Finite Element Wavelets

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Abstract

This paper give a construction of biorthogonal wavelets in the setting of P_1 finite elements spaces for general polygonal domain, with hierarchical triangulations. The construction is based on the so-called lifting scheme (or coarse grid correction) with a specific treatment for the nodes that are near the exceptional vertices and edges associated to the coarsest triangulation. Finite element spaces with both Dirichlet and Neumann boundary conditions are considered. The corresponding dual bases have positive Sobolev smoothness $\tilde{s}$, resulting in the possibility of characterizing H^s spaces for $-\tilde{s} < s < 3/2$. Although the analysis of this property is involved, the construction itself is simple and easy to implement.

1 Introduction

In recent years, there has been a growing interest for the use of wavelets in numerical simulation. The main motivation for the use of these tools in this context is twofold:

- The possibility to characterize function spaces from the numerical properties of wavelet coefficients results into diagonal preconditioners for elliptic operators when discretized in these bases, which can be viewed as a variant of multigrid additive (BPX) preconditioners.
- The ability of wavelet bases to adaptively represent functions that are piecewise smooth with a very high order of accuracy with respect to the number of parameters.

We refer to [6] and [2] for general surveys on the numerical analysis of wavelet methods, describing these aspects in more details. Let us simply mention that, in contrast to area such as statistical signal and image processing, where the efficiency of wavelet techniques for applications such as compression and denoising is established, these methods are still at an early stage in numerical simulation in which it still difficult to compare them with other more classical discretization, e.g. finite differences, finite elements or spectral methods. There is at least two reasons for this state of affair. First of all, while wavelets might have the ability to approximate the solution of a PDE with high accuracy and low complexity, it takes more work to actually build a resolution scheme which will indeed produce an approximation in such a compressed form (in the context of data compression, a simple thresholding algorithm does the job). Secondly, due to the versatility of numerical test problems that might be considered in benchmarking, a comparison of wavelets with more classical tools requires in particular that one can at least adapt the wavelet discretization to the inherent geometry of the problem at hand, which is not always an easy task.

On this second point, several progresses have been achieved recentely. There are basically two approaches which are currently being followed in order to build wavelet bases on general domains $\Omega \subset \mathbb{R}^d$: (i) domain decomposition into square patches and (ii) multilevel decomposition of finite element spaces. The first approach, as proposed in e.g. [1] or [7], exploits the well understood constructions of orthonormal or biorthogonal wavelets on unit cubes together with a proper "glueing" of the basis functions at the interfaces between subdomains. The second approach can be more tempting if one wants to combine the useful properties of wavelets with the structural simplicity of finite element spaces. In particular, one might be interested to modify or postprocess a given finite element code, by using a wavelet basis adapted to the corresponding finite element space.

In the two-dimensional case, this second approach can be summarized as follows: let Ω be a polygon, convex or not, with boundary $\partial\Omega$. We start from an initial coarse triangulation $\mathcal{T}_0$ of Ω and inductively define a triangulation $\mathcal{T}_j$ of resolution 2^{-j}, $j \geq 1$ by subdividing each triangle of $\mathcal{T}_{j-1}$ into four similar triangles. To such triangulations, we associate finite element spaces V_j. In the classical case of Lagrange P_n finite elements (continuous piecewise polynomials of total degree n), these spaces are nested ($V_j \subset V_{j+1}$) and one can easily define a first multiscale basis of V_J in the following way: we take the nodal functions at the coarsest level (which span V_0) to which we append the nodal functions of V_{j+1}, $j = 0, 1, \cdots, J-1$ that are associated to a Lagrange node which is not a node of V_j (which thus span for each j a complement space W_j of V_j into V_{j+1}). In the case of P_1 finite elements, the Lagrange nodes of V_j are simply the vertices of the triangulation $\mathcal{T}_j$ and this construction is the classical hierarchical basis, which was firstly considered in [15] for preconditioning purposes. Note that one can easily append an homogeneous Dirichlet boundary condition in this construction, provided that this condition occurs on a part of the boundary which is resolved by the coarsest grid. Such hierarchical bases can be considered as the simplest finite element wavelet bases. Their main disadvantage is that they fail to ensure the characterization of Sobolev spaces H^s by the usual norm equivalence for $s \leq d/2$, resulting in mediocre preconditioning properties for second order problems in dimension $d \geq 2$ and ill conditioned mass matrices in any dimension. This lack of stability is related to the fact that the decomposition in such bases is inherently related to the Lagrange interpolation operator I_j onto V_j, since the component of a function f in W_j is obtained as the difference $I_{j+1}f - I_jf$. The limitation by below on s expresses the fact that I_j is not well-defined on H^s for $s \leq d/2$, since the dual functionals that are used in the evaluation of I_j are Dirac distributions at the Lagrange nodes. Another defect of hierarchical basis functions is that in contrast to wavelets, they do not have vanishing moments, which limitates their ability ot compress stiffness matrices and data in the context of Galerkin discretizations.

The main idea in order to build more stable finite element wavelet bases is to start with a hierachical basis and apply a local correction process on the basis functions at each level, which generates a new complement space that should be in some sense more orthogonal to V_j than the initial W_j. This technique can be viewed as an application of the so-called *lifting scheme* introduced in [14] in the case of *coarse grid correction* of W_j by nodal functions in V_j, and of the more general *stable completion* technique of [4] for more general *fine grid correction* by nodal functions in V_{j+1}. While such constructions are usually simple to describe and implement, since the correction process is local, the analysis of the success of this strategy in terms of allowing a better range of Sobolev norm characterization is a more difficult task, which involves the understanding of the dual functions that are inherently built in place of the initial Dirac distributions. Typically, the new range is now limited by $s > -\tilde{s}$ where $\tilde{s}$ is the Sobolev smoothness of these dual functions, and the correction process should be made in such a way that $\tilde{s}$ is strictly positive.

Non orthogonal fine grid corrections, which are particularly attractive due to the very short support of the resulting wavelets, can be proved to result in strictly positive $\tilde{s}$ (see e.g. [10], [11]). However such results are essentially available in the restricted context of uniform triangulations, in which case the analysis benefits of shift invariance and fourier transform techniques, leaving aside the proper treatment of exceptional vertices, edges and boundary conditions, which is unavoidable when dealing with a polygonal domain. A possibility that leads to $\tilde{s} = 3/2$ is to use a fine grid correction which generates an orthogonal complement to V_j as in [12] or more generally to another lagrange finite element space $\tilde{V}_j$ which satisfies a minimal angle condition with respect to V_j, as proposed in [8]. In this case, the wavelets are constructed by an element by element orthogonalization procedure, which allows a proper treatment of exceptional edges and vertices. In both constructions, the dual basis is globally supported, which is not a practical problem for applications such as Galerkin discretizations.

The goal of the present paper is to provide a construction and an analysis for general polygonal domain based on coarse grid correction, i.e. the lifting scheme, which has the specificity of resulting in compactly supported dual functions. This is an advantage for applications where a fast decomposition algorithm is needed. Another advantage of this approach is that it will not require solving any local orthogonalization problem. We shall work in the setting of P_1 finite element which is both the simplest and most commonly used discretization for a $2D$ polygonal domain. However, one should view this as a "laboratory case", and be aware that the construction and analysis principle that we use could be adapted, with more technical work to other types of elements. For P_1 element, we can in addition exploit some existing results in the translation invariant setting: away from the exceptional vertices and edges, we obtain the same dual functions which have been introduced in [5]. These functions satisfy a classical refinement equation and are known to be in H^s for $s < 0.44$. The correction process is of course slighty adapted near the the exceptional vertices and edges, resulting in other dual functions which satisfy nonhomogeneous refinement equations. In all cases, the correction process is such that the resulting primal wavelets have at least their first moment vanishing, and uses at most the four neighbouring coarse grid functions.

This paper is organized as follows: in §2, we give some main notations for the hierarchical basis and the lifting scheme. We then classify the nodes of the triangulation $\mathcal{T}_j$ and define the lifting matrices accordingly. Then, in §3 we define and analyze the corresponding dual functions and study their smoothness. In §4 we prove their biorthogonality properties with respect to the nodal functions as well as their properties of polynomial exactness. Finally, we display in §5 the graphs of the various dual scaling functions obtained near exceptional points or edges.

The analysis in §3 and §4 is not straightforward - it require to study certain non-homogeneous refinement equations - but necessary to understand the stability properties of the resulting multiscale basis. In contrast, let us emphasize on the fact that the concrete implementation of this basis - i.e. the fast algorithm which connect the nodal and multiscale representation of a function in the finite element space - is very simple and only involves the lifting coefficients given in §2.

2 Construction

2.1 The lifting scheme

Using the same notations as in the introduction for the hierarchical triangulations $\mathcal{T}_j$ defined on the polygonal domain Ω, we also denote by T_j, the set of all vertices in the triangulation $\mathcal{T}_j$. We denote by $\partial\Omega_D$ the Dirichlet boundary, on which we impose that the functions of V_j vanish. We require that $\partial\Omega_D$ is a union of closed line segments on $\partial\Omega$ with endpoints in T_0. Let Γ_j be the set of nodes of T_j not on the Dirichlet boundary $\partial\Omega_D$. The corresponding spaces V_j are generated by the nodal basis $(\phi_{j,\gamma})_{\gamma\in\Gamma_j}$, where $\phi_{j,\gamma}$ is the unique continuous, piecewise affine function on each triangle of $\mathcal{T}_j$ such that $\phi_{j,\gamma}(\gamma') = \delta_{\gamma,\gamma'}$, $\gamma' \in T_j$. Obviously, we have $T_j \subset T_{j+1}$, $\Gamma_j \subset \Gamma_{j+1}$ and $V_j \subset V_{j+1}$. We shall sometimes refer to Γ_j as the *active nodes* of T_j in contrast to the *passive nodes* represented by $T_j \setminus \Gamma_j$. For a node $\gamma \in T_j$, let $\triangle_j(\gamma)$ be the collection of triangles in $\mathcal{T}_j$ with γ as a vertex and let $T_j(\gamma) := \Gamma_j \cap \triangle_j(\gamma) - \{\gamma\}$ be the neighbours of γ in Γ_j. Set $\Lambda_j = \Gamma_{j+1}\backslash\Gamma_j$. Then $T_{j+1}(\gamma) \subset \Lambda_j \subset \Gamma_{j+1}$ for $\gamma \in \Gamma_j$. The nodal functions $\phi_{j,\gamma}, \gamma \in \Gamma_j$ are *refinable* in the sense that $\phi_{j,\gamma}, \gamma \in \Gamma_j$ can be written as linear combination of $\phi_{j+1,\eta}, \eta \in \Gamma_{j+1}$. In fact,

$$\phi_{j,\gamma} = \phi_{j+1,\gamma} + \frac{1}{2}\sum_{\eta\in T_{j+1}(\gamma)} \phi_{j+1,\eta}. \tag{1}$$

The elements of the hierarchical wavelet basis are defined by

$$\psi^h_{j,\lambda} = \phi_{j+1,\lambda}, \tag{2}$$

for $\lambda \in \Lambda_j$. Then $\phi_{j+1,\eta}, \eta \in \Gamma_{j+1}$ can be obtained as a combination of the $\phi_{j,\gamma}, \gamma \in \Gamma_j$ and $\psi^h_{j,\lambda}, \lambda \in \Lambda_j$, according to

$$\phi_{j+1,\eta} = \begin{cases} \phi_{j,\eta} - \frac{1}{2}\sum_{\lambda\in T_{j+1}(\eta)} \psi^h_{j,\lambda}, & \eta \in \Gamma_j \\ \psi_{j,\eta}, & \eta \in \Lambda_j. \end{cases} \tag{3}$$

