Compactly Supported Distributional Solutions of Nonstationary Nonhomogeneous Refinement Equation

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Abstract

Let A be a matrix with the absolute values of all eigenvalues strictly larger than one, and let $\mathbf{Z}_0$ be a subset of $\mathbf{Z}$ such that $n \in \mathbf{Z}_0$ implies $n+1 \in \mathbf{Z}_0$. Denote the space of all compactly supported distributions by $\mathcal{D}'$, and the usual convolution between two compactly supported distributions f and g by $f * g$. For any bounded sequence G_n and $H_n, n \in \mathbf{Z}_0$ in $\mathcal{D}'$, define corresponding nonstationary nonhomogeneous refinement equation

$$\Phi_n = H_n * \Phi_{n+1}(A\cdot) + G_n \qquad \text{for all } n \in \mathbf{Z}_0, \qquad (*)$$

where $\Phi_n, n \in \mathbf{Z}_0$ is in a bounded set of $\mathcal{D}'$. The nonstationary nonhomogeneous refinement equation $(*)$ arises in the construction of wavelets on bounded domain, multiwavelets, and of biorthogonal wavelets on non-uniform meshes. In this paper, we study the existence problem of compactly supported distributional solutions $\Phi_n, n \in \mathbf{Z}_0$ of the equation $(*)$. In fact, we reduce the existence proobblem to finding a bounded solution $\tilde{F}_n$ of the linear equations

$$\tilde{F}_n - S_n \tilde{F}_n = \tilde{G}_n, \qquad \text{for all} \qquad n \in \mathbf{Z}_0,$$

where the matrices S_n and the vectors $\tilde{G}_n, n \in \mathbf{Z}_0$ can be contructed explicitly from H_n and G_n respectively.

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1 Introduction

Fix a dilation matrix A with the absolute values of its eigenvalues strictly larger than one, and fix a subset $\mathbf{Z}_0$ of $\mathbf{Z}$ such that $n \in \mathbf{Z}_0$ implies $n+1 \in \mathbf{Z}_0$. Define the *convolution* $f * g$ between two integrable functions f and g by

$$f * g = \int_{\mathbf{R}^d} f(\cdot - y) g(y) dy,$$

and the one between two compactly supported distributions by usual interpretation. Denote the space of all compactly supported distributions by $\mathcal{D}'$. The objective of this paper is to study the existence problem of compactly supported distributional solutions of the following equation

$$\Phi_n = H_n * \Phi_{n+1}(A\cdot) + G_n \quad \text{for all } n \in \mathbf{Z}_0, \tag{1.1}$$

where Φ_n and $G_n, n \in \mathbf{Z}_0$ are vector-valued distributions in a bounded set of $\mathcal{D}'$, and $H_n, n \in \mathbf{Z}_0$ are $N \times N$ matrix-valued distributions in a bounded set of $\mathcal{D}'$. Hereafter a vector-valued (matrix-valued) distribution belongs to $\mathcal{D}'$ means that its components (entries) belong to $\mathcal{D}'$. Note that any solution of the equation (1.1) can be written as the sum of a fixed solution of the equation (1.1) and a solution of the following equations

$$\Phi_n = H_n * \Phi_{n+1}(A\cdot) \quad \text{for all } n \in \mathbf{Z}_0. \tag{1.2}$$

So we call the equation (1.1) as *nonstationary nonhomogeneous refinement equations*, and the equation (1.2) as *nonstationary refinement equations*. The matrix-valued compactly supported distributions $H_n, n \in \mathbf{Z}_0$ and the vector-valued compactly supported distributions $G_n, n \in \mathbf{Z}_0$ are said to be the *masks* and the *nonhomogeneous terms* of the nonstationary nonhomogeneous refinement equation (1.1) respectively. Clearly the nonhomogeneous refinement equation

$$\Phi = \sum_{j \in \mathbf{Z}^d} c_j \Phi(2 \cdot - j) + G \tag{1.3}$$

in [8, 9, 21], and the continuous nonhomogeneous refinement equation

$$\Phi = \int_{\mathbf{R}^d} \Phi(A \cdot - y) d\mu(y) + G \tag{1.4}$$

in [16] are special cases of the nonstationary nonhomogeneous refinement equation (1.1), where $c_j, j \in \mathbf{Z}^d$ is a sequence of $N \times N$ matrices such that $c_j = 0$ for all but finitely many $j \in \mathbf{Z}^d$, and μ is an $N \times N$ matrix of finite Borel measures with compact support.

1.1 Motivation

The nonstationary nonhomogeneous refinement equations arise in the construction of wavelets on bounded domain, multiwavelets and biorthogonal wavelets on nonuniform meshes ([2, 4, 6, 11, 12, 15, 19, 20, 22]). All these inspire us to study systematically various properties of solutions of nonstationary nonhomogeneous refinement equations.

In the multiresolution approximation on the unit interval $[0, 1]$ proposed by Meyer ([20]) and Cohen, Daubechies and Vial ([4]), the approximation space of scale n is spanned by interior scaling functions, left boundary scaling functions and right boundary scaling functions. The interior scaling functions are scaling functions on the line with their support contained in $[0, 1]$, and these functions satisfy certain refinement equations. The left and right boundary scaling functions are modified from the restriction of usual scaling functions on the unit interval, and these boundary scaling functions satisfy the following nonhomogeneous refinement equations

$$\Phi = H\Phi(2\cdot) + G,$$

where H is an $N \times N$ matrix and G is a vector-valued compactly supported distribution.

