LINEAR DEPENDENCE OF THE SHIFTS OF VECTOR-VALUED COMPACTLY SUPPORTED DISTRIBUTIONS

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Abstract. For any vector-valued compactly supported distribution $F = (f_1, \ldots, f_N)^T$, let $i(F)$ be the space of all sequences of the form $(\ll F(\cdot + j), h \gg)_{j\in\mathbf{Z}^d}$, where h are compactly supported C^∞ functions, and let $\mathcal{K}(F)$ be the space of all linear dependence of the shifts of F, i.e.,

$$\mathcal{K}(F) = \Big\{(w(j))_{j\in\mathbf{Z}^d} : \ \textstyle\sum_{j\in\mathbf{Z}^d} w(j)^T F(\cdot - j) \equiv 0\Big\}.$$

The shift-invariant sequence space $i(F)$ is finitely generated and is spanned by $(F(x+j))_{j\in\mathbf{Z}^d}, x \in \mathbf{R}^d$ when F is continuous, and then $i(F)$ is easy to handle than $\mathcal{K}(F)$. In this paper, we show that $i(F)$ is the whole space of finitely supported sequences if and only if F has linear independent shifts, and that $\mathcal{K}(F)$ is the annihilator of $i(F)$. We also provide some methods, especially for refinable distributions F, to compute the shift-invariant space $i(F)$. Finally we apply the shift-invariant space $i(F)$ to establish the self-adaptive stability lemma for any compactly supported L^p function F, and to give a necessary and sufficient condition on the dependent ideal $i(F)$ for which a compactly supported (refinable) distribution can be decomposed as finite linear combination of the shifts of some compactly supported (refinable) distributions having linear independent shifts.

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1. Introduction

Let $\ell := \ell(\mathbf{Z}^d)$ be the space of all sequences on $\mathbf{Z}^d$, and $\ell_0 := \ell_0(\mathbf{Z}^d)$ be the space of all sequences $(d(j))_{j\in\mathbf{Z}^d}$ on $\mathbf{Z}^d$ with finite support, i.e., $d(j) = 0$ for all but finitely many $j \in \mathbf{Z}^d$. Denote their N copies by $(\ell)^N$ and $(\ell_0)^N$ respectively. Define *shift operators* $\tau_k, k \in \mathbf{Z}^d$, on the space $(\ell)^N$ by

$$\tau_k : \ (w(j))_{j\in\mathbf{Z}^d} \longmapsto (w(j-k))_{j\in\mathbf{Z}^d}.$$

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A linear space V of sequences is said to be *shift-invariant* if it is invariant under the shift operators $\tau_k, k \in \mathbf{Z}^d$. In this paper, a shift-invariant linear space of $(\ell_0)^N$ is said to be an *ideal*. The reason for the terminology of "ideal" is that for $N = 1$, a shift-invariant linear subspace of ℓ_0 is really an ideal of the ring ℓ_0 in algebraic terminology.

For any $W = (w(j))_{j\in\mathbf{Z}^d} \in (\ell)^N$ and $D = (d(j))_{j\in\mathbf{Z}^d} \in (\ell_0)^N$, define their action

$$(\ell)^N \times (\ell_0)^N \ni (W, D) \longmapsto D(W) \equiv W(D) \in \mathbf{C}$$

by

$$D(W) := \sum_{j\in\mathbf{Z}^d} d(j)^T w(-j) = \sum_{j\in\mathbf{Z}^d} w(j)^T d(-j) =: W(D).$$

Here and hereafter, A^T is the transpose of a vector (matrix) A. For a linear subspace $\mathcal{I}$ of $(\ell_0)^N$, define its *annihilator* $\mathcal{I}_\perp$ by

$$\mathcal{I}_\perp := \left\{W \in (\ell)^N : \ W(D) = 0 \quad \forall\ D \in \mathcal{I}\right\}$$

([3, 4]). Given a compactly supported distribution $F = (f_1, \ldots, f_N)^T$ on $\mathbf{R}^d$, define an $(\ell_0)^N$-valued distribution $\mathcal{L}(F)$ by

$$\mathcal{L}(F) := (F(\cdot + j))_{j\in\mathbf{Z}^d},$$

and a shift-invariant subspace $\mathcal{K}(F)$ of $(\ell)^N$ by

$$\mathcal{K}(F) := \left\{W \in (\ell)^N : \ W(\mathcal{L}(F)) \equiv 0\right\}.$$

The space $\mathcal{K}(F)$ contains all sequences of the linear dependence of the shifts of F, which is crucial in the study of redundancy of the system generated by the shifts of finitely many compactly supported distributions ([18]). We say that F has, or $f_1, \ldots, f_N$ have, *linear independent shifts* if $\mathcal{K}(F) = \{0\}$. The linear independent shifts are well-studied, and there is a long list of publications on the various characterizations and applications (see for instance [2, 9, 13, 16, 20, 23] and the survey paper [18]). In this paper, we introduce a sequence space to study the redundancy of the system generated by the shifts of compactly supported distributions, and apply it to establish a self-adaptive lemma and a decomposition of compactly supported (refinable) distributions.

Let us start from studying the dependent relation of a compactly supported distribution F, which is finite combinations of the shifts of some compactly supported distributions having linear independent shifts,

$$F = \sum_{i=1}^{M} \sum_{j\in\mathbf{Z}^d} p_i(j)\phi_i(\cdot - j), \tag{1.1}$$

where $\phi_1, \ldots, \phi_M$ are compactly supported distributions having linear independent shifts, and where $P_i = (p_i(j))_{j\in\mathbf{Z}^d}, 1 \le i \le M$, are sequences in $(\ell_0)^N$. We remark that the above decomposition does not restrict us because any compactly supported distribution has the decomposition of the form (1.1) theoretically.

Theorem 1.1. ([1]) *Let $F = (f_1, \ldots, f_N)^T$ be a vector-valued compactly supported distribution. Then F has the decomposition of the form (1.1). Moreover, the distributions $\phi_1, \ldots, \phi_M$ can be chosen to be finite linear combination of $hf_i(\cdot - j)$, where $1 \le i \le N, j \in \mathbf{Z}^d$, and h are compactly supported C^∞ functions.*

For the vector-valued compactly supported distribution F having the form (1.1), one may easily verify that

$$\mathcal{L}(F) = \sum_{i=1}^{M} \sum_{k\in\mathbf{Z}^d} \tau_k(P_i)\ \mathcal{L}(\phi_i)(\cdot + k), \tag{1.2}$$

which, together with the linear independent shifts of $\phi_1, \ldots, \phi_M$, imply that

$$W \in \mathcal{K}(F) \quad \text{if and only if} \quad W(\tau_k P_i) = 0 \quad \forall\ k \in \mathbf{Z}^d \ \text{ and } \ 1 \le i \le M. \tag{1.3}$$

This leads to the following important observations: $\mathcal{K}(F)$ is the annihilator of $\mathcal{I}(P_1, \ldots, P_M)$, i.e.,

$$\mathcal{K}(F) = \mathcal{I}(P_1, \ldots, P_M)_\perp, \tag{1.4}$$

where

$$\mathcal{I}(P_1, \ldots, P_M) := \text{spanned by } \ \tau_k P_i,\ 1 \le i \le M \ \text{ and } \ k \in \mathbf{Z}^d \tag{1.5}$$

(see also [13, Lemma 3.1] and [17, Proposition 3.2.7] for related results).

For the sequences $P_i = (p_i(j))_{j\in\mathbf{Z}^d}, 1 \le i \le M$, in (1.1), define $P_i(z) = \sum_{j\in\mathbf{Z}^d} p_i(j)z^{-j}, 1 \le i \le M$, and let z_0 be any common root of the Laurent polynomials $v^T P_1(z), \ldots, v^T P_M(z)$ in $(\mathbf{C}\backslash\{0\})^d$, where $v \in \mathbf{R}^N\backslash\{0\}$, then $(z_0^j v)_{j\in\mathbf{Z}^d} \in \mathcal{I}(P_1, \ldots, P_M)_\perp$ by direct computation, and hence $(z_0^j v)_{j\in\mathbf{Z}^d} \in \mathcal{K}(F)$ by (1.4). It is known that the set of roots of Laurent polynomials in high dimensions has complicated structure. So we believe that for a compactly supported distribution F on high dimensions, the shift-invariant space $\mathcal{K}(F)$ would have more complicated structure.

For a shift-invariant linear subspace $\mathcal{I}$ of $(\ell_0)^N$, we say that $\mathcal{I}$ is *generated by a subset* $\mathcal{J}$ *of* $(\ell_0)^N$, or $\mathcal{J}$ is a *generator* of $\mathcal{I}$, if $\mathcal{I}$ is the minimal shift-invariant linear subspace of $(\ell_0)^N$ containing all sequences in $\mathcal{J}$, and that $\mathcal{I}$ is *finitely generated* if there is a generator with finite cardinality. By Proposition A.1 in the appendix, every shift-invariant linear subspace of $(\ell_0)^N$ is finitely generated and closed in the usual topology of $(\ell_0)^N$.

Clearly, the shift-invariant space $\mathcal{I}(P_1,\ldots,P_M)$ in (1.5) is generated by $\{P_1,\ldots,P_M\}$. So the shift-invariant space $\mathcal{I}(P_1,\ldots,P_M)$ has simpler structure, and is easier to be handled than $\mathcal{K}(F)$. Moreover, in some applications such as certain quantity estimate of $D(\mathcal{L}(F))$ and decomposition of a refinable distribution, the shift-invariant space $\mathcal{I}(P_1,\ldots,P_M)$ is more useful than the shift-invariant space $\mathcal{K}(F)$ (see Theorems 4.2 and 5.3 for details).

Now let us introduce the problems considered in this paper. First we observe that the left hand side of (1.4) is independent of the decomposition (1.1). This arises the following problem naturally:

Problem 1 *Does the shift-invariant space* $\mathcal{I}(P_1,\ldots,P_M)$ *in (1.1) depend on* F *only?*

In this paper, an affirmative answer to Problem 1 is given (see Theorem 2.1 for detail). So we call the shift-invariant space $\mathcal{I}(P_1,\ldots,P_m)$ as the *dependent ideal* of F. As an application, we show that F has linear independent shifts if and only if $\mathcal{I}(P_1,\ldots,P_M)=(\ell_0)^N$ (Theorem 2.2). This gives a new time-domain characterization to the linear independent shifts of a vector-valued compactly supported distribution.

Given any compactly supported distribution F, the decomposition (1.1) in [1] is constructive, and the distributions $\phi_1,\ldots,\phi_M$ in the decomposition (1.1) have the same "smoothness" as $f_1,\ldots,f_N$ have, but the procedure to obtain that decomposition is complicated. So for a compactly supported distribution F, it is very necessary to find alternative ways to compute the shift-invariant space $\mathcal{I}(P_1,\ldots,P_M)$. This inspires us to consider the following problem:

Problem 2 *How to compute the shift-invariant space* $\mathcal{I}(P_1,\ldots,P_M)$ *for a given distribution* F*?*

In this paper, several approaches are provided for that computation (see Theorems 2.1, 3.1, 3.3, 3.2, 3.4 and 3.5 for details). The decomposition (1.1), the representation (3.3), and the linear space (3.5) is not new and has been used in the study of the existence of upper shift-invariant spaces, linear independent shifts, (locally) linear independent shifts in different situations(see for instance [1, 9, 11, 13, 20]). The advance made in this paper is that we show that the dependent ideal connects those techniques appeared and were used in different situations. We remark that for the case that F is continuous, the shift-invariant space $\mathcal{I}(P_1, \ldots, P_M)$ is spanned by $\mathcal{L}(F)(x), x \in \mathbf{R}^d$ (see Theorem 3.2). If, additionally, F is refinable, then $\mathcal{I}(P_1, \ldots, P_M)$ is a minimal subspace of $(\ell_0)^N$ which is invariant under operators $B_k, k \in \mathbf{Z}^d$, and which contains $\ll \mathcal{L}(F), G \gg$ (see Theorem 3.5 for details). A shift-invariant space generated by (but not containing) the vector $\ll \mathcal{L}(F), G \gg$ for some smooth refinable function G has been used in [15] to give a complete characterization of the L^p smoothness of compactly supported refinable distributions without stable assumption.

Finally we consider the applications of the shift-invariant space $i(F)$:

Problem 3 *Is the shift-invariant space $\mathcal{I}(P_1, \ldots, P_M)$ applicable?*

As the first application of the shift-invariant space $\mathcal{I}(P_1, \ldots, P_M)$, we establish a self-adaptive stability lemma (Theorem 4.2). The self-adaptive stability lemma plays an important role in the study of the convergence of cascade algorithm without linear independence assumption to the initial, the smoothness of refinable distributions with linear dependent shifts, and compactly supported solutions of nonhomogeneous refinement equations in Sobolev spaces with the nonhomogeneous term having linear dependent shifts([21]).

The second application given in this paper is to give a necessary and sufficient condition for the existence of the decomposition of the form (1.1) with $\phi_1, \ldots, \phi_M$ being (infinite) linear combinations of the shifts of F. For those functions $\phi_1, \ldots, \phi_M$, the shift-invariant space generated by $\phi_1, \ldots, \phi_M$ is the same as the one generated by components of F. Moreover if F is refinable, then $\phi_1, \ldots, \phi_M$ can be so chosen to be refinable too (see Theorem 5.3). This generalizes a result by Jia ([10]) for refinable distributions on the real line to higher dimensions.

