Local Polynomial Property and Linear Independence of Refinable Distributions

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Abstract

In this paper, local polynomial property, global linear independence, and local linear dependence of the convolution of a B-spline and a refinable distribution supported on a Cantor-like set are studied.

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1 Introduction and Main Result

Define Fourier transform $\hat{f}$ of an integrable function f by $\hat{f}(\xi) = \int_{\mathbf{R}} f(x)e^{-ix\xi}dx$ and interpret Fourier transform of a compactly supported distribution as usual. Fix an integer $M \geq 2$. In this paper, a compactly supported distribution f is said to be *refinable* if

$$f = \sum_{j\in\mathbf{Z}} c_j f(M\cdot -j) \tag{1}$$

and $\hat{f}(0) = 1$, where the sequence $\{c_j\}_{j\in\mathbf{Z}}$ has finite support, i.e., $c_j = 0$ for all but finite many $j \in \mathbf{Z}$, and satisfies $\sum_{j\in\mathbf{Z}} c_j = M$. Associated with the refinement equation (1) is the Laurent polynomial $H(z) := \frac{1}{M}\sum_{j\in\mathbf{Z}} c_j z^j$, which is known as the *symbol* of the refinement equation (1). The Fourier transform of both sides of the refinement equation (1) yields

$$\hat{f}(\xi) = H(e^{-i\xi/M})\hat{f}(\xi/M). \tag{2}$$

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We say that a compactly supported distribution f has *local polynomial property* if there is an open set $A \subset \text{supp}\, f$ such that $m(\text{supp}\, f - A) = 0$ and the restriction of f on any open interval contained in A coincides with some polynomial. Here $m(E)$ denotes the Lebesgue measure of a set E. For a refinable distribution, its analytic expression is useful in some practical applications, but we know little about it except B-splines ([7]). To some degree, a refinable function which is integrable and has local polynomial property can be regarded to have analytic expression, since analytic expression is given except on a set with Lebesgue measure zero. The local polynomial property was found in [1, 13] for the M band Daubechies' scaling functions $\phi_{M,N}$ with $N \leq M-1$ (see [1] for the precise statement).

We say that the integer shifts of a compactly supported distribution f are, or for simplicity f is, *globally linearly independent* if

$$\sum_{j\in\mathbf{Z}} e_j f(\cdot + j) \equiv 0 \quad \text{on} \quad \mathbf{R} \quad \text{implies} \quad e_j \equiv 0 \quad \forall\, j \in \mathbf{Z},$$

and *locally linearly independent* if for any open set A,

$$\sum_{j\in\mathbf{Z}} e_j f(\cdot + j) \equiv 0 \quad \text{on} \quad A \quad \text{and} \quad f(\cdot + j) \not\equiv 0 \quad \text{on} \quad A \quad \text{imply} \quad e_j = 0.$$

The global linear independence is a necessary condition for orthogonality, or biorthogonality, of a refinable function that we need at most times when a refinable distribution is used in the construction of various wavelets. The local linear independence plays an important role in spline interpolation as well as in nonlinear wavelet approximation (see [4, 6] and references therein). About the global and local linear independence of compactly supported refinable distributions, there is a long list of publications (see for instance [2, 6, 8, 9, 11, 12]). For a refinable distribution, its global linear independence is characterized by corresponding symbol in [8], but its local linear independence is **not** easy to be checked in general. When $M = 2$, it was proved that the local and global linear independence of refinable distributions are equivalent to each other (see [9] for refinable functions with biorthogonal dual and [11] for any refinable distributions). For refinable distributions with $M = 3$, it was pointed out in [2] that their local and global linear independence are not equivalent. Also some examples of refinable distributions, which are Hölder continuous, globally linearly independent but locally linearly dependent, are constructed in [2]. General discussion about the local linear independence of a refinable distribution can be found in [6, 12]. The local linear independence of any compactly supported distribution is characterized in [12].

Define the convolution of two integrable functions f and g by $f * g(x) = \int_{\mathbf{R}} f(x-y)g(y)dy$ and the one of two compactly supported distributions by usual interpretation.

