leq i(D+1) \leq (d_1+d_2+1)m - s = (D+1)\tilde{m} + (d_1+d_3+1-s)$. Therefore

$$\begin{aligned} & \#\{l \in [1, d_1+d_2+1] \cap \mathbb{Z} |\ lmh \in sh + [0, mh) + (D+1)h\mathbb{Z}\} \\ = \ & \sum_{i \in \mathbb{Z}} \#\{l \in [1, d_1+d_2+1] \cap \mathbb{Z} |\ lmh \in sh + [0, mh) + i(D+1)h\} \\ = \ & \sum_{i \in \mathbb{Z}} \#\{k_i \in [1, d_1+d_2+1] \cap \mathbb{Z}\} \\ = \ & \#([(1-s)/(D+1), \tilde{m} + (d_1+d_3+1-s)/(D+1)] \cap \mathbb{Z}). \end{aligned}$$

Counting the number of integers in the interval $[(1-s)/(D+1), \tilde{m} + (d_1+d_3+1-s)/(D+1)]$ proves the claim. □

We return to the proof of the equality (6.90). By Claim 6.9, we have inserted $\tilde{m}$ interval of length $1-a+\gamma$ and $m-\tilde{m}$ interval of length γ in the marked interval $[0, mh)$. So after performing the expansion, the mark located at mh on the line becomes the gap located at $mh+(m-\tilde{m})\gamma+\tilde{m}(1-a+\delta)$, which is equal to c_1 by (6.89). This completes the proof of the equality (6.90).

Next we show that

$$y_{d_1+d_3+1} = c_0 + a - 1 - \gamma. \tag{6.91}$$

By (6.28), we have inserted d_1 gaps of length $1-a+\delta$ and $(d_1+d_3+1)-d_1$ intervals of length γ in the marked interval $[0, (d_1+d_3+1)h)$. Therefore the mark located at $(d_1+d_3+1)h$ becomes

$$(d_1+d_3+1)h + d_1(b-a+\gamma) + (d_3+1)\gamma = c_0 + a - b - \gamma$$

after inserting gaps, where the last equality follows from (6.26). Hence (6.91) follows.

Then we prove by induction on $0 \le k \le D$ that

$$(R_{a,c})^k(c-c_0) + a\mathbb{Z} = y_{l(k)} + a\mathbb{Z},\ 0 \le k \le d_1+d_2, \tag{6.92}$$

and

$$(R_{a,c})^m(c_0+a-1-\gamma) + a\mathbb{Z} = y_{l(m+d_1+d_2)} + a\mathbb{Z},\ 1 \le m \le d_3+d_4+1, \tag{6.93}$$

where $l(k) \in (k+1)m+(D+1)\mathbb{Z}$. We remark that $l(0) = m, l(d_1+d_2+d_3+d_4+1) = l(D) = 0$ and $l(d_1+d_2) = d_1+d_3+1$, where the last equality follows from (6.25).

Proof of (6.92) and (6.93). The conclusion (6.92) for $k=0$ from (6.90) and the observation that $l(0) = m$. Inductively, we assume that the conclusion (6.92) holds for some $0 \le k \le d_1+d_2-1$. Then $l(k) \ne 0, d_1+d_3+1$ as $l(d_1+d_2) = d_1+d_3+1$ and $l(D) = 0$. If $0 < l(k) < d_1+d_3+1$, then

$$\begin{aligned} y_{l(k+1)} &= y_{l(k)} + (\tilde{m}+1)(1-a+\gamma) + (m-\tilde{m}-1)\gamma + mh \\ &= y_{l(k)} + c_1 + 1 - a \end{aligned} \tag{6.94}$$

if $l(k+1) - l(k) = m$, and

$$\begin{aligned} y_{l(k+1)} + a &= y_{l(k)} + (\tilde{m}+1)(1-a+\gamma) + (m-\tilde{m}-1)\gamma + mh \\ &= y_{l(k)} + c_1 + 1 - a \end{aligned} \tag{6.95}$$

if $l(k+1) - l(k) = m-(D+1)$, where (6.94) and (6.95) hold as we have inserted $\tilde{m}+1$ gaps of size $1-a+\gamma$ and $m-(\tilde{m}+1)$ gaps of size γ on $[l(k)h, (l(k)+m)h)$ by Claim 6.9. Also we obtain from (6.91) that $y_{l(k)} \in [0, c_0+a-1-\gamma)$ when $0 < l(k) < d_1+d_3+1$, which together with the inductive hypothesis implies that

$$\begin{aligned} (R_{a,c})^{k+1}(c-c_0) + a\mathbb{Z} &= R_{a,c}(y_{l(k)}) + a\mathbb{Z} \\ &= y_{l(k)} + c_1 + 1 - a + a\mathbb{Z}. \end{aligned} \tag{6.96}$$

Combining (6.94), (6.95) and (6.96) leads to

$$(R_{a,c})^{k+1}(c-c_0) + a\mathbb{Z} = y_{l(k+1)} + a\mathbb{Z} \tag{6.97}$$

if $0 < l(k) < d_1+d_3+1$.

Similarly if $d_1+d_3+1 < l(k) \le D$, we have that

$$y_{l(k+1)} - y_{l(k)} \in c_1 + a\mathbb{Z} \tag{6.98}$$

because we have inserted $\tilde{m}$ gaps of size $1-a+\delta$ and $m-\tilde{m}$ gaps of size δ on $[l(k)h,(l(k)+m)h)$ by Claim 6.9; and

$$(R_{a,c})^{k+1}(c-c_0)+a\mathbb{Z}=y_{l(k)}+c_1+a\mathbb{Z}, \tag{6.99}$$

since $y_{l(k)}\in[c_0,a)$ by (6.91). Combining (6.98) and (6.99) yields

$$(R_{a,c})^{k+1}(c-c_0)+a\mathbb{Z}=y_{l(k+1)}+a\mathbb{Z} \tag{6.100}$$

if $d_1+d_3+1<l(k)\le D$. Therefore we can proceed our inductive proof by (6.97) and (6.100). This completes the proof of the equalities in (6.92).

Notice that $y_{l(d_1+d_2)}=y_{d_1+d_3+1}=c_0+a-1-\delta$ by (6.91). Hence

$$\begin{aligned}(R_{a,c})^m(c_0+a-1-\delta)+a\mathbb{Z} &= (R_{a,c})^m(y_{l(d_1+d_2)})+a\mathbb{Z}\\ &= (R_{a,c})^{m+d_1+d_2}(y_{l(0)})+a\mathbb{Z}=(R_{a,c})^{m+d_1+d_2}(c-c_0)+a\mathbb{Z}\end{aligned} \tag{6.101}$$

for all $1\le m\le d_3+d_4+1$. Then we can follow the argument to prove (6.92) to show that (6.93) holds. □

Finally from (6.92) and (6.93) the mutually disjoint gaps we have inserted are $(R_{a,c})^k(c-c_0)+[0,1-a+\gamma)+a\mathbb{Z}, 0\le k\le d_1+d_2$, and $(R_{a,c})^m(c_0+a-1-\gamma)+[0,\gamma)+a\mathbb{Z}, 1\le m\le d_3+d_4+1$. Moreover

$$(R_{a,c})^{d_1+d_2}(c-c_0)+[0,1-a+\gamma)+a\mathbb{Z}=[c_0+a-1-\delta,c_0)+a\mathbb{Z}$$

by (6.91) and $l(d_1+d_2)=d_1+d_3+1$; and

$$\begin{aligned}&(R_{a,c})^{d_3+d_4+1}(c_0+a-1-\gamma)+[0,\gamma)+a\mathbb{Z}\\ =\ &(R_{a,c})^{D}(c_0+a-1-\gamma)+[0,\gamma)+a\mathbb{Z}=[0,\gamma)+a\mathbb{Z}\end{aligned}$$

by (6.101) and $l(D)=0$. Notice that the union of the above gaps is invariant under the transformation $R_{a,c}$ and contains the black holes of the transformations $R_{a,c}$ and $\tilde{R}_{a,c}$. Therefore its complement is the set $\mathcal{S}_{a,c}$ by Theorem 3.4, whose restriction on $[0,a)$ has Lebesgue measure $(D+1)h$. Thus the conclusion that $\mathcal{S}_{a,c}\neq\emptyset$ is established for this case.