The application of the dependent ideal to characterize stable shifts of globally supported distributions is given in [19].

The paper is organized as follows. In Section 2, we show that the shift-invariant space $\mathcal{I}(P_1, \ldots, P_M)$ depends on F only. In Section 3, several approaches are provided to compute the shift-invariant sequence space $\mathcal{I}(P_1, \ldots, P_M)$, especially for a refinable distribution. Sections 4 and 5 are devoted to the self-adaptive stability lemma and decompositions of a compactly supported (refinable) distribution. All proofs are gathered in Section 6. Some elementary properties of shift-invariant subspaces of $(\ell_0)^N$ are given in the appendix.

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2. Invariant of Decomposition and Linear Independent Shifts

In this section, we show that the shift-invariant space $\mathcal{I}(P_1, \ldots, P_M)$ is independent of the decomposition of the form (1.1), and also give a new time-domain characterization to linear independent shifts.

Let $\mathcal{D}$ be the space of all compactly supported C^∞ functions. Denote the action between a distribution and a compactly supported C^∞ function by $\ll \cdot, \cdot \gg$. Given any vector-valued compactly supported distribution $F = (f_1, \ldots, f_N)^T$, we define the *semi-convolution* operator $F*'$ on $(\ell)^N$ by

$$F *' W := W(\mathcal{L}(F)),$$

and let

$$i(F) := \{\ll \mathcal{L}(F), h \gg: \ h \in \mathcal{D}\}. \tag{2.1}$$

By direct computation, we have

$$W(\ll \mathcal{L}(F), h \gg) = \ll F *' W, h \gg .$$

So the space $i(F)$ can be thought of as the space of all continuous linear forms on $(\ell)^N$ induced by the semi-convolution $F*'$ and the continuous linear form $h \in \mathcal{D}$ on the space of distributions. One may verify that

$$\tau_k(\ll \mathcal{L}(F), h \gg) = \ll \mathcal{L}(F), h(\cdot + k) \gg,$$

where $W \in (\ell)^N, h \in \mathcal{D}$ and $k \in \mathbf{Z}^d$. Thus $i(F)$ is a shift-invariant subspace of $(\ell_0)^N$. In this paper, we show that for any distribution F having the form (1.1), $i(F)$ is just the shift-invariant sequence space generated by $P_1, \ldots, P_M$.

Theorem 2.1. *Let $F = (f_1, \ldots, f_N)^T$ be a vector-valued compactly supported distribution, and let $i(F)$ be defined by (2.1). If F has the decomposition (1.1) with $P_i = (p_i(j))_{j \in \mathbf{Z}^d} \in (\ell_0)^N, 1 \le i \le M$, and $\phi_1, \ldots, \phi_M$ having linear independent shifts, then*

$$\mathcal{I}(P_1, \ldots, P_M) = i(F),$$

where $\mathcal{I}(P_1, \ldots, P_M)$ is the shift-invariant sequence space generated by $\{P_1, \ldots, P_M\}$.

This gives an affirmative answer to Problem 1 because $i(F)$ is clearly independent of the decomposition of the form (1.1).

In this paper, the shift-invariant space $i(F)$ is said to be the *dependent ideal of* F. The reason to use the terminology of "dependent ideal" is that $\mathcal{K}(F)$ contains all linear dependence of the shifts of F, and that $\mathcal{K}(F)$ is the annihilator of $i(F)$ by (1.4), Theorem 1.1, and Theorem 2.1.

As a consequence of (1.4) and Theorem 2.1, we have the following new time-domain characterization of the linear independent shifts.

Theorem 2.2. *Let $F = (f_1, \ldots, f_N)^T$ be compactly supported. Then F has linear independent shifts if and only if $i(F) = (\ell_0)^N$.*

Similar time-domain characterization for the linear independent shifts was established in [9, 20]. In particular, they used the common roots of the Laurent polynomials with coefficient vectors in a time-domain space (similar to the one in (3.5)). When F is continuous, the characterization given in [9, 20] are equivalent to the one in Theorem 2.2 since the shift-invariant sequence space containing the finite dimensional space in the above two references is just the dependent ideal $i(F)$ in this paper.

3. Dependent Ideal

In this section, we provide several methods to compute dependent ideals $i(F)$.

A linear space V of distributions is said to be *shift-invariant* if it is invariant under the shift operators $f \longmapsto f(\cdot - j), j \in \mathbf{Z}^d$. For a compactly supported distribution $F = (f_1, \ldots, f_N)^T$, let

$$V(F) \equiv V(f_1, \ldots, f_N) := \{W(\mathcal{L}(F)) : \; W \in (\ell)^N\}.$$

Clearly $V(F)$ is shift-invariant. The space $V(F)$ is usually called the *shift-invariant space generated by* F, or *by* $f_1, \ldots, f_N$.

As mentioned early, the procedure to decompose a compactly supported distribution into the form (1.1) in [1] is universal but complicated. For the case that F is a compactly supported integrable function, one may construct a decomposition of the form (1.1) in a simple way. Let $V_|$ be the space of the restriction of functions in $V(F)$ on $[0,1]^d$. Then the space $V_|$ is finite dimensional, and hence there exists an M-dimensional basis $\phi_1, \ldots, \phi_M$ of $V_|$, where $M = \dim V_|$ is the dimension of the space $V_|$. From the above construction, the functions $\phi_1, \dots, \phi_M$ have linear independent shifts. Moreover, there exist sequences $P_i = (p_i(j))_{j\in\mathbf{Z}} \in (\ell_0)^N, 1 \le i \le M$, such that

$$F = \sum_{i=1}^{M} \sum_{j\in\mathbf{Z}^d} p_i(j)\phi_i(\cdot - j). \tag{3.1}$$

In the decomposition (3.1) for the integrable function F, we remark that the restriction of functions in F to unit cubes $j + [0,1]^d, j \in \mathbf{Z}^d$, are used, but such a restriction procedure cannot be generalized in a straightforward way to distributions that are not generated by functions.

By (3.1) and supp $\phi_1, \dots,$ supp $\phi_M \subset [0,1]^d$, we have the following three observations:

$$\mathcal{L}(F) = \sum_{i=1}^{M} P_i\phi_i \quad \text{on } [0,1]^d; \tag{3.2}$$

the family of sequences $P_1, \dots, P_M$ is unique by the linear independence of $\phi_1, \dots, \phi_M$ on $[0,1]^d$; and $\{P_1, \dots, P_M\}$ is a generator of the dependent ideal $i(F)$ by (3.1) and Theorem 2.1. This inspires us to consider similar decomposition to (3.2) on some open set for any compactly supported distribution F, and then to find a generator of the dependent ideal $i(F)$.

For a compactly supported distribution F and a bounded open set A, denote the space of the restriction of distributions in $V(F)$ on A by $V(F)|_A$, i.e.,

$$V(F)|_A := \{f|_A : \ f \in V(F)\}.$$

Then $V(F)|_A$ is a finite dimensional linear space, and hence there exist compactly supported distributions $\psi_i \in V(F), 1 \le i \le \dim V(F)|_A$, such that $\{\psi_i|_A : 1 \le i \le \dim V(F)|_A\}$ is a basis of $V(F)|_A$. We remark that not like the functions $\phi_1, \dots, \phi_M$ in (3.1), the above distributions $\psi_i, 1 \le i \le \dim V(F)|_A$, are not supported in A and have linear dependent shifts in general. But from the construction of $\psi_i, 1 \le i \le \dim V(F)|_A$, the shift-invariance of the space $V(F)$, and the boundedness of the open set A, there still exist sequences $E_i \in (\ell_0)^N, 1 \le i \le \dim V(F)|_A$, such that

$$\mathcal{L}(F) = \sum_{i=1}^{\dim V(F)|_A} E_i\psi_i \quad \text{on} \quad A. \tag{3.3}$$

In this paper, we use the above decomposition as an approach to compute the dependent ideal $i(F)$. We remark that the decomposition of a distribution into the form (3.3) has been used in studying the shift-invariant space $V(F)$ and the linear independent shifts of F (see for instance [1, 11, 13]).

Let $i(A,F)$ be the shift-invariant space generated by $E_i, 1 \le i \le \dim V(F)|_A$, in (3.3), i.e.,

$$i(A,F) := \text{spanned by } \ \tau_k E_i, \ 1 \le i \le \dim V(F)|_A \ \text{ and } \ k \in \mathbf{Z}^d.$$

Then we have

Theorem 3.1. *Let F be a vector-valued compactly supported distribution on $\mathbf{R}^d$, and A be a bounded open set with $A + \mathbf{Z}^d = \mathbf{R}^d$. Then*

$$i(A,F) = i(F).$$

From Theorem 3.1, we see that the space $i(F)$ is the "minimal" shift-invariant space in the sense that $i(F) \subset i(A,F)$ for any open set A such that $\mathbf{R}^d \backslash (A + \mathbf{Z}^d)$ is empty, and at the same times $i(F)$ is a "maximal" shift-invariant space since $i(A,F) \subset i(F)$ for any bounded open set A.

For the case that F is continuous, we have the following result about the dependent ideal $i(F)$:

Theorem 3.2. *Let F be a compactly supported distribution and $i(F)$ be its dependent shifts. Then*

$$i(F) = \text{spanned by } \mathcal{L}(F)(x), \ x \in \mathbf{R}^d. \tag{3.4}$$

For any bounded open set A and for any compactly supported distribution $F = (f_1, ..., f_N)^T$ on $\mathbf{R}^d$, we define two linear spaces $\mathcal{W}(A,F)$ and $\mathcal{S}(A,F)$ by

$$\mathcal{W}(A,F) := \Big\{ W \in (\ell)^N : \quad W(\mathcal{L}(F)) = 0 \quad \text{on} \quad A \Big\}$$

and

$$\mathcal{S}(A,F) := \Big\{ D \in (\ell_0)^N : \ W(D) = 0 \quad \forall \ W \in \mathcal{W}(A,F) \Big\}. \tag{3.5}$$

Similar time-domain linear spaces for refinable distributions were used to study the polynomial reproducibility, local and global linear independence, and convergence of cascade algorithm etc (see for instance [5, 9, 17, 20, 21]). In this paper, we use the above two time-domain linear spaces $\mathcal{S}(A,F)$ and $\mathcal{W}(A,F)$ to study the dependent ideal $i(F)$. For the case that F is continuous, one may easily verify that $\mathcal{S}(A,F)$ is spanned by $\mathcal{L}(F)(x), \ x \in A$. Then for any bounded open set A with $A + \mathbf{Z}^d = \mathbf{R}^d$, it follows from (3.4) that

$$i(F) = \ \text{spanned by} \quad \tau_k \mathcal{S}(A,F), \ k \in \mathbf{Z}^d.$$

The above consequence is also true for any vector-valued compactly supported distribution.

Theorem 3.3. *Let F be a compactly supported distribution on $\mathbf{R}^d$, and A be a bounded open set with $A+\mathbf{Z}^d=\mathbf{R}^d$. Then $i(F)$ is the shift-invariant space generated by $\mathcal{S}(A,F)$.*

Let $\ell_P := \ell_P(\mathbf{Z}^d)$ be the space of all sequences on $\mathbf{Z}^d$ with polynomial increase, and $(\ell_P)^N$ be N copies of ℓ_P. Define the Fourier series of a sequence $D=(d(j))_{j\in\mathbf{Z}^d}\in(\ell_P)^N$ by $\mathcal{F}(D):=\sum_{j\in\mathbf{Z}^d} d(j)e^{-ij\cdot}$, and let $\mathcal{F}^{-1}(f)$ be the Fourier sequence of a 2π periodic distribution f. As usual, we use $\hat f$ to denote the Fourier transform of a (vector-valued) tempered distribution f, which is defined by $\hat f(\xi)=\int_{\mathbf{R}^d} f(x)e^{-ix\xi}dx$ for the case that f is an integrable function. For any tempered distributions f and g such that $\hat f\overline{\hat g}$ is integrable on $\mathbf{R}^d$, define 2π periodization of $\hat f(\xi)\overline{\hat g(\xi)}$ by

$$[\hat f,\hat g](\xi):=\sum_{k\in\mathbf{Z}^d}\hat f(\xi+2k\pi)\overline{\hat g(\xi+2k\pi)}.$$

Then for any $g\in\mathcal{D}$ and compactly supported distribution f, it follows from Poisson summation formula that

$$[\hat f,\hat g](\xi)=\mathcal{F}(\ll \mathcal{L}(f),g\gg).$$

Thus the dependent ideal $i(F)$ can be also thought of as the space of all Fourier sequences of 2π periodization of $\widehat{F}\hat h, h\in\mathcal{D}$, i.e.,

$$i(F)=\Big\{\mathcal{F}^{-1}([\widehat{F},\hat h]):\ h\in\mathcal{D}\Big\}. \tag{3.6}$$

By the closedness of a linear subspace of $(\ell_0)^N$, the function class $\mathcal{D}$ in (3.6) can be replaced by the family of compactly supported distributions ν such that $\widehat{F}\widehat{\nu}$ is integrable on $\mathbf{R}^d$.