Let h_0 and h_1 be Schwartz functions with supp $\widehat{h}_0 \subset \{\xi : |\xi| \leq 1\}$, supp $\widehat{h}_1 \subset \{\xi : 1/2 \leq |\xi| \leq 2\}$, and $\widehat{h}_0(\xi) + \sum_{s=0}^{\infty} \widehat{h}_1(2^{-s}\xi) = 1$ for all $\xi \in \mathbf{R}$. The Besov space $B_{\infty}^{\alpha,\infty}, \alpha \in \mathbf{R}$, is the space of all tempered distributions f with

$$\|h_0 * f\|_{\infty} + \sup_{s\geq 0} 2^{s\alpha} \|h_{1,s} * f\|_{\infty} < \infty$$

where $\|\cdot\|_{\infty}$ is the usual L^{∞} norm of bounded functions, and where $h_{1,s} = 2^s h_1(2^s \cdot), 0 \leq s \in \mathbf{Z}$. The purpose of this paper is to study the local polynomial property, global linear independence, and local linear independence of the convolution of a B-spline and a refinable distribution supported on a Cantor-like set. Precisely, we have

Theorem 1 *Let $Q(z) = \sum_{j=0}^{M-1} d_j z^j / M$ for some sequence $\{d_j\}_{j=0}^{M-1}$, and assume that $\sum_{j=0}^{M-1} d_j = M$ and the number of nonzero coefficients d_j is at least 2 but at most $M-1$. Define $\phi_N, N \geq 0$, recursively by $\widehat{\phi}_0(\xi) = \prod_{j=0}^{\infty} Q(e^{-i2^{-j}\xi})$, and $\phi_{N+1} = \chi_{[0,1]} * \phi_N$. Then for any $N \geq 1$, we have*

- *(i) there exists $\alpha_0 \in \mathbf{R}$ such that ϕ_N belongs to the Besov space $B_{\infty}^{\alpha_0+N,\infty}$;*
- *(ii) ϕ_N is refinable;*
- *(iii) ϕ_N has local polynomial property;*
- *(iv) the integer shifts of ϕ_N are globally linearly independent;*
- *(v) the integer shifts of ϕ_N are locally linearly dependent.*

Given any $\tau > 0$, as a consequence of Theorem 1, for sufficiently large N, ϕ_N in Theorem 1 are refinable distributions in the space of all Hölder continuous functions with Hölder index τ, whose integer shifts are globally linearly independent but locally linearly dependent.

2 Proof

Define B-spline $B_N, N \geq 1$, by $\widehat{B_N}(\xi) = ((1 - e^{-i\xi})/(i\xi))^N$. Obviously $B_N, N \geq 1$, are refinable with corresponding symbol $((1 - z^M)/(M - Mz))^N$. The B-spline B_N can also be defined as the convolution of the characteristic function on $[0, 1]$ for N times,

$$B_N = \chi_{[0,1]} * \chi_{[0,1]} * \cdots * \chi_{[0,1]} \quad (N \text{ times}).$$

From the construction of ϕ_N, we have

$$\phi_N = B_N * \phi_0. \tag{3}$$

The Fourier transform of both sides of the equation (3) yields

$$\widehat{\phi}_N(\xi) = \widehat{B}_N(\xi)\widehat{\phi}_0(\xi). \tag{4}$$

Let $\alpha_0 \in \mathbf{R}$ be chosen so that $\phi_0 \in B_\infty^{\alpha_0,\infty}$. Such a real number α_0 exists since ϕ_0 is a compactly supported distribution. Then the first assertion follows from (4), and the facts that B_N are integrable, and that given any compactly supported C^∞ function h with supp $h \subset \{\xi : 1/4 \le |\xi| \le 4\}$, we have

$$\begin{aligned} &\int_{\mathbf{R}} \Big| \int_{\mathbf{R}} e^{ix\xi} h(2^{-s}\xi)\widehat{B}_N(\xi)d\xi\Big|dx \le 2^N \int_{\mathbf{R}} \Big| \int_{\mathbf{R}} e^{ix\xi} h(2^{-s}\xi)\xi^{-N} d\xi\Big|dx \\ \le\ & 2^{(1-s)N} \int_{\mathbf{R}} \Big| \int_{\mathbf{R}} e^{ix\xi} h(\xi)\xi^{-N} d\xi\Big|dx =: C2^{-sN} \end{aligned}$$

for all $0 \le s \in \mathbf{Z}$, where the positive constant $C := 2^N \int_{\mathbf{R}} |\int_{\mathbf{R}} e^{-ix\xi} h(\xi)\xi^{-N} d\xi|dx < \infty$ follows from our assumption on h.