Let h be the hat function defined by

$$h(x) = 1 - |x| \quad \text{if} \quad x \in [-1, 1] \quad \text{and} \quad h(x) = 0 \quad \text{if} \quad x \notin [-1, 1],$$

and w_c satisfy the nonhomogeneous refinement equation

$$w_c = h(2\cdot -1) + c(w_c(2\cdot) + w_c(2\cdot -1)),$$

where c is a constant. Then $(h, w_c)^T$ is a symmetric orthogonal continuous scaling vector when $c = -1/5$, and the pair $(h, w_c)^T$ and $(h, w_{\tilde{c}})^T$ leads to a family of symmetric biorthogonal scaling vectors when $\tilde{c} = (2c+1)/(5c-1)$ and $-1 < c < 1/7$ ([11, 12]). The perturbation of Daubechies' orthonormal scaling functions and wavelets in [15] is another example to use a solution of the nonhomogeneous refinement equation (1.3) and a usual scaling function to construct a new orthonormal scaling vector with any preassigned regularity.

The nonhomogeneous refinement equation is one of the cornerstones in the construction of biorthogonal wavelet basis on arbitrary triangulation of a polygon from hierarchical basis in [6], and of biorthogonal wavelet basis on one-dimensional non-uniform meshes by modifying cardinal primal refinable functions in [22]. The dual refinable function $\phi(x_1, x_2)$ in [6] at the intersection of different types of meshes satisfies a nonhomogeneous refinement equation of the form

$$\phi(x_1, x_2) = \sum_{j \in \mathbf{Z}} c_j \phi(2x_1 - j, 2x_2) + G(x_1, x_2),$$

and the one at exceptional node satisfies a nonhomogeneous refinement equation of the form

$$\phi(x_1, x_2) = \phi(2x_1, 2x_2) + G(x_1, x_2),$$

where $G(x_1, x_2)$ is a compactly supported function on $\mathbf{R}^2$.

1.2 Historical Sketch and Main Result

For the nonstationary nonhomogeneous refinement equations (1.1), the first and most elementary problem is the existence problem of its compactly supported distributional solutions. For the nonhomogeneous refinement equations (1.3) and (1.4), the existence problem of its compactly supported distributional solutions are discussed in [8, 9, 16, 21]. In particular, for the nonhomogeneous refinement equation (1.3), Strang and Zhou ([21]) characterized the existence of its compactly supported distributional solutions in term of $(c_j)_{j \in \mathbf{Z}^d}$ and the nonhomogeneous term G for one-dimensional and scale-valued case (i.e., $d = 1$ and $N = 1$), and Dinsenbacher and Hardin ([8, 9]) reduced the existence problem to finding polynomial solutions of a related nonhomogeneous polynomial refinement equation. For the continuous nonhomogeneous refinement equation (1.4), Jia, Jiang and Shen [16] gave some necessary and sufficient conditions to the existence of its compactly supported distributional solutions. For the nonstationary refinement equation (1.2), there is a much literatures for the existence, regularity and its applications to the constructions of nonstationary wavelets (see for instance [1, 5, 7, 10]).

For the nonstationary nonhomogeneous refinement equation (1.1), define corresponding cascade operators $T_n, n \in \mathbf{Z}_0$ by

$$T_n F = H_n * F(A \cdot). \tag{1.5}$$

In this paper, we reduce to the existence problem to finding appropriate initials

$F_n, n \in \mathbf{Z}_0$ of the cascade operators $T_n, n \in \mathbf{Z}_0$ such that

$$\sum_{m=1}^{\infty} T_n T_{n+1} \cdots T_{n+m-1} \tilde{G}_{n+m}$$

converges in distributional sense, where

$$\tilde{G}_n = G_n - F_n + T_n F_{n+1}, \quad n \in \mathbf{Z}_0.$$

Let τ_0 be the minimal nonnegative integer such that

$$\|\widehat{H}_n(0)\| \rho(A^{-T})^{\tau_0} \leq r_0 |\det A|$$

for all $n \in \mathbf{Z}_0$ and some constant $0 < r_0 < 1$ independent of $n \in \mathbf{Z}_0$, and let $\rho(B)$ denote the spectral radius of a matrix B. Precisely, we shall choose the initials $F_n, n \in \mathbf{Z}_0$ such that $F_n, n \in \mathbf{Z}_0$ is in a bounded set of $\mathcal{D}'$, and

$$\left|\widehat{G}_n(\xi) - \widehat{F}_n(\xi) + (\det A)^{-1} \widehat{H}_n(A^{-T}\xi) \widehat{F}_{n+1}(A^{-T}\xi)\right| \leq C|\xi|^{\tau_0} \quad \forall\, |\xi| \leq 1 \quad \text{and} \quad n \in \mathbf{Z}_0, \tag{1.6}$$

where $\hat{f}$ denotes the Fourier transform of a tempered distribution f, and C is a positive constant independent of n and ξ (see Theorem 2.1 for detail). Also we apply the assertions in Theorem 2.1 to study the stationary refinement equation

$$\Psi = H * \Psi(A\cdot). \tag{1.7}$$

Thus the exact dimension of the linear space of all compactly supported distributional solutions of the refinement equation (1.7) is computed (Corollary 3.1), and the existence of compactly supported distributional solution Ψ of (1.7) with $\widehat{\Psi}(0) \neq 0$ is also characterized (Corollary 3.3).