Theorem 3.4. *Let F be a vector-valued compactly supported distribution. Then*

$$i(F)=\{\mathcal{F}^{-1}([\widehat{F},\widehat{\nu}]):\ \nu \text{ are compactly supported and } \widehat{\nu}\widehat{F}\in L^1\}.$$

Theorem 3.4 would be helpful to verify whether a sequence belongs to $i(F)$ or not. In fact, it is used to select a convenient initial sequence of a minimal invariant subspace of $(\ell_0)^N$ (see Theorem 3.5 for details).

A vector-valued compactly supported distribution F is said to be *refinable* if there exist square matrices $c(j), j\in\mathbf{Z}^d$, such that $c(j)=0$ for all but finitely many $j\in\mathbf{Z}^d$, and such that

$$F=\sum_{j\in\mathbf{Z}^d}c(j)F(2\cdot-j) \tag{3.7}$$

([6, 8, 14]). The compactly supported distributions $f_1, \ldots, f_N$ are said to be refinable if $(f_1, \ldots, f_N)^T$ is. The sequence of matrices $c(j), j \in \mathbf{Z}^d$, and the matrix-valued trigonometric polynomial $H(\xi) := 2^{-d} \sum_{j\in\mathbf{Z}^d} c(j)e^{-ij\xi}$ are known as the *mask* and the *symbol* of the refinement equation (3.7), or of the refinable distribution F respectively.

Define operators on $(\ell_0)^N$ by

$$B_k := (c(2j - j' - k))_{j,j'\in\mathbf{Z}^d},\ k \in \mathbf{Z}^d. \tag{3.8}$$

Then it follows from (3.7) that

$$B_k\mathcal{L}(F) = \mathcal{L}(F)((\cdot + k)/2). \tag{3.9}$$

Thus for any $h \in \mathcal{D}$, we have

$$B_k(\ll \mathcal{L}(F), h \gg) = 2^d \ll \mathcal{L}(F), h(2 \cdot -k) \gg \in i(F),$$

which proves that $i(F)$ is invariant under $B_k, k \in \mathbf{Z}^d$, i.e.,

$$B_k i(F) \subset i(F) \quad \forall\ k \in \mathbf{Z}^d. \tag{3.10}$$

For $x_0 \in \mathbf{R}^d$ and the continuous refinable function F in (3.7), using (3.9) leads to

$$B_0^n B_k(\mathcal{L}(F)(x_0)) = F(2^{-n-1}k + 2^{-n-1}x_0) \quad \forall\ n \geq 0 \quad \text{and} \quad k \in \mathbf{Z}^d.$$

Then by (3.4), $i(F)$ is the minimal linear subspace of $(\ell_0)^N$ which is invariant under $B_k, k \in \mathbf{Z}^d$, and contains $\mathcal{L}(F)(x_0)$ for some $x_0 \in \mathbf{R}^d$. Similar results about the invariance of some time domain space generated by refinable distributions were obtained in [9, 20]. The above assertion about the dependent ideal $i(F)$ of a continuous refinable functions can be generalized to any compactly supported refinable distribution, but the initial sequence $\mathcal{L}(F)(x_0), x \in \mathbf{R}^d$, is replaced by $\ll \mathcal{L}(F), G \gg$ for some "good" distribution G.

Theorem 3.5. *Let F be a compactly supported nonzero distributional solution of the refinement equation (3.7), and let $B_k, k \in \mathbf{Z}^d$, be defined by (3.8). If G is a compactly supported distribution so chosen that*

(i) *$\widehat{G}\widehat{F}$ is integrable on $\mathbf{R}^d$, and that*
(ii) *for any $h \in \mathcal{D}$, there exist $h_n, n \geq 1$, such that h_n is finite linear combinations of $G(2^n \cdot -k), k \in \mathbf{Z}^d$, h_n are supported in a compact set independent of n, and $\lim_{n\to\infty} \|\widehat{F}(\widehat{h} - \widehat{h}_n)\|_1 = 0$,*

then $i(F)$ is the minimal shift-invariant subspace of $(\ell_0)^N$ which is invariant under $B_k, k \in \mathbf{Z}^d$, and contains the initial sequence $\ll \mathcal{L}(F), G \gg$.

Now let us make some remarks on the function G in the conditions (i) and (ii), and on the computation $\ll \mathcal{L}(F), G \gg$ in Theorem 3.5. Let F belong to Sobolev space $H^{-\alpha_0}$ for some α_0. In particular, any compactly supported distribution belongs to Sobolev space $H^{-\alpha}$ for sufficient large α. Thus by

$$\|\widehat{F}\widehat{G}\|_1 \le \|F\|_{H^{-\alpha_0}}\|G\|_{H^{\alpha_0}}, \tag{3.11}$$

the condition (i) is satisfied if G belongs to the Sobolev space H^{α_0}. Let $V_n(G)$ be the space spanned by $G(2^n\cdot -k), k\in \mathbf{Z}^d$, and define operators $P_n, n\ge 1$, from $\mathcal{D}$ to $V_n(G)$ by

$$P_n h = 2^{nd}\sum_{j\in\mathbf{Z}^d} \ll h, \tilde{G}(2^n\cdot -j) \gg G(2^n\cdot -j)\in V_n(G)$$

where $\tilde{G}$ is a compactly supported distribution. Then for any $h\in\mathcal{D}$, $P_n h$ belongs to $V_n(G)$ and is supported in a compact set independent of $n\ge 1$. If, additionally, $P_n h$ has good approximation to h, such as the H^{α_0} norm of $h-P_n h$ tends to zero as n tends to infinity, then for $F\in H^{-\alpha_0}$, by the Hölder inequality (3.11), $\|\widehat{F}(\widehat{h}-\widehat{P_n h})\|_1$ tends to 0 as n tends to infinity. So, roughly speaking, the conditions (i) an (ii) are satisfied if G is chosen sufficiently smooth and if the spaces $V_n(G), n\ge 1$, are approximative to $\mathcal{D}$ in certain Sobolev norm. Moreover, the projection of h to $V_n(G)$, or certain quasi-interpolant of h in $V_n(G)$ could be chosen as h_n.

Let G satisfy (i) and (ii) of Theorem 3.5. If, additionally, G is refinable,

$$G=\sum_{j\in\mathbf{Z}^d} g(j)G(2\cdot -j), \tag{3.12}$$

where $(g(j))_{j\in\mathbf{Z}^d}\in \ell_0$, then we can compute $\ll \mathcal{L}(F), G\gg$ from the masks of F and G in some cases. Set $\Psi(x) = \ll F(\cdot + x), G \gg$. Then Ψ is continuous by $\widehat{F}\widehat{G}\in L^1$, $\mathcal{L}(\Psi)(0) = \ll \mathcal{L}(F), G\gg$, and

$$\begin{aligned}\Psi(x) &= 2^{-d}\sum_{j,j'\in\mathbf{Z}^d} c(j)g(j') \ll F(2x-j+\cdot), G(\cdot - j')\gg \\ &= \sum_{j\in\mathbf{Z}^d}\Big(2^{-d}\sum_{j'\in\mathbf{Z}^d} c(j+j')d(j')\Big)\Psi(2x-j) \end{aligned} \tag{3.13}$$

by (3.7) and (3.12). Define an operator B on $(\ell_0)^N$ by

$$B=\Big(2^{-d}\sum_{k\in\mathbf{Z}^d} c(2j-j'+k)g(k)\Big)_{j,j'\in\mathbf{Z}^d}.$$

Then $\mathcal{L}(\Psi)(0)$ is an eigenvector of B with eigenvalue one by (3.13), and so is $\ll \mathcal{L}(F), G\gg$. This gives a way to compute $\ll \mathcal{L}(F), G\gg$ through finding eigenvectors of the operator B on $(\ell_0)^N$ for the case that G is chosen to be refinable and to satisfy (i) and (ii) of Theorem 3.5. Tensor product of B-splines and box splines with certain regularity are typical examples of functions G satisfying the above required properties.

In the application of characterization of L^p smoothness of refinable distribution F, the shift-invariant subspace $i_r(F)$ of the dependent ideal $i(F)$ was used in [21],

$$i_r(F) = \{\ll \mathcal{L}(F), h \gg: \quad h \in \mathcal{D} \quad \text{and} \quad \hat{h}(\xi) = O(|\xi|^r) \quad \text{as } \xi \to 0\},$$

where $r > 0$. For the compactly supported refinable distribution F, the shift-invariant subspace $i_r(F)$ is also the minimal shift-invariant space which is invariant under $B_k, k \in \mathbf{Z}^d$, and contains the vector $\ll \mathcal{L}(F), G^* \gg$ for some function G^* with certain regularity. Moreover, we may choose the certain finite linear combination of $G(2 \cdot -k), k \in \mathbf{Z}$ as G^*, where G is a compact supported refinable function with high regularity, such as the box spline with certain regularity. Such an idea has been used implicitly in [15] to give a first complete characterization of refinable distributions in Triebel-Lizorkin and Besov spaces without stable assumption.

4. Self-Adaptive Stability Lemma

In this section, we provide an estimate of the L^p norm of $D(\mathcal{L}(F))$ via a semi-norm of D without linear independent shift assumption on F (Theorem 4.2).

For $1 \le p \le \infty$, let $\ell^p := \ell^p(\mathbf{Z}^d)$ be the space of all p-summable sequences with usual ℓ^p norm $\|\cdot\|_p$, $(\ell^p)^N$ be N copies of ℓ^p, and $L^p := L^p(\mathbf{R}^d)$ be the space of all p-integrable functions with usual L^p norm $\|\cdot\|_p$. For any compactly supported L^p function F having linear independent shifts, the following quantity estimate is well known,

$$C^{-1}\|D\|_p \le \|D(\mathcal{L}(F))\|_p \le C\|D\|_p \quad \forall\ D \in (\ell^p)^N, \tag{4.1}$$

where C is a positive constant independent of D (see for instance [12]). The above L^p estimate of the sequence $D(\mathcal{L}(F))$ plays an important role in wavelet analysis as well as in the approximation by shift-invariant spaces. Recently there is much interest on estimating L^p norm, or even some norms in linear topological spaces such as Sobolev spaces, of $D(\mathcal{L}(F))$ without the linear independent assumption on F. The purpose of this section is to establish a quantity estimate of $D(\mathcal{L}(F))$ without linear independent assumption on F.

Let $1 \le p \le \infty$, $F = (f_1, \dots, f_N)^T$ be a compactly supported distribution, and let $\{E_1, \dots, E_r\}$ be a generator of the dependent ideal $i(F)$. Define a semi-norm $\|\!| \cdot |\!\|_p$ on $(\ell_0)^N$ adaptive to F by

$$\|\!|D|\!\|_p = \sum_{i=1}^{r} \big\|\{D(\tau_j E_i)\}_{j\in\mathbf{Z}^d}\big\|_p \quad \text{for any} \quad D \in (\ell_0)^N. \tag{4.2}$$

The semi-norm $\|\!| \cdot |\!\|_p$ in (4.2) is well-defined since for different generators of $i(F)$ the corresponding semi-norms are equivalent to each other.

Theorem 4.1. *Let $1 \le p \le \infty$, and let $\{E_1, \dots, E_{N_1}\}$ and $\{\tilde{E}_1, \dots, \tilde{E}_{N_2}\}$ be two generators of a shift-invariant linear subspace $\mathcal{I}$ of $(\ell_0)^N$. Then there exists a positive constant C such that for all $D \in (\ell_0)^N$,*

$$\begin{aligned} C^{-1}\sum_{i=1}^{N_1} \|\{D(\tau_j E_i)\}_{j\in\mathbf{Z}^d}\|_p &\le \sum_{i=1}^{N_2} \|\{D(\tau_j \tilde{E}_i)\}_{j\in\mathbf{Z}^d}\|_p \\ &\le C\sum_{i=1}^{N_1} \|\{D(\tau_j E_i)\}_{j\in\mathbf{Z}^d}\|_p. \end{aligned}$$

Using the semi-norm $\|\!| \cdot |\!\|_p$ in (4.2), we have the following self-adaptive stability lemma for any compactly supported L^p function F.

Theorem 4.2. *Let $1 \le p \le \infty$, $F = (f_1, \dots, f_N)^T$ be a compactly supported L^p function, and let the semi-norm $\|\!| \cdot |\!\|_p$ be defined by (4.2). Then there exists a positive constant C independent of D such that*

$$C^{-1}\|\!|D|\!\|_p \le \|D(\mathcal{L}(F))\|_p \le C\|\!|D|\!\|_p, \quad \forall\ D \in (\ell_0)^N. \tag{4.3}$$

We remark that the quantity estimate of $D(\mathcal{L}(F))$ in Theorem 4.2 is the same as the one in (4.1) if F has linear independent shifts, since the semi-norm $\|\!| \cdot |\!\|_p$ is equivalent to the usual ℓ^p norm at that situation.

5. Decomposition

We start this paper from a decomposition of the form (1.1). In this section, we return to that decomposition with additional properties for $\phi_1 \dots, \phi_M$, such as $\phi_1 \dots, \phi_M$ have linear independent shifts and belong to $V(F)$.

