Affine Similarity of Refinable Functions

In Memory of Professor Cheng M. T. with Great Respect

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Abstract

In this paper, we consider certain affine similarity of refinable functions and establish certain connection between some local and global properties of refinable functions, such as local and global linear independence, local smoothness and B-spline, local and global Hölder continuity.

1 Introduction and Main Result

In this paper, we study the relationship between some local and global properties of the compactly supported integrable function ϕ which satisfies

$$\phi(x) = \sum_{j=0}^{N} c_j \phi(Mx - j) \tag{1}$$

and

$$\int_{\mathbb{R}} \phi(x)dx = 1,$$

where $M \geq 2$ is a fixed integer and the sequence $c_j, j = 0, 1, \cdots, N$ satisfies

$$c_0 c_N \neq 0 \quad \text{and} \quad \sum_{j=0}^{N} c_j = M.$$

The equation (1) is known as a *refinement equation* and the solution of the refinement equation (1) is called as a *refinable function*. For the refinable function ϕ in (1), it is proved in [2] that ϕ is supported in $[0, N/(M-1)]$.

Associated to the refinement equation (1) is the polynomial H defined by

$$H(z) = \frac{1}{M}\sum_{j=0}^{N} c_j z^j, \tag{2}$$

which is known as its *symbol*. The polynomial $H(z)$ can be put into the factorized form

$$H(z) = \Big(\frac{1-z^M}{M-Mz}\Big)^{\zeta_0} R(z), \tag{3}$$

where ζ_0 is a nonnegative integer and the polynomial $R(z)$ does not have the factor $(1-z^M)/(1-z)$.

Let N' be the minimal integer larger than $N/(M-1)$. For the refinement equation (1), define

$$B_l = (c_{Mi-j+l})_{0\le i,j\le N'-1},\ l = 0, 1, \cdots, M-1 \tag{4}$$

and

$$\Phi = \Big(\phi(\cdot), \phi(\cdot+1), \ldots, \phi(\cdot+N'-1)\Big)^T \quad \text{on} \quad (0,1),$$

where we set $c_j = 0$ if $j < 0$ or $j > N$, and A^T as the transpose of A. Then by (1) we have

$$B_l\Phi = \Phi\Big(\frac{\cdot+l}{M}\Big) \quad \text{on} \quad (0,1), \quad \forall\, l = 0, 1, \ldots, M-1. \tag{5}$$

For the refinable function ϕ in (1), denote the set of all linear combinations of integer translates of ϕ by $V_0(\phi)$, and the set of all linear combination of $\phi(\cdot+j)$ with $j + (0,1) \cap (0, N/(M-1)) \ne \emptyset$ by $V_{(0,1)}(\phi)$. The linear space $V_0(\phi)$ is an integer shift invariant space and plays an important role in wavelet analysis. Obviously

$$V_0(\phi) = \Big\{\sum_{j\in\mathbf{Z}} d_j\phi(\cdot+j)\Big\}$$

and

$$V_{(0,1)}(\phi) = \Big\{ \sum_{j=0}^{N'-1} d_j \phi(\cdot + j) \Big\}.$$

Let Π_s be the set of polynomials with degree at most $s-1$ for $s \geq 1$, and let $\Pi_0 = \{0\}$. The set of restriction of all polynomials in Π_s on $(0,1)$ is denoted by Π_s^*. By straightforward computation, we obtain

$$\Pi_{\zeta_0} \subset V_0(\phi) \quad \text{and} \quad \sum_{j\in\mathbf{Z}} P(j)\phi(x+j) \in \Pi_{\zeta_0} \tag{6}$$

for any polynomial $P \in \Pi_{\zeta_0}$ when H has the factorized form (3).

We say that the integer translates of a compactly supported distribution f are *globally linearly independent* if

$$\sum_{j\in\mathbf{Z}} d_j f(\cdot + j) = 0 \quad \text{on} \quad \mathbb{R} \quad \text{implies} \quad d_j = 0 \quad \forall\, j \in \mathbb{Z}.$$

For $s \geq 0$, let $\mathcal{Q}_s$ be a family of functions on $(0,1)$ such that

(i) $f_1, f_2 \in \mathcal{Q}_s$ implies $af_1 + bf_2 \in \mathcal{Q}_s$ for any real numbers a and b.

(ii) $\mathcal{Q}_s \supset \Pi_s^*$.

(iii) $f \in \mathcal{Q}_s$ implies $f((\cdot + l)/M) \in \mathcal{Q}_s$ for any $l = 0, 1, \cdots, M-1$.