Case 4: There exist nonnegative integers d_1,d_2,d_3,d_4 and $\gamma=0$ satisfying (6.23)–(6.28).

In this case, we set

$$h=\frac{a-(d_1+d_2+1)(1-a)}{D+1}$$

and

$$m=\frac{(D+1)c_1+(d_1+d_3+1)(1-a)}{a},$$

and expand the real line $\mathbb{R}$ with marks at $h\mathbb{Z}$ by inserting gaps $[0,1-a)$ located at $lmh+(D+1)h\mathbb{Z}, 1\le l\le d_1+d_2+1$, and gaps of zero length otherwise, c.f. the fourth subfigure of Figure 1. Then after performing the above operation, the interval $[0,(D+1)h)$ becomes the interval $[0,a)$ with gaps $[y_i,y_i+h_i), 0\le i\le D$, where $0=y_0\le y_1\le\ldots\le y_D$ and $h_i\in\{1-a,0\}, 0\le i\le N-1$. We follow the argument in Case 3 to show that $y_m=c_1, y_{d_1+d_3}=c_0+a-1$ and by induction on $0\le k\le N-1$ that

$$(R_{a,c})^k(c-c_0)+a\mathbb{Z}=y_{l(k)}+a\mathbb{Z},\ 0\le k\le d_1+d_2,$$

and

$$(R_{a,c})^m(c_0)+a\mathbb{Z}=y_{l(m+d_1+d_2)}+a\mathbb{Z},\ 1\le m\le d_3+d_4+1,$$

where $l(k) \in (k+1)m + (D+1)\mathbb{Z}$. Thus the union of gaps of size $1-a$ is $\cup_{n=0}^{d_1+d_2}(R_{a,c})^n([c-c_0, c-c_0+1-a)+a\mathbb{Z}$ with $(R_{a,c})^{d_1+d_2}([c-c_0, c-c_0+1-a)+a\mathbb{Z}) = [c_0+a-1, c_0)+a\mathbb{Z}$. Therefore $\mathcal{S}_{a,c}$ is the complement of the above union of finite gaps and the sufficiency in the fifth case follows.

Case 5: There exist nonnegative integers d_1, d_2, d_3, d_4 and $\gamma \in (-\min(B_d/(D+1), a-c_0), 0)$ satisfying (6.23)–(6.28).

In this case, we define

$$h = \frac{a-(d_1+d_2+1)(1-a)}{D+1} + \gamma$$

and

$$m = \frac{(D+1)c_1 + (d_1+d_3+1)(1-a)}{a}.$$

We expand the real line $\mathbb{R}$ with marks at $h\mathbb{Z}$ by inserting gaps $[\gamma+a-1, 0)$ located at $lmh+(D+1)h\mathbb{Z}, 1 \le l \le d_1+d_2+1$, and gaps $[\gamma, 0)$ otherwise. After performing the above augmentation operation, the interval $[0, (D+1)h)$ with marks $[0, (D+1)h) \cap h\mathbb{Z}$ becomes the interval $[0, a)$ with gaps $[y_i+h_i, y_i), 0 \le i \le D$, where $0 < y_1 \le \ldots \le y_{D+1} = a$ and $h_i \in \{\gamma+a-1, \gamma\}, 1 \le i \le D+1$. We follow the argument used in Case 3 to show that $y_m = c_1+1-a, y_{d_1+d_3+1} = c_0-\gamma$ and for $0 \le k \le D$,

$$(R_{a,c})^k(c-c_0+1)+a\mathbb{Z} = y_{l(k)}+a\mathbb{Z},\ 0 \le k \le d_1+d_2,$$

and

$$(R_{a,c})^m(c_0-\gamma)+a\mathbb{Z} = y_{l(k)}+a\mathbb{Z},\ 1 \le m \le d_3+d_4+1,$$

where $l(k) \in (k+1)m+(D+1)\mathbb{Z}$. Therefore

$$\mathcal{S}_{a,c} = \mathbb{R}\backslash\Big(\big(\cup_{n=0}^{d_1+d_2}[y_{l(k)}+a-b+\gamma, y_{l(k)})+a\mathbb{Z}\big)\cup\big(\cup_{m=1}^{d_3+d_4+1}[y_{l(k)}+\delta, y_{l(k)})+a\mathbb{Z}\big)\Big),$$

whose restriction on $[0, a)$ has Lebesgue measure $(D+1)h > 0$. This proves the sufficiency for Case 4. □

CHAPTER 7

The *abc*-problem for Gabor Systems

In this chapter, we provide full classification of all pairs (a, c) of positive numbers of time-spacing and window-size parameters such that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ are Gabor frames for L^2.

Let us start from recalling some known classification of pairs (a, c), see for instance [**16, 23, 30**].

THEOREM 7.1. *Let $a, c > 0$, and let $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ be the Gabor system in* (1.1) *generated by the characteristic function on the interval* $[0, c)$. *Then the following statements hold.*

(I) *If $a > c$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is not a Gabor frame.*
(II) *If $a = c$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is a Gabor frame if and only if $a \le 1$.*
(IV) *If $a < c$ and $c \le 1$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is a Gabor frame.*
(III) *If $a < c, 1 < c$ and $a \ge 1$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is not a Gabor frame.*

The conclusions in Theorem 7.1 are illustrated in the red and low right-triangle green regions of Figure 1 below, where on the left subfigure we normalize the frequency-spacing parameter b to 1, while on the right subfigure we normalize the window-size parameter c to 1 and use the frequency-spacing parameter b as the y-axis, cf. Janssen's tie in [**30**].

Apply the equivalences in Theorem 2.1 and the explicit construction of the set $\mathcal{S}_{a,c}$ in Theorem 3.5, we take one step forward in the way to solve the *abc*-problem for Gabor systems.

THEOREM 7.2. *Let $0 < a < 1 < c$, and let $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ be the Gabor system in* (1.1) *generated by the characteristic function on the interval* $[0, c)$. *Set $c_0 = c - \lfloor c \rfloor$. Then the following statements hold.*

(V) *If $c_0 \ge a$ and $c_0 \le 1 - a$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is a Gabor frame.*
(VI) *If $c_0 \ge a$ and $c_0 > 1 - a$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is not a Gabor frame if and only if $a \in \mathbb{Q}$ and either*
   1) $c_0 > 1 - \gcd(\lfloor c \rfloor + 1, a)$ *and* $\gcd(\lfloor c \rfloor + 1, a) \ne (\lfloor c \rfloor + 1)\gcd(a, 1)$, *or*
   2) $c_0 > 1 - \gcd(\lfloor c \rfloor + 1, a) + \gcd(a, 1)$ *and* $\gcd(\lfloor c \rfloor + 1, a) = (\lfloor c \rfloor + 1)\gcd(a, 1)$.
(VII) *If $c_0 < a$ and $c_0 \le 1 - a$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is not a Gabor frame if and only if either*
   3) $c_0 = 0$*; or*
   4) $a \in \mathbb{Q}$, $0 < c_0 < \gcd(\lfloor c \rfloor, a)$ *and* $\gcd(\lfloor c \rfloor, a) \ne \lfloor c \rfloor \gcd(a, 1)$*; or*
   5) $a \in \mathbb{Q}$, $0 < c_0 < \gcd(\lfloor c \rfloor, a) - \gcd(a, 1)$ *and* $\gcd(\lfloor c \rfloor, a) = \lfloor c \rfloor \gcd(a, 1)$.