It follows that the space W^h_j spanned by $\psi^h_{j,\lambda}$, $\lambda \in \Lambda_j$ is a complement space of V_j into V_{j+1}. In particular, we have

$$V_J = V_0 \oplus W_0 \oplus \cdots \oplus W_{J-1}, \tag{4}$$

so that $\{\phi_{0,\gamma}, \gamma \in \Gamma_0\} \cup \{\psi^h_{j,\gamma}, \gamma \in \Lambda_j, 0 \le j < J\}$ is a basis of V_J. This hierarchical basis, can be viewed as a particular case of a biorthogonal wavelet basis, in the following sense. We define the *dual scaling functions*

$$\tilde{\phi}^h_{j,\gamma} = \delta_\gamma \tag{5}$$

for $\gamma \in \Gamma_j$, where δ_γ is the Dirac distribution at the node γ. For $\lambda \in \Lambda_j$ and $(\lambda_1, \lambda_2) \in T_j$ such that λ is the mid-point of these two nodes, we define the *dual wavelet* by

$$\tilde{\psi}^h_{j,\lambda} = \delta_\lambda - \frac{1}{2} \sum_{\lambda_i \in \Gamma_j, i=1,2} \delta_{\lambda_i}. \tag{6}$$

Then, the expansion of a function $f \in V_J$ in terms of the hierarchical basis can be expressed by

$$f = \sum_{\gamma \in \Gamma_0} \langle f, \tilde{\phi}^h_{0,\gamma} \rangle \phi_{0,\gamma} + \sum_{j=0}^{J-1} \sum_{\lambda \in \Lambda_j} \langle f, \tilde{\psi}^h_{\lambda,j} \rangle \psi^h_{\lambda,j}. \tag{7}$$

Furthermore $\{\phi_{j,\gamma}, \psi^h_{j,\lambda}\}$ and $\{\tilde{\phi}^h_{j,\gamma'}, \tilde{\psi}^h_{j,\lambda'}\}$ are *generalized biorthogonal* in the sense of the duality product between continuous functions and Radon measures

$$\left\{ \begin{array}{lcl} \langle \phi_{j,\gamma}, \tilde{\phi}^h_{j,\gamma'} \rangle & = & \delta_{\gamma\gamma'}, \quad \gamma, \gamma' \in \Gamma_j, \\ \langle \psi^h_{j,\lambda}, \tilde{\phi}^h_{j,\gamma'} \rangle & = & 0, \quad \lambda \in \tilde{\Gamma}_j, \gamma' \in \Gamma_j, \\ \langle \phi_{j,\gamma}, \tilde{\psi}^h_{j,\lambda'} \rangle & = & 0, \quad \gamma \in \Gamma_j, \lambda' \in \tilde{\Gamma}_j, \\ \langle \psi^h_{j,\lambda}, \tilde{\psi}^h_{j,\lambda'} \rangle & = & \delta_{\lambda,\lambda'}, \quad \lambda, \lambda' \in \Lambda_j, \end{array} \right.$$

It can also be checked that the spaces $\tilde{V}^h_j$ and $\tilde{W}^h_j$ spanned by the dual functions satisfy relations similar to (4). The hierachical basis is thus inherently tied to the interpolation operator P^h_j

$$P^h_j v = \sum_{\gamma \in \Gamma_j} \langle v, \tilde{\phi}^h_{j,\gamma} \rangle \phi_{j,\gamma}, \tag{8}$$

This restricts its application to the decompositions of Sobolev spaces H^α, since for $\alpha \leq 1$ these spaces are not embedded in C^0. In order to obtain a more stable decomposition for such spaces, we shall use the so-called lifting-scheme introduced in [14]. We briefly describe this technique in our specific context. Let

$$s^\gamma_{j,\eta} = \left\{ \begin{array}{ll} 1, & \eta = \gamma, \\ 1/2, & \eta \in T_{j+1}(\gamma), \\ 0, & \text{otherwise}, \end{array} \right.$$

for $\gamma \in \Gamma_j, \eta \in \Gamma_{j+1}$ and

$$d^{\lambda,h}_{j,\eta} = \left\{ \begin{array}{ll} 1, & \eta = \lambda, \\ 0, & \text{otherwise}, \end{array} \right.$$

for $\lambda \in \Lambda_j, \eta \in \Gamma_{j+1}$. With this notation, (1) and (2) are equivalent to

$$\begin{cases} \phi_{j,\gamma} &= \sum_{\eta\in\Gamma_{j+1}} s^{\gamma}_{j,\eta}\phi_{j+1,\eta}, \quad \gamma \in \Gamma_j, \\ \psi^h_{j,\lambda} &= \sum_{\eta\in\Gamma_{j+1}} d^{\lambda,h}_{j,\eta}\phi_{j+1,\eta}, \quad \gamma \in \Lambda_j. \end{cases}$$

Let

$$\tilde{s}^{\gamma,h}_{j,\eta} = \begin{cases} 1, & \eta = \gamma, \\ 0, & \text{otherwise}, \end{cases}$$

for $\gamma \in \Gamma_j, \eta \in \Gamma_{j+1}$ and

$$\tilde{d}^{l}_{j,\eta} = \begin{cases} 1, & \eta = \lambda, \\ -1/2, & \eta = \lambda_1, \\ -1/2, & \eta = \lambda_2, \lambda_2 \in \Gamma_j, \\ 0, & \text{otherwise}, \end{cases}$$

for $\lambda \in \Lambda_j, \eta \in \Gamma_{j+1}$, where λ_1, λ_2 are defined as for (6). Then $\tilde{\phi}^h_{j,\gamma}$ and $\tilde{\psi}^h_{j,\lambda}$ satisfy

$$\begin{cases} \tilde{\phi}^h_{j,\gamma} &= \sum_{\eta\in\Gamma_{j+1}} \tilde{s}^{\gamma,h}_{j,\eta}\tilde{\phi}^h_{j+1,\eta}, \\ \tilde{\psi}^h_{j,\gamma} &= \sum_{\eta\in\Gamma_{j+1}} \tilde{d}^{\gamma}_{j,\eta}\tilde{\phi}^h_{j+1,\eta}. \end{cases}$$

Defining the matrices, $A_j = (s^{\gamma}_{j,\eta})_{\gamma\in\Gamma_j,\eta\in\Gamma_{j+1}}$, $B_j = (d^{\lambda,h}_{j,\eta})_{\lambda\in\Lambda_j,\eta\in\Gamma_{j+1}}$, $\tilde{A}_j = (\tilde{s}^{\gamma,h}_{j,\eta})_{\gamma\in\Gamma_j,\eta\in\Gamma_{j+1}}$ and $\tilde{B}_j = (\tilde{d}^{l}_{j,\eta})_{\lambda\in\Lambda_j,\eta\in\Gamma_{j+1}}$, we see that the coordinates change from (resp. to) the basis $\{\phi_{j+1,\gamma}\}_{\gamma\in\Gamma_{j+1}}$ to (resp. from) $\{\phi_{j,\gamma}, \psi^h_{j,\lambda}\}_{\gamma\in\Gamma_j,\lambda\in\Lambda_j}$ are operated by the matrix $(\tilde{A}^T_j, \tilde{B}^T_j)$ (resp. $(A^T_j, B^T_j)^T$), so that we have

$$\begin{pmatrix} A_j \\ B_j \end{pmatrix} (\tilde{A}^T_j, \tilde{B}^T_j) = I. \tag{9}$$

Thus

$$\begin{pmatrix} A_j \\ B_j - C_j A_j \end{pmatrix} (\tilde{A}^T_j + \tilde{B}^T_j C_j, \tilde{B}^T_j) = I \tag{10}$$

for any matrix $C_j = (c^l_{j,\gamma})_{\lambda\in\Lambda_j,\gamma\in\Gamma_j}$, which we call as *lifting matrix*. The above identity expresses that we have modified the hierarchical functions $\psi^h_{j,\lambda}$ according to a coarse grid correction $\psi_j j, \lambda = \psi^h_{j,\lambda} - c^{\lambda}_{j,\gamma}\phi_{j,\gamma}$, resulting in a new complement space W_j. While, it is clear that we obtain another multiscale basis (which will also allow fast local computations, provided that C_j has a banded structure), understanding the stability of this new basis with respect to Sobolev spaces, requires that we understand the nature of the corresponding dual functions. For this, we set

$$\tilde{s}^{\gamma}_{j,\eta} = \tilde{s}^{\gamma,h}_{j,\eta} + \sum_{v\in\Lambda_j} c^{v}_{j,\gamma}\tilde{d}^{v}_{j,\eta}, \tag{11}$$

and

$$d^{\lambda}_{j,\eta} = d^{\lambda,h}_{j,\eta} - \sum_{\gamma\in\Gamma_j} c^{\lambda}_{j,\gamma}s^{\gamma}_{j,\eta}. \tag{12}$$

Then, while we clearly have

$$\phi_{j,\gamma} = \sum_{\eta\in\Gamma_{j+1}} s^{\gamma}_{j,\eta}\phi_{j+1,\eta} \text{ and } \psi_{j,\lambda} = \sum_{\eta\in\Gamma_{j+1}} d^{\lambda}_{j,\eta}\phi_{j+1,\eta}, \tag{13}$$

we expect the dual scaling functions and wavelets to be solutions of

$$\tilde{\phi}_{j,\gamma} = \sum_{\eta\in\Gamma_{j+1}} \tilde{s}^{\gamma}_{j,\eta}\tilde{\phi}_{j+1,\eta} \text{ and } \tilde{\psi}_{j,\lambda} = \sum_{\eta\in\Gamma_{j+1}} \tilde{d}^{\lambda}_{j,\eta}\tilde{\phi}_{j+1,\eta} \tag{14}$$

We shall thus build the lifting matrix in order to ensure that the solutions of the above equations are well defined and are more regular than the initial Dirac distributions. In the case of a uniform triangulation on the whole $\mathbb{R}^2$, the construction described in [5], which corresponds to a simple local lifting matrix, ensures that the dual functions are in H^s for $s \leq \tilde{s} \approx 0.44$. While we shall use this correction process in the regions where the triangulation is locally uniform, we shall need a particular treatment near the exceptional edges and vertices, as well as near the boundary $\partial\Omega$. Therefore, our first step is to classify the nodes of the triangulation into different categories.

2.2 Classification of Nodes

From the initial triangulation we obtain a union of line segments after deleting all common sides of two triangles of the initial triangulation such that the quadrangle composed by these two triangles is a parallelogram. We call the union of those line segments as the *frame* of the initial triangulation or the frame for simplicity. Thus the boundary $\partial\Omega$ is contained in the frame. This initial step is shown on Figure 1.

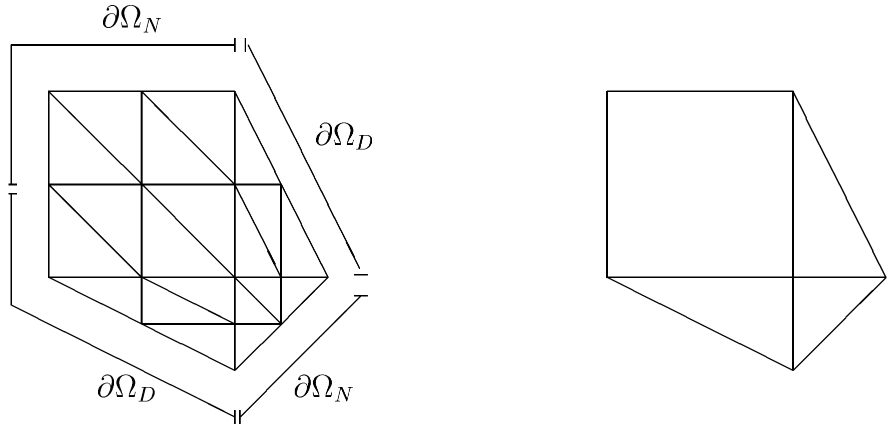


Figure 1. Initial triangulation (left) and frame (right)

We classify the nodes of T_j into three classes: exceptional nodes, frame nodes and inner nodes. A node is called an *exceptional node* if it lies on the intersection of different line segments of the frame or on the intersection of the the Neumann boundary $\partial\Omega\backslash\partial\Omega_D$ and the Dirichlet boundary $\partial\Omega_D$. Recall that $\partial\Omega_D$ is a union of closed line segments on $\partial\Omega$ with endpoints in T_0, so that all the exceptional nodes are in T_0. A node is called a *frame node* if it lies on the frame and if it is not an exceptional node. All other nodes are called *inner nodes*. The construction of the lifting matrix that we propose in the next subsection will use the above classification. We shall define this construction at a minimal refinement level which ensures that the exceptional nodes are "well separated" by the uniform mesh of frame and inner nodes. We express this as follows: the triangulation $\mathcal{T}_j$ of the polygon Ω is *lifting scheme separable* if for any node γ of the triangulation the following two conditions hold

(1) At most one exceptional node or exceptional boundary node belongs to $\Delta_j(\gamma)$.