The typical examples of $\mathcal{Q}_s$ are Π_s^*, the family of restriction on $(0,1)$ of all functions in $C^\infty(\mathbb{R})$, the family of functions with Hölder exponent at least $\beta > 0$ on $(0,1)$, and the family of p-integrable functions on $(0,1)$ for $p \geq 1$. We say that f has the *property* $\mathcal{Q}_s$ if the restriction of f on $(0,1)$ belongs to $\mathcal{Q}_s$. Obviously any polynomial with degree at most $s-1$ has the property $\mathcal{Q}_s$. For the refinable function ϕ in (1) and the property $\mathcal{Q}_s$, let

$$\mathcal{Q}_s(\phi) = \Big\{ f \in V_{(0,1)}(\phi);\ f \quad \text{has the property} \quad \mathcal{Q}_s \Big\}.$$

Then $\mathcal{Q}_s(\phi)$ is a finite dimensional linear subspace of $V_{(0,1)}(\phi)$.

For refinable functions, there are many properties useful in various applications, such as orthogonality, compact support, symmetry, smoothness, analytic expression, interpolation (see for instance [3]). It has been showed that Daubechies' scaling functions $\phi_{M,N}$ are orthonormal, Hölder continuous and unsymmetric except $\phi_{M,1}$, and B-splines B_k are in the Hölder class C^{k-1}, symmetric and not orthonormal except B_1, and have analytic expression (see [1] for precise definition of M band Daubechies' scaling functions $\phi_{M,N}$). In this paper, we shall prove the following result about $\mathcal{Q}_{\zeta_0}(\phi)$ and apply it to establish certain connection between some local and global properties for the refinable function ϕ.

Theorem 1 *Let ϕ be the compactly supported integrable function satisfying the refinement equation (1), its symbol H have the factorized form (3) and $\mathcal{Q}_{\zeta_0}(\phi)$ be defined as above. Assume that the integer translates of ϕ are globally linearly independent and $\sum_{l=0}^{M-1} B_l z_0^l$ is nonsingular for some complex number z_0. Then $\dim \mathcal{Q}_{\zeta_0}(\phi) = \zeta_0$ or N'.*

Under the assumption that the integer translates of ϕ are globally linearly independent and that $\sum_{l=0}^{M-1} B_l z_0^l$ is nonsingular for some complex number z_0, Theorem 1 can be interpreted as either all functions in $V_0(\phi)$ have the property $\mathcal{Q}_{\zeta_0}$ or any function which has the property $\mathcal{Q}_{\zeta_0}$ must coincide with some polynomial in Π_{ζ_0} on $(0, 1)$.

The paper is organized as follows. Theorem 1 is proved in Section 2 and some applications of Theorem 1 are given in Section 3. In fact, we shall choose the property $\mathcal{Q}_s$ as polynomial separated property, piecewise smoothness property and locally Hölder continuous property, and then apply Theorem 1 to obtain some connection between some local and global properties for refinable functions.

2 Proof of Theorem 1

In this section, we shall give the proof of Theorem 1. To prove it, we need some lemmas. For any linear subspace $\mathcal{V} \subset \mathbb{R}^n$, we identify it with a subspace $\mathcal{V}(z)$ of Π_n by

$$\mathcal{V}(z) = \Big\{(1, z, \cdots, z^{n-1})v;\ v \in \mathcal{V}\Big\}.$$

For any polynomial P and linear space $\mathcal{V} \subset \mathbb{R}^n$, define linear subspace $P(\mathcal{V}) \subset \mathbb{R}^{n+\deg P}$ by

$$P(\mathcal{V})(z) = P(z)\mathcal{V}(z).$$

For a linear subspace $\mathcal{V}$ of $\mathbb{R}^n$, define its *characteristic polynomial* $\chi(\mathcal{V})(z)$ as the common factor of all polynomials in $\mathcal{V}(z)$ with maximal degree and leading coefficient one.