The statement (V) in the above theorem is given in [**30**, Section 3.3.3.2]. The conclusions in Theorem 7.2 are illustrated in the green, yellow and purple regions of

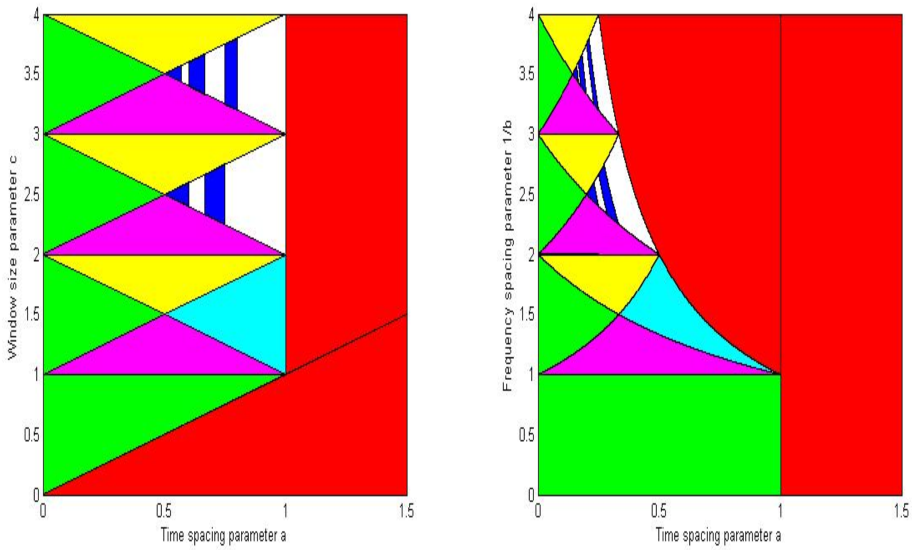


FIGURE 1. Left: Classification of pairs (a, c) such that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ are Gabor frames. Right: Classification of pairs (a, b) such that $\mathcal{G}(\chi_{[0,1)}, a\mathbb{Z} \times b\mathbb{Z})$ are Gabor frames.

Figure 1. In the green region, $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})$ are Gabor frames by Conclusion (V) of Theorem 7.2. In the yellow region, it follows from Conclusion (VI) of Theorem 7.2 that the set of pairs (a, c) such that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})$ are not Gabor frames contains needles (line segments) of lengths $\gcd(\lfloor c\rfloor + 1, p)/q - \{0, 1/q\}$ hanging vertically from the ceiling $\lfloor c\rfloor + 1$ at every rational time shift location $a = p/q$. In the purple region, we obtain from Conclusion (VII) of Theorem 7.2 that the set of pairs (a, c) such that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ are not Gabor frames contains floors $\lfloor c\rfloor \ge 2$ and also needles (line segments) of lengths $\gcd(\lfloor c\rfloor, p)/q - \{0, 1/q\}$ growing vertically from floors $\lfloor c\rfloor$ at every rational time shift location $a = p/q$.

Using the expression of the set $\mathcal{S}_{a,c}$ in Theorem 3.5, we can determine whether Gabor systems $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})$ corresponding to those pairs with either $c_1 \ge 1-2a$ or $c_1 = 0$ are frames for L^2.

THEOREM 7.3. *Let $0 < a < 1 < c$ and $1 - a < c_0 < a$, and let $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ be the Gabor system in* (1.1) *generated by the characteristic function on the interval $[0, c)$. Set $c_1 := \lfloor c\rfloor - \lfloor(\lfloor c\rfloor/a)\rfloor a$. Then the following statements hold.*

(VIII) *If $\lfloor c\rfloor = 1$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is a Gabor frame.*
(IX) *If $\lfloor c\rfloor \ge 2$ and $c_1 > 2a - 1$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is a Gabor frame.*
(X) *If $\lfloor c\rfloor \ge 2$ and $c_1 = 2a - 1$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is not a Gabor frame if and only if $a \in \mathbb{Q}$, $c_0 \le 1 - a + \gcd(a, 1)$ and $a = (\lfloor c\rfloor + 1)\gcd(a, 1)$.*
(XI) *If $\lfloor c\rfloor \ge 2$ and $c_1 = 0$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is not a Gabor frame if and only if $a \in \mathbb{Q}$, $c_0 \ge a - \gcd(a, 1)$ and $a = \lfloor c\rfloor\gcd(a, 1)$.*

The statement (VIII) in the above theorem can be found in [**23, 30**]. The conclusions in Theorem 7.3 are illustrated in the blue and dark blue regions of Figure 1.

Applying the parametrization of the maximal invariant set $\mathcal{S}_{a,c}$ in Theorem 5.5, we take another step forward in the direction to solve the *abc*-problem for Gabor systems.

THEOREM 7.4. *Let $0 < a < 1 < c, 1-a < c_0 < a, \lfloor c \rfloor \ge 2$, $0 < c_1 < 2a-1$ and $a \notin \mathbb{Q}$. Let $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})$ be the Gabor system in* (1.1) *generated by the characteristic function on the interval $[0,c)$. Then the following statement holds.*

(XII) *The Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})$ is not a frame for L^2 if and only if there exist nonnegative integers d_1 and d_2 such that*
  (a) $a \ne c-(d_1+1)(\lfloor c\rfloor+1)(1-a)-(d_2+1)\lfloor c\rfloor(1-a) \in a\mathbb{Z}$;
  (b) $\lfloor c\rfloor+(d_1+1)(1-a) < c < \lfloor c\rfloor+1-(d_2+1)(1-a)$; *and*
  (c) $\#E_{a,c} = d_1$, *where* $m = ((d_1+d_2+1)c_1 - c_0 + (d_1+1)(1-a))/a$ *and $E_{a,c}$ is given in* (5.10).

In the above theorem, we insert d_1 and d_2 holes contained in intervals $[0, c_0 + a-1)$ and $[c_0, a)$ respectively, and put marks at $\cup_{n=1}^{d_1+d_2+1}(n(c_1-m(1-a))+(a-(d_1+d_2+1)(1-a))\mathbb{Z})$.

Applying the characterization for $\mathcal{S}_{a,c} \ne \emptyset$ in Theorem 6.8, we reach the last step to solve the *abc*-problem for Gabor systems.

THEOREM 7.5. *Let $0 < a < 1 < c, 1-a < c_0 < a, \lfloor c\rfloor \ge 2, 0 < c_1 < 2a-1$ and $a \in \mathbb{Q}$. Let $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})$ be the Gabor system in* (1.1) *generated by the characteristic function on the interval $[0,c)$. The following statements hold.*

(XIII) *If $c \in \gcd(a,1)\mathbb{Z}$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})$ is not a Gabor frame if and only if the pair (a,c) satisfies one of the following three conditions:*
  6) $c_0 < \gcd(a,c_1)$ *and* $\lfloor c\rfloor(\gcd(a,c_1)-c_0) \ne \gcd(a,c_1)$.
  7) $1-c_0 < \gcd(a,c_1+1)$ *and* $(\lfloor c\rfloor+1)(\gcd(a,c_1+1)+c_0-1) \ne \gcd(a,c_1+1)$.
  8) *There exist nonnegative integers d_1,d_2,d_3,d_4 such that (a)* $0 < a-(d_1+d_2+1)(1-a) \in (D+1)\gcd(a,1)\mathbb{Z}$; *(b)* $(D+1)c_1+(d_1+d_3+1)(1-a) \in a\mathbb{Z}$; *(c)* $(d_1+d_2+1)((D+1)c_1+(d_1+d_3+1)(1-a))-(d_1+d_3+1)a \in (D+1)a\mathbb{Z}$; *(d)* $\gcd((D+1)c_1+(d_1+d_3+1)(1-a), (D+1)a) = a$; *(e)* $\#E^d_{a,c} = d_1$; *(f)* $c_0 = (d_1+1)(1-a)+(d_1+d_3+1)B_d/(D+1)+\gamma$ *for some some* $\gamma \in (-\min((a-(d_1+d_2+1)(1-a))/(D+1), a-c_0), \min((a-(d_1+d_2+1)(1-a))/(D+1), c_0+1-a))$; *and (g)* $|\gamma|+a/((D+1)\lfloor c\rfloor+(d_1+d_3+1)) \ne (a-(d_1+d_2+1)(a-1))/(D+1)$, *where* $D := d_1+d_2+d_3+d_4+1$ *and $E^d_{a,c}$ is defined by* (6.29).