(2) There are no two line segments of the frame which are parallel to each other and have nonempty intersection with $\Delta_j(\gamma)$.

The following result can be easily checked.

Proposition 2.1 *For any initial triangulation $\mathcal{T}_0$, all triangulations $\mathcal{T}_j$ are lifting scheme separable for $j \geq 2$.*

2.3 Lifting Matrix

In this subsection we shall construct the matrix

$$C_j = (c^v_{j,u})_{v \in \Lambda_j, u \in \Gamma_j}$$

in the lifting scheme for $j \geq 2$ in such way that $\mathcal{T}_j$ is lifting scheme separable. This will allow us to define the corrected space W_j for $j \geq 2$. Note that we can still define W_0 and W_1 by the usual hierarchical basis complement. For any $v \in \Lambda_j$ which is not on the boundary $\partial\Omega$, there exists two triangles $\triangle_1$ and $\triangle_2$ of the triangulation of scale j such that v lies at the midpoint of the common side of $\triangle_1$ and $\triangle_2$ (see Figure 4.a). For $v \in \Lambda_j \cap \partial\Omega$, there exists only one triangle $\triangle_1$ of the triangulation of scale j with v as the midpoint of the side of $\triangle_1$ on the boundary (see Figure 4.b).

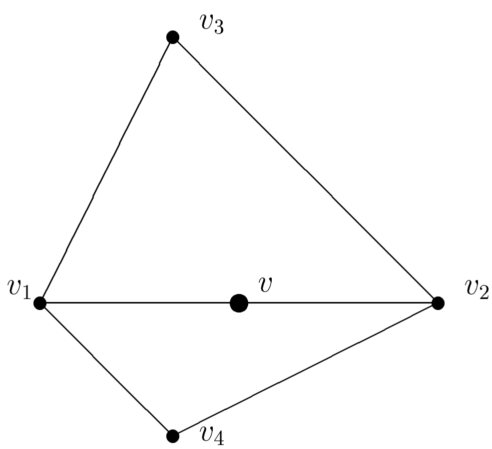


Figure 4.a.

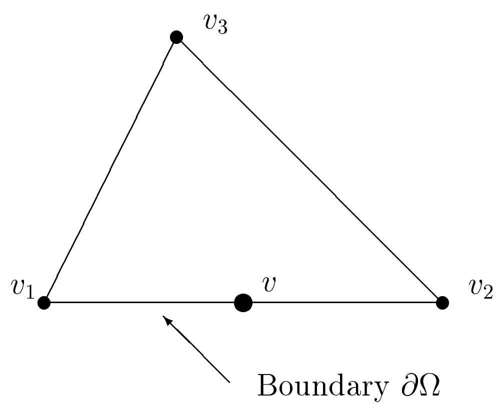


Figure 4.b.

We denote by v_i, $i = 1, 2, 3$ the vertices of $\triangle_1$ and by v_4 the extra vertex of $\triangle_2$ if $v \notin \partial\Omega$, as illustrated on Figure 4.a and 4.b. Then at least one of v_i is not on the boundary when $\mathcal{T}_j$ is lifting scheme separable. The correction process that we shall apply on the hierarchical basis function $\psi^h_{j,v}$ to derive $\psi_{j,\nu}$ will only involve these close neighbours of v on the coarse grid T_j: we shall set

$$c^v_{j,u} = 0,$$

when $u \neq v_i, i = 1,2,3,4$. It remains to define c^v_{j,v_i}, $i = 1,2,3,4$ when $v \notin \partial\Omega$ and $i = 1,2,3$ when $v \in \partial\Omega$. Several cases will be considered depending on the position of the nodes v_i with respect to the exceptional vertices and the frame. Hereafter we set

$$a(v,u) = \int_{\mathbb{R}^2} \phi_{j+1,v}(x)dx\big(\int_{\mathbb{R}^2} \phi_{j,u}(x)dx\big)^{-1}$$

for $v \in \Gamma_{j+1}$ and $u \in \Gamma_j$. We shall use the quantities $a(v,v_i)$ in order to define the lifting matrix. Note that when v is sufficiently away from the frame, we have $a(v,v_i) = 1/4$.

2.3.1 Correction near the exceptional points

Here, we assume that v is such that one of the v_i is an exceptional node. On the other hand, under our assumption $j \geq 2$, there is at most one exceptional node among the v_i, $i = 1,\cdots,4$, We denote by γ this node. By symmetry, we can assume that γ is either v_1 or v_3.

Case 1. γ is an active node, i.e. $\gamma \notin \partial\Omega_D$. In this case, we can use $\phi_{j,\gamma}$ to correct $\psi^h_{j,v}$. If $\gamma = v_1$, we define

$$c^v_{j,v_i} = \begin{cases} a(v,v_1), & i = 1 \\ 0, & i = 2,3,4. \end{cases}$$

If $\gamma = v_3$, we define

$$c^v_{j,v_i} = \begin{cases} a(v,v_3), & i = 3 \\ 0, & i = 1,2,4. \end{cases}$$

Case 2. γ is an passive node, i.e. $\gamma \in \partial\Omega_D$. In this case, we cannot use $\phi_{j,\gamma}$ to correct $\psi^h_{j,v}$. If $\gamma = v_1$, we define

$$c^v_{j,v_i} = \begin{cases} a(v,v_2), & i = 2 \\ 0, & i = 1,3,4. \end{cases}$$

If $\gamma = v_3$, we define

$$c^v_{j,v_i} = \begin{cases} a(v,v_4), & i = 4 \\ 0, & i = 1,2,3. \end{cases}$$

2.3.2 Correction near the frame

Here, we assume that v is not treated by the previous construction and that v is such that one of the v_i is a frame node. Under our assumption $j \geq 2$, there is at most two frame nodes among the v_i, $i = 1, \cdots, 4$.

Case 1 v_1 and v_2 are on the frame and are active nodes. In this case, we define

$$c_{j,v_i}^v = \begin{cases} \frac{1}{2}a(v, v_i), & i = 1, 2 \\ 0, & i = 3, 4. \end{cases}$$

Case 2 v_1 and v_3 are on the frame and are active nodes. In this case, we define

$$c_{j,v_i}^v = \begin{cases} a(v, v_1), & i = 1 \\ 0, & i = 2, 3, 4. \end{cases}$$

Case 3 v_1 and v_3 are on the frame and are passive nodes. In this case, we define

$$c_{j,v_i}^v = \begin{cases} a(v, v_2), & i = 2 \\ 0, & i = 1, 3, 4. \end{cases}$$

Case 4 v_3 is on the frame and is a passive node. In this case, we define

$$c_{j,v_i}^v = \begin{cases} \frac{5}{8}a(v, v_i), & i = 1, 2 \\ -\frac{1}{4}a(v, v_4), & i = 4 \\ 0, & i = 3. \end{cases}$$

Note that cases 2 and 3 apply by symmetry when (v_1, v_3) is replaced by (v_1, v_4), (v_2, v_3) or (v_2, v_4). Similarly v_3 could be replaced by v_4 in case 3.

2.3.3 Standard correction

For all other nodes $v \in \Lambda_j$, which are not treated by the two previous constructions, we define

$$c_{j,v_i}^v = \begin{cases} \frac{3}{4}a(v, v_i), & i = 1, 2, \\ -\frac{1}{4}a(v, v_i), & i = 3, 4, \end{cases}$$

which is exactly the correction corresponding to the construction in [5] for a uniform mesh. This completes the construction of lifting matrix $C_j, n \geq 0$ for lifting scheme admissible triangulation. For the lifting matrix C_j, we have

Remark 2.2 *Let the lifting matrix* $C_i = (c^v_{j,u})_{v\in\Lambda_j, u\in\Gamma_j}$ *be defined as above and let*

$$c^v_{j,u} = \tilde{c}^v_{j,u} a(v,u).$$

Then

$$\sum_{u\in\Gamma_j} \tilde{c}^v_{j,u} = 1, \text{ for all } v \in \Lambda_j.$$

3 Smoothness of the dual refinable functions

In this section, we shall define new refinable functions and wavelets from the previous lifting scheme and study their smoothness. Clearly, the new primal wavelets $\psi_{j,\lambda}$ are defined from (13) since the nodal functions $\phi_{j,\gamma}$ have been left unchanged. Also the new dual wavelets $\tilde{\psi}_{j,\lambda}$ are directly defined from the $\tilde{\phi}_{j,\gamma}$ according to (14). Thus our first task will be to study the existence and smoothness of the new dual refinable functions $\tilde{\phi}_{j,\gamma}$ such that the first equation in (14) hold. Note that solutions of such refinement equations are defined up to a multiplicative constant which is fixed by the biorthogonality constraint $\langle \phi_{j,\gamma}, \tilde{\phi}_{j,\eta}\rangle = \delta_{\gamma,\eta}$. In the following analysis, we shall give the values of the coefficients $\tilde{s}^\gamma_{j,\eta}$ in the dual refinement equations only when they are necessary for proving our results. Recall that the practical implementation of the new basis only requires the lifting coefficients which were given in §2. As in the construction of the lifting matrix, our study will distinguish the situation for the nodes γ away from the frame, near (or on) the frame and near (or on) the exceptional points.

3.1 Standard dual functions

Here we assume that $\gamma \in \Gamma_j$ is sufficiently away from the frame so that $\triangle_j(\gamma)$ does not intersect the frame. This also means that it takes at least two line segments in the triangulation $\mathcal{T}_j$ to connect γ with a point on the frame. We denote by S_j the set of such "standard nodes". Note that the frame defines a partition $(\Omega_m)_{m=1,\cdots,M}$ of Ω into connected components. In each of these connected components, the mesh T_j is uniform. We denote by $S^m_j = S_j \cap \Omega_m$ the corresponding partition of S_j. When $\gamma \in S^m_j$, the computation of the coefficients $\tilde{s}^\gamma_{j,\eta}$ of the refinement equation in (13) involves only the part of the lifting matrix constructed in Case 4 of §2.3.2 and in the standard correction of 2.3.3. One can easily check that the resulting refinement equation only involves at the next levels the functions $\tilde{\phi}_{j+1,\eta}$ for $\eta \in \triangle_j(\gamma) \cap \Gamma_{j+1}$, for which we also have $\eta \in S^m_{j+1}$. Moreover the coefficients $\tilde{s}^{\gamma}_{j,\eta}$ only depend on the relative position $\eta - \gamma$ on the mesh Γ_{j+1}. More precisely, after some computations we obtain

$$\tilde{\phi}_{j,\gamma} = \frac{7}{16}\tilde{\phi}_{j+1,\gamma} + \frac{3}{16}\sum_{\eta \in T_{j+1}(\gamma)} \tilde{\phi}_{j+1,\eta} - \frac{1}{32}\sum_{\eta \in T_j(\gamma)} \tilde{\phi}_{j+1,\eta} - \frac{1}{16}\sum_{\eta \in \tilde{T}_{j+1}(\gamma)} \tilde{\phi}_{j+1,\eta}, \tag{15}$$

where we have set $\tilde{T}_{j+1}(\gamma) = \Gamma_{j+1} \cap (\triangle_j(\gamma) \setminus \triangle_{j+1}(\gamma))$. This allows to look for a solution of the form

$$\tilde{\phi}_{j,\gamma} = 2^{2j}|\det(A_m)|\tilde{\phi}(A_m(2^j(\cdot - \gamma))), \tag{16}$$

where A_m is a linear transformation that maps a triangle of $\mathcal{T}_0 \cap \Omega_m$ (with one vertex being set to be the origin) onto the reference triangle $\{(0,0),(0,1),(1,0)\}$. The function $\tilde{\phi}$ is then the solution of the standard multivariate refinement equation

$$\tilde{\phi}(x) = \frac{7}{4}\tilde{\phi}(2x) + \frac{3}{4}\sum_{k \in K_1}\tilde{\phi}(2x-k) - \frac{1}{8}\sum_{k \in K_2}\tilde{\phi}(2x-k) - \frac{1}{4}\sum_{k \in K_3}\tilde{\phi}(2x-k), \tag{17}$$

with

$$\begin{aligned} K_1 &:= \{(0,1),(1,0),(-1,0),(0,-1),(1,-1),(-1,1)\}, \\ K_2 &:= \{(0,2),(2,0),(-2,0),(0,-2),(2,-2),(-2,2)\}, \\ K_3 &:= \{(1,1),(-1,-1),(-1,2),(-2,1),(1,-2),(2,-1)\}. \end{aligned}$$