1.3 Organization

The main result (Theorem 2.1) is proved in Section 2. We apply the main result to study stationary nonhomogeneous refinement equation (3.1) in Section 3 (see Corollaries 3.1 and 3.3). The conclusion of this paper is given in last section.

2 Main Result

To state our main result, we need to introduce some notation. For an integrable function f, its Fourier transform $\hat{f}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbf{R}^d} f(x) e^{-ix\xi} dx.$$

For a vector-valued or matrix-valued tempered distribution F, its Fourier transform $\widehat{F}$ is interpreted as usual. For any nonnegative integer τ, let

$$\mathbf{Z}^d_{+,\tau} = \Big\{(s_1,\ldots,s_d) \in \mathbf{Z}^d: \ s_1+\cdots+s_d \le \tau-1 \quad \text{and} \quad s_i \ge 0 \quad \forall\, 1 \le i \le d\Big\}$$

and $\mathbf{Z}^d_+ = \cup_{\tau\ge 1}\mathbf{Z}^d_{+,\tau}$. Define $g_{s,s'}$ for $s, s' \in \mathbf{Z}^d$ by

$$(A^{-T}\xi)^s = \sum_{s'\in\mathbf{Z}^d_+, |s'|=|s|} g_{s,s'}\xi^{s'}$$

if $s, s' \in \mathbf{Z}^d_+$ and $|s'| = |s|$, and $g_{s,s'} = 0$ if $|s| \ne |s'|$, or $s' \notin \mathbf{Z}^d_+$, or $s \notin \mathbf{Z}^d_+$.

Let τ_0 be the minimal nonnegative integer such that

$$\|\widehat{H}_n(0)\|\rho(A^{-T})^{\tau_0} \le r_0|\det A| < |\det A|, \tag{2.1}$$

for all $n \in \mathbf{Z}_0$ and some positive constant $r_0 < 1$ independent of $n \in \mathbf{Z}_0$. For any $n \in \mathbf{Z}_0$, write

$$\widehat{G}_n(\xi) = \sum_{s\in\mathbf{Z}^d_{+,\tau_0}} G_{n,s}\xi^s + O(|\xi|^{\tau_0}) \quad \text{as} \quad \xi \to 0, \tag{2.2}$$

and

$$(\det A)^{-1}\widehat{H}_n(\xi) = \sum_{s\in\mathbf{Z}^d_{+,\tau_0}} H_{n,s}\xi^s + O(|\xi|^{\tau_0}) \quad \text{as} \quad \xi \to 0, \tag{2.3}$$

where $B_1(\xi) = B_2(\xi) + O(|\xi|^{\tau_0})$ as $\xi \to 0$ means that there exists a positive constant C independent of ξ such that $|B_1(\xi) - B_2(\xi)| \le C|\xi|^{\tau_0}$ in a small neighborhood of the origin. Obviously $G_{n,s}, H_{n,s}, n \in \mathbf{Z}_0$ are bounded sequences for any $s \in \mathbf{Z}^d_{+,\tau_0}$ if G_n and $H_n, n \in \mathbf{Z}_0$ are in a bounded set of $\mathcal{D}'$.

Set

$$S_n = I - \Big(\sum_{t\in\mathbf{Z}^d_{+,\tau_0}} g_{t+s',s}H_{n,t}\Big)_{s,s'\in\mathbf{Z}^d_{+,\tau_0}} \tag{2.4}$$

and $\tilde{G}_n = (G_{n,s})_{s\in\mathbf{Z}^d_{+,\tau_0}}, n \in \mathbf{Z}_0$. When $d = 1, N = 1$ and $A = 2$,

$$S_n = \begin{pmatrix} I - H_{n,0} & 0 & \cdots & 0 & 0 \\ -\frac{1}{2}H_{n,1} & I - \frac{1}{2}H_{n,0} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2^{-\tau_0+2}H_{n,\tau_0-2} & -2^{-\tau_0+2}H_{n,\tau_0-3} & \cdots & I - 2^{-\tau_0+2}H_{n,0} & 0 \\ -2^{-\tau_0+1}H_{n,\tau_0-1} & -2^{-\tau_0+1}H_{n,\tau_0-2} & \cdots & -2^{-\tau_0+1}H_{n,1} & I - 2^{-\tau_0+1}H_{n,0}, \end{pmatrix}$$

and $\tilde{G} = (G_0, \cdots, G_{\tau_0-1})^T, n \in \mathbf{Z}_0$. In this paper, we shall prove the following result.

Theorem 2.1 *Let G_n and $H_n, n \in \mathbf{Z}_0$ be in a bounded set of $\mathcal{D}'$, and let $\tau_0, S_n, G_{n,s}, H_{n,s},\ n \in \mathbf{Z}_0, s \in \mathbf{Z}^d_{+,\tau_0}$ be defined as in (2.1) – (2.4). Assume that there exist positive constants C_0 and γ_0 such that*

$$\sup_{\xi\in\mathbf{R}^d} \|\widehat{H}_n(\xi)\| \le C_0 \quad \text{and} \quad \sup_{|\xi|\le 1} |\xi|^{-\gamma_0}\|\widehat{H}_n(\xi)-\widehat{H}_n(0)\| \le C_0 \quad \text{for all } n \in \mathbf{Z}_0. \quad (2.5)$$

Then the following statements are equivalent:

(i) There exist solutions $\Phi_n, n \in \mathbf{Z}_0$ of the nonstationary nonhomogeneous refinement equations (1.1) in a bounded set of $\mathcal{D}'$;