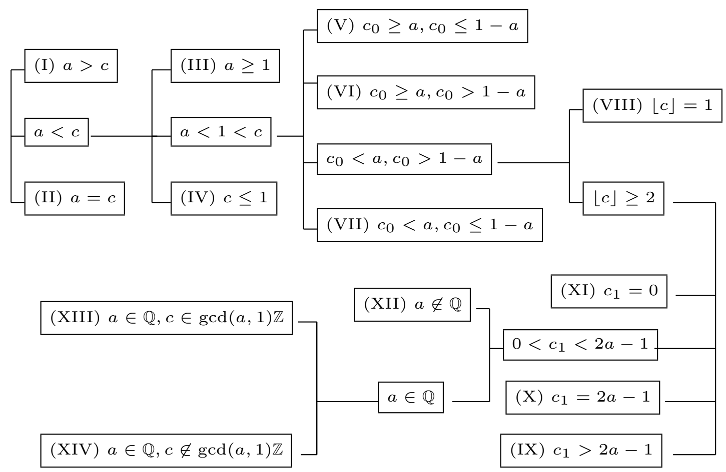
(XIV) *If $c \notin \gcd(a,1)\mathbb{Z}$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})$ is a Gabor frame if and only if both $\mathcal{G}(\chi_{[0,\tilde c)}, a\mathbb{Z}\times\mathbb{Z})$ and $\mathcal{G}(\chi_{[0,\tilde c+\gcd(a,1))}, a\mathbb{Z}\times\mathbb{Z})$ are Gabor frames, where $\tilde c = \lfloor c/\gcd(a,1)\rfloor \gcd(a,1)$.*

In Case 6) of Conclusion (XIII) in Theorem 7.5, the set $\mathcal{K}_{a,c}$ of marks is $(\gcd(a,c_1)-c_0)\mathbb{Z}$ and gaps inserted at marked positions have same length c_0. In Case 7) of Conclusion (XIII) in Theorem 7.5, $K_{a,c} = (\gcd(a,c_1+1)+c_0-1)\mathbb{Z}$ and gaps inserted at marks in $K_{a,c}$ are of size $1-c_0$. In Case 8) of Conclusion (XIII) in Theorem 7.5, $K_{a,c} = h\mathbb{Z}, Y_{a,c}(a) = (D+1)h$ and gaps inserted at marked positions $lmh+(D+1)h\mathbb{Z}, 1 \le l \le N$, have size $|1-a|+|\gamma|$ for $1 \le l \le d_1+d_2+1$ and $|\gamma|$ for $d_1+d_2+2 \le l \le N$, where $D = d_1+d_2+d_3+d_4+1, h = (a-(d_1+d_2+1)(1-a))/(D+1)-|\gamma|, m = ((D+1)c_1+(d_1+d_3+1)(1-a))/a$

and $\gamma = c_0 - (d_1+1)(1-a) - (d_1+d_3+1)(a-(d_1+d_2+1)(1-a))/(D+1)$. The statement (XIV) can be found in [**30**].

The conclusions of Theorems 7.4 and 7.5 are illustrated in the white region of Figure 1. It has rather complicated geometry for the set of pairs (a, c) in the white region such that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})$ are not Gabor frames. That set contains some needles (line segments) on the vertical line growing from rational time shift locations and few needle holes (points) on the vertical line located at irrational time shifts by Theorems 7.4 and 7.5.

Combining Theorems 7.1–7.5 gives a complete answer to the *abc*-problem for Gabor systems. The classification diagram of pairs (a, c) in Theorems 7.1–7.5 is presented below:



From Classifications (V)–(IX) and (XII) in Theorems 7.2–7.4, it confirms a conjecture in [**30**, Section 3.3.5]: *If $ab < 1 < bc, ab \notin \mathbb{Q}$ and $c \notin a\mathbb{Q} + \mathbb{Q}/b$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times b\mathbb{Z})$ is a Gabor frame for L^2.* This, together with Classification (IV) in Theorem 7.1 and the shift-invariance, implies that the range of density parameters a, b such that $\mathcal{G}(\chi_I, a\mathbb{Z}\times b\mathbb{Z})$ is a Gabor frame is a dense subset of the open region $\mathcal{U}_c := \{(a, b) : 0 < a < \max(1/b, c)\}$, where c is the length of the interval I.

7.1. Proofs

In this section, we give the proofs of Theorems 7.1–7.5.

Proof of Theorem 7.1. The conclusions in Theorem 7.1 can be found in [**30**, Sections 3.3.1.2, 3.3.1.4, 3.3.1.5, and 3.3.2.1]. We include a short proof for the convenience.

(I): The conclusion follows from the necessary condition (1.2) for a Gabor system to be a Gabor frame.

(II): The sufficiency holds since

$$\begin{aligned}\sum_{\phi\in\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})} |\langle f,\phi\rangle|^2 &= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |\langle f(\cdot+na)\chi_{[0,a)}, e^{-2\pi i m\cdot}\rangle|^2\\ &= \sum_{n\in\mathbb{Z}} \|f(\cdot+na)\chi_{[0,a)}\|_2^2 = \|f\|_2^2,\ f\in L^2,\end{aligned}$$

provided that $a \le 1$.

In the case that $a > 1$, we observe that $\{e^{-2\pi i m \cdot}\chi_{[0,a)} : m \in \mathbb{Z}\}$ is not a frame on $L^2([0,a))$ (the space of square-integrable functions on the interval $[0,a)$), and that

$$\langle f, \chi_{[0,a)}(\cdot - na)e^{-2\pi i m \cdot /b}\rangle = 0$$

for all $m \in \mathbb{Z}$, $0 \neq n \in \mathbb{Z}$ and $f \in L^2$ supported in $[0,a)$. Hence $\mathcal{G}(\chi_{[0,a)}, a\mathbb{Z} \times \mathbb{Z})$ is not a Gabor frame if $a > 1$, and thus the necessity follows.

(III): For any $f \in L^2$,

$$\begin{aligned}\sum_{\phi \in \mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})} |\langle f, \phi\rangle|^2 &= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |\langle f(\cdot + na)\chi_{[0,c)}, e^{-2\pi i m \cdot}\rangle|^2 \\ &= \sum_{n\in\mathbb{Z}} \|f(\cdot + na)\chi_{[0,c)}\|_2^2 \\ &= \int_{\mathbb{R}} |f(x)|^2 \big(\sum_{n\in\mathbb{Z}} \chi_{[0,c)}(x - na)\big)dx.\end{aligned}$$

This together with the observation that

$$\lfloor c/a \rfloor \le \sum_{n\in\mathbb{Z}} \chi_{[0,c)}(x - na) \le \lfloor c/a \rfloor + 1 \quad \text{for all } x \in \mathbb{R},$$

proves that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is a Gabor frame.

(IV): For $a > 1$, the non-frame property for the Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})$ holds by (1.4). Then it suffices to consider $a = 1$. In this case, the infinite matrix $\mathbf{M}_{a,c}(0)$ in (1.5) is a banded bi-infinite Toeplitz matrix $(A(\lambda - \mu))_{\mu,\lambda\in b\mathbb{Z}}$, where $A(\lambda) = 0$ if $\lambda \in \mathbb{Z}\backslash[0,c)$ and $A(\lambda) = 1$ if $\lambda \in \mathbb{Z}\cap[0,c)$. Take $\theta = \exp(-2\pi i/(k_0+1))$ and define $\mathbf{z}_\theta = (\theta^\lambda)_{\lambda\in\mathbb{Z}}$, where $k_0 = \lfloor c/a \rfloor \ge 1$ by our assumption. One may verify that $\mathbf{z}_\theta$ is a bounded sequence with $\mathbf{M}_{a,c}(0)\mathbf{z}_\theta = 0$. Thus $\mathbf{M}_{a,c}(0)$ does not have the ℓ^2-stability. This together Theorem 2.4 proves that $\mathcal{G}(\chi_{[0,c)}, \mathbb{Z} \times \mathbb{Z})$ is not a Gabor frame. □

Proof of Theorem 7.2. (V): By (3.2) and Theorem 2.1, it suffices to prove $\mathcal{S}_{a,c} = \emptyset$, which follows from the second statement of Theorem 3.5. We remark that the conclusion (V) was established in [**30**, Section 3.3.3.2].

