A Characterization of Compactly Supported Both m and n Refinable Distributions II

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Abstract

In this paper, a compactly supported distribution which is linearly independent, and both m and n refinable for some integer pair (m, n) satisfying $m^r \neq n^s$ for all positive integers r and s, is shown to be essentially a B-spline. Combining the characterization of compactly supported distributions which are linearly independent, and both m and n refinable for some integer pair (m, n) satisfying $m^r = n^s$ for some positive integers r and s in *Q. Sun and Z. Zhang, J. Approx. Theory, to appear*, we give a complete characterization of compactly supported distributions which are linearly independent, and both m and n refinable for some integer pair (m, n).

Keywords Refinable distribution, B-spline

AMS Subject Classification 42C15, 41A15

1 Introduction

For any integer $m \geq 2$, a compactly supported distribution f is said to be m *refinable* if $\hat{f}(0) = 1$ and

$$f = \sum_{k \in \mathbf{Z}} c_k f(m \cdot -k) \tag{1}$$

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for some finitely supported sequence $\{c_k\}_{k\in\mathbf{Z}}$. A refinable distribution in this paper means a compactly supported distribution which is m refinable for some $m \geq 2$. Here the *Fourier transform* $\hat{f}$ of an integrable function f is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx$$

and the one of a tempered distribution is understood by usual interpretation. Obviously an m refinable distribution is m^r refinable for any positive integer r. The refinable distribution is closely related to the construction of various wavelets and the limit function of a subdivision scheme (see [1], [3] and references therein).

Taking Fourier transform, the m refinability of a compactly supported f is equivalent to $\hat{f}(0) = 1$ and the existence of a trigonometrical polynomial $H(\xi)$ with $H(0) = 1$ such that

$$\hat{f}(m\xi) = H(\xi)\hat{f}(\xi). \tag{2}$$

For any $\tau \geq 0$, set

$$x_+^\tau = \begin{cases} x^\tau & x \geq 0 \\ 0 & x < 0. \end{cases}$$

A function g is said to be a *spline of degree* τ with knots $x_1 < x_2 < \cdots < x_N$ if g is linear combination of $(x - x_j)_+^\tau, 1 \leq j \leq N$. A function is said to be a *spline of degree* τ if it is a spline of degree τ with some knots $x_1 < x_2 < \cdots < x_N$, and to be *a spline of degree* τ *on an open interval* (a, b) if it equals the restriction of a spline of degree τ on (a, b).

For any integer $k \geq 0$, define B-spline B_k of degree $k - 1$ by

$$\widehat{B_k}(\xi) = \Big(\frac{1 - e^{-i\xi}}{i\xi}\Big)^k.$$

Obviously B_k is the delta distribution supported on the origin when $k = 0$, and B_k is a spline of degree $k - 1$ with knots $\{0, 1, \cdots, k\}$ when $k \geq 1$ (see [8]). For B-spline $B_k, k \geq 0$, there are various properties useful in some applications such as piecewise polynomial properties and total refinability. Here a compactly supported distribution is said to be *totally refinable* if it is m refinable for all $m \geq 2$ (See [2], [8] for the applications of total refinability).

As it is well known that there are some trade-off between various properties of refinable distributions. So it is an interesting problem to understand when a refinable distribution is essentially a B-spline (see for instance [4]-[7], [10], [11]). Lawton, Lee and Shen proved in [7] that a refinable distribution which is linearly independent and piecewise polynomial, is essentially a B-spline. In other words, B-splines are essentially the only compactly supported refinable distributions which are both linearly independent and piecewise polynomial. Here a compactly supported distribution f is said to be *linearly independent* if

$$\sum_{j\in\mathbf{Z}} d_j f(\cdot - j) \equiv 0 \quad \text{on} \quad \mathbb{R} \quad \text{implies} \quad d_j = 0 \quad \forall\, j \in \mathbb{Z}.$$

In [11], Sun and Zhang proved that B-splines are essentially the only compactly supported refinable distributions which are both linearly independent and totally refinable.

We say an integer pair (m, n) is of *type I* if there exist integer $p \geq 2$, and relatively prime positive integers r and s such that $m = p^r$ and $n = p^s$. Recall that a p refinable distribution must be p^r refinable. Then a compactly supported distribution which is both m and n refinable for an integer pair (m, n) of type I, need not to be a B-spline. In [11], Sun and Zhang proved that a compactly supported distribution which is linearly independent and both m and n refinable for an integer pair (m, n) of type I, must be p refinable, where $m = p^r$ and $n = p^s$ for some relatively prime positive integers r and s. Also it is shown in [11] that the B-splines are the only compactly supported distributions which are linearly independent and both m and n refinable for some types of integer pair (m, n). In this paper, we shall prove that a compactly supported distribution which is linearly independent and both m and n refinable for an integer pair (m, n) not of type I, is essentially a B-spline. Thus we give complete characterization of compactly supported distributions which are linearly independent and both m and n refinable for any integer pair (m, n).