(ii) There exist $F_n, n \in \mathbf{Z}_0$ in a bounded set of $\mathcal{D}'$ such that

$$|\widehat{G}_n(\xi)-\widehat{F}_n(\xi)+(\det A)^{-1}H_n(A^{-T}\xi)\widehat{F}_{n+1}(A^{-T}\xi)| \le C|\xi|^{\tau_0} \quad \forall\, |\xi| \le 1 \text{ and } n \in \mathbf{Z}_0, \quad (2.6)$$

where C is a positive constant independent of ξ and $n \in \mathbf{Z}_0$;

(iii) There exist $\tilde{F}_n = (F_{n,s})_{s\in\mathbf{Z}^d_{+,\tau_0}},\ n \in \mathbf{Z}_0$ such that $\tilde{F}_n, n \in \mathbf{Z}_0$ is a bounded sequence and satisfies the linear equation

$$\tilde{F}_n - S_n\tilde{F}_{n+1} = \tilde{G}_n \quad \forall\ n \in \mathbf{Z}_0, \quad (2.7)$$

i.e.

$$F_{n,s} - \sum_{t,s'\in\mathbf{Z}^d_{+,\tau_0}} g_{t+s',s}H_{n+1,t}F_{n+1,s'} = G_{n,s} \quad \forall\ s \in \mathbf{Z}^d_{+,\tau_0} \quad \text{and} \quad n \in \mathbf{Z}_0.$$

To prove Theorem 2.1, we need a pointwise estimate.

Lemma 2.1 *Let τ_0 and r_0 be as in (2.1). Assume that $H_n, n \in \mathbf{Z}_0$ satisfy (2.5). Then for any $\delta > 0$ there exists a positive constant C (dependent of δ and the constant C_0 and γ_0 in (2.5)) such that*

$$\begin{aligned} &\|\widehat{H}_n(A^{-T}\xi)\cdots\widehat{H}_{n+m-1}((A^{-T})^m\xi)\| \\ \le\ & C(1+|\xi|)^C|\det A|^m(r_0\rho(A^{-T})^{-\tau_0}+\delta)^m \quad \forall\ m \ge 1,\ n \in \mathbf{Z}_0 \quad \text{and} \quad \xi \in \mathbf{R}^d. \end{aligned}$$

Proof. Set $R_n(\xi) = (\det A)^{-1}\widehat{H}_n(\xi), n \in \mathbf{Z}_0$. By (2.5), there exists a positive constant C_1 such that

$$\|R_n(\xi)\| \le \min(C_1, r_0\rho(A^{-T})^{-\tau_0} + C_1|\xi|^{\gamma_0}) \quad \forall\ \xi \in \mathbf{R}^d \quad \text{and} \quad n \in \mathbf{Z}_0.$$

By the fact that A is a dilation matrix, there exist $0 < r_1 < r_2 < 1$ and a positive constant C_2 such that

$$C_2^{-1} r_1^m \le |(A^{-T})^m \xi| \le C_2 r_2^m \quad \text{for all} \quad |\xi| = 1 \quad \text{and} \quad m \ge 1.$$

Thus for any $\delta > 0$, any $\xi \in \mathbf{R}^d$ with its norm smaller than 2 and $m \ge 1$, we have

$$\begin{aligned} &\|R_n(A^{-T}\xi) \cdots R_{n+m-1}((A^{-T})^m \xi)\| \le \prod_{i=1}^{m} \Big(r_0 \rho(A^{-T})^{-\tau_0} + C_1 |(A^{-T})^i \xi| \Big) \\ \le\ & (r_0 \rho(A^{-T})^{-\tau_0} + \delta)^m \prod_{i=1}^{m} \Big(1 + 2C_1 C_2 (r_0 \rho(A^{-T})^{-\tau_0} + \delta)^{-1} r_2^i \Big) \le C_3 (r_0 \rho(A^{-T})^{-\tau_0} + \delta)^m, \end{aligned}$$

where C_3 is a positive constant independent of m and ξ.

For any $\xi \in \mathbf{R}^d$ with $|\xi| \ge 2$, let k_0 be the minimal positive integer such that $|A^{-k_0} \xi| \le 1$. Then there exists a constant C_4 such that

$$C_4^{-1} \ln |\xi| \le k_0 \le C_4 \ln |\xi| \quad \forall\ |\xi| \ge 2.$$

Therefore for $k_0 \ge m - 1$

$$\|R_n(A^{-T}\xi) \cdots R_{n+m-1}((A^{-T})^m \xi)\| \le C_1^m \le C(1 + |\xi|)^C (r_0 \rho(A^{-T})^{-\tau_0} + \delta)^m,$$

and for $k_0 \le m - 1$

$$\begin{aligned} & \|R_n(A^{-T}\xi) \cdots R_{n+m-1}((A^{-T})^m \xi)\| \\ \le\ & C_1^{k_0} \|R_{n+k_0}((A^{-T})^{k_0+1}\xi) \cdots R_{n+m-1}((A^{-T})^m \xi)\| \\ \le\ & C_3 C_1^{k_0} (r_0 \rho(A^{-T})^{-\tau_0} + \delta)^{m-k_0} \le C(1 + |\xi|)^C (r_0 \rho(A^{-T})^{-\tau_0} + \delta)^m, \end{aligned}$$

where C is a positive constant independent of n and $\xi \in \mathbf{R}^d$. Hence the assertion follows. □

Proof of Theorem 2.1. We divide the proof into four steps: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i).