By the definition of ϕ_0, $\widehat{\phi}_0(\xi) = Q(e^{-i\xi/2})\widehat{\phi}_0(\xi/2)$. Hence ϕ_0 is a refinable function with symbol $Q(z)$ by (2). This, together with (4), imply that ϕ_N is a refinable function with symbol $((1-z^M)/(M-Mz))^N Q(z)$. Therefore $\phi_N, N \ge 0$, are refinable.

Set $\mathcal{A} = \{0 \le j \le M-1 :\ d_j \ne 0\}$ and $\mathcal{F} = \{\sum_{n=1}^\infty M^{-n} i_n :\ i_n \in \mathcal{A}\}$. Then $2 \le \#\mathcal{A} \le M-1$, $\mathcal{F} \subset [0,1]$, $\mathcal{F}$ is a compact set, the Hausdorff dimension of $\mathcal{F}$ is $\ln \#\mathcal{A}/\ln M < 1$, and $M\mathcal{F} = \cup_{j\in\mathcal{A}}(j+\mathcal{F})$. Hence we have

$$(0,1)\backslash\mathcal{F} \quad \text{is an open set and } m((0,1)\backslash\mathcal{F}) = 1. \tag{5}$$

Taking derivatives of N-th order at both sides of (3) and using

$$D^N B_N = \sum_{j=0}^N (-1)^j \begin{pmatrix} N \\ j \end{pmatrix} \delta(\cdot - j),$$

we obtain

$$D^N \phi_N = \left(\sum_{j=0}^N (-1)^j \begin{pmatrix} N \\ j \end{pmatrix} \delta(\cdot - j)\right) * \phi_0 = \sum_{j=0}^N (-1)^j \begin{pmatrix} N \\ j \end{pmatrix} \phi_0(\cdot - j), \tag{6}$$

where δ denotes the usual delta distribution. By [5], ϕ_0 is supported in the set $\mathcal{F}$. Thus $D^N\phi_N$ is supported in $\cup_{j=0}^N (j+\mathcal{F})$ by (6). Hence the restriction of ϕ_N on any open interval contained in $\cup_{j=0}^N (j + (0,1)\backslash\mathcal{F})$ is a polynomial with its degree at most $N-1$. This, together with (5) and supp $\phi_N \subset [0, N+1]$, lead to the local polynomial property of ϕ_N.

To prove the global linear independence of ϕ_N, we need a characterization of global linear independence of a refinable distribution in [10].

Lemma 2 *Let f be a compactly supported distribution. Then the integer shifts of f are globally linearly independent if and only if the entire function $\hat{f}$ has no 2π-periodical zero, i.e., there does not exist a complex number z_0 such that $\hat{f}(z_0 + 2k\pi) = 0$ for all $k \in \mathbf{Z}$.*

Note that ϕ_0 is supported in $\mathcal{F}$, and is not reduced to $\{0, 1\}$. Then the integer shifts of ϕ_0 are globally linearly independent. Hence the entire function $\widehat{\phi}_0$ does not have any 2π periodical zero by Lemma 2. By (4), $\widehat{\phi}_N(0) = 1$ and

$$\{z \in \mathbf{C} : \widehat{\phi}_N(z) = 0\} \subset \{z \in \mathbf{C} : \widehat{\phi}_0(z) = 0\} \cup (2\pi\mathbf{Z}\backslash\{0\}). \tag{7}$$

Hence the global linear independence of the integer shifts of ϕ_N follows from (7), Lemma 2, and the global linear independence of the shifts of ϕ_0.