Let P_i and $\phi_i, 1 \le i \le M$, be as in the decomposition (1.1). By Theorem 2.1, $\{P_i, 1 \le i \le M\}$ is a generator of the dependent ideal $i(F)$. Conversely for any generator $\{Q_1, \dots, Q_r\}$ of the dependent ideal $i(F)$, it follows from (1.1) and

$$P_i = \sum_{i'=1}^{r} \sum_{k\in\mathbf{Z}^d} \alpha_{ii'}(k)\tau_k Q_{i'} \tag{5.1}$$

for some sequences $\alpha_{ii'} = (\alpha_{ii'}(k))_{k\in\mathbf{Z}^d} \in \ell_0$ that

$$F = \sum_{i=1}^{r} \sum_{j\in\mathbf{Z}^d} q_i(j)\psi_i(\cdot - j), \tag{5.2}$$

where we write $Q_i = (q_i(j))_{j\in\mathbf{Z}^d}$, and set

$$\psi_i = \sum_{i'=1}^{M} \sum_{k\in\mathbf{Z}^d} \alpha_{i'i}(k)\phi_{i'}(\cdot - k), \quad 1 \le i \le r. \tag{5.3}$$

The existence of the sequences $\alpha_{ii'}, 1 \le i \le M, 1 \le i' \le r$, follows from our assumption that both $\{Q_1, \ldots, Q_r\}$ and $\{P_1, \ldots, P_M\}$ are generators of the dependent ideal $i(F)$. This gives a decomposition of the form (5.2) for any generator of the dependent ideal $i(F)$. We note that given a generator $\{Q_1, \ldots, Q_r\}$ of the dependent ideal $i(F)$, the decomposition of the form (5.2) is not unique and the distributions $\psi_1, \ldots, \psi_r$ in (5.3) may have linear **dependent** shifts in general. For instance, one may easily verify that $\{Q_1, \ldots, Q_M, Q_{M+1}, \ldots, Q_{2M}\}$ with $Q_k = Q_{k+M} = P_k, 1 \le k \le M$, is a generator of the dependent ideal $i(F)$, and that F has the following decompositions:

$$F = \sum_{i=1}^{M} p_i(j)\phi_i(\cdot - j) = \frac{1}{2}\sum_{i=1}^{M} p_i(j)\phi_i(\cdot - j) + \frac{1}{2}\sum_{i=M+1}^{2M} p_i(j)\phi_{i-M}(\cdot - j),$$

where $p_i(j) = p_{i-M}(j)$ for $M+1 \le i \le 2M$ and $j \in \mathbf{Z}^d$.

In wavelet analysis, we are more interested in the decomposition (5.2) such that the shift-invariant space generated by the distributions $\psi_1, \ldots, \psi_r$ in the decomposition is the same as $V(F)$, i.e.,

$$V(F) = V(\psi_1, \ldots, \psi_r).$$

In the viewpoint of the shift-invariant space, $\{\psi_1, \ldots, \psi_r\}$ is a "good" generator of the shift-invariant space $V(F)$. All those inspires us to consider the following problem:

Problem *Given a compactly supported distribution F and a generator $\{Q_1, \ldots, Q_r\}$ of the dependent ideal $i(F)$. Is the decomposition of the form (5.2) unique, do the functions $\psi_1, \ldots, \psi_r$ have linear independent shifts and belong to $V(F)$?*

In order to solve the above problem, we introduce a new concept: *strongly linear independence*. We say that $D_1, \ldots, D_r \in (\ell_0)^N$ are strongly linearly independent if $\sum_{i=1}^{r} \sum_{k\in\mathbf{Z}^d} \gamma_i(k)\tau_k(D_i) = 0$ for some $(\gamma_i(k))_{k\in\mathbf{Z}^d} \in \ell_0, 1 \le i \le r$, implies that $\gamma_i(k) = 0$ for all $1 \le i \le r$ and $k \in \mathbf{Z}^d$.

Theorem 5.1. *Let $F = (f_1, \ldots, f_N)^T$ be a vector-valued compactly supported distribution, and let $i(F)$ be the dependent ideal of F. If $\{Q_i = (q_i(j))_{j\in\mathbf{Z}^d} : 1 \le i \le r\}$ is a strongly linear independent generator of the dependent ideal $i(F)$, then there exists only one family of compactly supported distributions $\psi_1, ..., \psi_r$ satisfying (5.2). Furthermore, those distributions have linear independent shifts, belong to $V(F)$ and satisfy*

$$V(F) = V(\psi_1, \ldots, \psi_r). \tag{5.4}$$

If F is refinable, then the above compactly supported distributions $\psi_1, \ldots, \psi_r$ are refinable too.

A shift-invariant space of $(\ell_0)^N$ with a strongly linear independent generator is said to be *quasi-principal*. By Proposition A.1, any shift-invariant linear space of $(\ell_0(\mathbf{Z}))^N$ is quasi-principal. Then as an easy consequence of Theorem 5.1, we have

Corollary 5.2. *Every vector-valued compactly supported distribution F on the real line is finite linear combinations of the shifts of some compactly supported distributions $\psi_1, \ldots, \psi_r \in V(F)$ having linear independent shifts. If, additionally, F is refinable, then the above compactly supported distributions $\psi_1, \ldots, \psi_r$ can be chosen to be refinable too.*

Similar result for scale-valued refinable functions with dilation 2 on the line was proved by Jia ([10, Theorem 5.3]).

The quasi-principal condition to the dependent ideal $i(F)$ in Theorem 5.1 is necessary if the distributions $\psi_1, \ldots, \psi_r \in V(F)$ have linear independent shifts.

Theorem 5.3. *Let F be a vector-valued compactly supported distribution and $i(F)$ be its dependent ideal. If there exist compactly supported distributions $\psi_1, \ldots, \psi_r$ such that they have linear independent shifts, belong to $V(F)$, and such that F is finite linear combinations of the shifts of $\psi_1, \ldots, \psi_r$, then $i(F)$ is quasi-principal.*

If we only need to find $\psi_1, \ldots, \psi_r$ having linear independent, the strongly linearly independent condition on the generator $\{Q_1, \ldots, Q_r\}$ is not necessary. For instance, the Zwart-Powell spline has a decomposition of the form (5.2) with $\psi_1, \ldots, \psi_r$ having linear independent shifts, but its dependent ideal is not quasi-principal.

Example 5.4. Let F be the refinable function with symbol

$$H(\xi_1,\xi_2)=\frac{1}{16}(1+e^{-i\xi_1})(1+e^{-i\xi_2})(1+e^{-i(\xi_1+\xi_2)})(e^{-i\xi_2}+e^{-i\xi_1}).$$

In fact, F is $(1,0)$-shift of the Zwart-Powell spline M_Ξ, and hence F is continuous and supported in $[0,3]^2$, where

$$\Xi=\begin{pmatrix} 1 & 0 & 1 & -1\\ 0 & 1 & 1 & 1\end{pmatrix}.$$

One may verify that the linear space spanned by the vector $(F(x+i,y+i'))_{i,i'=0,1,2}, (x,y)\in[0,1]^2$, is the orthogonal complement of the linear space spanned by $(1,-1,1,-1,1,-1,1,-1,1)$. Thus

$$i(F)=\Big\{(c(j_1,j_2))_{(j_1,j_2)\in\mathbf{Z}^2}\in\ell_0:\sum_{(j_1,j_2)\in\mathbf{Z}^2}c(j_1,j_2)(-1)^{j_1+j_2}=0\Big\}$$

and $i(F)$ is not a quasi-principal ideal. Therefore by Theorem 5.3, there does not exist a decomposition of the form (5.2) with $\psi_1,\ldots,\psi_r\in V(F)$ having linear independent shifts.

Divide the unit square $[0,1]^2$ into four triangles $D_i, 1\le i\le 4$,

$$\begin{cases} D_1=\{(x,y)\in[0,1]^2: y\le x\le 1-y\}\\ D_2=\{(x,y)\in[0,1]^2: \max(y,1-y)\le x\le 1\}\\ D_3=\{(x,y)\in[0,1]^2: x\le y\le 1-x\}\\ D_4=\{(x,y)\in[0,1]^2: \max(x,1-x)\le y\le 1\}.\end{cases}$$

Then F is the finite combinations of the shifts of $x^ky^l\chi_{D_i}(x,y), i=1,2,3,4$ and $(k,l)\in\Lambda_2$, where $\Lambda_2=\{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2)\}$, that is, F has a decomposition of the form (5.2), where we have used the fact that the restriction of Zwart-Powell spline on the shifts of D_i is a polynomial of degree at most 2. One may easily verify that the compactly supported functions $\phi_{k,l,i}, 1\le i\le 4, (k,l)\in\Lambda_2$, have linear independent shifts and are refinable.

6. Proofs

In this section, we gather all proofs. Let δ be the delta sequence on $\mathbf{Z}^d$, and let $e_i\in\mathbf{R}^N, 1\le i\le N$, be vectors with the i-th component one and other components zero.

To prove Theorem 2.1, we recall a result in [2, 23].

Lemma 6.1. ([2, 23]) *Let $F=(f_1,\ldots,f_N)^T$ be a vector-valued compactly supported distribution. If F has linear independent shifts, then there exist $h_1,\ldots,h_N\in\mathcal{D}$ such that*

$$\ll \mathcal{L}(F), h_i \gg = \delta e_i \quad \text{for all } 1\le i\le N.$$

Proof of Theorem 2.1. Let F have the form (1.1), and let P_i and $\phi_i, 1\le i\le M$, be as in (1.1). Then it follows from (1.2) that $\ll \mathcal{L}(F), h\gg\in \mathcal{I}(P_1,\ldots,P_M)$ for any $h\in\mathcal{D}$, which implies $i(F)\subset\mathcal{I}(P_1,\ldots,P_M)$.

By the linear independent shifts of $\phi_1,\ldots,\phi_M$ and Lemma 6.1, there exist $h_1,\ldots,h_M\in\mathcal{D}$ such that

$$\ll \phi_i(\cdot-k), h_{i'}(\cdot-k')\gg = \delta_{ii'}\delta_{kk'} \quad \forall\ 1\le i,i'\le M \quad \text{and} \quad k,k'\in\mathbf{Z}^d,$$

where $\delta_{ii'}$ and $\delta_{kk'}$ are Kronecker symbols. This, together with (1.2), imply that for any $1\le i\le M$,

$$P_i=\sum_{i'=1}^{M}\sum_{k\in\mathbf{Z}^d}\tau_k(P_{i'})\ll \phi_{i'}(\cdot+k), h_i\gg = \ll \mathcal{L}(F), h_i\gg\in i(F).$$

Then $\mathcal{I}(P_1,\ldots,P_M)\subset i(F)$ since $i(F)$ is shift-invariant. □

Proof of Theorem 2.2. If $i(F)=(\ell_0)^N$, then $\mathcal{K}(F)=\{0\}$ by (1.4), Theorem 1.1, and Theorem 2.1. Conversely, if $\mathcal{K}(F)=\{0\}$, then $f_1,\ldots,f_N$ have linear independent shifts. Hence $f_1,\ldots,f_N$ can be chosen as the functions ϕ_i in (1.1), and F has the decomposition of the form (1.1) with $P_i=\delta e_i, 1\le i\le N$, at this time. This, together with Theorem 2.1, lead to $i(F)=(\ell_0)^N$. □

To prove Theorem 3.1, we need the existence of dual functions for a family of linear independent distributions on some bounded open set. We include a proof for the completeness of this paper.

Lemma 6.2. *Let $\psi_1,\ldots,\psi_M$ be compactly supported distributions, and let A be a bounded open set. If $\psi_1,\ldots,\psi_M$ are linear independent on A, then there exist $h_1,\ldots,h_M\in\mathcal{D}$ with support in A such that*

$$\ll \psi_i, h_{i'}\gg\ = \delta_{ii'}, \quad \forall\ 1\le i,i'\le M.$$

Proof. We prove the assertion by induction on M. For $M=1$, $\psi_1\not\equiv 0$ on A by the linear independence of ψ_1 on A. Thus $\ll \psi_1, g\gg\ne 0$ for some $g\in\mathcal{D}$ supported in A. This proves the assertion for the case $M=1$ by letting $h_1=g/\ll g,\psi_1\gg$. Inductively we assume that the assertion holds for the case $M-1$, and start to prove the assertion for the case $M\ge 2$.