The smoothness of the compactly supported distributional solution to (17) has been studied in [5] (using the spectral radius of an associated tranfer operator), where it is shown that $\tilde{\phi}(x)$ is a compactly supported function in H^s for $s < 0.44$. As a consequence, the functions $\tilde{\phi}_{j,\gamma}$ are also in H^s for $s < 0.44$. The support of $\tilde{\phi}$ is the convex hull of K_2, so that $\tilde{\phi}_{j,\gamma}$ is supported in the convex hull of $T_{j-1}(\gamma)$, i.e. in $\Delta_{-1}(\gamma)$. In particular, it is supported in the uniform region

Ω_m if $\gamma \in S^m_j$. Moreover, when normalized according to $\int \tilde{\phi}(x)dx = 1$, $\tilde{\phi}$ is biorthogonal to the hat function ϕ associated with the reference mesh in the sense that $\langle \phi(\cdot - k), \tilde{\phi}(\cdot - l)\rangle = \delta_{k,l}$, for any k and l in $\mathbb{Z}^2$. Taking this normalization for $\tilde{\phi}$, and using that $\tilde{\phi}_{j,\gamma}$ is supported in Ω_m, we derive by a change of variable that

$$\langle \tilde{\phi}_{j,\tilde{\gamma}}, \phi_{j,\eta} \rangle = \delta_{\tilde{\gamma},\gamma} \text{ and } \int \tilde{\phi}_{j,\gamma} = 1, \text{ for } \tilde{\gamma} \in S_j, \gamma \in \Gamma_j. \tag{18}$$

3.2 Dual functions near the frame

Here, we assume that $\gamma \in \Gamma_j$ is such $\triangle_j(\gamma)$ intersects the frame. However, we assume that it is sufficiently away from the exceptional node in the sense that $\triangle_j(\gamma)$ does not contains an exceptional node, and that if $\tilde{\gamma}$ is an exceptional node, then $\triangle_j(\gamma)$ and $\triangle_j(\tilde{\gamma})$ have no common edge. We denote by F_j the set of such "frame or near frame nodes". Note that for $\gamma \in F_j$, there is a unique edge segment E of the frame such that $\triangle_j(\gamma)$ intersects E. Therefore, similarly to S_j, we have a partition $F_j := \cup_{p=1}^{P} F_j^p$ of F_j corresponding to the edge segments E_p, $p = 1, \cdots, P$ of the frame. In order to study the smoothness of the refinable functions near the frame, we shall distinguish three cases.

Case 1. $\gamma \in F_j^p$ is not on the frame, i.e. $\gamma \notin E_p$, and E_p is not a part of the Dirichlet boundary $\partial\Omega_D$. In this case, the computation of the coefficients $\tilde{s}_{j,\eta}^{\gamma}$ of the refinement equation in (13) involves only the part of the lifting matrix constructed in Case 1 and 2 of §2.3.2 and in the standard correction of §2.3.3. One can easily check that the resulting refinement equation only involves at the next levels functions $\tilde{\phi}_{j+1,\eta}$ for which we have $\eta \in S_{j+1}$ (and η in the same connected component Ω_m as γ). Therefore, we can directly derive from (13) and the result of §3.1 that $\tilde{\phi}_\gamma$ has H^s-smoothness for $s < 0.44$, as the standard dual functions. Moreover the coefficients $\tilde{s}_{j,\eta}^{\gamma}$ only depend on the relative position $\eta - \gamma$ on the mesh Γ_{j+1}. This allows to say that the dual scaling functions have the general

$$\tilde{\phi}_{j,\gamma} = 2^{2j} |\det(A_m)| \tilde{\phi}_1(A_m 2^j(\cdot - \gamma)), \tag{19}$$

where A_m is a linear transformation associated to the connected component Ω_m as in §3.1, and $\tilde{\phi}_1$ is a finite linear combination of the standard dual scaling functions $\tilde{\phi}(2 \cdot -k)$.

Case 2. $\gamma \in F_j^p$ is on the frame (and thus E_p is not a part of $\partial\Omega_D$). In this case, the computation of the coefficients $\tilde{s}_{j,\eta}^{\gamma}$ of the refinement equation in (13) involves only the part of the lifting matrix constructed in Case 1 and 2 of §2.3.2. One can easily check that the resulting refinement equation only involves at the next levels functions $\tilde{\phi}_{j+1,\eta}$ for which we have either $\eta \in F^p_{j+1}$ or $\eta \in S_{j+1}$. Moreover the coefficients $\tilde{s}^\gamma_{j,\eta}$ only depend on the relative position $\eta - \gamma$ on the mesh Γ_{j+1}. More precisely, after some computations, we obtain an equation of the form

$$\begin{aligned}\tilde{\phi}_{j,\gamma} = \tfrac{3}{8}\tilde{\phi}_{j+1,\gamma} + \tfrac{1}{8}(\tilde{\phi}_{j+1,\gamma_1} + \tilde{\phi}_{j+1,\gamma_2}) \quad & -\tfrac{1}{16}(\tilde{\phi}_{j+1,\gamma_3} + \tilde{\phi}_{j+1,\gamma_4}) \\ & + \textstyle\sum_{\eta \in N_{j+1}(\gamma)} \tilde{s}^\gamma_{j,\eta}\tilde{\phi}_{j+1,\eta},\end{aligned} \tag{20}$$

where we have set $\{\gamma_1, \gamma_2\} := E_p \cap T_{j+1}(\gamma)$ the two nearest neighbours of γ on the frame in Γ_{j+1} and $\{\gamma_3, \gamma_4\} := E_p \cap T_j(\gamma)$ the two next ones. The set $N_{j+1}(\gamma)$ represents the remaining coefficients for which $\tilde{\phi}_{j+1,\eta}$ is known to have smoothness H^s for $s < 0.44$ by the previous analysis of §3.1 and of Case 1 above. Therefore, we are not much interested in the exact values of the $\tilde{s}^\gamma_{j,\eta}$ for $\eta \in N_{j+1}(\gamma)$.

This allows to look for a solution of the form

$$\tilde{\phi}_{j,\gamma} = 2^{2j}|\det(B_p)|\tilde{\phi}_{2,p}(B_p 2^j(\cdot - \gamma)), \tag{21}$$

where B_p is a linear similarity transformation that maps a the line segment E_p (with one vertex being set to be the origin) onto the reference segment $\{(0,0),(0,1)\}$. The functions $\tilde{\phi}_{2,p}$ depend on the line segment E_p but all satisfy an equation of the type

$$\begin{aligned}\tilde{\phi}_{2,p}(x) \quad &= \tfrac{3}{2}\tilde{\phi}_{2,p}(2x) + \tfrac{1}{2}(\tilde{\phi}_{2,p}(2x-(1,0)) + \tilde{\phi}_{2,p}(2x+(1,0))) \\ & -\tfrac{1}{4}(\tilde{\phi}_{2,p}(2x-(2,0)) + \tilde{\phi}_{2,p}(2x+(2,0))) + G_p(x),\end{aligned} \tag{22}$$

where $G_p(x)$ is a compactly supported function which has H^s-smoothness for $s < 0.44$. Thus, we shall need to study the above *non homogeneous refinement equation* in order to understand the smoothness of the functions $\tilde{\phi}_{2,p}$. Such equations have been considered in various setting. For the purpose of our present analysis, we shall need a specific result from [13]. For $f \in L^1(\mathrm{I\!R^n})$, we define the Fourier L^p-smoothness exponent $s_p(f)$ by

$$s_p = s_p(f) := \sup\{s \in \mathrm{I\!R}\ (1+|\xi|)^s \hat{f}(\xi) \in L^p(\mathrm{I\!R}^n)\}. \tag{23}$$

In particular, we have seen that the standard dual scaling functions satisfy $s_2(\tilde{\phi}) \geq 0.44$. Note that since $\tilde{\phi}$ is also in L^1, we also obtain that $s_\infty(\tilde{\phi}) \geq 0$.

A simple manipulation of Hölder inequalities shows that for the intermediate values $2 < p < \infty$, we have

$$s_p(\tilde{\phi}) \geq \frac{2}{p} s_2(\phi) \geq \frac{0.88}{p}. \tag{24}$$

Consider now the general nonhomogeneous multivariate refinement equation in $\mathbb{R}^d$

$$\Phi(x) = \sum_{n \in \mathbb{Z}^d} c_n \Phi(2x - n) + G(x), \tag{25}$$

where G is a compactly supported function with some known smoothness and c_n a finite sequence. In the Fourier domain the equation has the form

$$\hat{\Phi}(\xi) = M(\xi/2)\hat{\Phi}(\xi/2) + \hat{G}(\xi), \tag{26}$$

with $M(\xi) := 2^{-d} \sum_n c_n e^{-in\xi}$ the usual symbol of the homogeneous refinement equation. Iterating (26), we see that a natural candidate to be the compactly supported solution of (25) is given by

$$\hat{\Phi}(\xi) = \sum_{j \geq 0} \hat{G}(2^{-j}\xi) \prod_{k=1}^{j} M(2^{-k}\xi). \tag{27}$$

In our case of interest, $d = 2$ and the summation in (25) is carried over the unidimensional set $(n, 0)$, $-N < n < N$. Therefore $M(\xi)$ is a trigonometric polynomial $M(\xi) = \frac{1}{2} H(\xi_1)$ of the first variable of $\xi = (\xi_1, \xi_2)$, where $H(\omega) := \frac{1}{2} \sum_{n \in \mathbb{Z}} c_{(n,0)} e^{-in\omega}$ is a one dimensional symbol. We now state and prove the result of [13] concerning the the convergence of the series (27) in the Sobolev spaces.

Proposition 3.1 *Let $1 \leq r \leq \infty$, $\alpha > 0$. Assume that G is a compactly supported function with $s_{2r}(G) \geq \alpha + 1 - 1/r$ and that there exists a constant C independent of j such that*

$$\Big(2^{-j} \int_{|\omega| \leq 2^j \pi} |H(\omega) \cdots H(2^{-j}\omega)|^{2r/(r-1)} d\xi\Big)^{(r-1)/r} \leq C 2^{-2j\alpha}.$$

Then for any $\beta < \alpha$ we have

$$\int_{\mathbb{R}^2} |2^{-j} \prod_{k=1}^{j} H(2^{-k}\xi_1)|^2 |\hat{G}(2^{-j}\xi)|^2 (1 + |\xi|)^{2\beta} d\xi \leq C 2^{-2(\alpha-\beta)j}.$$

Therefore, the series in (27) converges in H^β and it thus defines a compactly supported solution Φ to the nonhomogeneous refinement equation

$$\Phi(x) = \sum_{n\in\mathbb{Z}} c_{(n,0)}\Phi(2x-(n,0)) + G(x)$$

such that $s_2(\Phi) \geq \alpha$.