Finally, we prove the local linear dependence of the integer shifts of ϕ_N. For $0 \le k \le M^{n-1} - 1,\ i \in \{0, 1, \ldots, M-1\}\backslash\mathcal{A}$ and $n \ge 1$, set

$$A(k, i, n) = kM^{-n+1} + iM^{-n} + (0, M^{-n}).$$

Recall that $M\mathcal{F} = \cup_{j\in\mathcal{A}}(j + \mathcal{F})$. Thus

$$A(k, i, n) \subset (0, 1)\backslash\mathcal{F} \tag{8}$$

for any $0 \le k \le M^{n-1} - 1, i \in \{0, 1, \ldots, M-1\}\backslash\mathcal{A}$ and $n \ge 1$. This, together with (6) and supp $\phi_0 \subset \mathcal{F}$, imply that the restriction of ϕ_N on $A(k, i, n) + j$ is a polynomial with its degree at most $N - 1$ for any $j = 0, \ldots, N$. Hence there exists a nonzero sequence $\{v_j\}_{j=0}^N$ for any $A(k, i, n)$ such that $\sum_{j=0}^N v_j \phi_N(\cdot + j) \equiv 0$ on $A(k, i, n)$. Therefore the local linear dependence of ϕ_N reduces to the existence of an interval $A(k, i, n)$ such that

$$\phi_N(\cdot + j) \not\equiv 0 \quad \text{on} \quad A(k, i, n) \qquad \forall\ 0 \le j \le N. \tag{9}$$

For $N = 1$, ϕ_1 is a refinable function with symbol $Q(z) \times (1 - z^M)/(M - Mz)$. Then

$$\phi_1 = \frac{1}{M} \sum_{j=0}^{M-1} \sum_{s=0}^{M-1} d_j \phi_1(M \cdot - j - s) \tag{10}$$

and $\widehat{\phi}_1(0) = 1$. Moreover,

$$\text{supp } \phi_1 \subset [0, 2], \quad \sum_{j\in\mathbf{Z}} \phi_1(\cdot + j) = 1, \tag{11}$$

and ϕ_1 is constant-valued on $A(k, i, n)$, to be denote by $\alpha(k, i, n)$. Set

$$a_j = \frac{1}{M} \sum_{l<j} d_l \quad \text{and} \quad b_j = \frac{1}{M} d_j, \quad j \in \mathcal{A}.$$

Then for any $j \in \mathcal{A}$, by (10) and (11) we have

$$\alpha(jM^{n-1}+k,i,n+1) = a_j + b_j\alpha(k,i,n) \tag{12}$$

where $0 \le k \le M^{n-1}-1, i \in \{0,1,\ldots,M-1\}\backslash\mathcal{A}$, and $n \ge 1$. To prove (9), we need the following result about ϕ_1.

Lemma 3 *Let $\phi_N, \{d_j\}_{j=0}^{M-1}$ be as in Theorem 1, $\alpha(k,i,n)$ be defined as above, and let*

$$G = \{\alpha(k,i,n): \ 0 \le k \le M^{n-1}-1, i \in \{0,1,\ldots,M-1\}\backslash\mathcal{A}, n \ge 1\}.$$

Then $\#G = +\infty$.

For a moment, we assume that Lemma 3 holds, and continue our proof of the local linear dependence of the integer shifts of ϕ_N. Let $A(k_0,i_0,n_0)$ be chosen so that $\alpha(k_0,i_0,n_0) \notin N^{-1}\{0,1,\cdots,N\}$. The existence of such an open interval $A(k_0,i_0,n_0)$ follows from Lemma 3. Thus for any $j=0,\ldots,N$, by (11), we obtain

$$\begin{aligned}
& D^{N-1}\phi_N(\cdot+j) \\
= & \sum_{l=0}^{N-1}(-1)^l \left(\begin{array}{c} N-1 \\ l \end{array}\right)\phi_1(\cdot+j-l) \\
= & (-1)^{j-1}\left(\begin{array}{c} N-1 \\ j-1 \end{array}\right)\Big(1-\alpha(k_0,i_0,n_0)\Big) + (-1)^j\left(\begin{array}{c} N-1 \\ j \end{array}\right)\alpha(k_0,i_0,n_0) \\
= & (-1)^j\left(\begin{array}{c} N \\ j \end{array}\right)\Big(\alpha(k_0,i_0,n_0)-\frac{j}{N}\Big) \ne 0 \quad \text{on} \quad A(k_0,i_0,n_0).
\end{aligned}$$