Note that $\psi_1 \not\equiv 0$ on A, since, otherwise, $\psi_1, \ldots, \psi_M$ are linearly dependent on A. Then

$$\ll \psi_1, h_1^* \gg = 1 \tag{6.1}$$

for some $h_1^* \in \mathcal{D}$ with support in A. Define $\psi_i^* = \psi_i - \ll \psi_i, h_1^* \gg \psi_1, 2 \le i \le M$. Then $\psi_2^*, \ldots, \psi_M^*$ are linear independent on A, and

$$\ll \psi_i^*, h_1^* \gg = 0, \quad 2 \le i \le M \tag{6.2}$$

by (6.1). By the inductive hypothesis, there exist $h_2^*, \ldots, h_M^* \in \mathcal{D}$ supported in A such that

$$\ll \psi_i^*, h_{i'}^* \gg = \delta_{ii'} \quad \forall\ 2 \le i, i' \le M. \tag{6.3}$$

Define $\psi_1^* = \psi_1 - \sum_{i=2}^M \ll \psi_1, h_i^* \gg \psi_i^*$. Then

$$\ll \psi_1^*, h_i^* \gg = 0, \quad 2 \le i \le M \tag{6.4}$$

by (6.3). From the construction of $\psi_1^*, \ldots, \psi_M^*$, there exist $M \times M$ nonsingular matrices $B_1 = (a_{ii'})_{1 \le i,i' \le M}$ such that $\psi_i = \sum_{i'=1}^M a_{ii'} \psi_{i'}^*$ and $\psi_i^* = \sum_{i'=1}^M b_{ii'} \psi_{i'}, 1 \le i \le M$, where $(B_1)^{-1} = (b_{ii'})_{1 \le i,i' \le M}$. Then $h_i = \sum_{i'=1}^s \overline{b_{i'i}} h_{i'}^*, 1 \le i \le M$, satisfy the required properties by (6.1)–(6.4). □

Proof of Theorem 3.1. Let A be a bounded open set with $A + \mathbf{Z}^d = \mathbf{R}^d$, $k_0 = \dim V(F)|_A$, and let E_i and $\psi_i, 1 \le i \le k_0$, be as in (3.3). Therefore it suffices to prove

$$\ll \mathcal{L}(F), h \gg \in i(A, F) \quad \forall\ h \in \mathcal{D}, \tag{6.5}$$

and

$$E_i \in i(F) \quad \forall\ 1 \le i \le k_0. \tag{6.6}$$

By the assumption $A + \mathbf{Z}^d = \mathbf{R}^d$, there exists a function $g \in \mathcal{D}$ so that supp $g \subset A$ and $\sum_{k \in \mathbf{Z}^d} g(\cdot - k) \equiv 1$. Set $h_k = h(\cdot - k) g$. Then $h_k \in \mathcal{D}$, supp $h_k \subset A$ and $h = \sum_{k \in \mathbf{Z}^d} h_k(\cdot + k)$. This, together with (3.3), imply

$$\begin{aligned} \ll \mathcal{L}(F), h \gg &= \sum_{k \in \mathbf{Z}^d} \tau_k \ll \mathcal{L}(F), h_k \gg \\ &= \sum_{i=1}^{k_0} \sum_{k \in \mathbf{Z}^d} \tau_k(E_i) \ll \psi_i, h_k \gg \in i(A, F). \end{aligned}$$

Hence (6.5) is proved.

By Lemma 6.2 and the linear independence of $\psi_1,\dots,\psi_{k_0}$, there exist $h_1,\dots,h_{k_0}\in\mathcal{D}$, so that $h_1,\dots,h_{k_0}$ are supported in A, and $\ll \psi_i, h_{i'}\gg = \delta_{ii'}$ for all $1\le i,i'\le k_0$. This, together with (3.3), imply

$$E_i=\sum_{i'=1}^{k_0} E_{i'} \ll \phi_{i'}, h_i\gg = \ll \mathcal{L}(F), h_i\gg \in i(F) \quad \forall\, 1\le i\le k_0.$$

Hence (6.6) is proved. □

Proof of Theorem 3.2. By (2.1), any sequence in $i(F)$ belongs to the space spanned by $\mathcal{L}(F)(x), x\in\mathbf{R}^d$. Conversely, for any $x_0\in\mathbf{R}^d$, there exist $h_n\in\mathcal{D}, n\ge 1$, such that $\lim_{n\to\infty}\ll\mathcal{L}(F),h_n\gg=\mathcal{L}(F)(x_0)$ in the topology of $(\ell_0)^N$. Then $\mathcal{L}(F)(x_0)\in i(F)$ by the fact that $i(F)$ is closed in the topology of $(\ell_0)^N$ (see Proposition A.1 in the appendix). Thus the conclusion follows. □

Proof of Theorem 3.3 . Let $E_i, 1\le i\le \dim V(F)|_A$, be as in (3.3), and $\tilde{\mathcal{S}}(A,F)$ be the space spanned by those sequences, i.e.,

$$\tilde{\mathcal{S}}(A,F) := \text{spanned by}\quad E_i,\ 1\le i\le \dim V(F)|_A.$$

Then, by Theorem 3.1, it suffices to prove

$$\tilde{\mathcal{S}}(A,F)=\mathcal{S}(A,F) \tag{6.7}$$

for any open set A. Set $k_0=\dim V(F)|_A$, and let E_i and $\psi_i, 1\le i\le k_0$, be as in (3.3). Then the proof of (6.7) reduces to

$$W(E_i)=0 \tag{6.8}$$

for all $1\le i\le k_0$ and $W\in\mathcal{W}(A,F)$, and

$$W\in\mathcal{W}(A,F) \tag{6.9}$$

for any sequence $W\in(\ell)^N$ with $W(E)=0$ for all $E\in\tilde{\mathcal{S}}(A,F)$.

For any $W\in\mathcal{W}(A,F)$, it follows from (3.3) that $\sum_{i=1}^{k_0}W(E_i)\psi_i = W(\mathcal{L}(F))=0$ on A. Thus (6.8) follows from the linear independence of $\psi_1,\dots,\psi_{k_0}$ on A.

Let $W\in(\ell)^N$ satisfy $W(E)=0$ for all $E\in\tilde{\mathcal{S}}(A,F)$. Then $W(\mathcal{L}(F))=\sum_{i=1}^{k_0}W(E_i)\psi_i=0$ on A by (3.3). Then (6.9) follows. □

Proof of Theorem 3.4. Clearly it suffices to prove that $D=\mathcal{F}^{-1}([\widehat{F},\widehat{\nu}])\in i(F)$ for any compactly supported distribution ν with $\widehat{\nu}\widehat{F}\in L^1$. Fix $h_1\in\mathcal{D}$ with $\widehat{h}_1(0)=1$, and define ν_n by $\widehat{\nu_n}=\widehat{\nu}\widehat{h_1}(n\cdot), n\ge 1$. Then $\nu_n\in\mathcal{D}, n\ge 1$. Moreover ν_n are supported in a compact set independent of $n\ge 1$, and $\widehat{F}\widehat{\nu_n}$ converges to $\widehat{F}\widehat{\nu}$ in L^1 norm by the Lebesgue dominated theorem and the integrable hypothesis on $\widehat{F}\widehat{\nu}$. Therefore $[\widehat{F}, \widehat{\nu_n}]$ converges to $[\widehat{F}, \widehat{\nu}]$ in $L^1([0, 2\pi]^d)$ norm as n tends to infinity. This proves the convergence of $D_n, n \ge 1$, to D in the topology of $(\ell_0)^N$, where $D_n = \mathcal{F}^{-1}([\widehat{F}, \widehat{\nu_n}]), n \ge 1$. Note that $D_n \in i(F)$ by (3.6) and $\nu_n \in \mathcal{D}$, and recall that $i(F)$ is closed in the topology of $(\ell_0)^N$ by Proposition A.1. Then the limit D of $D_n, n \ge 1$, belongs to $i(F)$ too. □

Proof of Theorem 3.5. Let V be the minimal shift-invariant subspace of $(\ell_0)^N$ which is invariant under $B_k, k \in \mathbf{Z}^d$, and contains the sequence $\ll \mathcal{L}(F), G \gg$. By the assumption on G and Theorem 3.4, we have $\ll \mathcal{L}(F), G \gg \in i(F)$. This, together with (3.10), lead to $V \subset i(F)$. Then it remains to prove

$$\ll \mathcal{L}(F), h \gg \in V \quad \forall\ h \in \mathcal{D}. \tag{6.10}$$

Let $D_0 = \ll \mathcal{L}(F), G \gg$, h_n be chosen as in (ii), and define $D_n = \ll \mathcal{L}(F), h_n \gg$, $n \ge 1$. By (3.9), we have

$$B_0^{n-1} B_k D_0 = 2^{nd} \ll \mathcal{L}(F), G(2^n \cdot -k) \gg .$$

Thus $D_n, n \ge 1$, are finite combination of $B_0^{n-1} B_k D_0, k \in \mathbf{Z}^d$, and hence belong to V. On the other hand, by the assumption on G, D_n converges to $\ll \mathcal{L}(F), h \gg$ in the topology of $(\ell_0)^N$ as n tends to infinity. Hence (6.10) follows from the closedness of V in the topology of $(\ell_0)^N$. □

Proof of Theorem 4.1. Obviously it suffices to prove the first inequality in (4.3). Actually, the second inequality follows from the first one by changing the roles of $\{E_1, \ldots, E_{N_1}\}$ and $\{\tilde{E}_1, \ldots, \tilde{E}_{N_2}\}$. By the definition of a generator, there exist $\alpha_{ii'} = (\alpha_{ii'}(j))_{j \in \mathbf{Z}^d} \in \ell_0$ such that

$$E_i = \sum_{i'=1}^{N_2} \sum_{j \in \mathbf{Z}^d} \alpha_{ii'}(j) \tau_j(\tilde{E}_{i'}), \quad 1 \le i \le N_1. \tag{6.11}$$

Therefore there exists a positive constant C by (6.11) such that

$$\begin{aligned} \sum_{i=1}^{N_1} \|\{D(\tau_j(E_i))\}_{j \in \mathbf{Z}^d}\|_p &\le \sum_{i=1}^{N_1} \sum_{i'=1}^{N_2} \sum_{j' \in \mathbf{Z}^d} |\alpha_{ii'}(j')|\ \|\{D(\tau_{j+j'}(\tilde{E}_{i'}))\}_{j \in \mathbf{Z}^d}\|_p \\ &\le C \sum_{i'=1}^{N_2} \|\{D(\tau_j(\tilde{E}_{i'}))\}_{j \in \mathbf{Z}^d}\|_p \quad \forall\ D \in (\ell_0)^N. \end{aligned}$$

This proves the first inequality of (4.3) and completes the proof. □

Proof of Theorem 4.2. Let P_i and $\phi_i, 1 \le i \le M$, be as in (1.1). Then by (1.2), (4.1), and the linear independent shifts of $\phi_1, \ldots, \phi_M$, there exists a positive constant C such that for all $D \in (\ell_0)^N$,
(6.12)
$$C^{-1} \sum_{i=1}^{M} \|(D(\tau_j P_i))_{j\in\mathbf{Z}^d}\|_p \le \|D(\mathcal{L}(F))\|_p \le C \sum_{i=1}^{M} \|(D(\tau_j P_i))_{j\in\mathbf{Z}^d}\|_p.$$
Therefore (4.3) follows from (6.12), Theorem 2.1, and Theorem 4.1. □

The proof of Theorem 5.1 is very technical, and we need several lemmas including a well-known result about linear independent shifts, a property of strongly linear independent sequences, a lemma about the sequences $\alpha_{ii'}$ in (5.1), a characterization of strongly linear independence, and an existence result of the inverse sequence of a nonzero finitely supported sequence.

Lemma 6.3. ([13, 16]) *Let F be a vector-valued compactly supported distribution on $\mathbf{R}^d$. Then F has linear independent shifts if and only if the matrix $(\widehat{F}(\xi + 2j\pi))_{j\in\mathbf{Z}^d}$ is of full rank for any $\xi \in \mathbf{C}^d$.*

Lemma 6.4. *Let $Q_1, \ldots, Q_r \in (\ell_0)^N$ be strongly linearly independent. If*
$$\sum_{i=1}^{r} \mathcal{F}(Q_i)(\xi)\widehat{g}_i(\xi) = 0 \tag{6.13}$$
for some compactly supported distributions $g_1, \ldots, g_r$, then $g_i = 0$ for all $1 \le i \le r$.

Proof. For any $y \in \mathbf{R}^d$ and $h \in \mathcal{D}$, multiplying $e^{iy\xi}\widehat{h}(\xi)$ at both sides of (6.13) and then taking 2π periodization, we obtain
$$\sum_{i=1}^{r} \mathcal{F}(Q_i)(\xi)[\widehat{g}_i, e^{-iy\cdot}\widehat{h}](\xi) = 0 \quad \forall\ \xi \in \mathbf{R}^d.$$
This, together with the strongly linear independence of $Q_1, \ldots, Q_r$, implies that for all $1 \le i \le r$,
$$[\widehat{g}_i, e^{-iy\cdot}\hat{h}] = e^{iy\xi} \sum_{k\in\mathbf{Z}^d} e^{2kiy}\widehat{g}_i(\xi + 2k\pi)\overline{\hat{h}(\xi + 2k\pi)} = 0.$$
Therefore, $\widehat{g}_i(\xi)\overline{\widehat{h}(\xi)} = 0$ on $\mathbf{R}^d$ since y is chosen arbitrarily. Hence the assertion follows because $h \in \mathcal{D}$ is chosen arbitrarily too. □

Lemma 6.5. *Let F be a vector-valued compactly supported distribution having the decomposition (1.1), $\{Q_i = (q_i(j))_{j\in\mathbf{Z}^d}, 1 \le i \le r\}$ be a strongly linear independent generator of $i(F)$, and let the sequences $\alpha_{ii'}$ be defined as* *in (5.1). Then the matrix* $(\mathcal{F}(\alpha_{ii'})(\xi))_{1\le i\le r, 1\le i'\le M}$ *has full rank* $r \le M$ *for any* $\xi \in \mathbf{C}^d$.