Proof Setting $A_j = \int_{\mathbb{R}^2} |\prod_{k=1}^{j} H(2^{-k}\xi_1)|^2 |\hat{G}(2^{-j}\xi)|^2(1+|\xi|)^{2\beta}d\xi$, we want to prove that $A_j \leq 2^{2j(1+\beta-\alpha)}$. We define

$$\tilde{G}_{\beta,j}(\xi) := \sum_{k\in\mathbb{Z}^2} |\hat{G}(\xi+2k\pi)|^2(1+2^j|\xi+2k\pi|)^{2\beta}.$$

Then

$$\begin{aligned}
A_j &= \int_{|\xi_1|,|\xi_2|\leq 2^j\pi} |\prod_{k=1}^{j} H(2^{-k}\xi_1)|^2 \tilde{G}_{\beta,j}(2^{-j}\xi)d\xi \\
&\leq C2^{2j(1-1/r-\alpha)}(\int_{|\xi_1|,|\xi_2|\leq 2^j\pi} |\tilde{G}_{\beta,j}(2^{-j}\xi)|^r d\xi)^{1/r} \\
&= C2^{2j(1-\alpha)}(\int_{|\xi_1|,|\xi_2|\leq \pi} |\tilde{G}_{\beta,j}(\xi)|^r d\xi)^{1/r} \\
&= C2^{2j(1-\alpha)}(\int_{|\xi_1|,|\xi_2|\leq \pi} |\tilde{G}_{\beta,j}(\xi)|^r d\xi)^{1/r} \\
&= C2^{2j(1-\alpha)}(\int_{|\xi_1|,|\xi_2|\leq \pi}(\sum_{k\in\mathbb{Z}^2} |\hat{G}(\xi+2k\pi)|^2(1+2^j|\xi+2k\pi|)^{2\beta})^r d\xi)^{1/r} \\
&\leq C2^{2j(1+\beta-\alpha)}(\int_{|\xi_1|,|\xi_2|\leq \pi}(\sum_{k\in\mathbb{Z}^2} |\hat{G}(\xi+2k\pi)|^2(1+|\xi+2k\pi|)^{2\beta})^r d\xi)^{1/r}
\end{aligned}$$

where we have used Hölder inequality and the assumption on the symbol H. Remarking that for any sequence $(e_n)_{n\in\mathbb{Z}^2}$, $r \geq 1$ and $\delta > 0$, we have (by Hölder inequality)

$$(\sum_{n\in\mathbb{Z}^2} |e_n|)^r \leq C \sum_{n\in\mathbb{Z}^2} |e_n|^r (1+|2n\pi|)^{2(r-1)+\delta},$$

with $C = C(r,\delta)$, we thus obtain

$$\begin{aligned}
A_j &\leq C2^{2j(1+\beta-\alpha)}(\int_{|\xi_1|,|\xi_2|\leq \pi} \sum_{k\in\mathbb{Z}^2} |\hat{G}(\xi+2k\pi)|^{2r}(1+|\xi+2k\pi|)^{2r\beta+2(r-1)+\delta}d\xi)^{1/r} \\
&= C2^{2j(1+\beta-\alpha)}(\int_{\mathbb{R}^2} |\hat{G}(\xi)|^{2r}(1+|\xi|)^{2r\beta+2(r-1)+\delta}d\xi)^{1/r}.
\end{aligned}$$

Taking $0 < \delta < r(\alpha-\beta)$, the last integral is finite according to the assumptions which concludes the proof. □

In the setting of (22), the symbol $H(\xi)$ coming from the homogeneous part of the equation turns out to be well known: it correspond to the univariate refinable function $\tilde{\varphi}_{2,2}$ dual to the B-spline $\varphi_2(x) = (1-|x|)_+$, as constructed in [3]. In particular, it is known that we have

$$\int_{|\omega|\leq 2^j\pi} |H(\omega)\cdots H(2^{-j}\omega)|^p d\omega < C(p),$$

for some constant $C(p)$ independent of j for all $p > 1$ (this reflects the fact that the function $\hat{\tilde{\varphi}}_{2,2}(\omega) = \prod_{k>0} H(2^{-k}\omega)$ is in L^p for all $p > 1$). This allows to apply Proposition 3.1 with an α and some r such that $s_{2r}(\tilde{\phi}) \geq \alpha+1-1/r$ and $2\alpha \leq 1-1/r$. As we already noted $s_{2r}(\tilde{\phi}) \geq s_2(\tilde{\phi})/r$, so it suffices that $s_2(\tilde{\phi})/r \geq \alpha+1-1/r$ and $2\alpha \leq 1-1/r$. Imposing $2\alpha = 1-1/r$, we obtain the maximal value of α by solving $s_2(\tilde{\phi})(1-2\alpha) = 3\alpha$. We thus obtain that

$$s_2(\tilde{\phi}_{2,p}) \geq \frac{s_2(\tilde{\phi})}{3+2s_2(\tilde{\phi})} \approx 0.114.$$

Case 3. $\gamma \in F_j^p$ and E_p is a part of the Dirichlet boundary $\partial\Omega_D$. In this case, the computation of the coefficients $\tilde{s}_{j,\eta}^{\gamma}$ of the refinement equation in (13) involves only the part of the lifting matrix constructed in Case 3 and 4 of §2.3.2. One can easily check that the resulting refinement equation only involves at the next level functions $\tilde{\phi}_{j+1,\eta}$ for which either $\eta \in F_{j+1}^p$ or $\eta \in S_{j+1}$. Moreover the coefficients $\tilde{s}_{j,\eta}^{\gamma}$ only depend on the relative position $\eta-\gamma$ on the mesh Γ_{j+1}. More precisely, after some computations, we obtain an equation of the form

$$\tilde{\phi}_{j,\gamma} = \frac{1}{4}(\tilde{\phi}_{j+1,\gamma_1} + \tilde{\phi}_{j+1,\gamma_2}) + \sum_{\eta\in N_{j+1}(\gamma)} \tilde{s}_{j,\eta}^{\gamma}\tilde{\phi}_{j+1,\eta}, \tag{28}$$

where we have set $\{\gamma_1,\gamma_2\} := F_{j+1}^p \cap T_{j+1}(\gamma)$ the two nearest neighbours of γ near the frame in Γ_{j+1}. The set $N_{j+1}(\gamma)$ represents the remaining coefficients for which $\tilde{\phi}_{j+1,\eta}$ is known to have smoothness H^s for $s < 0.44$ by the previous analysis of §3.1 and of Case 1 above. Here again, we are not much interested in the exact values of the $\tilde{s}_{j,\eta}^{\gamma}$ for $\eta \in N_{j+1}(\gamma)$. This allows to look for a solution of the form

$$\tilde{\phi}_{j,\gamma} = 2^{2j}|\det(B_p)|\tilde{\phi}_{3,p}(B_p 2^j(\cdot-\gamma)), \tag{29}$$

where B_p is a linear similarity transformation that maps the line segment E_p (with one vertex being set to be the origin) onto the reference segment

$\{(0,0),(1,0)\}$. The functions $\tilde{\phi}_{3,p}$ depend on the line segment E_p but all satisfy an equation of the type

$$\tilde{\phi}_{2,p}(x) = \tilde{\phi}_{2,p}(2x) + \tilde{\phi}_{2,p}(2x-(1,0)) + G_p(x), \tag{30}$$

where $G_p(x)$ is a compactly supported function which belongs to H^s for $s < 0.44$. Here the symbol $H(\xi)$ coming from the homogeneous part of the equation is even more simple than in the previous case since it correspond $\varphi_0 = \chi_{[0,1]}$. In particular, it is known that we have

$$\int_{|\omega|\leq 2^j\pi} |H(\omega)\cdots H(2^{-j}\omega)|^p d\omega < C(p),$$

for some constant $C(p)$ independent of j for all $p > 1$ (this reflects the fact that $\hat{\varphi}_0$, which behaves like a cardinal sine, is in L^p for all $p > 1$). We thus reach the same conclusion as in the previous case, i.e.

$$s_2(\tilde{\phi}_{3,p}) \geq \frac{s_2(\tilde{\phi})}{3+2s_2(\tilde{\phi})} \approx 0.114.$$

3.3 Dual functions near the exceptional points

It remains to treat the case of the nodes γ which are close to the exceptional nodes in the sense that they are not in $S_j \cup F_j$. For such a $\gamma \in \Gamma_j \setminus (S_j \cup F_j)$ there exists a unique exceptional node $\tilde{\gamma}$ such that either $\tilde{\gamma} \in \triangle_j(\gamma)$ or $\triangle_j(\gamma)$ and $\triangle_j(\tilde{\gamma})$ have a common edge. In order to study the smoothness of $\tilde{\phi}_{j,\gamma}$, we shall again distinguish two cases.

Case 1. $\tilde{\gamma}$ is an active node (i.e. $\tilde{\gamma} \in \Gamma_j$) and $\gamma \neq \tilde{\gamma}$. In this case, the computation of the coefficients $\tilde{s}^{\gamma}_{j,\eta}$ of the refinement equation in (13) involves only the part of the lifting matrix constructed in Case 1 of §2.3.1, Case 1 and Case 2 of §2.3.2 and in the standard correction of §2.3.3. One can easily check that the resulting refinement equation only involves at the next levels functions $\tilde{\phi}_{j+1,\eta}$ for which we have $\eta \in S_{j+1} \cup F_{j+1}$. Therefore, we can directly derive from (13) and the result of §3.1 and §3.2 that $\tilde{\phi}_\gamma$ has H^s-smoothness for $s < 0.114$ as the previously constructed dual functions. Because we are near an exceptional point there is no translation invariant formula for these scaling functions, however they are invariant with respect to a change of scale: setting $\tilde{\gamma} := 0$ as the origin, they will have the general form

$$\tilde{\phi}_{j,\gamma} = 2^{2j}\tilde{\phi}_{\gamma_0}(2^j\cdot), \tag{31}$$

where $\gamma_0 = 2^j\gamma$ and $\tilde{\phi}_{\gamma_0}$ is the corresponding dual scaling function at scale $j = 0$.

Case 2. $\gamma = \tilde{\gamma}$ is an active node or $\tilde{\gamma}$ is a passive node (i.e. $\tilde{\gamma} \notin \Gamma_j$). In this case, if we set $\tilde{\gamma} := 0$ as the origin, one can easily check that the refinement equation in (13) has the form

$$\tilde{\phi}_{j,\gamma} = \frac{1}{4}\tilde{\phi}_{j+1,\gamma/2} + \sum_{\eta \in N_{j+1}(\gamma)} \tilde{s}^{\gamma}_{j,\eta}\tilde{\phi}_{j+1,\eta}, \tag{32}$$

where the set $N_{j+1}(\gamma)$ corresponds to functions $\tilde{\phi}_{j+1,\eta}$ which have been treated in the previous cases, and therefore have H^s-smoothness for $s < 0.114$. The fact that $\tilde{s}^{\gamma}_{j,\gamma/2} = 1/4$ is immediate in the case where $\tilde{\gamma}$ is a passive node, in view of the construction in Case 2 of §2.3.1. In the case where $\gamma = \tilde{\gamma}$ is an active node, the construction in Case 1 of §2.3.1 shows that

$$\tilde{s}^{\tilde{\gamma}}_{j,\tilde{\gamma}} = 1 - \frac{1}{2}\sum_{\eta \in T_{j+1}(\gamma)} a(\gamma, \eta). \tag{33}$$

Therefore, we can look for a solution of the form

$$\tilde{\phi}_{j,\gamma} = 2^{2j}\tilde{\phi}_{\gamma_0}(2^j\cdot), \tag{34}$$

where $\gamma_0 = 2^j\gamma$ and $\tilde{\phi}_{\gamma_0}$ satisfies a nonhomogeneous refinement equation of the form

$$\tilde{\phi}_{\gamma_0}(x) = \tilde{\phi}_{\gamma_0}(2x) + G_{\gamma_0}(x), \tag{35}$$

where $G_{\gamma_0}(x)$ is a compactly supported function which belongs to H^s for $s < 0.114$. Here we do not need a sophisticated analysis, since it is immediate to check that the series

$$\tilde{\phi}_{\gamma_0}(x) = \sum_{j \geq 0} G_{\gamma_0}(2^j x), \tag{36}$$

converges in H^s for $s < 0.114$ and thus defines a compactly supported solution to (35) with the same smoothness.

In summary, all dual scaling functions have some positive Sobolev smoothness and moreover can be derived by a change of scale from a finite number of compactly supported functions corresponding to each case. This allows to state the following result.