This proves (9). Hence it remains to prove Lemma 3. On the contrary, either $\#G=1$ or $1<\#G<\infty$. If $\#G=1$, then it follows from (12) that

$$c = a_j + b_j c \quad \forall\, j \in \mathcal{A}, \tag{13}$$

where we set $G=\{c\}$. Hence $c \ne 0$ by letting j in (13) as the minimal positive index in $\mathcal{A}$. Writing (13) in matrix form leads to

$$\left(\begin{array}{cccccc}
1 & 0 & 0 & \cdots & \cdots & 0 \\
h & 1 & 0 & & & \vdots \\
h & h & 1 & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \\
h & h & \ddots & h & 1 & 0 \\
h & h & \cdots & h & h & 1
\end{array}\right)
\left(\begin{array}{c}
d_{j_1} \\ d_{j_2} \\ \vdots \\ \vdots \\ d_{j_{\#\mathcal{A}-1}} \\ d_{j_{\#\mathcal{A}}}
\end{array}\right)
= M \left(\begin{array}{c}
1 \\ 1 \\ \vdots \\ \vdots \\ 1 \\ 1
\end{array}\right), \tag{14}$$

where $h = c^{-1}$, $\mathcal{A} = \{j_k, 1 \le k \le \#\mathcal{A}\}$ and $0 \le j_1 < j_2 < \cdots < j_{\#\mathcal{A}} \le M-1$. Therefore it follows from (14), $\#\mathcal{A} \ge 2$, and $d_{j_k} \ne 0$ for all $1 \le k \le \#\mathcal{A}$ that $h \ne 1$ and

$$d_{j_k} = M(1-h)^{k-1} \quad \forall\, 1 \le k \le \#\mathcal{A}.$$

Summing d_{j_k} over $1 \le k \le \#\mathcal{A}$ and using $h \ne 1$ and $\#\mathcal{A} \ge 2$, we obtain

$$\sum_{j=0}^{M-1} d_j = \sum_{k=1}^{\#\mathcal{A}} d_{j_k} = Mh^{-1}(1-(1-h)^{\#\mathcal{A}}) \ne M,$$

which contradicts to the assumption $\sum_{j=0}^{M-1} d_j = M$.

If $1 < \#G < \infty$, then there exist two triples (k_1, i_1, n_1) and (k_2, i_2, n_2) such that

$$\alpha(k_1, i_1, n_1) \ne \alpha(k_2, i_2, n_2). \tag{15}$$

Let $\gamma \in \mathcal{A}$ be chosen that $d_\gamma \ne 0, M$. The existence of such an index γ follows from $\sum_{j=0}^{M-1} d_j = M$ and $\#\mathcal{A} \ge 2$. By (12), we have

$$\begin{aligned} & \alpha\Big(k_s + \gamma\frac{M^{n_s+n} - M^{n_s-1}}{M-1}, i_s, n+n_s+1\Big) \\ = \;& a_\gamma + b_\gamma \alpha\Big(k_s + \gamma\frac{M^{n_s+n-1} - M^{n_s-1}}{M-1}, i_s, n+n_s\Big), \quad s = 1,2, \end{aligned} \tag{16}$$

for $n = 0, 1, 2, \cdots$. By $\#G < \infty$, there exist positive integers N_{1s} and N_{2s} such that $N_{1s} \ne N_{2s}$ and

$$\begin{aligned} & \alpha\Big(k_s + \gamma\frac{M^{n_s+N_{1s}} - M^{n_s-1}}{M-1}, i_s, N_{1s}+n_s+1\Big) \\ = \;& \alpha\Big(k_s + \gamma\frac{M^{n_s+N_{2s}} - M^{n_s-1}}{M-1}, i_s, N_{2s}+n_s+1\Big), \quad s = 1,2. \end{aligned} \tag{17}$$

Combining (16) and (17), we obtain

$$a_\gamma = (1-b_\gamma)\alpha(k_1, i_1, n_1)$$

and

$$a_\gamma = (1-b_\gamma)\alpha(k_2, i_2, n_2),$$

which contradicts to (15), since $b_\gamma = d_\gamma/M \ne 1$. This completes the proof of Lemma 3, and hence Theorem 1. $\square$

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