In the next section, we give the main result and its proof. The last section is devoted to remarks.

2 Main Result and Proof

In this section, we shall prove the following result.

Theorem 1 *Let the integer pair (m, n) be not of type I and let the compactly supported distribution ϕ be linearly independent. If ϕ is both m and n refinable, then there exist integers $k \geq 0$ and $s \in \mathbb{Z}$ such that $(n-1)s/(m-1) \in \mathbb{Z}$ and $\phi = B_k(\cdot - s/(m-1))$.*

To prove Theorem 1, we need some lemmas. For any compactly supported distribution ϕ, there exists an integer α such that ϕ is continuous linear functional on

$$C^\alpha(\mathbb{R}) = \left\{f;\ f, f', \cdots, f^{(\alpha)} \quad \text{are continuous}\right\}.$$

By (1) and the straightforward computation, we have

Lemma 2 *If ϕ is a compactly supported continuous linear functional of $C^{\alpha}(\mathbb{R})$ and $\hat{\phi}(0) = 1$, then $\phi_{\alpha'}(x) = \langle \phi, (x - \cdot)_+^{\alpha'} \rangle$ is a continuous function of x, $\phi_{\alpha'}(x)$ is a polynomial of degree α' on (K, ∞), and $\phi_{\alpha'}^{(\alpha'+1)} = \phi$ in distributional sense, where α' is an integer strictly larger than α and K satisfies* $\operatorname{supp} \phi \cap (K, \infty) = \emptyset$. *Furthermore $\phi_{\alpha'}$ satisfies the refinement equation*

$$\phi_{\alpha'} = m^{-(\alpha'+1)} \sum_{k=N_1}^{N_2} c_k \phi_{\alpha'}(m \cdot -k)$$

if ϕ satisfies

$$\phi = \sum_{k=N_1}^{N_2} c_k \phi(m \cdot -k),$$

where $c_{N_1} c_{N_2} \neq 0$.

Lemma 3 *Let $m, n \geq 2$ be two integers and ϕ be a nonzero distribution supported on $[K, \infty)$ for some K. Assume that ϕ satisfies the refinement equation*

$$\phi = \sum_{k=N_1}^{N_2} c_k \phi(m \cdot -k) \tag{3}$$

and

$$\phi = \sum_{k=\tilde{N}_1}^{\tilde{N}_2} \tilde{c}_k \phi(n \cdot -k), \tag{4}$$

where $c_{N_1} c_{N_2} \neq 0$ and $\tilde{c}_{\tilde{N}_1} \tilde{c}_{\tilde{N}_2} \neq 0$. Then

$$\frac{N_1}{m-1} = \frac{\tilde{N}_1}{n-1}.$$

Proof From (3), (4) and the assumption $\operatorname{supp} \phi \subset [K, \infty)$, it follows that $\operatorname{supp} \phi \subset [\max(\frac{N_1}{m-1}, \frac{\tilde{N}_1}{n-1}), \infty)$. On the contrary, we may assume that $\frac{N_1}{m-1} < \frac{\tilde{N}_1}{n-1}$. Let K_1 be the maximal number such that $\operatorname{supp} \phi \subset [\frac{N_1}{m-1} + K_1, \infty)$. Then $0 < K_1 < \infty$ since $\operatorname{supp} \phi \subset [\frac{\tilde{N}_1}{n-1}, \infty)$ and $\phi \not\equiv 0$. Write (3) as

$$\phi = c_{N_1}^{-1} \Big(\phi(m^{-1} \cdot + m^{-1} N_1) - \sum_{k=1}^{N_2 - N_1} c_{k+N_1} \phi(\cdot - k) \Big). \tag{5}$$

Then the right hand side of (5) is supported in $[\frac{N_1}{m-1} + \min(mK_1, K_1 + 1), \infty)$, which is a contradiction. ♠

Lemma 4 *Let g be a spline of degree τ with knots $x_1 < x_2 < \cdots < x_N$. If g is a nonzero polynomial of degree α on some open subset of (x_N, ∞). Then τ is an integer larger than α.*

