THE MATRIX-VALUED RIESZ LEMMA AND LOCAL ORTHONORMAL BASES IN SHIFT-INVARIANT SPACES

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Abstract. We use the matrix-valued Fejér-Riesz lemma for Laurent polynomials to characterize when a univariate shift-invariant space has a local orthonormal shift-invariant basis, and we apply the above characterization to study local dual frame generators, local orthonormal bases of wavelet spaces, and MRA-based affine frames. Also we provide a proof of the matrix-valued Fejér-Riesz lemma for Laurent polynomials.

1. Introduction

Let ℓ_0 denote the set of all real-valued finitely supported sequences $u = (u[k])_{k\in\mathbb{Z}}$ on $\mathbb{Z}$. For $u \in \ell_0$ the **symbol** $U(z)$ is the Laurent polynomial defined by $\sum_{k\in\mathbb{Z}} u[k]z^k$. If $u \in \ell_0^{N\times N'}$ then the symbol $U(z)$ is similarly defined. We let $\mathcal{L}$ denote the set of all such Laurent polynomials

$$\mathcal{L} := \left\{ \sum_{k\in\mathbb{Z}} u[k]z^k \,\middle|\, (u[k])_{k\in\mathbb{Z}} \in \ell_0 \right\}.$$

Let $\mathbb{T}$ denote the unit circle $\{z \in \mathbb{C} : \ |z| = 1\}$. If $P \in \mathcal{L}$ is such that $P(z) \geq 0$ for $z \in \mathbb{T}$ then there is some $Q \in \mathcal{L}$ such that $P(z) = Q(z)Q(1/z)$, or equivalently, such that $P(z) = |Q(z)|^2$ for $z \in \mathbb{T}$ (since Q is a Laurent polynomial with real coefficients it follows that

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$\overline{Q(z)} = Q(1/z)$ for all $z \neq 0$.) This factorization is known as the (real-coefficient) Fejér-Riesz Lemma ([7, 30]) and many different approaches to that factorization have been developed (see [15, 28] and references therein). The Fejér-Riesz lemma plays an important role in classical function theory (see [27] and references therein), the prediction problem ([18, 26]), and the construction of compactly supported orthonormal wavelets with arbitrary regularity ([6, 7]).

Here we are interested in a generalization of the Fejér-Riesz lemma to matrix-valued Laurent polynomials and its applications to the study of wavelets and framelets. Recall that a square matrix $A := (a_{ij})_{1\le i,j\le N} \in \mathbb{C}^{N\times N}$ is said to be **positive semi-definite** if

$$\bar{x}^T A x = \sum_{1\le i,j\le N} \bar{x}_i a_{ij} x_j \ge 0 \quad \forall\, x = (x_1, \ldots, x_N)^T \in \mathbb{C}^N \backslash \{0\},$$

and A is said to be **positive definite** if $\bar{x}^T A x > 0$ for all $x \in \mathbb{C}^N\backslash\{0\}$. A matrix $A(z) \in \mathcal{L}^{N\times N}$ is said to be **positive (semi-)definite on** $\mathbb{T}$ if $A(z)$ is positive (semi-)definite for any $z \in \mathbb{T}$. For a matrix $B(z) \in \mathcal{L}^{N\times N}$, it is obvious that the matrix $B(1/z)^T B(z)$ is positive semi-definite on $\mathbb{T}$ since $\overline{B(z)}^T = B(1/z)^T$ for $z \in \mathbb{T}$. The converse problem is the subject of the well-known matrix-valued Fejér-Riesz lemma.

Theorem 1.1. *Let $N \ge 1$. Suppose that $A(z) \in \mathcal{L}^{N\times N}$ is positive semi-definite on $\mathbb{T}$. Then there exists a $B(z) \in \mathcal{L}^{N\times N}$ such that*

$$A(z) = B(1/z)^T B(z). \tag{1.1}$$

If $A(z)$ is positive definite on $\mathbb{T}$ then Theorem 1.1 may be found in [13, 18, 26, 31]. If the matrix coefficients of B are allowed to have complex components then a factorization of the form (1.1) (with $B(1/z)^T$ replaced by $B_*(z) = \sum_n \overline{b[n]}^T z^{-n}$) is known in great generality (see [27, Theorem 6.6]). In applications to wavelets and shift-invariant spaces, however, one requires real coefficients. We thank L. Baratchart and C. A. Micchelli for pointing out to us that semidefinite (real-coefficient) case of Theorem 1.1 follows from results for the "continuous time spectral factorization" given in [12, 32, 33]. In particular, the Fejér-Riesz lemma in Theorem 1.1 and the "continuous time spectral factorization" given in [32] are equivalent to each other ([24, 33]). We also thank one of the anonymous referee for providing references [2, 5, 19, 20, 25] from the EE literature in which Theorem 1.1 appears under the name "discrete time spectral factorization".

The first topic of this paper concerns the characterization of shift invariant spaces with local orthonormal bases or with local dual frame generators in shift-invariant spaces. The matrix-valued Fejér-Riesz lemma plays essential roles in those characterizations. Another application of the matrix-valued Fejér-Riesz lemma is on the construction of tight frames from a given multiresolution with dilation $M \geq 2$ ([4]).

The second topic of this paper is to provide a constructive proof for the matrix-valued Fejér-Riesz lemma. This proof was developed before the authors became aware of prior works in the EE literature referenced above. We include the proof in order to make the paper self-contained and for the convenience of the readers. This proof uses only the scalar-valued Fejér-Riesz factorization and some linear algebra techniques. It proceeds in two steps: (1) first obtain a factorization of the form (1.1) for a matrix $B_{\rm rat}(z)$ with rational polynomial entries, and (2) find a rational polynomial matrix $E(z)$ such that (a) $E(1/z)^T E(z) = I$ and (b) $B(z) = E(z)B_{\rm rat}(z)$ is Laurent polynomial.

The paper is organized as follows. In Section 2, we apply Theorem 1.1 to characterize when a shift-invariant space has a local generator with orthonormal shifts (this is part of a project focusing on refinable shift-invariant spaces *cf.*, [16], [17]). In Section 3, we apply Theorem 2.1 to study the local dual frame generators of a shift-invariant space, local orthonormal wavelet bases and MRA-based affine frames. In Section 4, we give a constructive proof of Theorem 1.1.

2. Local orthonormal bases

In this section, we study the problem of when a local finitely generated shift-invariant space has a local generator with orthonormal shifts. The main theorem of this section is Theorem 2.1.