(VI): ($\Longrightarrow$) By Theorem 2.1,

$$\mathcal{D}_{a,c} \neq \emptyset, \tag{7.1}$$

which together with the supset property (3.2) implies that $\mathcal{S}_{a,c} \neq \emptyset$. Hence

$$c_0 > 1 - \gcd(\lfloor c \rfloor + 1, a) \tag{7.2}$$

and

$$\mathcal{S}_{a,c} = [-\gcd(\lfloor c \rfloor + 1, a), c_0 - 1) + \gcd(\lfloor c \rfloor + 1, a)\mathbb{Z} \tag{7.3}$$

by Theorem 3.5. Recall from the set $\mathcal{D}_{a,c}$ can be obtained from the maximal invariant set $\mathcal{S}_{a,c}$ given in Theorem 2.3, we have that

$$\mathcal{D}_{a,c} = \mathcal{S}_{a,c} \cap \big(\cup_{\lambda=1}^{\lfloor c \rfloor} (\mathcal{S}_{a,c} - \lambda)\big). \tag{7.4}$$

Combining (7.1), (7.2), (7.3) and (7.4) leads to the necessity.

($\Longleftarrow$) In this case,

$$\mathcal{S}_{a,c} = [-\gcd(\lfloor c \rfloor + 1, a), c_0 - 1) + \gcd(\lfloor c \rfloor + 1, a)\mathbb{Z}$$

by Theorem 3.5; and

$$\mathcal{D}_{a,c} = \mathcal{S}_{a,c} \cap \big(\cup_{\lambda=1}^{\lfloor c \rfloor} (\mathcal{S}_{a,c} - \lambda)\big)$$

by Theorem 2.3. Therefore $\mathcal{D}_{a,c} \neq \emptyset$, which proves the sufficiency by Theorem 2.1.

(VII): The conclusion can be proved by following the arguments used in the proof of the conclusion (VI), except showing $\mathcal{D}_{a,c} = \mathcal{S}_{a,c} = \mathbb{R}$ for $c_0 = 0$, and replacing (7.3) and (7.4) by

$$\mathcal{S}_{a,c} = [c_0, \gcd(\lfloor c \rfloor, a)) + \gcd(\lfloor c \rfloor, a)\mathbb{Z}$$

and

$$\mathcal{D}_{a,c} = \mathcal{S}_{a,c} \cap \big(\cup_{\lambda=1}^{\lfloor c \rfloor -1} (\mathcal{S}_{a,c} - \lambda)\big)$$

for $c_0 > 0$. □

Proof of Theorem 7.3. (VIII): The conclusion follows from the results in [**30**, Section 3.3.3.5, 3.3.3.6 and 3.3.4.3]. We include a different proof here. Suppose on the contrary that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is not a Gabor frame. Then by Theorem 2.1 there exist $t \in \mathbb{R}$ and $(\mathbf{x}(\lambda))_{\lambda \in \mathbb{Z}} \in \mathcal{B}^0$ such that

$$\sum_{\lambda \in \mathbb{Z}} \chi_{[0,c)}(t - \mu + \lambda)\mathbf{x}(\lambda) = 2 \text{ for all } \mu \in a\mathbb{Z}. \tag{7.5}$$

By the assumption $\lfloor c \rfloor = 1$ and $c > 1$, given any $t \in \mathbb{R}$ and $\mu \in a\mathbb{Z}$, the equality $\chi_{[0,c)}(t - \mu + \lambda) = 1$ holds for at most two distinct $\lambda \in \mathbb{Z}$. This together with (7.5) that $\mathbf{x}(\lambda) = 1$ for all $\lambda \in \mathbb{Z}$, and also that

$$t - \mu \notin [c, 2) + \mathbb{Z} \text{ for all } \mu \in a\mathbb{Z}. \tag{7.6}$$

If $a \notin \mathbb{Q}$, then there exists $\mu_0 \in a\mathbb{Z}$ by the density of the set $a\mathbb{Z} + \mathbb{Z}$ in $\mathbb{R}$ such that $t - \mu_0 \in [c, 2) + \mathbb{Z}$, which contradicts to (7.6).

If $a \in \mathbb{Q}$, then $a = p/q$ for some positive coprime integers p and q. Hence

$$t \notin [c, 2) + \mathbb{Z}/q = \mathbb{R},$$

where the first conclusion follows from (7.6) and the equality holds as $2 - c = 1 - c_0 > 1 - a \geq 1/q$ by the assumption $0 < 1 - a < c_0 < a$. This is a contradiction.

(IX): The conclusion follows from Conclusion (v) of Theorem 3.5, the supset property (3.2) and Theorem 2.1.

(X): By Conclusion (vi) of Theorem 3.5, we have that

$$\mathcal{S}_{a,c} = [0, c_0 + a - 1) + a\mathbb{Z}. \tag{7.7}$$

From the assumption on c_1 it follows that $a \in \mathbb{Q}$. We write $a = p/q$ for some coprime integers p and q. Clearly $p \geq 2$ as $1 - a < c_0 < a$. By the assumption that $c_1 = 2a - 1$, we have that $\lfloor c \rfloor + 1 \in p\mathbb{Z}$, which implies that

$$R_{a,c}(t) - t \in a\mathbb{Z} \quad \text{for all } t \in \mathcal{S}_{a,c} = [0, c_0 + a - 1) + a\mathbb{Z}.$$

This together with Theorems 2.1 and 2.3 implies that the Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is a frame of L^2 if and only if $\mathcal{D}_{a,c} = \emptyset$ if and only if

$$([0, c_0 + a - 1) + a\mathbb{Z}) \cap ([0, c_0 + a - 1) + \lambda + a\mathbb{Z}) = \emptyset \quad \text{for all} \quad \lambda \in [1, \lfloor c \rfloor] \cap \mathbb{Z}.$$

Observe that

$$([0, c_0 + a - 1) + a\mathbb{Z}) \cap ([0, c_0 + a - 1) + \lambda + a\mathbb{Z}) = [0, c_0 + a - 1) + a\mathbb{Z} \neq \emptyset$$

for $\lambda = p \in [1, \lfloor c \rfloor] \cap \mathbb{Z}$ provided that $\lfloor c \rfloor \ge p$, and also that

$$([0, c_0 + a - 1) + a\mathbb{Z}) \cap ([0, c_0 + a - 1) + \lambda + a\mathbb{Z}) = [1/q, c_0 + a - 1) + a\mathbb{Z} \neq \emptyset$$

for $\lambda = k \in [1, \lfloor c \rfloor] \cap \mathbb{Z}$ where $1 \le k \le p - 1$ is the unique integer such that $qk - 1 \in p\mathbb{Z}$, provided that $\lfloor c \rfloor + 1 = p$ and $c_0 + a - 1 > 1/q$. Therefore the assumptions that $\lfloor c \rfloor + 1 = p$ and $c_0 + a - 1 \le 1/q$ are necessary for the Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ being a frame of L^2. On the other hand, if $\lfloor c \rfloor + 1 = p$ and $c_0 + a - 1 \le 1/q$, one may verify that

$$\begin{aligned} & ([0, c_0 + a - 1) + a\mathbb{Z}) \cap ([0, c_0 + a - 1) + \lambda + a\mathbb{Z}) \\ = \; & ([0, c_0 + a - 1) + a\mathbb{Z}) \cap ([0, c_0 + a - 1) + k(\lambda)/q + a\mathbb{Z}) = \emptyset \end{aligned}$$

for all $\lambda \in [1, \lfloor c \rfloor] \cap \mathbb{Z}$, where $k(\lambda)$ is the unique integer in $[1, p - 1]$ such that $k(\lambda)/q - \lambda \in a\mathbb{Z}$. Therefore the assumptions that $\lfloor c \rfloor + 1 = p$ and $c_0 + a - 1 \le 1/q$ is also sufficient for the Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ to be a frame for L^2.