Theorem 3.2 *For all $j \geq 0$ and $\gamma \in \Gamma_j$, the function $\tilde{\phi}_{j,\gamma}$ is in H^s for $s < 0.114$. Moreover, it is locally suppported, like the function $\phi_{j,\gamma}$, in a ball $\{|x-\gamma| \leq C2^{-j}\}$ with C independent of j and γ. We also have for $0 \leq s < 3/2$ and $0 \leq \tilde{s} < 0.114$ the estimates*

$$\|\phi_{j,\gamma}\|_{H^s} \leq C2^{j(s-1)} \text{ and } \|\tilde{\phi}_{j,\gamma}\|_{H^{\tilde{s}}} \leq C2^{j(\tilde{s}+1)}, \tag{37}$$

with C independent of j and γ.

4 Biorthogonality and polynomial exactness

In this section, we shall prove that the new functions $(\phi_{j,\gamma}, \tilde{\phi}_{j,\gamma}, \psi_{j,\lambda}, \tilde{\psi}_{j,\lambda})$ satisfy the same biorthogonality relations as those stated in §2.1 for the hierarchical basis. We shall also prove that the dual multiresolution analysis $\tilde{V}_j$ reproduces constants. Together with the smoothness properties of the scaling functions, such properties imply that the new wavelet basis allows to characterize Sobolev spaces H^s for $-0.114 < s < 3/2$ (see e.g. [6] or [2] for an introduction to the general mechanism of characterizing smoothness through wavelet decompositions).

4.1 Biorthogonality

We already have seen with (18) in §3.1 that for $\gamma \in \Gamma_j$ and $\tilde{\gamma} \in S_j$, we have $\langle \phi_{j,\gamma}, \tilde{\phi}_{j,\tilde{\gamma}} \rangle = \delta_{\gamma,\tilde{\gamma}}$. In order to show that this property holds for all $\tilde{\gamma} \in \Gamma_j$, we shall progress as for the smoothness analysis of the previous section. At first, we remark that for $l \geq 0$ and $\gamma, \tilde{\gamma} \in \Gamma_j$, the functions $\phi_{j,\gamma}$ and $\tilde{\phi}_{j,\tilde{\gamma}}$ are linear combination of $\phi_{j+l,\eta}, \eta \in \Gamma_{j+l}$ and $\tilde{\phi}_{j+l,\eta}, \eta \in \Gamma_{j+l}$, according to

$$\phi_{j,\gamma} = \sum_{\eta \in \Gamma_{j+l}} s^{l,\gamma}_{j,\eta} \tilde{\phi}_{j+l,\eta} \text{ and } \tilde{\phi}_{j,\tilde{\gamma}} = \sum_{\eta \in \Gamma_{j+l}} \tilde{s}^{l,\tilde{\gamma}}_{j,\eta} \tilde{\phi}_{j+l,\eta}. \tag{38}$$

The coefficients $s_{j,\eta}^{l,\gamma}$ and $\tilde{s}_{j,\eta}^{l,\tilde{\gamma}}$ are obtained by iteration of the refinement equations, which amounts in the subdivision algorithms

$$s_{j,\eta}^{l+1,\gamma} = \sum_{\mu\in\Gamma_{j+l}} s_{j+l,\eta}^{\mu} s_{j,\mu}^{l,\gamma} \text{ and } \tilde{s}_{j,\eta}^{l+1,\tilde{\gamma}} = \sum_{\mu\in\Gamma_{j+l}} \tilde{s}_{j+l,\eta}^{\mu} \tilde{s}_{j,\mu}^{l,\tilde{\gamma}}, \tag{39}$$

with the inital fundamental data $s_{j,\eta}^{0,\gamma} = \delta_{\gamma,\eta}$ and $\tilde{s}_{j,\eta}^{0,\tilde{\gamma}} = \delta_{\tilde{\gamma},\eta}$. Clearly, we have at the first step $s_{j,\eta}^{1,\gamma} = s_{j+1,\eta}^{\gamma}$ and $\tilde{s}_{j,\eta}^{1,\tilde{\gamma}} = \tilde{s}_{j+1,\eta}^{\tilde{\gamma}}$. Form (10), we know the discrete biorthogonality relation

$$\sum_{\eta\in\Gamma_{j+1}} s_{j+1,\eta}^{\gamma} \tilde{s}_{j+1,\eta}^{\tilde{\gamma}} = \delta_{\gamma,\tilde{\gamma}}. \tag{40}$$

It is easy to check (by induction on l using formula (39) that this generalize to

$$\sum_{\eta\in\Gamma_{j+l}} s_{j,\eta}^{l,\gamma} \tilde{s}_{j,\eta}^{l,\tilde{\gamma}} = \delta_{\gamma,\tilde{\gamma}}. \tag{41}$$

4.1.1 Biorthogonality near the frame

Assuming that $\tilde{\gamma} \in F_j^p$ for some p, we consider the three cases that were discussed in §3.2.

Case 1. $\tilde{\gamma} \in F_j^p$ is not on the frame, i.e. $\tilde{\gamma} \notin E_p$, and E_p is not a part of the Dirichlet boundary $\partial\Omega_D$. In this case, we can write

$$\tilde{\phi}_{j,\tilde{\gamma}} = \sum_{\eta\in S_{j+1}} \tilde{s}_{j+1,\eta}^{\tilde{\gamma}} \tilde{\phi}_{j+1,\eta}, \tag{42}$$

since $\tilde{s}_{j+1,\eta}^{\tilde{\gamma}}$ is zero when $\eta \notin S_{j+1}$. Using the refinement equation for the primal function, we thus obtain

$$\begin{aligned}\langle \phi_{j,\gamma}, \tilde{\phi}_{j,\tilde{\gamma}} \rangle &= \textstyle\sum_{\eta\in\Gamma_{j+1},\tilde{\eta}\in S_{j+1}} s_{j+1,\eta}^{\gamma} \tilde{s}_{j+1,\tilde{\eta}}^{\tilde{\gamma}} \langle \phi_{j+1,\eta}, \tilde{\phi}_{j+1,\tilde{\eta}} \rangle \\ &= \textstyle\sum_{\eta\in S_{j+1}} s_{j+1,\eta}^{\gamma} \tilde{s}_{j+1,\eta}^{\tilde{\gamma}} = \delta_{\gamma,\tilde{\gamma}},\end{aligned}$$

where we have used the known biorthogonality $\langle \phi_{j+1,\eta}, \tilde{\phi}_{j+1,\tilde{\eta}} \rangle = \delta_{\eta,\tilde{\eta}}$ for $\tilde{\eta} \in S_{j+1}$ and the discrete biorthogonality.

Case 2. $\tilde{\gamma} \in F_j^p$ is on the frame. In this case, according to the remarks in §3.2, we can split the generalized refinement equation (38) according to

$$\tilde{\phi}_{j,\tilde{\gamma}} = \sum_{\eta \in E_{j+l}^p} \tilde{s}_{j,\eta}^{l,\tilde{\gamma}} \tilde{\phi}_{j+l,\eta} + \sum_{\eta \in N_{j+l}} \tilde{s}_{j,\eta}^{l,\tilde{\gamma}} \tilde{\phi}_{j+l,\eta}, \tag{43}$$

where $E_{j+l}^p = \Gamma_{j+l} \cap E_p$ are the mesh point on the frame and N_{j+l} is a subset of $S_{j+l} \cup F_{j+l}^p \setminus E_{j+l}^p$. We now write

$$\begin{aligned} \langle \phi_{j,\gamma}, \tilde{\phi}_{j,\tilde{\gamma}} \rangle &= \textstyle\sum_{\eta \in \Gamma_{j+l}, \tilde{\eta} \in \Gamma_{j+l}} s_{j,\eta}^{l,\gamma} \tilde{s}_{j,\tilde{\eta}}^{l,\tilde{\gamma}} \langle \phi_{j+l,\eta}, \tilde{\phi}_{j+l,\tilde{\eta}} \rangle \\ &= \textstyle\sum_{\eta \in \Gamma_{j+l}, \tilde{\eta} \in N_{j+l}} s_{j,\eta}^{l,\gamma} \tilde{s}_{j,\tilde{\eta}}^{l,\tilde{\gamma}} \langle \phi_{j+l,\eta}, \tilde{\phi}_{j+l,\tilde{\eta}} \rangle + \sum_{\eta \in \Gamma_{j+l}, \tilde{\eta} \in E_{j+l}^p} s_{j,\eta}^{l,\gamma} \tilde{s}_{j,\tilde{\eta}}^{l,\tilde{\gamma}} \langle \phi_{j+l,\eta}, \tilde{\phi}_{j+l,\tilde{\eta}} \rangle \\ &= \textstyle\sum_{\eta \in N_{j+l}} s_{j,\eta}^{l,\gamma} \tilde{s}_{j,\eta}^{l,\tilde{\gamma}} + \sum_{\eta \in \Gamma_{j+l}, \tilde{\eta} \in E_{j+l}^p} s_{j,\eta}^{l,\gamma} \tilde{s}_{j,\tilde{\eta}}^{l,\tilde{\gamma}} \langle \phi_{j+l,\eta}, \tilde{\phi}_{j+l,\tilde{\eta}} \rangle \\ &= \delta_{\gamma,\tilde{\gamma}} + \textstyle\sum_{\eta \in \Gamma_{j+l}, \tilde{\eta} \in E_{j+l}^p} s_{j,\eta}^{l,\gamma} \tilde{s}_{j,\tilde{\eta}}^{l,\tilde{\gamma}} (\langle \phi_{j+l,\eta}, \tilde{\phi}_{j+l,\tilde{\eta}} \rangle - \delta_{\eta,\ t\eta}), \end{aligned}$$

where we have used the known biorthogonality $\langle \phi_{j+l,\eta}, \tilde{\phi}_{j+l,\tilde{\eta}} \rangle = \delta_{\eta,\tilde{\eta}}$ when $\tilde{\eta} \in N_{j+l}$ (from the standard case and the previous Case 1), together with the generalized discrete biorthogonality (41). In order to prove that $\langle \phi_{j,\gamma}, \tilde{\phi}_{j,\tilde{\gamma}} \rangle = \delta_{\gamma,\tilde{\gamma}}$, we shall simply show that the second term in the last expression tends to zero as l goes to $+\infty$. For this we first note that the coefficients $s_{j,\eta}^{l,\gamma} = \phi_{j,\gamma}(\eta)$ are uniformly bounded by 1. In view of (20), the coefficients $\tilde{s}_{j,\tilde{\eta}}^{l,\tilde{\gamma}}$, $\tilde{\eta} \in E_{j+l}^p$, are obtained by iteration of the homogeneous part of the refinement equation, so that standard arguments yield

$$\sum_{\tilde{\eta} \in E_{j+l}^p} |\tilde{s}_{j,\tilde{\eta}}^{l,\tilde{\gamma}}|^2 \leq (2\pi)^{-1} 2^{-3l} \int_{-2^j\pi}^{2^j\pi} |\prod_{k=1}^{l} H(2^{-k\omega})|^2 d\omega,$$

where $H(\omega)$ is the symbol associated to tht homogeneous part in (22). We have already remarked that $\int_{-2^j\pi}^{2^j\pi} |\prod_{k=1}^{l} H(2^{-k\omega})|^2 d\omega$ is uniformly bounded independently of l so that, remarking that the above sum has $\mathcal{O}(2^l)$ non-zero terms and using Schwarz inequality, we obtain

$$\sum_{\tilde{\eta} \in E_{j+l}^p} |\tilde{s}_{j,\tilde{\eta}}^{l,\tilde{\gamma}}| \leq C 2^{-l}.$$

From the local support properties and the L^2-bounds of the primal and dual refinable functions expressed in Theorem 3.2, there exists a uniform constant C such that

$$\sum_{\eta \in \Gamma_{j+l}} |\langle \phi_{j+l,\eta}, \tilde{\phi}_{j+l,\tilde{\eta}} \rangle| \leq C,$$

independently of j, l and $\tilde{\eta}$. Therefore, we conclude that

$$|\sum_{\eta\in\Gamma_{j+l},\tilde{\eta}\in E^p_{j+l}} s^{l,\gamma}_{j,\eta}\tilde{s}^{l,\tilde{\gamma}}_{j,\tilde{\eta}}(\langle\phi_{j+l,\eta},\tilde{\phi}_{j+l,\tilde{\eta}}\rangle-\delta_{\eta,\tilde{\eta}})|\leq C2^{-l},$$

and thus tends to 0 as l goes to $+\infty$.