We call a finite length (row) vector $\Phi = (\phi_1, \dots, \phi_N)$ of (compactly supported) functions $\phi_i \in L^2(\mathbb{R})$ a (**local**) **generator**. If Φ is a local generator and $c = (c_i[k])_{k\in\mathbb{Z}, i=1,\dots,N} \in \ell^{N\times 1}$, the **semi-discrete convolution** $\Phi *' c$ is defined by

$$\Phi *' c := \sum_{k\in\mathbb{Z}} \Phi(\cdot - k)c[k] = \sum_{k\in\mathbb{Z}} \sum_{i=1}^{N} \phi_i(\cdot - k)c_i[k].$$

If $c \in \ell^{N\times N'}$, the semi-discrete convolution is similarly defined. The shift invariant space generated by Φ is the space consisting of arbitrary linear combinations of the integer translates of the components of Φ, i.e.,

$$S(\Phi) := \{\Phi *' c \mid c \in \ell^{N\times 1}\}.$$

A space $V \subset L^2_{\rm loc}(\mathbb{R})$ is called a **local finitely generated shift-invariant (local FSI)** space if $V = S(\Phi)$ for some local generator Φ.

For generators $\Phi = (\phi_1, \dots, \phi_N) \subset L^2(\mathbb{R})^N$ and $\Psi = (\psi_1, \dots, \psi_{N'}) \subset L^2(\mathbb{R})^{N'}$, we let $\langle \Phi, \Psi\rangle$ denote the $N\times N'$ matrix $(\langle \phi_n, \psi_{n'}\rangle)_{1\le n\le N, 1\le n'\le N'}$, where $\langle f, g\rangle$ denotes the standard inner product of f and g in $L^2(\mathbb{R})$.

If the components of Φ and Ψ are compactly supported, we define the **Gram sequence** $g_{\Phi,\Psi} \in \ell_0^{N\times N'}$ by

$$g_{\Phi,\Psi}[k] := \langle \Phi, \Psi(\cdot + k)\rangle, \ k \in \mathbb{Z},$$

and the **Gram symbol** $G_{\Phi,\Psi} \in \mathcal{L}^{N\times N'}$ by

$$G_{\Phi,\Psi}(z) := \sum_{k\in\mathbb{Z}} g_{\Phi,\Psi}[k]z^k = \sum_{k\in\mathbb{Z}} \langle \Phi, \Psi(\cdot + k)\rangle z^k.$$

One may verify by direct calculation that

$$G_{\Phi*'u,\Psi*'v}(z) = U(1/z)^T G_{\Phi,\Psi}(z)V(z) \tag{2.1}$$

whenever $u \in \ell_0^{N,L}$ and $v \in \ell_0^{N',L'}$.

If Φ is a local generator such that the only $c \in \ell^{N\times 1}$ such that $\Phi *' c = 0$ is $c \equiv 0$, then we say that Φ has **linearly independent shifts**. If the collection $\{\phi_i(\cdot - j) \mid 1 \le i \le N,\ j \in \mathbb{Z}\}$ of integer shifts of the components of Φ forms an orthonormal system, then we say Φ has **orthonormal shifts**. It is well known that any local generator with orthonormal shifts has linearly independent shifts. Let $G_\Phi(z) := G_{\Phi,\Phi}(z)$. The Gram symbol $G_\Phi(z)$ characterizes when the integer shifts of Φ form an orthonormal basis, precisely, $G_\Phi(z) = I$ if and only Φ has orthonormal shifts.

The following theorem, whose proof is postponed to the end of this section, characterizes when an FSI space has a local orthonormal basis.

Theorem 2.1. *Suppose $\Phi = (\phi_1, \ldots, \phi_N)$ consists of compactly supported functions in $L^2(\mathbb{R})$, i.e., Φ is a local generator, and further suppose Φ has linearly independent shifts. Then $S(\Phi) = S(\Psi)$ for some local generator Ψ having orthonormal shifts if and only if*

$$\det G_\Phi(z) = c, \ z \in \mathbb{C}' := \mathbb{C}\backslash\{0\} \tag{2.2}$$

for some nonzero constant c.

Remark 2.2. In [10], Donovan et. al. characterized shift invariant spaces V such that $V = S(\Phi)$ for some generator Φ having orthonormal shifts, satisfying a certain local linear independence condition, and having support in [-1,1]. Theorem 2.1 is therefore a generalization of the characterization given in [10].

Remark 2.3. T. N. T. Goodman has used essentially the same condition as (2.2) in his construction of a family of piecewise polynomial refinable generators with orthogonal shifts [14].

2.1. **Examples.** The following are two examples to use the Gram symbol to construct a local generator with orthonormal shifts. The first example was first constructed in [9].

Example 1. Let $\phi_1(x) := (1-|x|)^+$ and $\phi_2(x) := (x(1-x))^+(2x-1+a)$, where a is a constant to be determined and f^+ denotes the positive part of f. Since ϕ_1, $\phi_1(\cdot-1)$, and ϕ_2 are linearly independent on $[0,1]$, it follows that $\Phi = (\phi_1, \phi_2)$ has linearly independent shifts for any a. We then find

$$G_\Phi(z) = \frac{1}{25200}((66+210a^2) + (27-35a^2)(z+z^{-1}))$$

(we used the computer algebra system *Mathematica* to assist with the calculations) showing that $S(\Phi)$ has some generator with orthonormal shifts if and only if $a = \pm\sqrt{27/35}$. Factoring $G_\Phi(z)$ we obtain an orthonormal generator (here we take $a = -\sqrt{27/35}$) shown in Figure 1.

Let M be an integer larger than or equal to 2, and suppose $\Phi = (\phi_1, \ldots, \phi_N)$ is a local generator. Let $D_M S(\Phi)$ denote the M-dilation

FIGURE 1. The orthonormal generator $\Phi = (\phi_1, \phi_2)$ consisting of two piecewise quadratic functions in $C^1(\mathbb{R})$ from Example 1. One component has support [-1,1] and the other has support [0,1].

of $S(\Phi)$, *i.e.*,

$$D_M S(\Phi) = \{f(M\cdot) \mid \; f \in S(\Phi)\}\,. \tag{2.3}$$

If there exists some $u \in \ell_0^{N\times N}$ such that

$$\Phi(\cdot/M) = \Phi *' u \tag{2.4}$$

then we say that Φ is M**-refinable**, or **refinable** for short ([29]). Next, we use the idea of **intertwining multiresolution analysis** from [10] and the Gram symbol to construct a refinable (dilation 3) piecewise quadratic, continuously differentiable generator.