(XI) By Conclusion (vii) of Theorem 3.5,

$$\mathcal{S}_{a,c} = [c_0, a) + a\mathbb{Z}. \tag{7.8}$$

Now we can apply similar argument used in the proof of the conclusion (X) of this theorem. From the assumption that $c_1 = 0$, it follows $a = p/q$ for some coprime integers p and q with $p \ge 2$ and $\lfloor c \rfloor \in p\mathbb{Z}$. By (7.8) and Theorems 2.1 and 2.3, we can show that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is a frame of L^2 if and only if $([c_0, a) + a\mathbb{Z}) \cap ([c_0, a) + \lambda + a\mathbb{Z}) = \emptyset$ for all $\lambda \in [1, \lfloor c \rfloor - 1] \cap \mathbb{Z}$. Then the desired necessary condition for the Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ being a frame of L^2 follows from the observation that

$$([c_0, a) + a\mathbb{Z}) \cap ([c_0, a) + p + a\mathbb{Z}) = [c_0, a) + a\mathbb{Z} \neq \emptyset$$

if $\lfloor c \rfloor \ge p + 1$, and that

$$([c_0, a) + a\mathbb{Z}) \cap ([c_0, a) + k + a\mathbb{Z}) = [c_0, a - 1/q) + a\mathbb{Z} \neq \emptyset$$

if $\lfloor c \rfloor = p$ and $a - c_0 > 1/q$ where $1 \le k \le p - 1$ is the unique integer such that $qk + 1 \in p\mathbb{Z}$. The sufficiency for the conditions that $\lfloor c \rfloor = p$ and $a - c_0 \le 1/q$ holds as

$$([c_0, a) + a\mathbb{Z}) \cap ([c_0, a) + \lambda + a\mathbb{Z}) = ([c_0, a) + a\mathbb{Z}) \cap ([c_0, a) - k(\lambda)/q + a\mathbb{Z}) = \emptyset$$

for all $\lambda \in [1, \lfloor c \rfloor] \cap \mathbb{Z}$, where $k(\lambda)$ is the unique integer in $[1, p - 1]$ such that $k(\lambda)/q + \lambda \in a\mathbb{Z}$. □

Proof of Theorem 7.4. (XII): We observe that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is not a Gabor frame if and only if $\mathcal{D}_{a,c} \neq \emptyset$ if and only if $\mathcal{S}_{a,c} \neq \emptyset$ and (3.4) does not hold if and only if the pair (a, c) satisfies (5.7), (5.8), (5.9) and

$$c - (\lfloor c \rfloor + 1)(d_1 + 1)(1 - a) - \lfloor c \rfloor (d_2 + 1)(1 - a) \neq a.$$

In the above argument, the first equivalence holds by Theorem 2.1, the second one follows from (3.2) and Theorem 3.3, and the last one is obtained from Theorem 5.5 and the observation that (3.4) holds if and only if

$$c - (\lfloor c \rfloor + 1)(d_1 + 1)(1 - a) - \lfloor c \rfloor (d_2 + 1)(1 - a) = a$$

as there are d_1 holes of length $1 - a$ in $\mathcal{S}_{a,c} \cap [0, c_0 + a - 1)$ and d_2 holes of length $1 - a$ in $\mathcal{S}_{a,c} \cap [c_0, a)$ by Theorem 5.2. □

PROOF OF THEOREM 7.5. (XIII): By (3.2) and Theorems 2.1 and 3.3, we see that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})$ is not a Gabor frame if and only if $\mathcal{S}_{a,c}\neq\emptyset$ and

$$(\lfloor c\rfloor+1)|\mathcal{S}_{a,c}\cap[0,c_0+a-1)|+\lfloor c\rfloor|\mathcal{S}_{a,c}\cap[c_0,a)|\neq a.$$

For the case that the pair (a,c) satisfies the first condition in Theorem 6.8, it follows from the argument used in the proof of Theorem 6.8 that

$$\mathcal{S}_{a,c}\cap[0,c_0+a-1)=\emptyset$$

and

$$\mathcal{S}_{a,c}\cap[c_0,a)=\cup_{i=0}^{N}[c_0,\gcd(a,c_1))+i\gcd(a,c_1),$$

where $N+1=a/\gcd(a,c_1)$. Hence (3.4) holds if and only if

$$(N+1)\lfloor c\rfloor(\gcd(a,c_1)-c_0)=a$$

if and only if

$$\lfloor c\rfloor(\gcd(a,c_1)-c_0)=\gcd(a,c_1).$$

For the case that the triple (a,c) satisfies the second condition in Theorem 6.8,

$$\mathcal{S}_{a,c}\cap[c_0,a)=\emptyset$$

and

$$\mathcal{S}_{a,c}\cap[0,c_0+a-1)=\cup_{i=0}^{N}[0,\gcd(c_1+1,a)-1+c_0)+i\gcd(c_1+1,a),$$

where $N=a/\gcd(c_1+1,a)-1$. Hence (3.4) holds if and only if

$$(N+1)(\lfloor c\rfloor+1)(\gcd(c_1+1,a)-1+c_0)=a$$

if and only if

$$(\lfloor c\rfloor+1)(\gcd(c_1+1,a)-1+c_0)=\gcd(c_1+1,a).$$

For the case that the pair (a,c) satisfies the third condition in Theorem 6.8, there are d_1+d_3+1 intervals of length h contained in $[0,c_0+1-a)$ and d_2+d_4+1 intervals of length h contained in $[c_0,a)$, where

$$h+|\gamma|=B_d/(D+1)$$

and

$$B_d=a-(d_1+d_3+1)(1-a).$$

Therefore (3.4) holds if and only if

$$(\lfloor c\rfloor+1)(d_1+d_2+1)h+\lfloor c\rfloor(d_2+d_4+1)h=a$$

if and only if

$$h=\frac{a}{(D+1)\lfloor c\rfloor+(d_1+d_3+1)}$$

if and only if

$$\frac{a}{(D+1)\lfloor c\rfloor+(d_1+d_3+1)}+|\gamma|=\frac{B_d}{D+1}.$$

Therefore the conclusion (XIII) holds by Theorem 6.8.

(XIV): This conclusion is given in [**30**, Section 3.3.6.1]. Here is a different proof using the set $\mathcal{D}_{a,c}$. ($\Longrightarrow$) Write $a=p/q$ for some coprime integers p and q. For any $t_0\in\mathcal{D}_{a,\lfloor qc\rfloor/q}\cap\mathbb{Z}/q\neq\emptyset$, there exists $\mathbf{x}=(\mathbf{x}(\lambda))_{\lambda\in\mathbb{Z}}\in\mathcal{B}^0$ such that

$$\sum_{\lambda\in\mathbb{Z}}\chi_{[0,\lfloor qc\rfloor/q)}(t_0-\mu+\lambda)\mathbf{x}(\lambda)=2$$

for all $\mu \in a\mathbb{Z}$. Thus

$$\sum_{\lambda\in\mathbb{Z}} \chi_{[0,c)}(t_0 + c - \lfloor qc\rfloor/q - \mu + \lambda)\mathbf{x}(\lambda) = 2 \quad \text{for all } \mu \in a\mathbb{Z},$$

as

$$\chi_{[0,c)}(t + c - \lfloor qc\rfloor/q) = \chi_{[0,\lfloor qc\rfloor/q)}(t)$$

for all $t \in \mathbb{Z}/q$. This proves that

$$c - \lfloor qc\rfloor/q + \mathcal{D}_{a,\lfloor qc\rfloor/q} \cap \mathbb{Z}/q \subset \mathcal{D}_{a,c}. \tag{7.9}$$

Therefore $\mathcal{G}(\chi_{[0,\lfloor qc\rfloor/q)}, a\mathbb{Z}\times\mathbb{Z})$ is a Gabor frame by (1.25), (7.9), Theorem 2.1, and the assumption that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})$ is a Gabor frame.

Similarly we notice that

$$\mathcal{D}_{a,(\lfloor qc\rfloor+1)/q} \cap \mathbb{Z}/q \subset \mathcal{D}_{a,c} \tag{7.10}$$

because

$$\chi_{[0,c)}(t) = \chi_{[0,(\lfloor qc\rfloor+1)/q)}(t)$$

for all $t \in \mathbb{Z}/q$. Hence $\mathcal{G}(\chi_{[0,(\lfloor qc\rfloor+1)/q)}, a\mathbb{Z}\times\mathbb{Z})$ is a Gabor frame by (1.25), (7.10), Theorem 2.1, and the assumption that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})$ is a Gabor frame.