Proof Observe that g is an analytic function on (x_N, ∞). Then g is a polynomial of degree α on (x_N, ∞) by the assumption. Write

$$g(x) = \sum_{j=1}^{k} d_j (x - x_j)_+^\tau.$$

Then

$$g(x) = a_0 x^\tau + x^\tau \sum_{n=1}^{\infty} a_n \frac{(-1)^n x^{-n}}{n!} \prod_{i=0}^{n-1} (\tau - i),$$

where $a_n = \sum_{j=1}^{k} d_j x_j^n$. This shows that τ is a nonnegative integer larger than α. ♠

Lemma 5 ([7]) *Let ϕ be a spline of degree $k \geq 0$ with compact support. If ϕ is m refinable and linearly independent, then there exists an integer s such that $\phi = B_{k+1}(\cdot - s/(m-1))$.*

Now we start to prove Theorem 1.

Proof of Theorem 1 Let ϕ be a compactly supported distribution being linearly independent and satisfying the refinement equations (3) and (4). Let $\tau_0 \geq 1$ be an integer such that ϕ is a continuous linear functional of $C^{\tau_0 - 1}(\mathbb{R})$. Set

$$\tilde{\phi}_{\tau_0}(x) = \langle \phi, (x + \frac{N_1}{m-1} - \cdot)_+^{\tau_0} \rangle.$$

Then $\tilde{\phi}_{\tau_0}$ is continuous and supported in $[0, \infty)$, and satisfies the refinement equations

$$\tilde{\phi}_{\tau_0} = m^{-\tau_0 - 1} \sum_{k=0}^{N_2 - N_1} c_{k+N_1} \tilde{\phi}_{\tau_0}(m \cdot -k) \tag{6}$$

and

$$\tilde{\phi}_{\tau_0} = n^{-\tau_0 - 1} \sum_{k=0}^{\tilde{N}_2 - \tilde{N}_1} \tilde{c}_{k+\tilde{N}_1} \tilde{\phi}_{\tau_0}(n \cdot -k)$$

by Lemmas 2 and 3. Thus

$$\begin{cases} \tilde{\phi}_{\tau_0}(x) = A\tilde{\phi}_{\tau_0}(mx) & x \in [0, 1/m] \\ \tilde{\phi}_{\tau_0}(x) = B\tilde{\phi}_{\tau_0}(nx) & x \in [0, 1/n], \end{cases} \tag{7}$$

where $A = m^{-\tau_0-1}c_{N_1} \neq 0$ and $B = n^{-\tau_0-1}\tilde{c}_{\tilde{N}_1} \neq 0$. Iterating (7), we get

$$\tilde{\phi}_{\tau_0}(x) = A^a B^b \tilde{\phi}_{\tau_0}(m^a n^b x), \tag{8}$$

when $x, m^a n^b x \in [0,1]$ and $a, b \in \mathbb{Z}$.

For any integer pair (m,n) being not of type I, $\ln n/\ln m$ is irrational. Thus the set $\{a + b\frac{\ln n}{\ln m}; a, b \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Hence

$$\{m^a n^b; \quad a, b \in \mathbb{Z}\} \quad \text{is dense in } \mathbb{R}_+. \tag{9}$$

Here $\mathbb{R}_+$ denotes the set of all positive numbers.

Let $x_0, t_0 \in (0,1)$ be chosen as $\tilde{\phi}_{\tau_0}(x_0) \neq 0$ and $\tilde{\phi}_{\tau_0}(t_0 x_0) \neq 0$. The existences of x_0 and t_0 follow from $\tilde{\phi}_{\tau_0} \not\equiv 0$ on $[0,1]$ and the continuity of $\tilde{\phi}_{\tau_0}$. By (9), there exist sequences $a_j, b_j \in \mathbb{Z}$ of even integers such that $\lim_{j\to\infty} |a_j| = \lim_{j\to\infty} |b_j| = \infty$ and $\lim_{j\to\infty} m^{a_j} n^{b_j} = t_0$. By (8), $\lim_{j\to\infty} A^{a_j} B^{b_j}$ exists and is nonzero. Hence

$$\frac{\ln|A|}{\ln m} = \frac{\ln|B|}{\ln n}. \tag{10}$$

Set $\tau = -\frac{\ln|A|}{\ln m}$. Combining (8) and (10), we obtain

$$\tilde{\phi}_{\tau_0}(x) = (m^a n^b)^{-\tau} \tilde{\phi}_{\tau_0}(m^a n^b x) \quad \forall\ 0 \le x, m^a n^b x \le 1 \text{ and } a, b \in 2\mathbb{Z}. \tag{11}$$