Example 2. Let ϕ_1 be the C^1 piecewise quadratic B-spline on [0,3]:

$$\phi_1(x) = \begin{cases} x^2/2 & \text{if } x \in [0,1] \\ 3/4 - (x-3/2)^2 & \text{if } x \in [1,2] \\ (x-3)^2 & \text{if } x \in [2,3] \\ 0 & \text{otherwise.} \end{cases}$$

One may easily verify that $V = S(\phi_1)$ is refinable for any integer dilation M, that is, $V \subset D_M V$. Suppose $\phi_2 \in D_M V$ and let $\Phi = (\phi_1, \phi_2)$. The observation in [10] is that $S(\Phi)$ is refinable since $V \subset S(\Phi) \subset$

$D_M V \subset D_M S(\Phi)$ (the last inclusion follows by applying D_M to the first inclusion). In this example, we choose $M = 3$ and

$$\phi_2 = \phi_1(3\cdot) + c_1\phi_1(3\cdot -1) + c_2\phi_1(3\cdot -2) + c_3\phi_1(3\cdot -3)$$

so that the support ϕ_2 is in [0,2]. Then (again with the aid of *Mathematica*) we find

$$\det G_\Phi(z) = \frac{1}{10497600}\left(\alpha(z^{-3}+z^3) + 6\beta(z^{-2}+z^2) + 3\gamma(z^{-1}+z) + 4\delta\right)$$

where

$$\alpha = -29\,{c_1}^2 - 6462\,c_3 + c_2\,\left(1968 - 29\,c_2 + 93\,c_3\right) + c_1\,\left(93 - 599\,c_2 + 1968\,c_3\right),$$

$$\beta = -1077 + 692\,{c_1}^2 + c_2\,\left(7089 + 692\,c_2\right) + 30498\,c_3 - 7371\,c_2\,c_3 - 1077\,{c_3}^2 + c_1\,\left(-7371 - 3379\,c_2 + 7089\,c_3\right),$$

$$\gamma = 60996 + 46931\,{c_1}^2 + 46931\,{c_2}^2 + 6\,c_3\,\left(22147 + 10166\,c_3\right) - 3\,c_2\,\left(22492 + 19857\,c_3\right) - c_1\,\left(59571 + 89431\,c_2 + 67476\,c_3\right),$$

and

$$\delta = 117082\,{c_1}^2 + 117082\,{c_2}^2 - 3\,c_2\,\left(36859 + 26039\,c_3\right) - c_1\,\left(78117 + 44957\,c_2 + 110577\,c_3\right) + 9\,\left(11253 + c_3\,\left(10166 + 11253\,c_3\right)\right).$$

Setting α, β, γ equal to zero and solving numerically we find 4 sets of solutions for c_1, c_2, and c_3. In Figure 2 we show the resulting orthonormal generator for the numerical solution $c_1 = 3.247727$, $c_2 = 0.991937$ and $c_3 = 0.456933$.

FIGURE 2. A refinable (dilation 3) orthonormal generator $\Phi = (\phi_1, \phi_2)$ consisting of two piecewise quadratic functions in $C^1(\mathbb{R})$ from Example 1.

2.2. **Proof of Theorem 2.1.** To our aim, we need a result for local generators of finitely generated shift-invariant spaces (see for instance [21, 22])

Lemma 2.4. *Suppose $\Phi = (\phi_1, \dots, \phi_N)$ is a local generator with linearly independent shifts. Then*

(a) *$G_\Phi(z)$ is positive definite on $\mathbb{T}$; and*

(b) *if $\Psi = (\psi_1, \dots, \psi_{N'})$ is a local generator having linearly independent shifts such that $S(\Phi) = S(\Psi)$ then $N = N'$ and there exists some matrix-valued sequence $u \in \ell_0^{N\times N}$, such that $\Phi = \Psi *' u$ and such that $\det U(z)$ is a nonzero monomial.*

Now we reach the stage to start the proof of Theorem 2.1.

Proof of Theorem 2.1. We first prove the necessity of the condition (2.2). Suppose Ψ is a local generator with orthonormal shifts such that $S(\Phi) = S(\Psi)$. Then Ψ has linearly independent shifts and by Lemma 2.4, there is some $u \in \ell_0^{N\times N}$ such that $\Phi = \Psi *' u$ and $\det U(z) = cz^j$ for some $c \in \mathbb{R}\backslash\{0\}$ and $j \in \mathbb{Z}$. By (2.1), we have

$$G_\Phi(z) = G_{\Psi*'u,\Psi*'u} = U(1/z)^T G_\Psi(z) U(z) = U(1/z)^T U(z). \tag{2.5}$$

Taking determinants of both sides of Equation 2.5 gives

$$\det G_\Phi(z) = \det U(1/z)^T \det U(z) = (cz^{-j})(cz^j) = c^2 \neq 0.$$

This proves the necessity.

Conversely, suppose (2.2) holds. Since Φ has linearly independent shifts, we have that G_Φ is positive definite on $\mathbb{T}$ by Lemma 2.4. By Theorem 1.1, $G_\Phi(z) = B(1/z)^T B(z)$ for some $B(z) \in \mathcal{L}^{N\times N}$. Therefore $\det G_\Phi(z) = \det B(1/z) \det B(z)$ is a nonzero constant which implies that $\det B(z)$ is a nonzero monomial. Using the cofactor formula for the inverse of $B(z)$, we have $B(z)^{-1} \in \mathcal{L}^{N\times N}$. In this case we let b^{-1} denote the sequence in $\ell_0^{N\times N}$ whose symbol is $B(z)^{-1}$. Let $\Psi = \Phi *' b^{-1}$. Clearly, Ψ consists of compactly supported functions in $S(\Phi)$ and, since $\Phi = \Psi *' b$, we have $S(\Psi) = S(\Phi)$. By direct computation, $G_\Psi(z) = B(1/z)^{-T} G_\Phi(z) B(z)^{-1} = I$, showing that Ψ has orthonormal shifts. □

3. Local dual frame generators

In this section, we apply Theorem 2.1 to study the local dual frame generators for the finitely generated shift-invariant spaces, the local orthonormal basis of wavelet spaces, and MRA-based affine frames. The main results of this section are Theorems 3.1, 3.3 and 3.5.