($\Longleftarrow$) Suppose that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times\mathbb{Z})$ is not a Gabor frame. Then $\mathcal{D}_{a,c} \neq \emptyset$ by Theorem 2.1. Take any $t \in \mathcal{D}_{a,c}$, one may verify that

$$\lfloor qt\rfloor/q \in \mathcal{D}_{a,(\lfloor qc\rfloor+1)/q}$$

if $t - \lfloor qt\rfloor/q > c - \lfloor qc\rfloor/q$, and

$$\lfloor qt\rfloor/q \in \mathcal{D}_{a,\lfloor qc\rfloor/q}$$

otherwise. Therefore either $\mathcal{G}(\chi_{[0,(\lfloor qc\rfloor+1)/q)}, a\mathbb{Z}\times\mathbb{Z})$ or $\mathcal{G}(\chi_{[0,\lfloor qc\rfloor/q)}, a\mathbb{Z}\times\mathbb{Z})$ is not a Gabor frame by Theorem 2.1, which is a contradiction. □

APPENDIX A

Algorithm

In this appendix, we provide a finite-step algorithm to verify whether the Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times b\mathbb{Z})$ is a Gabor frame for any given triple of (a,b,c) of positive numbers.

Given a triple (a,b,c), we divide two cases $ab \notin \mathbb{Q}$ and $ab \in \mathbb{Q}$ to verify whether $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times b\mathbb{Z})$ is a Gabor frame for L^2. First we normalize the frequency-spacing parameter b to 1 by defining $a = ab, c = bc$ and $b = 1$. Set $c_0 = c - \lfloor c \rfloor$ and $c_1 = c - c_0 - \lfloor (c-c_0)/a \rfloor a$. We set $Gabor = 1$ if the Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z}\times b\mathbb{Z})$ is a Gabor frame for L^2 and $Gabor = 0$ otherwise.

Algorithm for $a \notin \mathbb{Q}$, Part I, based on Theorems 7.1 and 7.2:

```
if a > c, Gabor = 0;
elseif a = c
    if a ≤ 1, Gabor = 1;
    else, Gabor = 0; end
else % a < c
    if a ≥ 1, Gabor = 0;
    elseif c ≤ 1, Gabor = 1;
    % The value of Gabor is obtained from Theorem 7.1.
    else   % 0 < a < 1 < c
         If c_0 ≥ a, Gabor = 1;
         elseif c_0 ≤ 1 − a,    % 0 < a < 1 < c, c_0 < a
              if c_0 = 0, Gabor = 0;
              else, Gabor = 1; end
              % The value of Gabor is obtained from Theorem 7.2.
         else, do algorithm part 2;
         end   % 0 < a < 1 < c and 1 − a < c_0 < a
    end
end
```

Algorithm for $a \notin \mathbb{Q}$, Part II, based on Theorems 3.3, 5.2, 7.3 and 7.4:

```
if ⌊c⌋ = 1, Gabor = 1;
else   % 0 < a < 1 < c, 1 − a < c_0 < a and ⌊c⌋ ≥ 2
    if c_1 > 2a − 1, Gabor = 1;
    % c_1 ≠ 2a − 1 and c_1 ≠ 0 as a ∉ Q
    else   % 0 < a < 1 < c, 1 − a < c_0 < a, ⌊c⌋ ≥ 2, 0 < c_1 < 2a − 1.
         s1 = c_0 + a − 1; s2 = a − c_0;
         Hole = c_1; N = ⌊a/(1 − a)⌋;
         for n = 1 : N
```

```
                if Hole < c_0+2*a-2, Hole = Hole+⌊c⌋+1-⌊(Hole+
                ⌊c⌋ + 1)/a⌋ * a and s1 = s1 - 1 + a;
                elseif Hole < c_0 + a - 1, s1 = -a and break;
                elseif Hole = c_0 + a - 1, break;
                elseif Hole < c_0, s2 = -a and break;
                elseif Hole < 2 * a - 1, Hole = Hole + ⌊c⌋ - ⌊(Hole +
                ⌊c⌋)/a⌋ * a and s2 = s2 - 1 + a;
                else, s2 = -a and break
                end
                % s1 = |S_{a,c} ∩ [0, c_0 + a - 1)| and s2 = |S_{a,c} ∩ [c_0, a)| if
                S_{a,c} ≠ ∅; and either s1 < 0 or s2 < 0 if S_{a,c} = ∅
                by Theorem 5.2
            m = (⌊c⌋) + 1) * s1 + ⌊c⌋ * s2;
            if s1 < 0, Gabor = 1;
            elseif s2 < 0, Gabor = 1;
            elseif m = a, Gabor = 1;   % by Theorem 3.3
            else, Gabor = 0; end
        end
    end
```

Now consider the algorithm for $a \in \mathbb{Q}$. Write $a = p/q$ for some coprime integers p and q. Recall that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ is a Gabor frame if and only if both $\mathcal{G}(\chi_{[0,\lfloor qc \rfloor/q)}, a\mathbb{Z} \times \mathbb{Z})$ and $\mathcal{G}(\chi_{[0,(\lfloor qc \rfloor+1)/q)}, a\mathbb{Z} \times \mathbb{Z})$ are Gabor frames [**30**]. So in the following algorithm, we assume that $c \in \mathbb{Z}/q$.

Algorithm for $a = p/q \in \mathbb{Q}$ and $c \in \mathbb{Z}/q$, Part III, based on Theorems 7.1 and 7.2:

```
    if a > c, Gabor = 0;
    elseif a = c
        if a ≤ 1, Gabor = 1;
        else, Gabor = 0; end
    else    % a < c
        if a ≥ 1, Gabor = 0;
        elseif c ≤ 1, Gabor = 1;
        % The value of Gabor is obtained from Theorem 7.1.
        else    % 0 < a < 1 < c
            If c_0 ≥ a,
                if c_0 > 1-gcd(⌊c⌋+1, p)/q and gcd(⌊c⌋+1, p) ≠ ⌊c⌋+1,
                Gabor = 0;
                elseif c_0 > 1 - gcd(⌊c⌋ + 1, p)/q + 1/q and gcd(⌊c⌋ +
                1, p) = ⌊c⌋ + 1, Gabor = 0;
                else, Gabor = 1; end
            elseif c_0 ≤ 1 - a    % 0 < a < 1 < c, c_0 < a
                if c_0 = 0, Gabor = 0;
                elseif c_0 < gcd(⌊c⌋, p)/q and gcd(⌊c⌋, p) ≠ ⌊c⌋, Gabor =
                0;
                elseif c_0 < gcd(⌊c⌋, p)/q - 1/q and gcd(⌊c⌋, p) = ⌊c⌋,
                Gabor = 0;
                else, Gabor = 1; end
```

```
            % The value of Gabor is obtained from Theorem 7.2.
        else do algorithm part 4;
        end
    end
end
```