Proof. Let $P_1, \ldots, P_M$ be as in (1.1). Then $\{P_1, \ldots, P_M\}$ is a generator of the dependent ideal $i(F)$ by Theorem 2.1. Hence there exist sequences $\beta_{i'i} = (\beta_{i'i}(k))_{k\in\mathbf{Z}^d} \in \ell_0$ such that

$$Q_{i'} = \sum_{i=1}^{M}\sum_{k\in\mathbf{Z}^d} \beta_{i'i}(k)\tau_k(P_i),$$

which is equivalent to

$$\mathcal{F}(Q_{i'}) = \sum_{i=1}^{M} \mathcal{F}(\beta_{i'i})\mathcal{F}(P_i). \tag{6.14}$$

On the other hand,

$$\mathcal{F}(P_i) = \sum_{i'=1}^{r} \mathcal{F}(\alpha_{ii'})\mathcal{F}(Q_{i'}) \tag{6.15}$$

by (5.1). Then combining (6.14) and (6.15), we obtain

$$\mathcal{F}(Q_{i'}) = \sum_{i,i''=1}^{M} \mathcal{F}(\beta_{i'i})\mathcal{F}(\alpha_{ii''})\mathcal{F}(Q_{i''}).$$

By the assumption on $Q_1, \ldots, Q_r$ and Lemma 6.4, we have

$$\sum_{i=1}^{M} \mathcal{F}(\beta_{i'i})(\xi)\mathcal{F}(\alpha_{ii''})(\xi) = \delta_{i'i''} \quad \forall\ 1 \le i', i'' \le r \ \text{ and } \ \xi \in \mathbf{C}^d.$$

Hence the assertion follows. □

To state the characterization of strongly linear independence, we introduce $\mathcal{Z}$-transform of a sequence in $(\ell)^N$. For a sequence $W = (w(k))_{k\in\mathbf{Z}^d} \in \ell$, define the formal series $\sum_{k\in\mathbf{Z}^d} w(k)z^{-k}$ as its $\mathcal{Z}$-transform $\mathcal{Z}(W)(z)$. Denote the space of $\mathcal{Z}$-transforms of all sequences in ℓ_0 and in $(\ell_0)^N$ by $\mathcal{L}$ and $\mathcal{L}^N$ respectively. For two sequences $U := (u(k))_{k\in\mathbf{Z}^d} \in \ell$ and $V := (v(k))_{k\in\mathbf{Z}^d} \in \ell_0$, we define their convolution $W := U * V := (w(k))_{k\in\mathbf{Z}^d}$ by

$$w(k) := \sum_{k'\in\mathbf{Z}^d} u(k')v(k-k'), \quad k \in \mathbf{Z}^d.$$

One may easily verify that $\mathcal{Z}(U*V)(z) = \mathcal{Z}(U)(z)\mathcal{Z}(V)(z)$ for all $U, V \in \ell_0$. So we define the product $\mathcal{Z}(U)(z)\mathcal{Z}(V)(z)$ of $\mathcal{Z}$-transform of $U \in \ell$ and $V \in \ell_0$ as the $\mathcal{Z}$-transform of their convolution, i.e.,

$$\mathcal{Z}(U)(z)\mathcal{Z}(V)(z) := \mathcal{Z}(U*V)(z).$$

Lemma 6.6. *Let $Q_i \in (\ell_0)^N, 1 \le i \le r$. Then $Q_i, 1 \le i \le r$, are strongly linearly independent if and only if there exist $B(z) \in \mathcal{L}^{r\times N}$ and $0 \ne a(z) \in \mathcal{L}$ such that*

$$B(z)(\mathcal{Z}(Q_1)(z), \ldots, \mathcal{Z}(Q_r)(z)) = a(z)I_r, \tag{6.16}$$

where I_r is the $r \times r$ unit matrix.

Proof. First the sufficiency. Take any $R_i(z) \in \mathcal{L}, 1 \le i \le r$, satisfying $\sum_{i=1}^r \mathcal{Z}(Q_i)(z)R_i(z) = 0$. Then it suffices to prove

$$R_i(z) \equiv 0, \ 1 \le i \le r. \tag{6.17}$$

By (6.16), the $N \times r$ matrix $Q(z) := (\mathcal{Z}(Q_1)(z), \ldots, \mathcal{Z}(Q_r)(z))$ is of full rank r for all $z \in (\mathbf{C}\backslash\{0\})^d$ with $a(z) \ne 0$. Therefore $R_i(z) = 0, 1 \le i \le r$, for all $z \in (\mathbf{C}\backslash\{0\})^d$ with $a(z) \ne 0$, which leads to (6.17).

Then the necessity. We claim that for any nonzero $N \times r$ matrix $Q(z)$ of Laurent polynomials, there exist $0 \ne a(z) \in \mathcal{L}$ and $1 \le s \le r$ such that

$$a(z)Q(z) = A_1(z)A_2(z)A_3(z) \tag{6.18}$$

where $A_1(z) \in \mathcal{L}^{N\times N}$ and $A_3(z) \in \mathcal{L}^{r\times r}$ satisfy $\det A_1(z) \det A_3(z) \not\equiv 0$, and $A_2(z) = \begin{pmatrix} A_4(z) & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}^{N\times r}$ for some diagonal matrix $A_4(z) \in \mathcal{L}^{s\times s}$ with $\det A_4(z) \not\equiv 0$. In one dimension, the above decomposition follows from the usual Smith decomposition (see for instance [22, p.150]). The above decomposition in high dimensions can be established by using elementary transforms: interchange two rows(columns); multiply a row by a nonzero Laurent polynomial; and add a polynomial multiple of a row(column) to another row(column). So we omit the detail of the proof here. Let $Q(z) := (\mathcal{Z}(Q_1)(z), \ldots, \mathcal{Z}(Q_r)(z))$ has the decomposition (6.18). Then $s = r$ by the strongly linear independence of $Q_i, 1 \le i \le r$, which leads to $A_2(z) = \begin{pmatrix} A_4(z) \\ 0 \end{pmatrix}$ for some $r \times r$ matrix $A_4(z)$ with $\det A_4(z) \not\equiv 0$. For

$$B(z) := A_3(z)^*(A_4(z)^*, 0)A_1(z)^*, \tag{6.19}$$

we have

$$\begin{aligned} B(z)Q(z) &= A_3(z)^*(A_4(z)^*, 0)A_1(z)^*A_1(z)\begin{pmatrix} A_4(z) \\ 0 \end{pmatrix} A_3(z) \\ &= \det A_1(z) \det A_4(z) \det A_3(z) I_r, \end{aligned}$$

where $A(z)^*$ is the adjoint matrix of $A(z)$. Hence (6.16) follows by letting $B(z)$ be as in (6.19) and $a(z) = \det A_1(z) \det A_4(z) \det A_3(z)$. $\square$

To find the inverse sequence of a nonzero finitely supported sequence, we need introduce a sequence space $\ell_+(\mathbf{Z}^d)$, which is closed under the convolution. Let $\ell_+(\mathbf{Z})$ be the space of all sequences $U := (u(k))_{k\in\mathbf{Z}}$ such that $u(k) = 0$ for all $k \le N$, where $N \in \mathbf{Z}$ depends on the sequence U. Inductively we define $\ell_+(\mathbf{Z}^d), d \ge 2$, to be the space of all sequences $U := (u(k))_{k\in\mathbf{Z}^d}$ such that $(u(k', k_d))_{k'\in\mathbf{Z}^{d-1}} \in \ell_+(\mathbf{Z}^{d-1})$ for all $k_d \in \mathbf{Z}$, and such that $u(k', k_d) = 0$ for all $k_d \le N$ and $k' \in \mathbf{Z}^{d-1}$, where $N \in \mathbf{Z}$ depends on U. For two sequences $U := (u(k))_{k\in\mathbf{Z}^d} \in \ell_+(\mathbf{Z}^d)$ and $V := (v(k))_{k\in\mathbf{Z}^d} \in \ell_+(\mathbf{Z}^d)$, we define their convolution $W = U * V := (w(k))_{k\in\mathbf{Z}^d} \in \ell_+(\mathbf{Z}^d)$ by

$$w(k) := \sum_{k'\in\mathbf{Z}^d} u(k')v(k-k'), \quad k \in \mathbf{Z}^d.$$

The above sum is well defined since for any $k \in \mathbf{Z}^d$, only finite many of $u(k')v(k-k'), k' \in \mathbf{Z}^d$, are nonzero. Similar to the product of $\mathcal{Z}$-transform of two sequences in ℓ and ℓ_0, we define the product of $\mathcal{Z}$-transform of two sequences in $\ell_+(\mathbf{Z}^d)$ by the $\mathcal{Z}$-transform of their convolution, i.e.,

$$\mathcal{Z}(U)(z)\mathcal{Z}(V)(z) := \mathcal{Z}(U * V)(z)$$

for $U, V \in \ell_+(\mathbf{Z}^d)$.

Lemma 6.7. *Let $0 \ne V := (v(k))_{k\in\mathbf{Z}^d} \in \ell_0(\mathbf{Z}^d)$. Then there exists $W := (w(k))_{k\in\mathbf{Z}^d} \in \ell_+(\mathbf{Z}^d)$ such that*

$$W * V = \delta \tag{6.20}$$

where $\delta := (\delta(k))_{k\in\mathbf{Z}^d}$ is the usual delta sequence defined by $\delta(0) = 1$ and $\delta(k) = 0$ otherwise.

Proof. We prove the assertion by the induction on the dimension d. For $d = 1$, write

$$\mathcal{Z}(V)(z) = az^{L_1}(1 + z^{-1}Q(z^{-1})),$$

where $a \in \mathbf{R}$, $L_1 \in \mathbf{Z}$ and $Q(z)$ is a polynomial. Define $U_n := (u_n(k))_{k\in\mathbf{Z}}$ by

$$\mathcal{Z}(U_n)(z) = a^{-1}z^{-L_1}(-z^{-1}Q(z^{-1}))^n, \ n \ge 0.$$

One may verify that $u_n(k) = 0$ for all $k \le n + L_1 - 1$. Thus $W := \sum_{n=0}^{\infty} U_n$ is well defined, belongs to $\ell_+(\mathbf{Z})$, and

$$\begin{aligned} &\mathcal{Z}(V)(z)\mathcal{Z}(W)(z) \\ = \ & az^{L_1}(1 + z^{-1}Q(z^{-1}))a^{-1}z^{-L_1}\sum_{n=0}^{\infty}(-z^{-1}Q(z^{-1}))^n = 1, \end{aligned}$$

which leads to (6.20) for $d = 1$. Inductively we assume that the assertion holds for any nonzero sequence in $\ell_0(\mathbf{Z}^{d-1})$. Take any nonzero sequence $V \in \ell_0(\mathbf{Z}^d)$, we write

$$\mathcal{Z}(V)(z', z_d) = z^L(a(z'^{-1}) + z_d^{-1} Q(z'^{-1}, z_d^{-1}))$$

where $L \in \mathbf{Z}^d$, $a(z')$ is a nonzero polynomial of z', and $Q(z', z_d)$ is a polynomial of z' and z_d. By the inductive hypothesis, there exists a sequence $W_1 \in \ell_+(\mathbf{Z}^{d-1})$ such that $a(z'^{-1})\mathcal{Z}(W_1)(z') = 1$. Thus $R(z', z_d) := z_d^{-1} Q(z'^{-1}, z_d^{-1})\mathcal{Z}(W_1)(z')$ is the $\mathcal{Z}$-transform of a sequence in $\ell_+(\mathbf{Z}^d)$. Define $U_n = (u_n(k))_{k\in\mathbf{Z}^d} \in \ell_+(\mathbf{Z}^d), n \ge 0$, by

$$\mathcal{Z}(U_n)(z) = z^{-L}\mathcal{Z}(W_1)(z')(-1)^n R(z', z_d)^n.$$

One may verify that $u_n(k', k_d) = 0$ for all $k_d \le n + L_d$ and $k' \in \mathbf{Z}^{d-1}$, where $L = (L', L_d) \in \mathbf{Z}^d$. Therefore $W := \sum_{n=0}^\infty U_n$ is well defined, belongs to $\ell_+(\mathbf{Z}^d)$, and

$$\begin{aligned} \mathcal{Z}(V)(z)\mathcal{Z}(W)(z) &= z^L(a(z'^{-1}) + z_d^{-1} Q(z'^{-1}, z_d^{-1}))z^{-L}\mathcal{Z}(W_1)(z') \\ &\quad \times \sum_{n=0}^\infty (-1)^n \Big(z_d^{-1} Q(z'^{-1}, z_d^{-1})\mathcal{Z}(W_1)(z')\Big)^n \\ &= (1 + R(z', z_d)) \sum_{n=0}^\infty (-1)^n R(z', z_d)^n = 1, \end{aligned}$$

where $z = (z', z_d)$. Thus W is a sequence in $\ell_+(\mathbf{Z}^d)$ satisfying (6.20). □

Proof of Theorem 5.1. The existence of the decomposition of the form (5.2) follows from the assumption on $Q_1, \ldots, Q_M$. Let $\psi_1, \ldots, \psi_r$ be defined by (5.3). Then it suffices to prove the uniqueness of the decomposition of the form (5.2), $\psi_1, \ldots, \psi_r$ have linear independent shifts, belong to $V(F)$, and satisfy (5.4).