3.1. **Local dual frame generators.**

For a Hilbert space H, the collection $\{e_\lambda, \lambda \in \Lambda\}$ is called a **frame** of H if there exist positive constants A and B such that

$$A\|f\|^2 \le \sum_{\lambda\in\Lambda} |\langle f, e_\lambda\rangle|^2 \le B\|f\|^2, \quad \forall\ f \in H, \tag{3.1}$$

where $\|\cdot\|$ and $\langle \cdot\ ,\ \cdot\rangle$ are the norm and inner product on H. For a frame $\{e_\lambda, \lambda \in \Lambda\}$ of H, there exists another frame $\{\tilde e_\lambda, \lambda \in \Lambda\}$ of H, which is known as the **dual frame**, such that

$$f = \sum_{\lambda\in\Lambda} \langle f, \tilde e_\lambda\rangle e_\lambda = \sum_{\lambda\in\Lambda} \langle f, e_\lambda\rangle \tilde e_\lambda \quad \forall\ f \in H.$$

We say that Φ is a (**local**) **frame generator** if the collection $\{\phi_n(\cdot - k) \mid k \in \mathbb{Z}, n = 1, \ldots, N\}$ of integer translates of the components of (compactly supported) Φ is a frame of the shift-invariant space $S_2(\Phi)$,

$$S_2(\Phi) = \left\{ \Phi *' c \mid c \in (\ell^2)^{N\times 1} \right\}.$$

If $\tilde{\Phi}$ and Φ are (local) frame generators, and if the following reconstruction formulas

$$f = \sum_{k\in\mathbb{Z}} \sum_{n=1}^{N} \langle f, \tilde{\phi}_n(\cdot - k)\rangle \phi_n(\cdot - k) \tag{3.2}$$

and

$$g = \sum_{k\in\mathbb{Z}} \sum_{n=1}^{N} \langle g, \phi_n(\cdot - k)\rangle \tilde{\phi}_n(\cdot - k) \tag{3.3}$$

hold for all $f \in S_2(\Phi)$ and $g \in S_2(\tilde{\Phi})$ respectively, then we say that Φ and $\tilde{\Phi}$ are **(local) dual frame generators**. We remark that the above reconstruction formulas (3.2) and (3.3) hold for all $f \in S(\Phi)$ and $g \in S(\tilde{\Phi})$ respectively when Φ and $\tilde{\Phi}$ have compact support.

If Φ is a local frame generator, then we may choose a dual frame which consists of the shifts of some functions $\tilde{\phi}_1, \ldots, \tilde{\phi}_N$ in $S_2(\Phi)$ such that Φ and $\tilde{\Phi}$ are dual frame generators, where $\tilde{\Phi} := (\tilde{\phi}_1, \ldots, \tilde{\phi}_N)$ ([1]). Moreover, the FSI space $S_2(\tilde{\Phi})$ is the same as the original FSI space $S_2(\Phi)$, *i.e.*, $S_2(\Phi) = S_2(\tilde{\Phi})$. Usually, the above functions $\tilde{\phi}_i, 1 \le i \le N$, are **not** compactly supported, and hence Φ and $\tilde{\Phi}$ are not local dual frame generators.

The first problem to be considered in this section is when $S_2(\Phi)$ has local dual frame generators. As an application of Theorem 2.1, we show that $S_2(\Phi)$ has local dual frame generators if and only if it has a local orthonormal generator.

Theorem 3.1. *Suppose that $\Phi = (\phi_1, \dots, \phi_N)$ is a local generator, and suppose further that Φ has linearly independent shifts. Then the following statements are equivalent:*

(i) *$S_2(\Phi) = S_2(\Psi)$ for some local generator Ψ with orthonormal shifts.*

(ii) *there exist two local generators Θ and $\tilde{\Theta}$ consisting of compactly supported functions in $S_2(\Phi)$ such that $\{\theta(\cdot - k) : \theta \in \Theta, k \in \mathbb{Z}\}$ and $\{\tilde{\theta}(\cdot - k) : \tilde{\theta} \in \Theta, k \in \mathbb{Z}\}$ are dual frames of $S_2(\Phi)$.*

Remark 3.2. Note that if Φ has linear independent shifts, then Φ is M-refinable if and only if $S(\Phi) \subset D_M S(\Phi)$. Therefore if the function Φ in Theorem 3.1 is assumed to be M-refinable additionally, then the function Ψ chosen in (i) of Theorem 3.1 is M-refinable too.

Proof of Theorem 3.1. Obviously it suffices to prove the implication of (ii) $\Longrightarrow$ (i). Suppose $\Theta = (\theta_1, \dots, \theta_L)$ and $\tilde{\Theta} = (\tilde{\theta}_1, \dots, \tilde{\theta}_L)$ are local dual frame generators of the shift-invariant space $S_2(\Phi)$. Then $\Theta = \Phi *' a$ for some sequence $a \in \ell_0^{N\times L}$, where we have used the facts that $\Theta \subset S_2(\Phi)$, Φ has linear independent shifts and Θ has compact support. Applying the reconstruction formula to the components of Φ implies

$$\Phi = \Theta *' g_{\tilde{\Theta},\Phi} = \Phi *' (a * g_{\tilde{\Theta},\Phi}) = \Phi *' g_{\tilde{\Theta}*'\bar{a}^T,\Phi}.$$

Since Φ has linearly independent shifts, this shows that

$$G_{\tilde{\Phi},\Phi}(z) = I, \tag{3.4}$$

where $\tilde{\Phi} := \tilde{\Theta} *' \bar{a}^T$. By $\tilde{\Phi} \in S_2(\Phi)$ and the linear independent shifts of Φ, we have $\tilde{\Phi} = \Phi *' u$ for some $u \in \ell_0^{N\times N}$. Thus

$$G_{\Phi,\tilde{\Phi}}(z) = G_{\Phi,\Phi*'u}(z) = G_{\Phi}(z)U(z). \tag{3.5}$$

Combining (3.4) and (3.5) implies that $\det G_\Phi(z)$ is a nonzero monomial. This together with Theorem 2.1 leads to the existence of local generator of $S_2(\Phi)$ with orthonormal shifts. □

3.2. **Local orthonormal wavelet basis.** Suppose $\Phi = (\phi_1, \dots, \phi_N)$ is an M-refinable local generator. Then $S_2(\Phi) \subset D_M S_2(\Phi)$, where $D_M S_2(\Phi)$ is defined in (2.3). Denote the orthogonal complement of the space $S_2(\Phi)$ in $D_M S_2(\Phi)$ by W_0, which is known as the wavelet space in the wavelet literature ([7, 29]). It is well known that if Φ is a local generator with orthonormal shifts, then $W_0 = D_M S_2(\Phi) \ominus S_2(\Phi)$ has a generator Ψ of compactly supported functions with orthonormal shifts. Therefore in the case that Φ has orthonormal shifts and is refinable, we can find local dual generators for the shift-invariant space W_0.

The second problem to be considered in this section is the corresponding converse problem.