(i)$\Longrightarrow$(ii): Let $\Phi_n, n \in \mathbf{Z}_0$ be in a bounded set of $\mathcal{D}'$ and satisfy the nonstationary nonhomogeneous refinement equation (1.1). Taking Fourier transform at each side of (1.1),

$$\widehat{\Phi}_n(\xi) = (\det A)^{-1} \widehat{H}_n(A^{-T}\xi) \widehat{\Phi}_{n+1}(A^{-T}\xi) + \widehat{G}_n(\xi) \quad \text{for all } n \in \mathbf{Z}_0. \tag{2.8}$$

Then (2.6) follows by letting $F_n = \Phi_n, n \in \mathbf{Z}_0$.

(ii) $\Longrightarrow$(iii): Let $F_n, n \in \mathbf{Z}_0$ be in a bounded set of $\mathcal{D}'$ and satisfy (2.6). Write

$$\widehat{F}_n(\xi) = \sum_{s \in \mathbf{Z}^d_{+,\tau_0}} F_{n,s}\xi^s + O(|\xi|^{\tau_0}) \quad \text{as} \quad \xi \to 0.$$

Note that $F_{n,s}$ is the Fourier transform of $(-i)^{|s|}\phi_s F_n$ for any $n \in \mathbf{Z}_0$ and $s \in \mathbf{Z}^d_{+,\tau_0}$, where $\phi_s(x) = x^s/s!$. Then $(F_{n,s})_{s\in\mathbf{Z}^d_{+,\tau_0}}, n \in \mathbf{Z}_0$ is a bounded sequence. By direct computation, we have

$$\begin{aligned} & \widehat{G}_n(\xi) - \widehat{F}_n(\xi) + (\det A)^{-1}\widehat{H}_n(A^{-T}\xi)\widehat{F}_{n+1}(A^{-T}\xi) \\ = & \sum_{s\in\mathbf{Z}^d_{+,\tau_0}} (G_{n,s} - F_{n,s})\xi^s + \sum_{t',s'\in\mathbf{Z}^d_{+,\tau_0}} H_{n,t'}F_{n+1,s'}(A^{-T}\xi)^{s'+t'} + O(|\xi|^{\tau_0}) \\ = & \sum_{s\in\mathbf{Z}^d_{+,\tau_0}} \Big(G_{n,s} - F_{n,s} + \sum_{s',t'\in\mathbf{Z}^d_{+,\tau_0}} g_{s'+t',s}H_{n,t'}F_{n+1,s'}\Big)\xi^s + O(|\xi|^{\tau_0}) \quad \text{as} \quad \xi \to 0. \end{aligned}$$

Thus

$$\sum_{s\in\mathbf{Z}^d_{+,\tau_0}} \Big(G_{n,s} - F_{n,s} + \sum_{s',t'\in\mathbf{Z}^d_{+,\tau_0}} g_{s'+t',s}H_{n,t'}F_{n+1,s'}\Big)\xi^s = 0$$

by (2.6). Hence $\tilde{F} = (F_{n,s})_{s\in\mathbf{Z}^d_{+,\tau_0}}$ satisfies the linear equation (2.7), and (iii) follows.

(iii)$\Longrightarrow$(ii): Let $\tilde{F}_n = (F_{n,s})_{s\in\mathbf{Z}^d_{+,\tau_0}}, n \in \mathbf{Z}_0$ be a bounded sequence and satisfy the linear equation (2.7). Define F_n by

$$\widehat{F}_n(\xi) = \sum_{s\in\mathbf{Z}^d_{+,\tau_0}} F_{n,s}\xi^s, \quad n \in \mathbf{Z}_0.$$

Then $F_n, n \in \mathbf{Z}_0$ is in a bounded set of $\mathcal{D}'$. By using the same procedure as in (ii)$\Longrightarrow$(iii) and the fact that $H_n, G_n, n \in \mathbf{Z}_0$ are in a bounded set of $\mathcal{D}'$, there exists a positive constant C such that

$$|\widehat{G}_n(\xi) - \widehat{F}_n(\xi) + (\det A)^{-1}\widehat{H}_n(A^{-T}\xi)\widehat{F}_{n+1}(A^{-T}\xi)| \le C|\xi|^{\tau_0} \quad \forall\, |\xi| \le 1 \quad \text{and} \quad n \in \mathbf{Z}_0.$$

(ii)$\Longrightarrow$(i): For the nonstationary nonhomogeneous refinement equation (1.1), define corresponding cascade operators $T_n, n \in \mathbf{Z}_0$, by

$$T_n F = H_n * F(A\cdot).$$

Let $F_n, n \in \mathbf{Z}_0$ be in a bounded set of $\mathcal{D}'$ and satisfy (2.6). For $n \in \mathbf{Z}_0$ and $m \ge 1$, set

$$\tilde{G}_n = G_n - F_n + H_n * F_{n+1}(A\cdot) = G_n - F_n + T_n F_{n+1}$$

and

$$G_{n,m} = T_n T_{n+1} \cdots T_{n+m-1}\tilde{G}_{n+m}. \tag{2.9}$$

Then for any $n \in \mathbf{Z}_0$ and $m \geq 1$,

$$G_{n,m} \quad \text{is supported in a compact set} \quad K \quad \text{independent of} \quad m \quad \text{and} \quad n, \tag{2.10}$$

and

$$\widehat{G}_{n,m}(\xi) = (\det A)^{-m}\widehat{H}_n(A^{-T}\xi)\cdots\widehat{H}_{n+m-1}((A^{-T})^m\xi)\widehat{\tilde{G}}_{n+m}((A^{-T})^m\xi) \tag{2.11}$$

by taking Fourier transform at each side of (2.9). By the assumption on G_n, F_n and $H_n, n \in \mathbf{Z}_0$, there exists a positive constant C_1 such that

$$|\widehat{F}_n(\xi) + |\widehat{G}_n(\xi)| + \|\widehat{H}_n(\xi)\| \leq C_1(1+|\xi|)^{C_1} \quad \text{for all} \quad \xi \in \mathbf{R}^d \quad \text{and} \quad n \in \mathbf{Z}_0.$$