Case 3. $\tilde{\gamma}\in F^p_j$ and E_p is a part of the Dirichlet boundary $\partial\Omega_D$. This case is treated exactly in the same way as the previous one, since it is described by a similar non-homogeneous refinement equation.

4.1.2 Biorthogonality near the exceptional points

Assuming that $\tilde{\gamma}$ is not in $S_j\cup F_j$, we again consider the two cases that were treated in §3.3, denoting by γ^* the exceptional node such that either $\gamma^*\in\triangle_j(\tilde{\gamma})$ or $\triangle_j(\tilde{\gamma})$ and $\triangle_j(\gamma^*)$ have a common edge.

Case 1. γ^* is an active node (i.e. $\gamma^*\in\Gamma_j$) and $\tilde{\gamma}\neq\gamma^*$. In this case we can write

$$\tilde{\phi}_{j,\tilde{\gamma}}=\sum_{\eta\in S_{j+1}\cup F_{j+1}}\tilde{s}^{\tilde{\gamma}}_{j+1,\eta}\tilde{\phi}_{j+l,\eta},\tag{44}$$

since $\tilde{s}^{\tilde{\gamma}}_{j+1,\eta}$ is zero when $\eta\notin S_{j+1}\cup F_{j+1}$. Using the refinement equation for the primal function, we thus obtain

$$\begin{aligned}\langle\phi_{j,\gamma},\tilde{\phi}_{j,\tilde{\gamma}}\rangle &= \textstyle\sum_{\eta\in\Gamma_{j+1},\tilde{\eta}\in S_{j+1}\cup F_{j+1}} s^{\gamma}_{j+1,\eta}\tilde{s}^{\tilde{\gamma}}_{j+1,\tilde{\eta}}\langle\phi_{j+1,\eta},\tilde{\phi}_{j+1,\tilde{\eta}}\rangle\\ &= \textstyle\sum_{\eta\in S_{j+1}\cup F_{j+1}} s^{\gamma}_{j+1,\eta}\tilde{s}^{\tilde{\gamma}}_{j+1,\eta}=\delta_{\gamma,\tilde{\gamma}},\end{aligned}$$

where we have used the known biorthogonality $\langle\phi_{j+1,\eta},\tilde{\phi}_{j+1,\tilde{\eta}}\rangle=\delta_{\eta,\tilde{\eta}}$ for $\tilde{\eta}\in S_{j+1}\cup F_{j+1}$ and the discrete biorthogonality.

Case 2. $\tilde{\gamma}=\gamma^*$ is an active node or γ^* is a passive node (i.e. $\gamma^*\notin\Gamma_j$). In this case, if we set $\gamma^*:=0$ as the origin, the particular form (32) of the refinement equation show that we can split (38) into

$$\tilde{\phi}_{j,\tilde{\gamma}}=4^{-l}\tilde{\phi}_{j+l,2^{-l}\tilde{\gamma}}+\sum_{\eta\in N_{j+l}(\tilde{\gamma})}\tilde{s}^{l,\tilde{\gamma}}_{j,\eta}\tilde{\phi}_{j+l,\eta},\tag{45}$$

where the set $N_{j+l}(\tilde{\gamma})$ corresponds to functions $\tilde{\phi}_{j+l,\eta}$ which have been treated in the various previous cases and are therefore known to be biorthogonal to the $\phi_{j+l,\eta}$, $\eta \in \Gamma_{j+l}$. We now write

$$\begin{aligned}
\langle \phi_{j,\gamma}, \tilde{\phi}_{j,\tilde{\gamma}} \rangle &= \textstyle\sum_{\eta\in\Gamma_{j+l},\tilde{\eta}\in\Gamma_{j+l}} s^{l,\gamma}_{j,\eta} \tilde{s}^{l,\tilde{\gamma}}_{j,\tilde{\eta}} \langle \phi_{j+l,\eta}, \tilde{\phi}_{j+l,\tilde{\eta}} \rangle \\
&= \textstyle\sum_{\eta\in\Gamma_{j+l},\tilde{\eta}\in N_{j+l}} s^{l,\gamma}_{j,\eta} \tilde{s}^{l,\tilde{\gamma}}_{j,\tilde{\eta}} \langle \phi_{j+l,\eta}, \tilde{\phi}_{j+l,\tilde{\eta}} \rangle + 4^{-l} \sum_{\eta\in\Gamma_{j+l}} s^{l,\gamma}_{j,\eta} \langle \phi_{j+l,\eta}, \tilde{\phi}_{j+l,2^{-l}\tilde{\gamma}} \rangle \\
&= \textstyle\sum_{\eta\in N_{j+l}} s^{l,\gamma}_{j,\eta} \tilde{s}^{l,\tilde{\gamma}}_{j,\eta} + 4^{-l} \sum_{\eta\in\Gamma_{j+l}} s^{l,\gamma}_{j,\eta} \langle \phi_{j+l,\eta}, \tilde{\phi}_{j+l,2^{-l}\tilde{\gamma}} \rangle \\
&= \delta_{\gamma,\tilde{\gamma}} + 4^{-l} \textstyle\sum_{\eta\in\Gamma_{j+l}} s^{l,\gamma}_{j,\eta} (\langle \phi_{j+l,\eta}, \tilde{\phi}_{j+l,2^{-l}\tilde{\gamma}} \rangle - \delta_{\gamma,2^{-l}\tilde{\gamma}}),
\end{aligned}$$

where we have used the known biorthogonality $\langle \phi_{j+l,\eta}, \tilde{\phi}_{j+l,\tilde{\eta}} \rangle = \delta_{\eta,\tilde{\eta}}$ when $\tilde{\eta} \in N_{j+l}$, together with the generalized discrete biorthogonality (41). Here it is immediate that the second term in this last expression is bounded by

$$|4^{-l} \sum_{\eta\in\Gamma_{j+l}} s^{l,\gamma}_{j,\eta} (\langle \phi_{j+l,\eta}, \tilde{\phi}_{j+l,2^{-l}\tilde{\gamma}} \rangle - \delta_{\eta,2^{-l}\tilde{\gamma}})| \leq C4^{-l},$$

ant thus tends to 0 as l goes to $+\infty$. We thus have proved $\langle \phi_{j,\gamma}, \tilde{\phi}_{j,\tilde{\gamma}} \rangle = \delta_{\gamma,\tilde{\gamma}}$ for all $\gamma, \tilde{\gamma} \in \Gamma_j$. As in the usual construction of biorthogonal wavelet basis, we can then derive more general biorthogonality relation for the functions $\{\phi_{j,\gamma}, \psi_{j,\lambda}, \tilde{\phi}_{j,\gamma}, \tilde{\psi}_{j,\lambda}\}$, by combining the definition of these functions in terms of the $\phi_{j+1,\eta}$ and $\tilde{\phi}_{j+1,\eta}$, $\eta \in \Gamma_{j+1}$ and using the discrete biorthogonality relations in (10). We summarize these statements below.

Theorem 4.1 *The functions* $(\phi_{j,\gamma}, \psi_{j,\lambda}, \tilde{\phi}_{j,\gamma}, \tilde{\psi}_{j,\lambda})$ *are biorthogonal, in the sense that*

$$\left\{ \begin{array}{lcll}
\langle \phi_{j,\gamma}, \tilde{\phi}_{j,\tilde{\gamma}} \rangle &=& \delta_{\gamma,\tilde{\gamma}}, & \gamma, \tilde{\gamma} \in \Gamma_j, \\
\langle \psi_{j,\lambda}, \tilde{\phi}_{j,\tilde{\gamma}} \rangle &=& 0, & \lambda \in \Lambda_j, \tilde{\gamma} \in \Gamma_j, \\
\langle \phi_{j,\gamma}, \tilde{\psi}_{j,\tilde{\lambda}} \rangle &=& 0, & \gamma \in \Gamma_j, \tilde{\lambda} \in \Lambda_j, \\
\langle \psi_{j,\lambda}, \tilde{\psi}_{j,\tilde{\lambda}} \rangle &=& \delta_{\lambda,\tilde{\lambda}}, & \lambda, \tilde{\lambda} \in \Lambda_j, \\
\langle \psi_{j,\lambda}, \tilde{\psi}_{\tilde{j},\tilde{\lambda}} \rangle &=& \delta_{j,\tilde{j}} \delta_{\lambda,\tilde{\lambda}}, & \lambda, \tilde{\lambda} \in \Lambda_j,\ j, \tilde{j} > 0.
\end{array} \right.$$

Remark There are some flexibility in the construction of the lifting matrix, resulting in more general dual functions with the same type of analysis. In particular, we may have used more general corrections near the frame, by taking

$$C^v_{j,v_i} = \left\{ \begin{array}{ll}
(1-t)a(v, v_1), & i = 1 \\
ta(v, v_3), & i = 3 \\
0, & \text{otherwise}
\end{array} \right.$$

for Case 2 in Section 2.3.2 and

$$C^v_{j,v_i} = \begin{cases} (1-s)a(v,v_2), & i=2 \\ sa(v,v_4), & i=4 \\ 0, & \text{otherwise,} \end{cases}$$

new correction for Case 3 in Section 2.3.2, where t and s are parameters that only depend on the frame where v_1 and v_3 lies. Obviously the correction near the frame with the parameters $t = 0$ and $s = 0$ are the one we have discussed. With this more general lifting matrix, we may still discuss the smoothness and biorthogonality of dual refinable functions $\tilde{\phi}_{j,\gamma}$ by similar procedures, with the equations (20) and (28) replaced by

$$\tilde{\phi}_{j,\gamma} = \frac{3+2t}{8}\tilde{\phi}_{j+1,\gamma}+\frac{1}{8}(\tilde{\phi}_{j+1,\gamma_1}+\tilde{\phi}_{j+1,\gamma_2})-\frac{1+4t}{16}(\tilde{\phi}_{j+1,\gamma_3}+\tilde{\phi}_{j+1,\gamma_4})+\sum_{\eta\in N_{j+1}(\gamma)}\tilde{s}^{\gamma}_{j,\eta}\tilde{\phi}_{j+1,\eta}$$

and

$$\tilde{\phi}_{j,\gamma} = \frac{1-s}{4}(\tilde{\phi}_{j+1,\gamma_1}+\tilde{\phi}_{j+1,\gamma_2})+\frac{s}{4}(\tilde{\phi}_{j+1,\gamma_3}+\tilde{\phi}_{j+1,\gamma_4})+\sum_{\eta\in N_{j+1}(\gamma)}\tilde{s}^{\gamma}_{j,\eta}\tilde{\phi}_{j+1,\eta}$$

where $\{\gamma_3,\gamma_4\} =: F^p_{j+1}\cap\triangle_j(\gamma)\cap(\Lambda_{j+1}\backslash T_{j+1}(\gamma))$. Similarly the equations (22) and (30) should be replaced by

$$\begin{aligned} \tilde{\phi}_{2,p}(x) &= (\frac{3}{2}+2t)\tilde{\phi}_{2,p}(2x)+\frac{1}{2}\Big(\tilde{\phi}_{2,p}(2x-(1,0))+\tilde{\phi}_{2,p}(2x+(1,0))\Big) \\ &\quad -(\frac{1}{4}+t)\Big(\tilde{\phi}_{2,p}(2x-(2,0))+\tilde{\phi}_{2,p}(2x+(2,0))\Big)+G_p(x) \end{aligned}$$

and

$$\begin{aligned} \tilde{\phi}_{2,p}(x) &= (1-s)\tilde{\phi}_{2,p}(2x)+(1-s)\tilde{\phi}_{2,p}(2x-(1,0)) \\ &\quad +s\tilde{\phi}_{2,p}(2x+(1,0))+s\tilde{\phi}_{2,p}(2x-(2,0))+G_p(x). \end{aligned}$$

Then, one can check that for a choice of parameters such that

$$-\frac{2\sqrt{2}+3}{8} < t < \frac{4\sqrt{2}-5}{12} \quad\text{and}\quad -\frac{\sqrt{2}-1}{3} < s < \frac{\sqrt{2}+1}{4},$$

the same estimates apply as in the analysis for $s = t = 0$, so that the dual functions have smoothness $\tilde{s} \geq 0.114$.