Theorem 3.3. *Let $M \geq 2$ and suppose $\Phi = (\phi_1, \dots, \phi_N)$ is an M-refinable local generator with linearly independent shifts. If there exists a local orthonormal basis of $W_0 := D_M S_2(\Phi) \ominus S_2(\Phi)$, then there exists a local orthonormal basis of $S_2(\Phi)$.*

As an easy consequence of Theorems 3.1 and 3.3, we have

Corollary 3.4. *Let $M \geq 2$, and Φ be a local generator. Assume that Φ is M-refinable and has linear independent shifts. Then the existences of local dual frame generators of $S_2(\Phi)$ and of $D_M S_2(\Phi) \ominus S_2(\Phi)$ are equivalent to each other.*

Proof of Theorem 3.3. Let Ψ has orthonormal shifts and $S_2(\Psi) = W_0$. By $W_0 \subset D_M S_2(\Phi)$,

$$\Psi(\cdot/M) = \Phi *' h^w \tag{3.6}$$

for some matrix sequence $h^w = (h^w_{mn}[k])_{k\in\mathbb{Z}} \in \ell_0^{N\times N'}$. Set $\omega_M = e^{2\pi i/M}$ and let $H^w(z)$ denote the symbol of h^w. Then it follows from (3.6) that

$$\Psi((x+j)/M) = \sum_k \Phi(x-k)h[k+j] = \Phi *' h[\cdot + j].$$

One may verify directly that

$$M^2 G_\Psi(z^M) = \sum_{j=0}^{M-1} G_{\Psi(\frac{\cdot+j}{M})}(\omega_M^j z).$$

Then the previous two equations together with (2.1) show that

$$\sum_{j=0}^{M-1} H^w(z^{-1}\omega_M^{-j})^T G_\Phi(z\omega_M^j) H^w(z\omega_M^j) = M^2 G_\Psi(z^M) = M^2 I. \tag{3.7}$$

By the refinability of Φ,

$$\Phi(\cdot/M) = \Phi *' h^s \tag{3.8}$$

for some sequence $h^s = (h^s_{nn'}[k])_{k\in\mathbb{Z}} \in \ell_0^{N\times N}$. Then, as in (3.7), the refinement equation (3.8) implies

$$\sum_{j=0}^{M-1} H^s(z^{-1}\omega_M^{-j})^T G_\Phi(z\omega_M^j) H^s(z\omega_M^j) = M^2 G_\Phi(z^M), \tag{3.9}$$

where $H^s(z)$ denotes the symbol of h^s. Furthermore, using the same argument as the one in the proof of the equation (3.7), we obtain

$$\sum_{j=0}^{M-1} H^w(z^{-1}\omega_M^{-j})^T G_\Phi(z\omega_M^j) H^s(z\omega_M^j) = M^2 G_{\Psi,\Phi}(z^M) = 0. \tag{3.10}$$

Set

$$H(z) := \begin{pmatrix} H^s(z) & H^w(z) \\ \vdots & \vdots \\ H^s(z\omega_M^{M-1}) & H^w(z\omega_M^{M-1}) \end{pmatrix}.$$

Then combining (3.7), (3.9) and (3.10), we obtain

$$H(z^{-1})^T \mathrm{diag}\left(G_\Phi(z), \ldots, G_\Phi(z\omega_M^{M-1})\right) \overline{H(z)}$$
$$= M^2 \mathrm{diag}\left(G_\Phi(z^M), I, \ldots, I\right). \tag{3.11}$$

Note that $D_M S(\Phi)$ is a shift-invariant space generated by the integer shifts of $\Phi_M = \{\phi(M \cdot -j) \mid j \in \{0, 1, \cdots, M-1\}, \phi \in \Phi\}$ and that Φ_M has linearly independent shifts since Φ does. Thus by Lemma 2.4, $N' = (M-1)N$, which implies that $H(z)$ is an $MN \times MN$ square matrix. Taking the determinant of both sides of the equation (3.11), we have

$$\prod_{j=0}^{M-1} \det G_\Phi(z\omega_M^j) \times \det H(z) \times \det H(1/z)$$
$$= \quad M^{2MN} \det G_\Phi(z^M). \tag{3.12}$$

Noting that $\prod_{j=0}^{M-1} \det G_\Phi(z\omega_M^j)$ and $\det G_\Phi(z^M)$ are Laurent polynomials of the same degree leads to

$$\det H(z) \times \det H(1/z) = c \neq 0. \tag{3.13}$$

Let $p(z) = \det G_\Phi(z)$, then (3.12) and (3.13) show that $p(a^M) = 0$ whenever $p(a) = 0$ for some $a \in \mathbb{Z}$. Hence, any zero of $p(z) = \det G_\Phi(z)$ must lie on $\mathbb{T}$. By the linearly independent shifts of Φ, $G_\Phi(z)$ is positive definite on $\mathbb{T}$, and hence $\det G_\Phi(z) > 0$ for all $z \in \mathbb{T}$ showing that p has no zeros in $\mathbb{C}'$. Therefore $\det G_\Phi(z)$ is a nonzero monomial and the assertion follows from Theorem 2.1 and the fact that $G_\Phi(z) = G_\Phi(z^{-1})$. □

3.3. **MRA-based affine frames.** Let ϕ be a compactly supported M-refinable (scale-valued) function in L^2, and have linear independent shifts. Then there exists a local generator $\Psi = (\psi_1, \ldots, \psi_L)$ with

$\psi_l \in D_M S_2(\phi), 1 \le l \le L$, such that $\{\psi_{l;j,k} : \ j,k \in \mathbb{Z}, 1 \le l \le L\}$ is a tight frame of $L^2(\mathbb{R})$, *i.e.*,

$$f = \sum_{j,k\in\mathbb{Z}} \sum_{l=1}^{L} \langle f, \psi_{l;j,k}\rangle \psi_{l;j,k} \quad \forall\ f \in L^2(\mathbb{R}) \tag{3.14}$$

where $\psi_{l;j,k} := M^{j/2}\psi_l(M^j \cdot -k)$ (see [3] for $M = 2$ and [4] for $M \ge 3$). The functions $\psi_l, 1 \le l \le L$, are **MRA-based affine frames**, here the MRA(multiresolution) is generated by the refinable function ϕ (see [3, 8, 23] for more information on MRA-based affine frames).

The third problem to be considered in this section is whether the above functions $\psi_l, 1 \le l \le L$, can be chosen in $D_M S_2(\phi) \ominus S_2(\phi)$.