Lemma 2 *Let ϕ be the refinable function in (1), H be its symbol and $B_l, l = 0, 1, \cdots, M-1$ be as in (4). Assume that $\sum_{l=0}^{M-1} B_l z_0^l$ is nonsingular for some complex number z_0 and that the integer translates of ϕ are globally linearly independent. Then $\mathcal{V} \neq \{0\}$ is a linear subspace of $\mathbb{R}^{N'}$ invariant under $B_l, l = 0, 1, \ldots, M-1$, i.e.,*

$$B_l \mathcal{V} \subset \mathcal{V}, \quad \forall\ l = 0, 1, \ldots, M-1$$

if and only if

$$\chi(\mathcal{V})(z) = (z-1)^\kappa$$

for some integer κ less than ζ_0 and

$$\mathcal{V} = \chi(\mathcal{V})(\mathbb{R}^{\dim \mathcal{V}}),$$

where $\dim \mathcal{V}$ *denotes the dimension of the linear space $\mathcal{V}$. Further*

$$\kappa + \dim \mathcal{V} = N'.$$

Lemma 2 follows easily from the characterization of global linear independence of a refinable function by its symbol in [4, 7] and the characterization of an invariant subspace under $B_l, l = 0, 1, \ldots, M-1$ in [10].

Lemma 3 ([4, 7]) *Let ϕ be the refinable function in (1) and H be its symbol. Then the integer translates of ϕ are globally linearly independent if and only if there do not exist Laurent polynomials R_1 and R_2 such that $R_1(1) \neq 0$, R_1 is not a monomial and*

$$H(z) = R_1(z^M)/R_1(z) \times R_2(z).$$

Lemma 4 ([10]) *Let H and $B_l, l = 0, 1, \cdots, M-1$ be as in (2) and (4). Assume that $\sum_{l=0}^{M-1} B_l z_0^l$ is nonsingular for some complex number z_0. Then $\mathcal{V} \neq \{0\}$ is a linear subspace of $\mathbb{R}^{N'}$ invariant under $B_l, l = 0, 1, \ldots, M-1$ if and only if*

$$\mathcal{V} = \chi(\mathcal{V})(\mathbb{R}^{\dim \mathcal{V}})$$

and there exists a polynomial $R_3(z)$ such that

$$H(z) = \chi(\mathcal{V})(z^M)/\chi(\mathcal{V})(z) \times R_3(z).$$

Let

$$\mathcal{W}(\phi) = \Big\{ d \in \mathbb{R}^{N'};\ d^T\Phi \in \mathcal{Q}_{\zeta_0}(\phi) \Big\}.$$

Lemma 5 *Let $\phi, H(z), B_l, l = 0, 1, \cdots, M-1$ be as in Lemma 2 and let $\mathcal{Q}_{\zeta_0}(\phi)$ and $\mathcal{W}(\phi)$ as above. Assume that $\sum_{l=0}^{M-1} B_l z_0^l$ is nonsingular for some complex number z_0 and that the integer translates of ϕ are globally linearly independent. Then we have*

$$\dim \mathcal{W}(\phi) = \dim \mathcal{Q}_{\zeta_0}(\phi).$$

Proof. Set

$$\mathcal{S}_0(\phi) = \Big\{ d \in \mathbb{R}^{N'};\ d^T\Phi = 0 \quad \text{on} \quad (0,1) \Big\}.$$

Then it suffices to prove that

$$\mathcal{S}_0(\phi) = \{0\}.$$

By (5), we have

$$B_l^T \mathcal{S}_0(\phi) \subset \mathcal{S}_0(\phi) \quad \forall\ 0 \le l \le M-1.$$

As a consequence, the orthogonal complement of $\mathcal{S}_0(\phi)$ in $\mathbb{R}^{N'}$ is invariant under $B_l, 0 \le l \le M-1$. This together with Lemma 2 imply that either

$$\mathcal{S}_0(\phi) = \{0\}$$

or $\mathcal{S}_0(\phi)$ is spanned by

$$v_t = (1, 2^t, \cdots, N'^t)^T, \quad 0 \le t \le s_0$$

for some integer $s_0 \le \zeta_0 - 1$. Note that $v_t^T\Phi$ is a nonzero polynomial on $(0,1)$ for any $0 \le t \le \zeta_0 - 1$ by (6). Then

$$v_t \notin \mathcal{S}_0(\phi) \quad \forall\ 0 \le t \le \zeta_0 - 1.$$

This shows that $\mathcal{S}_0(\phi) = \{0\}$. Hence the assertion follows. $\Box$