At first the uniqueness of the decomposition (5.2). Let $\varphi_1, ..., \varphi_r$ be another family of compactly supported distributions such that $F = \sum_{i=1}^r \sum_{j\in\mathbf{Z}^d} q_i(j)\varphi_i(\cdot - j)$. Substituting the above formula for F into (5.2), and then taking Fourier transform, we obtain

$$\sum_{i=1}^r \mathcal{F}(Q_i)(\xi)(\widehat{\psi}_i(\xi) - \widehat{\varphi}_i(\xi)) = 0 \tag{6.21}$$

Then $\psi_i = \varphi_i, 1 \le i \le r$, by the strongly linear independence of $Q_1, \ldots, Q_r$ and Lemma 6.4. This proves the uniqueness.

Secondly the linear independent shifts of $\psi_1, \dots, \psi_r$. Let $\phi_1, \dots, \phi_M$ be as in (1.1), set $\Psi = (\psi_1, \dots, \psi_r)^T$ and $\Phi = (\phi_1, \dots, \phi_M)^T$. Taking Fourier transform at both sides of the equation (5.3), we have

$$\widehat{\Psi} = (\mathcal{F}(\alpha))^T \widehat{\Phi},$$

where we set $\mathcal{F}(\alpha) = (\mathcal{F}(\alpha_{ii'}))_{1 \le i \le M, 1 \le i' \le r}$. Thus

$$\big(\widehat{\Psi}(\xi + 2j\pi)\big)_{j \in \mathbf{Z}^d} = (\mathcal{F}(\alpha)(\xi))^T \big(\widehat{\Phi}(\xi + 2j\pi)\big)_{j \in \mathbf{Z}^d}. \tag{6.22}$$

By the linear independent shifts of Φ and Lemma 6.3, we obtain

$$\operatorname{rank}\big(\widehat{\Phi}(\xi + 2j\pi)\big)_{j \in \mathbf{Z}^d} = M. \tag{6.23}$$

By Lemma 6.5, $\mathcal{F}(\alpha)(\xi)$ has full rank $r \le M$ for any $\xi \in \mathbf{C}^d$. This, together with (6.22) and (6.23), lead to

$$\operatorname{rank}\ \big(\widehat{\Psi}(\xi + 2j\pi)\big)_{j \in \mathbf{Z}^d} = r \quad \forall\ \xi \in \mathbf{C}^d.$$

Therefore $\psi_1, \dots, \psi_r$ have linear independent shifts by Lemma 6.3.

Thirdly $\psi_1, \dots, \psi_r \in V(F)$. Taking Fourier transform at both sides of the equation (5.2) gives

$$\widehat{F}(\xi) = \sum_{i=1}^{r} \mathcal{Z}(Q_i)(e^{i\xi}) \widehat{\psi}_i(\xi). \tag{6.24}$$

By the strongly linearly independent assumption on $Q_1, \dots, Q_r$ and Lemma 6.6, there exist $0 \ne a(z) \in \mathcal{L}$ and $b_i(z) \in \mathcal{L}^N, 1 \le i \le r$, such that for any $1 \le i, i' \le r$, it holds

$$b_i(z)^T Q_{i'}(z) = \begin{cases} a(z) & \text{if } i = i', \\ 0 & \text{otherwise.} \end{cases}$$

Then the functions $g_i, 1 \le i \le r$, defined by

$$\widehat{g}_i(\xi) = b_i(e^{-i\xi})^T \widehat{F}(\xi), \tag{6.25}$$

satisfy

$$a(e^{-i\xi}) \widehat{\psi}_i(\xi) = \widehat{g}_i(\xi). \tag{6.26}$$

Let $W := (w(j))_{j \in \mathbf{Z}^d} \in \ell_+(\mathbf{Z}^d)$ be a sequence satisfying

$$a(z) \mathcal{Z}(W)(z) = 1. \tag{6.27}$$

The existence of the above sequence w follows from Lemma 6.7. Combining (6.26) and (6.27) gives $\psi_i = \sum_{j \in \mathbf{Z}} w(j) g_i(\cdot - j)$, which together with (6.25) implies that ψ_i is (infinite) linear combination of the shifts of F. This proves $\psi_1, \dots, \psi_r \in V(F)$.

Fourthly we prove (5.4). Set $\Psi=(\psi_1,\ldots,\psi_r)^T$. By (5.2), we have $V(F)\subset V(\Psi)$. Then it suffices to prove

$$V(\Psi)\subset V(F). \tag{6.28}$$

Take any sequence $V:=(v(j))_{j\in\mathbf{Z}^d}\in\ell(\mathbf{Z}^d)$, and decompose it as the sum of $V_\epsilon\in\ell_\epsilon(\mathbf{Z}^d)$,

$$V=\sum_{\epsilon\in\{0,1\}^d}V_\epsilon, \tag{6.29}$$

where for any $\epsilon\in\{0,1\}^d$ we define

$$\ell_\epsilon(\mathbf{Z}^d)=\{(c(j))_{j\in\mathbf{Z}^d}:\ (c((-1)^\epsilon j))_{j\in\mathbf{Z}^d}\in\ell_+(\mathbf{Z}^d)\}.$$

In fact, the sequences $V_\epsilon:=(v_\epsilon(j))_{j\in\mathbf{Z}^d}$ can be defined by $v_\epsilon(j)=2^{-l(j)}v(j)$ for all j with all components of $(-1)^j$ being nonnegative, and $v_\epsilon(j)=0$ otherwise, where $l(j)$ is the cardinality of the set $\{1\le k\le d: j_k=0\}$ for $j=(j_1,\ldots,j_d)\in\mathbf{Z}^d$. By the proof of the assertion $\psi_1,\ldots,\psi_r\in V(F)$, there exist sequences $W_\epsilon:=(w_\epsilon(j))_{j\in\mathbf{Z}^d}\in(\ell_\epsilon(\mathbf{Z}^d))^{r\times N}$ such that

$$\Psi=\sum_{j\in\mathbf{Z}^d}w_\epsilon(j)F(\cdot-j). \tag{6.30}$$

Combining (6.29) and (6.30), we get

$$\begin{aligned}\sum_{j\in\mathbf{Z}^d}v(j)^T\Psi(\cdot-j)&=\sum_{\epsilon\in\{0,1\}^d}\sum_{j,j'\in\mathbf{Z}^d}v_\epsilon(j')^Tw_\epsilon(j-j')F(\cdot-j)\\&=\sum_{\epsilon\in\{0,1\}^d}\sum_{j\in\mathbf{Z}^d}\Big(\sum_{j'\in\mathbf{Z}^d}v_\epsilon(j')^Tw_\epsilon(j-j')\Big)F(\cdot-j)\in V(F),\end{aligned}$$

where we have used the fact that $\left(\sum_{j'\in\mathbf{Z}^d}v_\epsilon(j')^Tw_\epsilon(j-j')\right)_{j\in\mathbf{Z}^d}\in\ell_\epsilon(\mathbf{Z}^d)$. This proves (6.28) and hence completes the proof of (5.4).

Finally we prove the refinability of $\psi_1,\ldots,\psi_r$. For any space V of distributions, let

$$D_2V=\{f(2\cdot): f\in V\}.$$

By the refinability of F, we have $V(F)\subset D_2V(F)$. This together with (5.4) leads to $V(\psi_1,\ldots,\psi_r)\subset D_2(\psi_1,\ldots,\psi_r)$. Thus there exists sequences $(c(j))_{j\in\mathbf{Z}}\in(\ell)^{r\times r}$ such that

$$\Psi=\sum_{j\in\mathbf{Z}^d}c(j)\Psi(2\cdot-j). \tag{6.31}$$

By the linear independent shifts of Ψ and Lemma 6.1, the sequence $(c(j))_{j\in\mathbf{Z}^d}$ has finite support. This together with (6.31) proves the refinablity of Ψ. $\square$

Proof of Theorem 5.3. Let F have the decomposition (5.2) with $Q_i := (q_i(j))_{j\in\mathbf{Z}^d} \in (\ell_0)^N$ and $\psi_1, \dots, \psi_r \in V(F)$ having linear independent shifts. By Theorem 2.1, $\{Q_1, \dots, Q_r\}$ is a generator of the dependent ideal $i(F)$. By the decomposition in (6.18), there exists $0 \neq a(z) \in \mathcal{L}$, $A_1(z) \in \mathcal{L}^{N\times N}$, $A_3(z) \in \mathcal{L}^{r\times r}$ and a diagonal matrix $A_4(z) \in \mathcal{L}^{s\times s}$ such that $\det A_1(z) \det A_3(z) \det A_4(z) \not\equiv 0$ and

$$a(z)\mathcal{Z}(Q)(z) = A_1(z) \begin{pmatrix} A_4(z) & 0 \\ 0 & 0 \end{pmatrix} A_3(z), \tag{6.32}$$

where $1 \leq s \leq r$ and $Q := (q(j))_{j\in\mathbf{Z}^d} = \big((q_1(j), \dots, q_r(j))\big)_{j\in\mathbf{Z}^d} \in (\ell_0)^{N\times r}$. By Lemma 6.6, it suffices to prove

$$s = r. \tag{6.33}$$

By the assumption $\psi_1, \dots, \psi_r \in V(F)$, there exists a matrix-valued sequence $P := (p(j))_{j\in\mathbf{Z}^d} \in (\ell)^{r\times N}$ such that

$$\Psi = \sum_{j\in\mathbf{Z}} p(j)F(\cdot - j), \tag{6.34}$$

where $\Psi = (\psi_1, \dots, \psi_r)^T$. By (5.2), we have

$$F = \sum_{j\in\mathbf{Z}} q(j)\Psi(\cdot - j). \tag{6.35}$$

Combining (6.34) and (6.35) gives

$$\Psi = \sum_{j,j'\in\mathbf{Z}^d} p(j')q(j - j')\Psi(\cdot - j).$$

This together with the linear independent shifts of Ψ leads to

$$\sum_{j'\in\mathbf{Z}^d} p(j')q(j - j') = I_r\delta_j \quad \forall\ j \in \mathbf{Z}^d.$$

Thus

$$\mathcal{Z}(P)(z)\mathcal{Z}(Q)(z) = I_r. \tag{6.36}$$

Substituting the decomposition (6.32) into (6.36), and multiplying $A_3(z)$ and its adjoint $A_3(z)^*$ at both sides from the left and right, we obtain

$$A_3(z)\mathcal{Z}(P)(z)A_1(z) \begin{pmatrix} A_4(z) & 0 \\ 0 & 0 \end{pmatrix} = a(z)I_r.$$

This leads to (6.33) and completes the proof of Theorem 5.3. $\square$

Appendix A. Shift-Invariant Linear Subspaces of $(\ell_0)^N$

In the appendix, we give some basic properties of shift-invariant subspaces of $(\ell_0)^N$. Those properties are known for the scale-valued case, i.e., $N = 1$ (see [7] for instance). We include a proof here for the completeness of this paper.

Proposition A.1. *Every shift-invariant linear subspace of $(\ell_0(\mathbf{Z}^d))^N$ is closed and finitely generated. Furthermore if $d = 1$, then it is quasi-principal.*

Let $\mathcal{L}$ be the space of all Laurent polynomials on $(\mathbf{C}\backslash\{0\})^d$ and let $\mathcal{L}^N$ be N copies of $\mathcal{L}$. We say that a linear subspace of $\mathcal{L}^N$ is an *ideal* if it is invariant under multiplying $z^k, k \in \mathbf{Z}^d$. An ideal $\mathcal{J}$ of $\mathcal{L}^N$ is said to be *finitely generated* if there exist $Q_s(z) \in \mathcal{J}, 1 \le s \le N_1$, such that any $R(z) \in \mathcal{J}$ can be written as $R(z) = \sum_{s=1}^{N_1} R_s(z)Q_s(z)$, where $R_1(z), \ldots, R_{N_1}(z) \in \mathcal{L}$, and to be *quasi-principal* if there exist a generator $\{p_1(z), \ldots, p_{N_2}(z)\}$ is of $\mathcal{J}$ such that $\sum_{s=1}^{N_2} R_s(z)p_s(z) = 0$ and $R_s(z) \in \mathcal{L}$ imply $R_s(z) = 0$ for all $1 \le s \le N_2$.

For any $D = (d(j))_{j\in\mathbf{Z}^d} \in (\ell_0)^N$, define $D(z) = \sum_{j\in\mathbf{Z}^d} d(j)z^{-j}$. Then the map $D \longmapsto D(z)$ establishes a one-to-one correspondence between $(\ell_0)^N$ and $\mathcal{L}^N$. For any $\mathcal{I} \subset (\ell_0)^N$, let $\mathcal{I}(z)$ be the space of corresponding Laurent polynomials. Then $\mathcal{I}$ is an ideal of $(\ell_0)^N$ if and only if $\mathcal{I}(z)$ is an ideal of $\mathcal{L}^N$, $\mathcal{I}$ is finitely generated if and only if $\mathcal{I}(z)$ is finitely generated, and $\mathcal{I}$ is quasi-principal if and only if $\mathcal{I}(z)$ is quasi-principal.

Proof of Proposition A.1. Let $\mathcal{I}$ be a shift-invariant subspace of $(\ell_0)^N$. Then for any finite set $K \subset \mathbf{Z}^d$, $\mathcal{I} \cap (\ell_K)^N$ is finite dimensional and closed, where $(\ell_K)^N$ is the space of all $\mathbf{R}^N$-valued sequences supported in K. Then the closedness of $\mathcal{I}$ in the topology of $(\ell_0)^N$ follows.