Theorem 3.5. *Let ϕ be a compactly supported M-refinable (scale-valued) function in L^2, and have linear independent shifts. If there exist compactly supported functions $\psi_1, \ldots, \psi_L \in D_M S_2(\phi) \ominus S_2(\phi)$ such that $\{\psi_{l;j,k} : \ j,k \in \mathbb{Z}, 1 \le l \le L\}$ is a tight frame of $L^2(\mathbb{R})$, then there exist compactly supported functions $\tilde\psi_1, \ldots, \tilde\psi_{M-1} \in D_M S_2(\phi) \ominus S_2(\phi)$ such that $\{\tilde\psi_{l;j,k} : \ j,k \in \mathbb{Z}, 1 \le l \le M-1\}$ is an orthonormal basis of $L^2(\mathbb{R})$.*

Proof. For any $f \in D_M S_2(\phi) \ominus S_2(\phi)$, it follows from the reconstruction formula (3.14) that

$$f = \sum_{l=1}^{L} \langle f, \psi_l(\cdot - k)\rangle \psi_l(\cdot - k). \tag{3.15}$$

Thus $\{\psi_l(\cdot - k) : 1 \le l \le L, k \in \mathbb{Z}\}$ is a tight frame of the shift invariant space $D_M S_2(\phi) \ominus S_2(\phi)$. Therefore by Theorem 3.3 and Corollary 3.4, there exists a local generator of $S_2(\phi)$ with orthonormal shifts. Hence the assertion follows. $\square$

4. The real matrix-valued Fejér-Riesz Theorem

In this section, we give a constructive proof of Theorem 1.1. In particular, we will prove the following strong version of Theorem 1.1 instead.

Theorem 4.1. *Let $N \geq 1$. Suppose that $A(z) = \sum_{j=-L}^{L} a_j z^j \in \mathcal{L}^{N\times N}$ is positive semi-definite on $\mathbb{T}$. Then there exists a $B(z) = \sum_{j=0}^{L} b_j z^j \in \mathcal{L}^{N\times N}$ such that*

$$A(z) = B(1/z)^T B(z). \tag{4.1}$$

To this aim, we first factorize positive semi-definite matrices $A(z) \in \mathcal{L}^{N\times N}$ in the field $\mathcal{R}^{N\times N}$, where $\mathcal{R}$ denotes the field of rational Laurent polynomials with real coefficients

$$\mathcal{R} := \left\{ \frac{p(z)}{q(z)} \middle| \; p(z), q(z) \in \mathcal{L} \quad \text{and} \quad q(z) \not\equiv 0 \right\}.$$

We say that $A(z) \in \mathcal{R}^{N\times N}$ is **positive semi-definite** on $\mathbb{T}$ if $A(z)$ is positive semi-definite for all $z \in \mathbb{T}$ where $A(z)$ is defined, that is, except for poles of A. By Gauss elimination on the matrix and the scalar-valued Fejér-Riesz lemma ([7, Lemma 6.1.3]), we have

Proposition 4.2. *Suppose $A(z) \in \mathcal{L}^{N\times N}$ is positive semi-definite on $\mathbb{T}$. Then $A(z) = C(1/z)^T C(z)$ for some $C(z) \in \mathcal{R}^{N\times N}$.*

Clearly if a rational polynomial has no nonzero pole, then it is a Laurent polynomial. Therefore it suffices to remove the poles of the rational polynomial $C(z) \in \mathcal{R}^{N\times N}$ in Proposition 4.2, and the product $C(1/z)^T C(z)$ is preserved at the same time.

A matrix $E(z) \in \mathcal{L}^{N\times N}$ (or in $\mathcal{R}^{N\times N}$) is called **paraunitary** if $E(1/z)^T E(z) = I$, where I is the appropriate sized identity matrix.

Proposition 4.3. *Suppose that* $C(z) \in \mathcal{R}^{N\times N}$ *is such that* $C(1/z)^T C(z) \in \mathcal{L}^{N\times N}$. *Then* $E(z)C(z) \in \mathcal{L}^{N\times N}$ *for some paraunitary matrix* $E(z) \in \mathcal{R}^{N\times N}$.

Let $\mathcal{P}$ denote the space of all polynomials with real coefficients, and $\mathcal{P}^{N\times N'}$ be the space of $N\times N'$ matrices with entries in $\mathcal{P}$. For a moment, we assume that Proposition 4.3 hold, and start to prove Theorem 4.1.

Proof of Theorem 4.1. By Proposition 4.2, there exists $C(z) \in \mathcal{R}^{N\times N}$ such that $A(z) = C(1/z)^T C(z)$. By Proposition 4.3, $B(z) := E(z)C(z) \in \mathcal{L}^{N\times N}$ for some paraunitary matrix $E(z) \in \mathcal{R}^{N\times N}$ and

$$B(1/z)^T B(z) = C(1/z)^T E(1/z)^T E(z)C(z) = A(z).$$

We set

$$\mathcal{P}_A = \{B(z) \in \mathcal{P}^{N\times N} : \ B(1/z)B(z) = A(z)\}.$$

Then $\mathcal{P}_A \neq \emptyset$, since $z^k B(z) \in \mathcal{P}_A$ for some integer $k \in \mathbb{Z}$. Let $B_1 \in \mathcal{P}_A$ be so chosen that it has minimal degree in the class of all matrix-valued polynomials in $\mathcal{P}_A$. Denote the degree of B_1 by $\tilde{L}$. If $\tilde{L} \le L$, then the proof is done. Suppose, on the contrary, that $\tilde{L} > L$. Write $B_1(z) = \sum_{j=0}^{\tilde{L}} b_j z^j$. By direct calculation,

$$B_1(1/z)^T B_1(z) = \sum_{j=-\tilde{L}}^{\tilde{L}} \alpha_j z^j \tag{4.2}$$

and

$$\alpha_{\tilde{L}} = b_0^T b_{\tilde{L}}. \tag{4.3}$$

Combining (4.2), (4.3), and the assumption $B_1 \in \mathcal{P}_A$ leads to

$$b_0^T b_{\tilde{L}} = 0. \tag{4.4}$$

Let Q be the projection operator onto the linear space spanned by the columns of the matrix $b_{\tilde{L}}$. Then it follows from (4.4) and the construction of the matrix Q that

$$Qb_{\tilde{L}} = b_{\tilde{L}} \quad \text{and} \quad Qb_0 = 0. \tag{4.5}$$

Define $Q(z) = (I - Q) + Qz^{-1}$. Then $\tilde{B}_1(z) := Q(z)B_1(z)$ is a matrix-valued polynomial with degree at most $\tilde{L} - 1$, because

$$\begin{aligned} \tilde{B}_1(z) &= (I - Q + z^{-1}Q)(b_0 + b_1 z + \cdots + b_{\tilde{L}} z^{\tilde{L}}) \\ &= (I - Q)(b_0 + b_1 z + \cdot + b_{\tilde{L}-1} z^{\tilde{L}-1}) + Q(b_1 + \cdot + \cdots + b_{\tilde{L}} z^{\tilde{L}-1}) \end{aligned}$$

by (4.5). Also the matrix $\tilde{B}_1(z)$ satisfies $\tilde{B}_1(1/z)^T \tilde{B}_1(z) = A(z)$ since $Q(1/z)^T Q(z) = I$. Therefore $\tilde{B}_1$ is a polynomial in $\mathcal{P}_A$ having degree at most $\tilde{L} - 1$, which is a contradiction. □

The rest of this section is devoted to the proof of Proposition 4.3. To do so, we need two elementary results about matrices.