This together with (2.6) leads to

$$|\widehat{\tilde{G}}_n(\xi)| \leq C_2|\xi|^{r_0}(1+|\xi|)^{C_2} \quad \text{for all} \quad \xi \in \mathbf{R}^d \quad \text{and} \quad n \in \mathbf{Z}_0,$$

where C_2 is a positive constant independent of $n \in \mathbf{Z}_0$ and $\xi \in \mathbf{R}^d$. It is easy to check that

$$|(A^{-T})^m\xi| \leq C_3(1+m)^N\rho(A^{-T})^m|\xi| \quad \text{for all} \quad m \geq 1 \quad \text{and} \quad \xi \in \mathbf{R}^d,$$

where C_3 is a positive constant independent of $m \geq 1$ and $\xi \in \mathbf{R}^d$. Thus by (2.11) and Lemma 2.1,

$$|\widehat{G}_{n,m}(\xi)| \leq C(1+|\xi|)^C\Big(\frac{r_0+1}{2}\Big)^m \quad \text{for all } \xi \in \mathbf{R}^d \quad \text{and} \quad m \geq 1,$$

where C is a positive constant independent of n, m and ξ. Thus $\sum_{m=1}^M \widehat{G}_{n,m}(\xi)$ converges uniformly on any compactly set and in distributional sense, i.e.,

$$\lim_{M\to\infty}\int_{\mathbf{R}^d}\sum_{m=M+1}^{\infty}\widehat{G}_{n,m}(\xi)f(\xi)d\xi = 0 \quad \text{for any Schwartz function } f.$$

Hence $\sum_{m=1}^M G_{n,m}$ converges in distributional sense, too. Denote the limit of $\sum_{m=1}^M G_{n,m}$ by $\sum_{m=1}^\infty G_{n,m}$. Note that

$$\Big|\sum_{m=1}^M \widehat{G}_{n,m}(\xi)\Big| \leq \sum_{m=1}^M |\widehat{G}_{n,m}(\xi)| \leq C(1+|\xi|)^C$$

for some positive constant C independent of $n \in \mathbf{Z}_0$ and $M \geq 1$. This together with (2.10) implies that $\sum_{m=1}^\infty G_{n,m}, n \in \mathbf{Z}_0$, is in a bounded set of $\mathcal{D}'$.

Set

$$\Phi_n = G_n + T_nF_{n+1} + \sum_{m=1}^\infty G_{n,m} \quad n \in \mathbf{Z}_0. \tag{2.12}$$

Then $\Phi_n, n \in \mathbf{Z}_0$ is in a bounded set of $\mathcal{D}'$. By (2.9),

$$G_{n,m} = T_n G_{n+1,m-1} \quad \forall\, n \in \mathbf{Z}_0 \quad \text{and} \quad m \geq 2.$$

Thus

$$\begin{aligned}
\Phi_n &= G_n + T_n F_{n+1} + G_{n,1} + T_n\Big(\sum_{m=2}^{\infty} G_{n+1,m-1}\Big) \\
&= G_n + T_n\Big(G_{n+1} + T_{n+1}F_{n+2} + \sum_{m=1}^{\infty} G_{n+1,m}\Big) \\
&= G_n + T_n\Phi_{n+1} = H_n * \Phi_{n+1}(A\cdot) + G_n.
\end{aligned}$$

Hence $\Phi_n, n \in \mathbf{Z}_0$, is in a bounded set of $\mathcal{D}'$ and satisfies the nonstationary nonhomogeneous refinement equation (1.1). $\square$

Remark 2.1 For any κ, let $\mathcal{D}'_\kappa$ be the set of all compactly supported tempered distributions with finite $\|\cdot\|_{\mathcal{D}'_\kappa}$, where

$$\|f\|_{\mathcal{D}'_\kappa} = \sup_{\xi \in \mathbf{R}^d} |\hat{f}(\xi)|(1+|\xi|)^{-\kappa}.$$

Then $\mathcal{D}'_\kappa$ is a linear topological subspace of $\mathcal{D}'$. By the proof of Theorem 2.1 and letting κ be sufficiently large, $G_{n,m} \in \mathcal{D}'_\kappa$ and

$$\|G_{n,m}\|_{\mathcal{D}'_\kappa} \leq C 2^{-m\delta} \quad \text{for all} \quad n \in \mathbf{Z}_0 \quad \text{and} \quad m \geq 1,$$

where C and δ are positive constants independent of n and m. In other words, the cascade sequence $G_{n,m}, m \geq 1$ converges in $\mathcal{D}'_\kappa$ exponentially. Such an idea can be used in finding solutions of the nonstationary nonhomogeneous refinement equation (1.1) in certain linear topological spaces, such as the space of all p-integrable functions, Besov spaces, Bessel potential spaces and Triebel-Lizorkin spaces.