Secondly we prove that $\mathcal{I}$ is finitely generated. Let $\mathcal{J} = \mathcal{I}(z)$. Then it clearly suffices to prove that $\mathcal{J}$ is finitely generated. We prove the above assertion by induction on N. For $N = 1$, it is known that $\mathcal{J}$ is finitely generated ([7]). Inductively we assume that the assertion holds for $N - 1$, and start to prove the assertion for $N \ge 2$. Define

(A.1)
$$\mathcal{J}_1 = \{q_1(z) : \ (q_1(z), \ldots, q_N(z))^T \in \mathcal{J} \ \text{ for some } q_2(z), \ldots, q_N(z) \in \mathcal{L}\}$$

and

$$\mathcal{J}_2 = \{(q_2(z), \ldots, q_N(z))^T : \ (0, q_2(z), \ldots, q_N(z))^T \in \mathcal{J}\}. \tag{A.2}$$

It is easy to check that $\mathcal{J}_1$ and $\mathcal{J}_2$ are ideals of $\mathcal{L}$ and $\mathcal{L}^{N-1}$ respectively. If $\mathcal{J}_1 = \{0\}$ or $\mathcal{J}_2 = \{0\}$, then the assertion follows easily from the inductive hypothesis. So we may assume that $\mathcal{J}_1 \neq \{0\}$ and $\mathcal{J}_2 \neq \{0\}$ hereafter. By inductive hypothesis, there exist $p_1(z), \ldots, p_{N_1}(z) \in \mathcal{J}_1$ and $p_{N_1+1}(z), \ldots, p_{N_2}(z) \in \mathcal{J}_2$ such that $\mathcal{J}_1$ and $\mathcal{J}_2$ are generated by $\{p_1(z), \ldots, p_{N_1}(z)\}$ and $\{p_{N_1+1}(z), \ldots, p_{N_2}(z)\}$ respectively. By the definition of $\mathcal{J}_1$, there exist $P_1(z), \ldots, P_{N_1}(z) \in \mathcal{J}$ such that the first component of $P_s(z)$ is $p_s(z), 1 \leq s \leq N_1$. Set $P_t(z) = (0, p_t(z)^T)^T, N_1+1 \leq t \leq N_2$. Then $P_1(z), \cdots, P_{N_2}(z) \in \mathcal{J}$. Now it remains to prove that $\{P_1(z), \cdots, P_{N_2}(z)\}$ is a generator of $\mathcal{J}$. Let $Q(z) \in \mathcal{J}$ and $q_1(z)$ be its first component. Then $q_1(z) \in \mathcal{J}_1$, which leads to the existence of $R_1(z), \ldots, R_{N_1}(z) \in \mathcal{L}$ such that $q_1(z) = \sum_{s=1}^{N_1} R_s(z) p_s(z)$. Hence the first component of $Q(z) - \sum_{s=1}^{N_1} R_s(z) P_s(z)$ is zero, and $Q(z) - \sum_{s=1}^{N_1} R_s(z) P_s(z)$ belongs to $\mathcal{J}_2$. Thus $Q(z) - \sum_{s=1}^{N_1} R_s(z) P_s(z) = \sum_{t=N_1+1}^{N_2} R_t(z) P_t(z)$ for some $R_{N_1+1}(z), \ldots, R_{N_2}(z) \in \mathcal{L}$. This proves that $\{P_1(z), \ldots, P_{N_2}(z)\}$ is a generator of $\mathcal{J}$, and hence $\mathcal{J}$ is finitely generated.

Finally, we prove that $\mathcal{I}$ is quasi-principal for the case $d = 1$. Clearly, it suffices to prove that for $d = 1$, $\mathcal{J} = \mathcal{I}(z)$ is quasi-principal. We prove the assertion by induction on $N \geq 1$. For $N = 1$, $\mathcal{J}$ is a Laurent polynomial ideal. Then there exists $p(z) \in \mathcal{J}$ such that $\mathcal{J}$ is generated by $p(z)$. Hence the assertion holds for $N = 1$. Inductively, we assume that the assertion holds for $N - 1$, and start to prove the assertion for $N \geq 2$. Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be as in (A.1) and (A.2). If $\mathcal{J}_1 = \{0\}$ or $\mathcal{J}_2 = \{0\}$, then the assertion follows easily from the inductive hypothesis. So we may assume that $\mathcal{J}_1 \neq \{0\}$ and $\mathcal{J}_2 \neq \{0\}$ hereafter. By the inductive hypothesis, there exist $p_1(z) \in \mathcal{J}_1$ and $p_2(z), \ldots, p_{N_1}(z) \in \mathcal{J}_2$ such that $\mathcal{J}_1$ and $\mathcal{J}_2$ are generated by $p_1(z)$ and $\{p_2(z), \ldots, p_{N_1}(z)\}$ respectively, and such that $p_2(z), \ldots, p_{N_1}(z)$ are strongly linearly independent. Define $P_1(z), \ldots, P_{N_1}(z)$ as the ones in the proof of the existence of a generator of $\mathcal{J}$ with finite cardinality. Then $\{P_1(z), \ldots, P_{N_1}(z)\}$ is a generator of $\mathcal{J}$ by the proof there. Hence it remains to prove strongly linear independence of $P_s(z), 1 \leq s \leq N_1$. Let $R_1(z), \ldots, R_{N_1}(z)$ be any Laurent polynomials such that $\sum_{s=1}^{N_1} R_s(z) P_s(z) = 0$. Then $R_1(z) = 0$ since the first component of $\sum_{s=1}^{N_1} R_s(z) P_s(z)$ is $R_1(z) p_1(z)$. Substituting $R_1(z) = 0$ into $\sum_{s=1}^{N_1} R_s(z) P_s(z) = 0$ leads to $\sum_{s=2}^{N_1} R_s(z) P_s(z) = 0$, which, together with strongly linear independence of $p_2(z), \ldots, p_{N_1}(z)$, implies that $R_s(z) = 0$ for all $2 \leq s \leq N_1$. This proves the strongly linear independence of $P_1, \ldots, P_{N_1}$, and hence $\mathcal{J}$ is quasi-principal. □

Proposition A.2. *Let $\mathcal{N}$ be a counterable set, and $\mathcal{I}$ be a closed linear subspace of $\ell_0(\mathcal{N})$. Then $(\mathcal{I}_\perp)_\perp = \mathcal{I}$*

Proof. Let $\mathcal{K} = \mathcal{I}_\perp$. Clearly we have

$$\mathcal{I} \subset (\mathcal{I}_\perp)_\perp. \tag{A.3}$$

Suppose, on the contrary, that $(\mathcal{I}_\perp)_\perp \neq \mathcal{I}$. Then there exists $b \in (\mathcal{I}_\perp)_\perp \backslash \mathcal{I}$ by (A.3). Let $\mathcal{I}_0$ be the space spanned by $\mathcal{I}$ and b. It follows from the closedness of $\mathcal{I}_0$ that every element x in $\mathcal{I}_0$ can be uniquely written as $x = a + \lambda b$ for some $a \in \mathcal{I}$ and $\lambda \in \mathbf{R}$. Thus we may define a linear function l_0 from $\mathcal{I}_0$ to $\mathbf{R}$ by $l_0(x) = \lambda$ if $x = a + \lambda b \in \mathcal{I}_0$. Let $e_n, n \geq 1$ be the standard basis of $\ell_0(\mathcal{N})$. Let $\mathcal{I}_n$ be the space spanned by $\mathcal{I}_0$ and $e_1, \ldots, e_n$. Clearly we have the following inclusion:

$$\mathcal{I}_0 \subset \mathcal{I}_1 \subset \mathcal{I}_2 \subset \cdots.$$

Let $n_k, k \geq 1$, be chosen so that $\mathcal{I}_0 = \mathcal{I}_l$ for $l < n_1$, and $\mathcal{I}_{n_k} = \mathcal{I}_n \neq \mathcal{I}_{n_{k+1}}$ for all $n_k < n < n_{k+1}, 1 \leq k$. Now we define $l_k, k \geq 1$, from $\mathcal{I}_{n_k}$ to $\mathbf{R}$ by $l_k(x) = l_{k-1}(y)$ if $x = y + \lambda e_{n_k}$ for some $y \in \mathcal{I}_{n_{k-1}}$ and $\lambda \in \mathbf{R}$. The linear function l_k is well-defined since $e_{n_k} \notin \mathcal{I}_{n_{k-1}}$ and $\mathcal{I}_{n_{k-1}}$ is closed. Now we define a linear function l_∞ on $\ell_0(\mathcal{N})$ by $l_\infty(x) = l_k(x)$ if $x \in \mathcal{I}_{n_k}$. Clearly the above linear function is well defined from the construction of the linear functions ℓ_k and the fact that for any $x \in \ell_0(\mathcal{N})$, there exists $N \geq 1$ such that x belongs to the space spanned by $e_1, \ldots, e_N$. Since $\ell(\mathcal{N})$ is the dual of $\ell_0(\mathcal{N})$. Therefore there exists $c \in \ell(\mathcal{N})$ such that

$$l(x) = \langle x, c \rangle \quad \text{for all } x \in \ell_0(\mathcal{N}).$$

From the definition of the linear functional ℓ, we see that $\langle x, c \rangle = 0$ for all $x \in \mathcal{I}$, which implies that $c \in \mathcal{I}_\perp$. This is a contradiction, since $b \in (\mathcal{I}_\perp)_\perp$ and $\langle b, c \rangle = 1$. □

References

[1] A. Aldroubi and Q. Sun, Locally finite dimensional shift-invariant spaces in R^d, *Proc. Amer. Math. Soc.*, **130**(2002), 2641-2654.

[2] A. Ben-Artzi and A. Ron, On the integer translates of a compactly supported function: dual bases and linear projectors, *SIAM J. Math. Anal.*, **21**(1990), 1550-1562.

[3] S. K. Berberian, *Lectures in Functional Analysis and Operator Theory*, Springer-Verlag, 1974.

[4] C. de Boor and A. Ron, Polynomial ideals and multivariate spline, In *Multivariate Approximation Theory IV*, C. Chui, W. Schempp, and K. Zeller (eds.), ISNM Vol.90 Birkhäuser Verlag (Basel), 1989, 31-40.

[5] A. Cavaretta, W. Dahmen and C. A. Micchelli, Stationary subdivision, *Memoir Amer. Math. Soc.*, **453**(1991), 1-186.

[6] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics, 61., SIAM, Philadelphia, PA, 1992.

[7] R. Fröberg, *An Introduction to Gröbner Bases*, John Wiley & Sons, 1997.

[8] T. N. T. Goodman, S. L. Lee and W.-S. Tang, Wavelets in wandering subspaces, *Trans. Amer. Math. Soc.*, **338**(1993), 639-654.

[9] T. A. Hogan and R. Q. Jia, Dependency relations among the shifts of a multivariate refinable distribution, *Constr. Approx.*, **17**(2001), 19-37.

[10] R. Q. Jia, Subdivision schemes in L_p spaces. *Adv. Comput. Math.*, **3**(1995), 309-341.

[11] R. Q. Jia, Shift-invariant spaces and linear operator equations, *Israel Math. J.*, **103**(1998), 259-288.

[12] R. Q. Jia and C. A. Micchelli, Using the refinement equation for the construction of pre-wavelets II: powers of two, In *Curve and Surfaces*, P. J. Laurent, A. Le Méhauté and L. L. Schumaker eds., Academic Press, 1991, pp. 209-246.

[13] R. Q. Jia and C. A. Micchelli, On linear independence of integer translates of a finite number of functions, *Proc. Edinburgh Math. Soc.*, **36**(1992), 69-75.

[14] R. Long, W. Chen, S. Yuan, Wavelets generated by vector multiresolution analysis, *Appl. Comput. Harmon. Anal.*, **4**(1997), 317-350.

[15] B. Ma and Q. Sun, Compactly supported refinable distribution in Triebel-Lizorkin spaces and Besov spaces, *J. Fourier Anal. Appl.*, **5**(1999), 87-104.

[16] A. Ron, A necessary and sufficient condition for the linear independence of the integer translates of a compactly supported distribution, *Constr. Approx.*, **5**(1989), 297-308.

[17] A. Ron, Wavelets and their associated operators, In *Approximation Theory IX*, C. K. Chui and L. L. Schumaker eds., Vanderbilt University Press, Nashville, pp. 1-35.

[18] A. Ron, Introduction to shift-invariant spaces I: linear independence, In *Multivariate Approximation and Applications*, A. Pinkus, D. Leviatan, N. Dyn, and D. Levin eds., Cambridge University Press, Cambridge, 2001.

[19] Q. Sun, Stability of the shifts of global supported distributions, *J. Math. Anal. Appl.*, **261**(2001), 113-125.

[20] Q. Sun, Linear independence of the integer translates of compactly supported distributions and refinable vectors, *Applied Math.- A Journal of Chinese University*, **16B**(2001), 381-396.

[21] Q. Sun, Convergence of cascade algorithms and smoothness of refinable distributions, *Chinese Ann. Math.*, to appear.

[22] B. W. Suter, *Multivariate and Wavelet Signal Processing*, Academic Press, 1997.

[23] K. Zhao, Global linear independence and finitely supported dual basis, *SIAM J. Math. Anal.*, **23**(1992), 1352-1355.

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