Lemma 4.4. *Suppose $0 \neq \lambda \in \mathbb{R}$, and W is a linear subspace of $\mathbb{C}^N$. Then there exist $N \times N$ matrices Q_1, Q_2 and Q_3 with real entries so that*

(i) $Q_1 + 2Q_2 + Q_3 = I$;

(ii) $(Q_1 + i\lambda Q_2)w = 0$ *for all* $w \in W$;

(iii) $(Q_3 - i\lambda Q_2)v = 0$ *for all* $v \in W^{\perp}$.

Proof. For the case that the space W is null or full, i.e., $W = \{0\}$ and $\mathbb{C}^N$, the matrices Q_1, Q_2 and Q_3 can be chosen trivially. For instance, for the case that $W = \{0\}$, we may let $Q_1 = I$ and $Q_2 = Q_3 = 0$. So we assume that $W \neq \{0\}$ and $\mathbb{C}^N$ hereafter. Let B be a matrix whose columns form an orthonormal basis of $W^{\perp}$. Then the ℓ^2 (operator) norm $\|B\|$ of B satisfies $\|B\|\le 1$, where $\|B\|:=\max_{x\in\mathbb{C}^N,\,\|x\|=1}\|Bx\|$ and $\|x\|^2=x^Tx$. Thus the matrix

$$\begin{pmatrix} I & -\frac{1}{1-\lambda i}\bar{B}^T\bar{B}\\ -\frac{1}{1+\lambda i}B^TB & I\end{pmatrix}$$

is nonsingular since $\|(1+\lambda i)^{-1}B^TB\|\le (1+\lambda^2)^{-1/2}\|B\|^2<1$. Let (X,Y) be the unique solution of the linear equation

(4.6)
$$(X,Y)\begin{pmatrix} I & -\frac{1}{1-\lambda i}\overline{B}^T\overline{B}\\ -\frac{1}{1+\lambda i}B^TB & I\end{pmatrix}=\left(\frac{\lambda i}{1+\lambda i}B,-\frac{\lambda i}{1-\lambda i}\overline{B}\right).$$

It is easy to check that $Y=\overline{X}$. This together with (4.6) leads to

$$\frac{\lambda i}{1+\lambda i}B-X+\frac{1}{1+\lambda i}\overline{X}B^TB=0. \tag{4.7}$$

Define

$$\begin{cases} Q_1=\frac{1}{2}(X\overline{B}^T+\overline{X}B^T)\\ Q_2=\frac{1}{2\lambda i}(X\overline{B}^T-\overline{X}B^T)\\ Q_3=I-Q_1-2Q_2.\end{cases} \tag{4.8}$$

Clearly $Q_1+2Q_2+Q_3=I$, and all entries of Q_1,Q_2,Q_3 are real. By (4.8),

$$(Q_1+i\lambda Q_2)w=X\overline{B}^Tw=0\quad\forall\ w\in W,$$

where we have used the fact that all columns of B belong to $W^\perp$ to obtain the last equality. By (4.7), we obtain

$$\begin{aligned}(Q_3-i\lambda Q_2)B &= B-\left(1+\frac{1}{\lambda i}\right)X\overline{B}^TB+\frac{1}{\lambda i}\overline{X}B^TB\\ &= B-\left(1+\frac{1}{\lambda i}\right)X+\frac{1}{\lambda i}\overline{X}B^TB=0.\end{aligned}$$

This proves that $(Q_3-i\lambda Q_2)v=0$ for all $v\in W^\perp$. □

Lemma 4.5. *Let $a \notin \mathbb{T}$ be a complex number with nonzero imaginary part* $\mathrm{Im}(a)$*, set* $\lambda = 2\mathrm{Im}(a)/(1-|a|^2)$*, and let* Q_1, Q_2, Q_3 *be* $N \times N$ *matrices with real-coefficients. Define the matrix* $E(z) \in \mathcal{R}^{N\times N}$ *by*

$$E(z) = Q_1 + \Big(\frac{z-a}{1-za} + \frac{z-\bar a}{1-z\bar a}\Big)Q_2 + \frac{z-a}{1-za} \times \frac{z-\bar a}{1-z\bar a}Q_3.$$

If $Q_1 + 2Q_2 + Q_3 = I$ *and* $(Q_1 + i\lambda Q_2)(Q_3 + i\lambda Q_2)^T = 0$*, then*

$$E(1/z)^T E(z) = I. \tag{4.9}$$

Proof. Set $S(z) = E(z)E(1/z)^T$. Then one may verify that $a, \bar a, 1/a, 1/\bar a$ are not poles of $S(z)$. Thus the matrix $T(z) := (1-za)(1-z\bar a)(z-a)(z-\bar a)(S(z)-I)$ of polynomials with their degrees at most 4 satisfies

$$T(a) = T(a^{-1}) = T(\bar a) = T(\bar a^{-1}) = 0. \tag{4.10}$$

Note that $S(1) = (Q_1+2Q_2+Q_3)(Q_1+2Q_2+Q_3)^T = I$, which implies $T(1) = 0$. This together with (4.10) shows that $T(z) \equiv 0$. Therefore $S(z) \equiv I$, and $E(z)$ is a paraunitary matrix. □

For $T(z) \in \mathcal{R}^{N\times N}$ and $a \in \mathbb{C}' := \mathbb{C}\backslash\{0\}$, let $k(T,a)$ denote the order (possibly zero) of the pole of $T(z)$ at a (by order of the pole we mean the maximum of the order of the poles of the entries of $T(z)$ at a). If $a \in \mathbb{C}'$ and $a \neq \pm 1$, then we find it convenient to express the Laurent expansion about a in terms of the variable $w = (z-a)/(1-za)$:

$$T(z) = \left(\frac{z-a}{1-za}\right)^{-k(T,a)} \left(R_{T,a} + \left(\frac{z-a}{1-za}\right) Q_{T,a}(z)\right) \tag{4.11}$$

for some $R_{T,a} \in \mathbb{C}^{N\times N}\backslash\{0\}$ and some $Q_{T,a}(z) \in \mathcal{R}^{N\times N}$ that is analytic in a neighborhood of a.

Next we start to prove Proposition 4.3.