Remark 2.2 By the proof of Theorem 2.1, $T_n T_{n+1} \cdots T_{n+m} \Phi_{n+m}$ tends to zero in distributional sense as m tends to infinity if $\Phi_n, n \in \mathbf{Z}_0$ is in a bounded set of $\mathcal{D}'$ satisfying

$$|\widehat{\Phi}_n(\xi)| \leq C|\xi|^{\tau_0} \quad \text{for all} \quad |\xi| \leq 1 \quad \text{and} \quad n \in \mathbf{Z}_0,$$

where C is a positive constant independent of n and ξ. Thus there is one-to-one correspondence between compactly supported distribtuional solutions $\Phi_n, n \in \mathbf{Z}_0$ of (1.1) in a bounded set and bounded solutions $\tilde{F}_n, n \in \mathbf{Z}_0$ of the linear equation (2.7).

3 Applications

In this section, we apply the results in the previous sections to investigate the following stationary nonhomogeneous refinement equation

$$\Phi = H * \Phi(A\cdot) + G \tag{3.1}$$

and the stationary refinement equation

$$\Psi = H * \Psi(A\cdot). \tag{3.2}$$

For the nonhomogeneous refinement equation (3.2), there is a much large literatures for the existence and regularity, and as well as the applications of its solutions ([3, 11, 13, 14, 15, 17, 18]). For the nonhomogeneous refinement equation (3.1), let τ_0 be the minimal nonnegative integer satisfying $\rho(\widehat{H}(0))\rho(A^{-T})^{\tau_0} < |\det A|$. Write

$$(\det A)^{-1}\widehat{H}(\xi) = \sum_{s\in\mathbf{Z}^d_{+,\tau_0}} H_s\xi^s + O(|\xi|^{\tau_0}) \quad \text{as} \quad \xi \to 0$$

and

$$S = I - \Big(\sum_{t\in\mathbf{Z}^d_{+,\tau_0}} g_{t+s',s}H_t\Big)_{s,s'\in\mathbf{Z}^d_{+,\tau_0}},$$

where we define $H_s = 0$ when $s \notin \mathbf{Z}^d_{+,\tau_0}$. By Theorem 2.1 and Remark 2.2, we have

Corollary 3.1 *Let G be a vector-valued compactly supported distribution, H be a $N\times N$ matrix-valued compactly supported distribution such that $\widehat{H}(\xi)$ is bounded, and let τ_0, S and $H_s, s\in\mathbf{Z}^d_{+,\tau_0}$ be as above. Then the following statements are equivalent:*

(i) There exists a compactly supported distribution Φ satisfying the stationary nonhomogeneous refinement equation (3.1);

(ii) There exists a compactly supported distribution F satisfying

$$\widehat{G}(\xi) - \widehat{F}(\xi) + (\det A)^{-1}\widehat{H}(A^{-T}\xi)\widehat{F}(A^{-T}\xi) = O(|\xi|^{\tau_0}) \quad \text{as} \quad \xi \to 0;$$

(iii) The rank of S is the same as the one of its augmented matrix $\tilde{S}$, i.e.,

$$r(S) = r(\tilde{S}),$$

where $\tilde{S} = (S, \tilde{G})$ is the augmented matrix of S, $\tilde{G} = (G_s)_{s\in\mathbf{Z}^d_{+,\tau_0}}$ and

$$\widehat{G}(\xi) = \sum_{s\in\mathbf{Z}^d_{+,\tau_0}} G_s\xi^s + O(|\xi|^{\tau_0}) \quad \text{as} \quad \xi \to 0.$$

For the stationary refinement equation (3.2), the dimension of the linear space of all compactly supported distributional solutions of the refinement equation (3.2) is $N\beta(\tau_0, d) - r(S)$*, where* $\beta(\tau_0, d)$ *denotes the dimension of the space of all polynomials in* $\mathbf{R}^d$ *with degree at most* $\tau_0 - 1$*.*

Remark 3.1 Note that S is a block lower triangular matrix with diagonal blocks

$$I - (\det A)^{-1}(g_{s,s'}\widehat{H}(0))_{|s|=|s'|=l}, \quad 0 \le l \le \tau_0 - 1.$$

Then S is singular if and only if one of diagonal blocks is singular. Hence by Corollary 3.1, there exists nonzero compactly supported distributional solution of (3.2) if and only if $I - (\det A)^{-1}(g_{s,s'}\widehat{H}(0))_{|s|=|s'|=l}$ is singular for some $0 \le l \le \tau_0 - 1$. For the refinement equation

$$\Psi = \sum_{j \in \mathbf{Z}^d} c_j \Psi(2 \cdot -j), \tag{3.3}$$

where $c_j = 0$ for all but finitely many $j \in \mathbf{Z}^d$, the assertion above was proved in [17, 23].