By (9), (11) and the continuity of $\tilde{\phi}_{\tau_0}(x)$, we get

$$\tilde{\phi}_{\tau_0}(x) = t^{-\tau} \tilde{\phi}_{\tau_0}(tx) \quad \forall\ t, x \in (0,1].$$

This together with the continuity of $\tilde{\phi}_{\tau_0}$ lead to

$$\tilde{\phi}_{\tau_0}(x) = Cx^\tau \quad \forall\ x \in [0,1]$$

and $\tau > 0$. This shows that $\tilde{\phi}_{\tau_0}$ is a spline of degree $\tau > 0$ on $(-\infty, 1)$.

Let K_1 be the maximal number such that $\tilde{\phi}_{\tau_0}$ is a spline of degree τ on $(-\infty, \tilde{K}_1)$ for all $\tilde{K}_1 < K_1$. Then $K_1 \ge 1$. Write (6) as

$$\tilde{\phi}_{\tau_0} = c_{N_1}^{-1}\Big(m^{\tau_0+1}\tilde{\phi}_{\tau_0}(m^{-1}\cdot) - \sum_{k=1}^{N_2-N_1} c_{k+N_1}\tilde{\phi}_{\tau_0}(\cdot - k)\Big). \tag{12}$$

It is easy to be checked that the right hand side of (12) is a spline of degree τ on $(-\infty, \min(m\tilde{K}_1, \tilde{K}_1 + 1))$ for all $\tilde{K}_1 < K_1$. Thus $K_1 = \infty$ by $K_1 \ge 1$. This implies that $\tilde{\phi}_{\tau_0}$ is a spline of degree τ on $(-\infty, K)$ for any $K > 0$.

Notice that supp $\phi \subset [N_1/(m-1), N_2/(m-1)]$. This together with Lemma 2 imply that $\tilde{\phi}_{\tau_0}$ is a polynomial of degree of τ_0 on $((N_2-N_1)/(m-1), \infty)$. Thus τ is an integer larger than τ_0 by Lemma 4 and $\tilde{\phi}_{\tau_0}$ is a spline of degree τ.

Recall that

$$\phi = \tilde{\phi}_{\tau_0}^{(\tau_0+1)}(\cdot - \frac{N_1}{m-1})$$

by Lemma 2. Then ϕ is a spline of degree of $\tau - \tau_0 - 1$ when $\tau > \tau_0$ and linear combination of delta distributions on finite knots when $\tau = \tau_0$. Hence $\phi(\cdot + \frac{N_1}{m-1}) = B_{\tau-\tau_0}$ by Lemma 5 when $\tau > \tau_0$. Similarly by the proof of Lemma 5, we have $\phi(\cdot + \frac{N_1}{m-1}) = B_0$ when $\tau = \tau_0$.

By taking Fourier transform, we obtain

$$\hat{\phi}(\xi) = e^{-iN_1/(m-1)\xi} \times \Big(\frac{1-e^{-i\xi}}{i\xi}\Big)^{\tau-\tau_0}.$$

Hence $N_1(n-1)/(m-1)$ is an integer by (2) and the n refinability of ϕ. ♠

3 Remarks

We say that a Laurent polynomial P is m *closed* if $P(z^m)/P(z)$ is still a Laurent polynomial. When the linear independence of ϕ in Theorem 1 is left out, we have

Theorem 6 *Let the integer pair (m,n) be not of type I, and ϕ be a compactly supported distribution. Then ϕ is both m and n refinable if and only if there exists an integer s such that $s(n-1)/(m-1)$ is an integer, and a B-spline B_k and a sequence $\{d_j\}_{j\in\mathbf{Z}}$ with finite length such that $(1-z)^k \sum_{j\in\mathbf{Z}} d_j z^j$ is both m and n closed, and*

$$\phi = \sum_{j\in\mathbf{Z}} d_j B_k(\,\cdot - \frac{s}{m-1} - j).$$

Obviously Theorem 6 follows from Theorem 1 and the following lemma.

Lemma 7 ([11]) *Let $m, n \geq 2$ be two integers, and let compactly supported distribution ϕ be both m and n refinable. Then there exist a compactly supported distribution ϕ_1 and a sequence $\{d_j\}_{j\in\mathbf{Z}}$ with finite length such that ϕ_1 is linearly independent, both m and n refinable, and satisfies*

$$\phi = \sum_{j\in\mathbf{Z}} d_j \phi_1(\cdot - j).$$

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