Proof of Proposition 4.3. If $C(z)$ has no pole in $\mathbb{C}' := \mathbb{C}\backslash\{0\}$, then the assertion follows by letting $E(z) = I$. Otherwise, suppose $C(z)$ has a pole at $a \in \mathbb{C}'$ with $k(C,a) > 0$ (recall the definitions of $k(C,a)$ and $R_{C,a}$ given in (4.11)). Then it suffices to construct a paraunitary factor $E(z)$ to cancel the pole of $C(z)$ at a. By $C(1/z)^T C(z) \in \mathcal{L}^{N\times N}$, we have

$$R_{C,1/a}^T R_{C,a} = 0. \tag{4.12}$$

Note that $R_{C,1/a} = \overline{R_{C,a}}$ for $|a| = 1$. Thus $a \notin \mathbb{T}$ by (4.12).

First suppose that a is real. Note that the orthogonal projection P onto the column space of $R_{C,a}$ satisfies $PR_{C,a} = R_{C,a}$ and $PR_{C,1/a} = 0$. Then the matrix $E(z) := I - P + \frac{z-a}{1-za}P$ is a paraunitary matrix in $\mathcal{R}^{N\times N}$. Let $\tilde{C}(z) := E(z)C(z)$. We now show that $k(\tilde{C}, w) \le k(C, w)$ for all $w \in \mathbb{C}'$ with strict inequality for $w = a$, i.e., $k(\tilde{C}, a) < k(C, a)$. Since $E(z)$ is analytic for $z \ne 1/a$ we have $k(\tilde{C}, w) \le k(C, w)$ for any $w \ne 1/a$. In the neighborhood of $z = 1/a$, we have

$$\begin{aligned} \tilde{C}(z) &= \left(\frac{z-1/a}{1-z/a}\right)^{-k(C,1/a)} \Big(R_{C,1/a} + PQ_{C,1/a}(z) \\ &\quad + \Big(\frac{1-za}{z-a}\Big)(I-P)Q_{C,1/a}(z)\Big), \end{aligned} \tag{4.13}$$

which leads to $k(\tilde{C}, 1/a) \le k(C, 1/a)$. Lastly in the neighborhood of $z = a$, we obtain

$$\tilde{C}(z) = \left(\frac{z-a}{1-za}\right)^{-k(C,a)+1} \left(R_{C,a} + \left(I - P + \frac{z-a}{1-za}P\right) Q_{C,a}(z)\right),$$

which implies that $\tilde{C}(z)$ has a pole at a of order at most $k(C,a) - 1$.

Now suppose $\mathrm{Im}(a) \ne 0$. We again construct a paraunitary factor $E(z)$ to cancel the pole of $C(z)$ at a, however the construction is more complex in the complex case. Similarly we write

$$\begin{cases} C(z) = \left(\frac{z-a}{1-za}\right)^{-k(C,a)} \left(R_{C,a} + \left(\frac{z-a}{1-za}\right) Q_{C,a}(z)\right) \\ C(z) = \left(\frac{z-1/a}{1-z/a}\right)^{-k(C,1/a)} \left(R_{C,1/a} + \left(\frac{1-za}{z-a}\right) Q_{C,1/a}(z)\right), \end{cases}$$

where $k(C,a), k(C,a^{-1}) > 0$, and $0 \neq R_{C,a}, R_{C,1/a} \in \mathbb{C}^{N\times N}$. Let W be the complex space spanned by all columns of $R_{C,a}$, and let Q_1, Q_2, Q_3 be square matrices as in Lemma 4.4 with $\lambda = 2\mathrm{Im}(a)/(1-|a|^2) \neq 0$. Therefore the kernel of $Q_3 - i\lambda Q_2$ contains $W^\perp$, which implies that the range of $(Q_3 + i\lambda Q_2)^T$ is contained in W. This together with the second equation in Lemma 4.4 leads to

$$(Q_1 + i\lambda Q_2)(Q_3 + i\lambda Q_2)^T = 0. \tag{4.14}$$

Define

$$E(z) := Q_1 + \Big(\frac{z-a}{1-za} + \frac{z-\bar a}{1-z\bar a}\Big)Q_2 + \frac{z-a}{1-za} \times \frac{z-\bar a}{1-z\bar a}Q_3.$$

Then it follows from (4.14), Lemmas 4.4 and 4.5 that $E(z)$ is a paraunitary matrix in $\mathcal{R}^{N\times N}$. Set $\tilde C(z) := E(z)C(z)$. Recall that $E(z)$ has poles only at $1/a$ and $1/\overline{a}$. Then it remains to prove that $k(\tilde C, a) < k(C,a)$ and $k(\tilde C, \bar a) < k(C, \bar a)$, and that $k(\tilde C, 1/a) \le k(C, 1/a)$ and $k(\tilde C, 1/\bar a) \le k(C, 1/\bar a)$. Note that $k(R,w) = k(R,\bar w)$ for any $R \in \mathcal{R}^{N\times N}$ and nonzero complex number w. Therefore it suffices to compare the orders of the pole of $\tilde C(z)$ and $C(z)$ for $z = a$ and a^{-1}. In the neighborhood of $z = a$,

$$\begin{aligned} \tilde C(z) &= \left(\frac{z-a}{1-za}\right)^{-k(C,a)} \left\{(Q_1 + i\lambda Q_2)R_{C,a} + \left(\frac{z-a}{1-za}\right)R(z)\right\} \\ &= \left(\frac{z-a}{1-za}\right)^{-k(C,a)+1} R(z) \end{aligned}$$

for some $R(z) \in \mathcal{R}^{N\times N}$ analytic at a neighborhood of $z = a$, where the last equality follows from Lemma 4.4 and the fact that every column of $R_{C,a}$ belongs to W. This proves that $k(\tilde C, a) < k(C,a)$.

By (4.12), each column of $R_{C,1/a}$ belongs to $\overline{W^\perp}$. Therefore

$$(Q_3 + i\lambda Q_2)R_{C,1/a} = 0 \tag{4.15}$$

by Lemma 4.4. In the neighborhood of $z = a^{-1}$,

$$\begin{aligned}\tilde{C}(z) &= \left(\frac{1-za}{z-a}\right)^{-k(C,1/a)-1}\left\{(Q_2+(i\lambda)^{-1}Q_3)R_{C,1/a}+\frac{1-za}{z-a}R(z)\right\}\\ &= \left(\frac{1-za}{z-a}\right)^{-k(C,1/a)}R(z)\end{aligned}$$

for some $R(z) \in \mathcal{R}^{N\times N}$ analytic on a neighborhood of $z = 1/a$, where we have used (4.15) to obtain the last equation. This shows that $k(\tilde{C}, 1/a) \le k(C, 1/a)$. □

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