The nonhomogeneous refinement equation below is an important type of nonhomogeneous refinement equations

$$\Phi = H\Phi(2\cdot) + G, \tag{3.4}$$

where H is an $N \times N$ matrix. Such a nonhomogeneous refinement equation appeared in [4, 6, 19, 20, 22]. Note that $H = U^{-1}TU$ for some nonsingular matrix U and block diagonal matrix $T = \mathrm{diag}(E(\lambda_1), \cdots, E(\lambda_{l_0}))$, where

$$E(\lambda_l) = \begin{pmatrix} \lambda_l & & & 0 \\ 1 & \lambda_l & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_l \end{pmatrix}, \quad 1 \le l \le l_0.$$

Then the nonhomogeneous refinement equation (3.4) is essentially the combination of the following two types of nonhomogeneous refinement equations,

Type I: $\Phi = \lambda\Phi(2\cdot) + G, \quad N = 1;$

Type II: $\Phi = E(\lambda)\Phi(2\cdot) + G.$

By Corollary 3.1, we have

Corollary 3.2 *Let nonhomogeneous refinement equations of type I and of type II be defined as above. Then*

(i) The nonhomogeneous refinement equation of type I is solvable in $\mathcal{D}'$ if and only if $\lambda \neq 2^{l+d}$ for all nonnegative integers l, or $\lambda = 2^{l+d}$ and $D^l G(0) = 0$ for some nonnegative integer l.

(ii) The nonhomogeneous refinement equation of type II is solvable in $\mathcal{D}'$ if and only if $\lambda \neq 2^{l+d}$ for all nonnegative integer l, or $\lambda = 2^{l+d}$ and $D^l G_1(0) = 0$, where G_1 denotes the first component of G.

In some applications, the compactly supported distribution Ψ of the refinement equation (3.2) need satisfy $\hat{\Psi}(0) \neq 0$. It is obvious that $\hat{\Psi}(0)$ is an eigenvector of $\widehat{H}(0)$ with eigenvalue $\det A$. However, such a solution does not exist in general even when $\det A$ is an eigenvalue of $\widehat{H}(0)$. For example, $(0, \gamma\delta')^T, \gamma \in \mathbf{C}$ are all compactly supported distributional solutions of the following refinement equation

$$\Psi = \begin{pmatrix} 2 & 0 \\ -2 & 4 \end{pmatrix} \Psi(2\cdot) + \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \Psi(2\cdot -1),$$

where δ' denotes the derivative of the delta distribution, but

$$\tilde{H}(0) = \begin{pmatrix} 2 & 0 \\ -2 + 2e^{-i\xi} & 4 \end{pmatrix}$$

has eigenvalue 2.

Corollary 3.3 *Let $H_s, s \in \mathbf{Z}^d_{+,\tau_0}$ and S be as above. Set*

$$S^* = I - \Big(\sum_{t \in \mathbf{Z}^d_{+,\tau_0}} g_{t+s',s} H_t \Big)_{s,s' \in \mathbf{Z}^d_{+,\tau_0} \setminus \mathbf{Z}^d_{+,1}}.$$

Then there exist a compactly supported distributional solution Ψ of the refinement equation (3.2) with $\hat{\Psi}(0) \neq 0$ if and only if $r(S) \leq r(S^*) + N - 1$.

Proof. $\Longleftarrow$**:** On the contrary, $r(S) = r(S^*) + N$. Then $I - (\det A)^{-1}\widehat{H}(0)$ is nonsingular since

$$S = \begin{pmatrix} I - (\det A)^{-1}\widehat{H}(0) & 0 \\ B^* & S^* \end{pmatrix}$$

for some matrix B^*. Recall that

$$\widehat{H}(0)\hat{\Psi}(0) = \det A\ \hat{\Psi}(0)$$

for any compactly supported distributional solution Ψ of the refinement equation (3.2). Thus $\hat{\Psi}(0) = 0$, which is a contradiction.

$\Longrightarrow$**:** Assume that $r(S) \le r(S^*) + N - 1$. Note that the dimension of the linear space of solutions $\tilde{F}$ of the linear equation $S\tilde{F} = 0$ with the first block zero is $N(\beta(\tau_0, d) - 1) - r(S^*)$, and that the dimension of the linear space of solutions $\tilde{F}$ of the linear equation $S\tilde{F} = 0$ is $N\beta(\tau_0, d) - r(S)$. Thus by the assumption there exists $\tilde{F} = (F_s)_{s \in \mathbf{Z}^d_{+,\tau_0}}$ satisfying $S\tilde{F} = 0$ and $F_0 \ne 0$. Let compactly supported distributions F and Ψ be defined by $\widehat{F}(\xi) = \sum_{s\in\mathbf{Z}^d_+} F_s \xi^s$ and

$$\hat{\Psi}(\xi) = \lim_{m\to\infty} (\det A)^{-m} \widehat{H}(A^{-T}\xi) \cdots \widehat{H}((A^{-T})^m \xi) \widehat{F}((A^{-T})^m \xi).$$

Then Ψ satisfies the refinement equation (3.2) and $\hat{\Psi}(0) = F_0 \ne 0$. □

Remark 3.2 The assertion in Corollary 3.3 was proved in [17, 23] for the refinement equation (3.3) under additional assumption that S^* is of full rank.

4 Conclusion

In this paper, we reduce the existence of compactly supported distributional solutions of the nonstationary nonhomogeneous refinement equations,

$$\Phi_n = H_n * \Phi_{n+1}(A\cdot) + G_n \quad \text{for all} \quad n \in \mathbf{Z}_0, \tag{4.1}$$

to finding appropriate initials $F_n, n \in \mathbf{Z}_0$ of the cascade operators $T_n, n \in \mathbf{Z}_0$, where

$$T_n F = H_n * F(A\cdot).$$

Further we can construct the appropriate initials $F_n, n \in \mathbf{Z}_0$ through solving certain linear equations. The ideas in the proof of our main result can be used in finding solutions of nonstationary nonhomogeneous refinement equation (4.1) in certain topological spaces, such as Besov spaces and Bessel potential spaces.

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