Algorithm for $a = p/q \in \mathbb{Q}$ and $c \in \mathbb{Z}/q$, Part IV, based on Theorems 3.3, 6.3, 6.4, 6.5,7.3 and 7.4:

```
if ⌊c⌋ = 1, Gabor = 1;
else    % 0 < a < 1 < c, 1 − a < c_0 < a and ⌊c⌋ ≥ 2
    if c_1 > 2a − 1, Gabor = 1;
    elseif c_1 = 2a − 1
        if c_0 ≤ 1 − a + 1/q and ⌊c⌋ + 1 = p, Gabor = 0;
        else Gabor = 1; end
    elseif c_1 = 0
        if c_0 ≤ a − 1/q and ⌊c⌋ = p, Gabor = 0;
        else Gabor = 1; end
    else    % 0 < a < 1 < c, 1 − a < c_0 < a, ⌊c⌋ ≥ 2, 0 < c_1 < 2a − 1.
        s1 = c_0 + a − 1; s2 = a − c_0;
        Hole1 = c_1; Hole2 = c_1 + 1 − a; D = p;
        for n = 1 : D + 1
            if Hole1 < c_0 + a − 1, Hole2 = min(Hole2, c_0 + a − 1);
            holelength = Hole2 − Hole1; Hole1 = Hole1 + ⌊c⌋ + 1 −
            ⌊(Hole1 + ⌊c⌋ + 1)/a⌋a; Hole2 = Hole1 + holelength;
            and s1 = s1 − holelength;
            elseif Hole2 ≤ c_0, break
            elseif Hole2 ≤ a, Hole1 = max(Hole1, c_0); holelength =
            Hole2 − Hole1; Hole1 = Hole1 + ⌊c⌋ − ⌊(Hole1 + ⌊c⌋)/a⌋a;
            Hole2 = Hole1 + holelength; and s2 = s2 − holelength;
            else, s1 = −a and break;
            end
            % s1 = |S_{a,c} ∩ [0, c_0 + a − 1)| and s2 = |S_{a,c} ∩ [c_0, a)| if
            S_{a,c} ≠ ∅; and s1 < 0 if S_{a,c} = ∅ by Theorems 6.3,
            6.4 and 6.5
        end
        m = (⌊c⌋) + 1) ∗ s1 + ⌊c⌋) ∗ s2;
        if s1 < 0, Gabor = 1;
        elseif s2 < 0, Gabor = 1;
        elseif m = a, Gabor = 1;    % by Theorem 3.3
        else, Gabor = 0;
        end
    end
end
```

APPENDIX B

Uniform sampling of signals in a shift-invariant space

An interesting problem in sampling in shift-invariant spaces is to identify generators ϕ and sampling-shift lattices $a\mathbb{Z} \times b\mathbb{Z}$ such that any signal f in the shift-invariant space

$$V_2(\phi, b\mathbb{Z}) := \Big\{ \sum_{\lambda \in b\mathbb{Z}} d(\lambda)\phi(t-\lambda) : \sum_{\lambda \in b\mathbb{Z}} |d(\lambda)|^2 < \infty \Big\} \tag{B.1}$$

can be stably recovered from its equally-spaced samples $f(t_0 + \mu), \mu \in a\mathbb{Z}$, for *arbitrary* initial sampling position t_0, i.e., there exist positive constants A and B such that

$$A\|f\|_2 \le \Big(\sum_{\mu \in a\mathbb{Z}} |f(t_0+\mu)|^2 \Big)^{1/2} \le B\|f\|_2 \tag{B.2}$$

for all $f \in V_2(\phi, b\mathbb{Z})$ and $t_0 \in \mathbb{R}$. For fixed initial sampling position t_0, the stability requirement (B.2) is well studied, see [**2, 4, 44, 46, 47, 49**]. On the other hand, for arbitrary initial sampling position t_0 it is known only for few generators ϕ. For instance, the classical Whittaker-Shannon-Kotel'nikov sampling theorem states that (B.2) holds for signals bandlimited to $[-\sigma, \sigma]$ if and only if $a \le b = \pi/\sigma$. For the uniform spline generator $\underbrace{\chi_{[0,b)} * \cdots * \chi_{[0,b)}}_{n \text{ times}}$, obtained by convoluting the characteristic function on $[0, b)$ for n times, (B.2) holds if and only if $a < b$ [**1, 42, 46**]. In this appendix, we consider the range problem of sampling-shift pairs (a, b) for any given generator χ_I, the characteristic function on an interval I, such that the stability requirement (B.2) holds.

We say that $\{\phi(\cdot - \lambda) : \lambda \in b\mathbb{Z}\}$ is a *Riesz basis* for the shift-invariant space $V_2(\phi, b\mathbb{Z})$ if there exist positive constants A and B such that

$$A\Big(\sum_{\lambda \in b\mathbb{Z}} |d(\lambda)|^2 \Big)^{1/2} \le \Big\| \sum_{\lambda \in b\mathbb{Z}} d(\lambda)\phi(\cdot - \lambda) \Big\|_2 \le B\Big(\sum_{\lambda \in b\mathbb{Z}} |d(\lambda)|^2 \Big)^{1/2} \tag{B.3}$$

for all square-summable sequences $(d(\lambda))_{\lambda \in b\mathbb{Z}}$. For an interval $I = [d, c+d)$, $\{\chi_I(\cdot - \lambda) : \lambda \in b\mathbb{Z}\}$ is a Riesz basis for the shift-invariant space $V_2(\chi_I, b\mathbb{Z})$ except that $2 \le c/b \in \mathbb{Z}$. Therefore except that $2 \le c/b \in \mathbb{Z}$, one may easily verify that any signal f in $V_2(\chi_I, b\mathbb{Z})$ can be stably recovered from its equally-spaced samples $f(t_0+\mu), \mu \in a\mathbb{Z}$, for any initial sampling position t_0 if and only if infinite matrices $\mathbf{M}_{a/b,c/b}(t), t \in \mathbb{R}$, in (1.5) have the uniform stability property (2.4), c.f. [**2, 47, 49**]. This together with the characterization of frame property of the Gabor system $\mathcal{G}(\chi_I, a\mathbb{Z} \times \mathbb{Z})$ in [**38**] leads to the following equivalence between our sampling problem associated with the box generator χ_I and the *abc*-problem for Gabor systems.

PROPOSITION B.1. *Let $a, b > 0$ and I be an interval with length $c > 0$. Except that $2 \le c/b \in \mathbb{Z}$, the following two statements are equivalent.*

(i) *Any signal f in the shift-invariant space $V_2(\chi_I, b\mathbb{Z})$ can be stably recovered from equally-spaced samples $f(t_0 + \mu), \mu \in a\mathbb{Z}$, for arbitrary initial sampling position $t_0 \in \mathbb{R}$.*

(ii) *$\mathcal{G}(\chi_I, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame for L^2.*

If $I = [d, c + d)$ with $2 \le c/b \in \mathbb{Z}$, then the shift-invariant space $V_2(\chi_I, b\mathbb{Z})$ is not closed in L^2, but its closure is the shift-invariant space generated by $\chi_{I'}$ where $I' = [d, b + d)$. Therefore for the case that $I = [d, c + d)$ with $2 \le c/b \in \mathbb{Z}$, any signal f in $V_2(\chi_I, b\mathbb{Z})$ can be stably recovered from equally-spaced samples $f(t_0 + \mu), \mu \in a\mathbb{Z}$, for any initial sampling position $t_0 \in \mathbb{R}$ if and only if any signal f in $V_2(\chi_{[d,b+d)}, b\mathbb{Z})$ can be stably recovered from equally-spaced samples $f(t_0 + \mu), \mu \in a\mathbb{Z}$, for any initial sampling position $t_0 \in \mathbb{R}$ if and only if $a \le b$. This together with Theorems 7.1–7.5 and Proposition B.1 provides the full classification of sampling-shift lattices $a\mathbb{Z} \times b\mathbb{Z}$ such that any signal f in $V_2(\chi_I, b\mathbb{Z})$ can be stably recovered from equally-spaced samples $f(t_0 + \mu), \mu \in a\mathbb{Z}$, for any initial sampling position $t_0 \in \mathbb{R}$.

Our results indicate that it is almost equivalent to the *abc*-problem for Gabor systems, and hence geometry of the range of sampling-shift parameters could be very complicated. We remark that two statements in Proposition B.1 are not equivalent for the case that $2 \le c/b \in \mathbb{Z}$ and $a \le b$. The reason is that under that assumption on the triple (a, b, c), $\mathcal{G}(\chi_I, a\mathbb{Z} \times \mathbb{Z}/b)$ is not a Gabor frame by Theorems 7.1 and 7.2, while any signal f in $V_2(\chi_I, b\mathbb{Z})$ can be stably recovered from equally-spaced samples $f(t_0 + \mu), \mu \in a\mathbb{Z}$, for any initial sampling position $t_0 \in \mathbb{R}$.

Oversampling, i.e., $a < b$, helps for perfect reconstruction of band-limited signals and spline signals from their equally-spaced samples [**1, 2, 47**]. Our results indicate that oversampling does **not** always implies the stability of sampling and reconstruction process for signals in the shift-invariant space $V_2(\phi, b\mathbb{Z})$.

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