4.2 Polynomial exactness

We conclude by some properties that are easily derived from our construction, in particular the fact that the dual scaling functions reproduce constants. Let $V_j, W_j, \tilde{V}_j, \tilde{W}_j$ be the spaces spanned by $\{\phi_{j,\gamma}, \gamma \in \Gamma_j\}, \{\psi_{j,\lambda}, \lambda \in \Lambda_j\}, \{\tilde{\phi}_{j,\gamma}, \gamma \in \Gamma_j\}, \{\tilde{\psi}_{j,\lambda}, \lambda \in \Lambda_j\}$ respectively. Theorem 4.1 allows to say that the operators defined by

$$\begin{aligned}
P_j v &= \sum_{\gamma \in \Gamma_j} \langle v, \tilde{\phi}_{j,\gamma} \rangle \phi_{j,\gamma}, \\
Q_j v &= \sum_{\lambda \in \Lambda_j} \langle v, \tilde{\psi}_{j,\lambda} \rangle \psi_{j,\lambda}, \\
\tilde{P}_j v &= \sum_{\gamma \in \Gamma_j} \langle v, \phi_{j,\gamma} \rangle \tilde{\phi}_{j,\gamma}, \\
\tilde{Q}_j v &= \sum_{\lambda \in \Gamma_j} \langle v, \psi_{j,\lambda} \rangle \tilde{\psi}_{j,\lambda}
\end{aligned}$$

are projectors onto each of these spaces with the usual relations $\tilde{P}_j = P_j^*$, $\tilde{Q}_j = Q_j^*$ and

$$P_{j+1} = P_j + Q_j, \quad \text{and} \quad \tilde{P}_{j+1} = \tilde{P}_j + \tilde{Q}_j.$$

From the local support properties and the L^2-bounds of the primal and dual refinable functions expressed in Theorem 3.2, these projectors are uniformly stable in $L^2(\Omega)$ in the sense that

$$\|P_j\| + \|\tilde{P}_j\| + \|Q_j\| + \|\tilde{Q}_j\| \le C,$$

with C independent of j.

From the density of the spaces V_j in $L^2(\Omega)$ and the uniform L^2-stability of P_j, we obtain that

$$\lim_{j\to\infty} \|f - P_j f\|_{L^2(\Omega)} = 0,$$

for all $f \in L^2(\Omega)$ (we simply remark that if P is a projector onto some subspace V, we always have $\|f - Pf\| \le (1 + \|P\|) \inf_{g \in V} \|f - g\|$). The same holds for $\tilde{P}_j$ by the following argument: if a function $f \in L^2(\Omega)$ is such that $\langle f, \tilde{\phi}_{j,\gamma} \rangle = 0$ for all $j > 0$ and $\gamma \in \Gamma_j$, then $P_j f = 0$ for all $j > 0$ so that $f = 0$ by letting j go to $+\infty$. Therefore the spaces $\tilde{V}_j$ are dense in $L^2(\Omega)$ so that

$$\lim_{j\to\infty} \|f - \tilde{P}_j f\|_{L^2(\Omega)} = 0,$$

for all $f \in L^2(\Omega)$. From the construction of the lifting matrix in §2, we remark that $\int \psi_{j,\lambda} = 0$ for all $\lambda \in \Lambda_j$. Therefore $\tilde{Q}_j 1 = 0$ for all $j > 0$, so that

$$\tilde{P}_j 1 = \tilde{P}_{j+1} 1 = \cdots = \lim_{l \to +\infty} \tilde{P}_l 1 = 1.$$

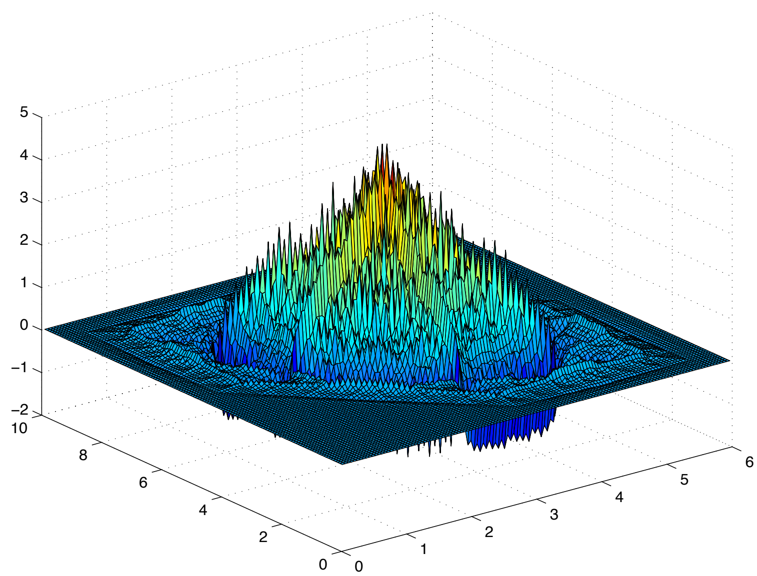
Therefore $\tilde{P}_j$ reproduces constant functions. Note that in contrast, P_j does not fullfill this property in the neighborhood of $\partial\Omega_D$, due to the homogeneous condition imposed on the functions of V_j. We still have $P_j 1 = 1$ at some distance of order 2^{-j} from $\partial\Omega_D$, which also means that $\int \tilde{\phi}_{j,\gamma} = 1$ and $\int \tilde{\psi}_{j,\lambda} = 0$ for all $\gamma \in \Gamma_j$ and $\lambda \in \Lambda_j$ sufficiently away from $\partial\Omega_D$.

5 Annex

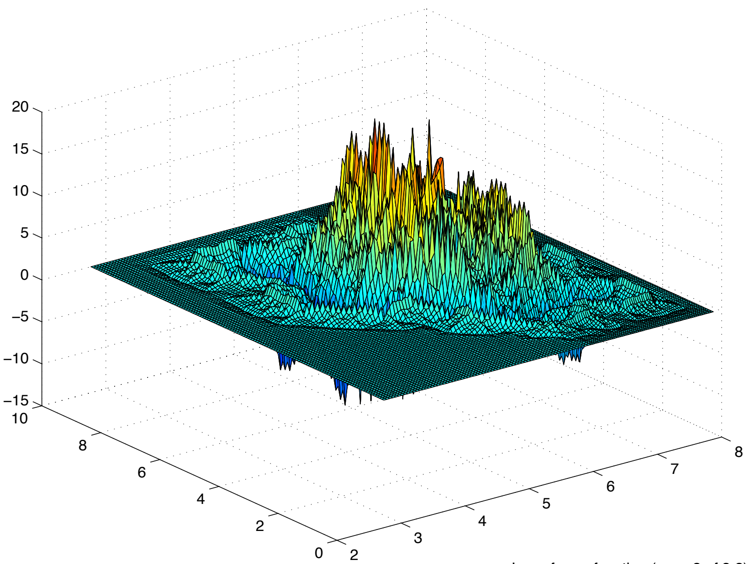
In this annex, we display the graphs of various dual scaling functions. We start by the standard function described in §3.1. which can also be found in [5].

We next display the special functions near the frame described in §3.2. Since the dual functions of case 1 are simple combinations of standard ones, we only present two functions of case 2, respectively corresponding to γ on the inner frame and on the Neumann boundary, and one function of case 3, corresponding to γ on the Dirichlet boundary. It is interesting to note that the aspect of the trace of the Neumann and Dirichlet boundary dual functions are very similar to the 1D functions $\tilde{\varphi}_{2,2}$ and $\chi_{[0,1]}$, which exactly correspond to the homogeneous part of their refinement equation.

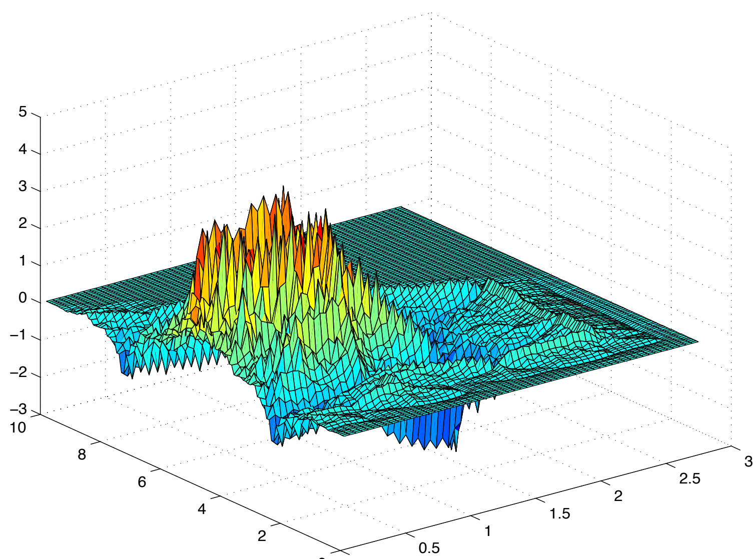
We end by the graphs of the special functions near the exceptional points. Since the dual functions of case 1 are simple combinations of standard or frame ones, we only present one function of case 2, corresponding to an exceptional inner node γ.



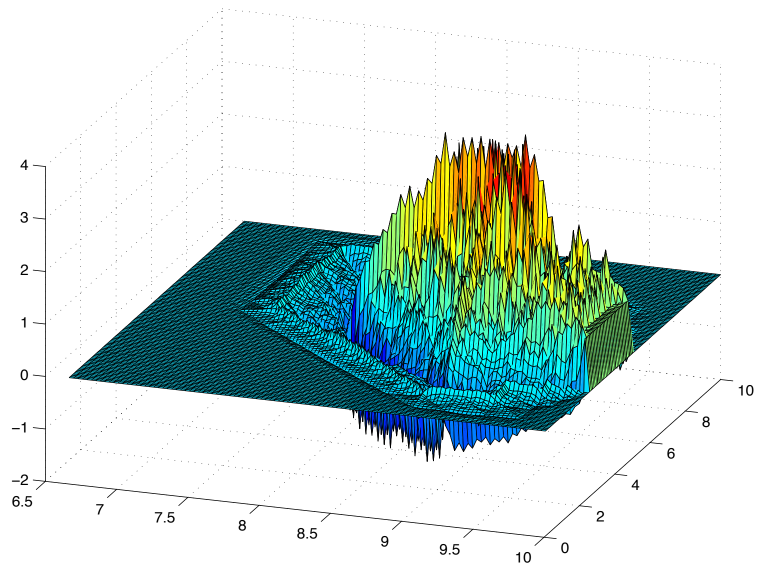
Standadr dual function (3.1)



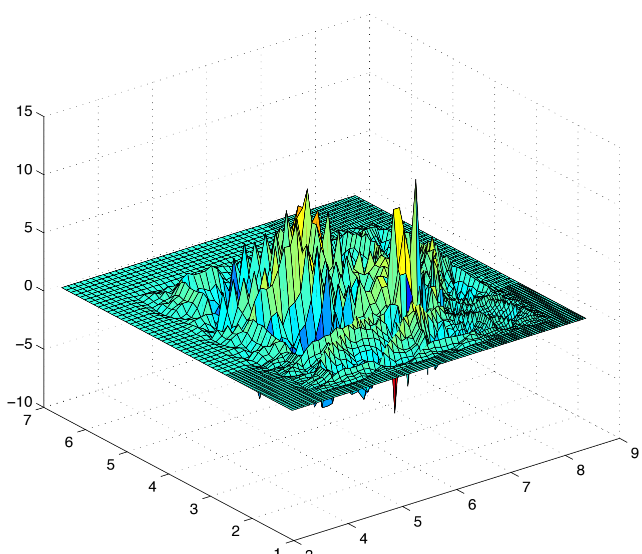
Inner frame function (case 2 of 3.2)



Neumann function case 2 of 3.2)



Dirichlet function (case 3 of 3.2)



Exceptional inner node function (case 2 of 3.3

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