Proof of Theorem 1. From (6) it follows that $\Pi_{\zeta_0}^* \subset \mathcal{Q}_{\zeta_0}(\phi)$. Thus

$$\dim\ \mathcal{Q}_{\zeta_0}(\phi) \ge\ \dim\ \Pi_{\zeta_0}^* = \zeta_0.$$

This together with Lemma 5 imply that

$$\dim\ \mathcal{W}(\phi) = \dim\ \mathcal{Q}_{\zeta_0}(\phi) \geq \zeta_0.$$

By (5), (iii) in the definition of $\mathcal{Q}_{\zeta_0}$ and the assumption on $B_l, l = 0, 1, \cdots, M-1$, we obtain

$$B_l^T \mathcal{W}(\phi) \subset \mathcal{W}(\phi), \quad \forall\ l = 0, 1, \cdots, M-1. \tag{7}$$

Let $\mathcal{V}(\phi)$ be the orthogonal complement space of $\mathcal{W}(\phi)$ in $\mathbb{R}^{N'}$. Then

$$\dim \mathcal{V}(\phi) = N' - \dim \mathcal{W}(\phi) \leq N' - \zeta_0 \tag{8}$$

and

$$B_l \mathcal{V}(\phi) \subset \mathcal{V}(\phi) \quad \forall\ l = 0, 1, \cdots, M-1.$$

Thus either

$$\mathcal{V}(\phi) = \{0\}$$

or

$$\mathcal{V}(\phi) = \chi(\mathcal{V}(\phi))(\mathbb{R}^{\dim \mathcal{V}(\phi)}) \quad \text{and} \quad \chi(\mathcal{V}(\phi))(z) = (z-1)^\kappa$$

for some integer $\kappa \leq \zeta_0$ by Lemma 2. This leads to either

$$\dim\ \mathcal{V}(\phi) = 0 \tag{9}$$

or

$$\dim\ \mathcal{V}(\phi) = N' - \kappa \geq N' - \zeta_0. \tag{10}$$

Combining (8)-(10), we get

$$\dim \mathcal{W}(\phi) = N' \quad \text{or} \quad \zeta_0.$$

Hence the assertion follows from Lemma 5. □

3 Remarks

In this section, we shall apply Theorem 1 to establish polynomial separation property for $V_0(\phi)$, and to show that a locally smooth refinable function must be a spline, and that a locally Hölder continuous refinable function has certain global Hölder continuity under certain additional assumption. In some sense, it shows certain affine similarity of refinable functions.

Remark 1 (Polynomial Separation Property) In [6], Lemarie introduced *separation property of Malgouyres* of the space $V_0(\phi)$, which means that any function $f \in V_0(\phi)$ vanishing on an open set (a, b) is linear combination of $\phi(\cdot + j)$ which are supported in $(-\infty, E(a)]$ or in $[-E(-b), \infty)$. Hereafter $E(a)$ denotes the integral part of a real number a. For $M = 2$, Lemarie proved that $V_0(\phi)$ has the separation property of Malgouyres under the assumption that there is biorthogonal dual (see [6] for precise statement). We may use Theorem 1 to consider more general property. We say that a function f in $V_0(\phi)$ is *polynomial separated* if $f \in C^\infty(a, b)$ for some interval (a, b) implies that there exist a polynomial P_f such that $f - P_f$ is linear combination of $\phi(\cdot + j)$ which are supported in $(-\infty, E(a)]$ or in $[-E(-b), \infty)$. Obviously $f \in V_0(\phi)$ has separation property of Malgouyres if f is polynomial separated. From the definition of polynomial separation, we see that there exist a polynomial P_f and two functions $f_1, f_2 \in V_0(\phi)$ such that $\text{supp } f_1 \subset (-\infty, E(a)]$, $\text{supp } f_2 \subset [-E(-b), \infty)$ and $f - P_f = f_1 + f_2$, if $f \in V_0(\phi)$ is smooth on (a, b) and polynomial separated. We say that the space $V_0(\phi)$ has *polynomial separation property* if f is polynomial separated for any $f \in V_0(\phi)$.

Theorem 6 *Let ϕ and $B_l, l = 0, 1, \cdots, M - 1$ be as in (1) and (4). Assume that the integer translates of ϕ are globally linearly independent and that $B_l, l = 0, 1, \cdots, M-1$ are nonsingular. Then $V_0(\phi)$ has polynomial separation property.*

For $M = 2$, it is proved that B_0 and B_1 are nonsingular if the integer translates of ϕ are globally linearly independent ([8, 10]). Therefore $V_0(\phi)$ has polynomial separation property if the integer translates of ϕ are globally linearly independent for $M = 2$. This generalizes the separation property of Malgouyres of $V_0(\phi)$ in [6].

To prove Theorem 6, we need to introduce some concepts and an assertion in [9]. For $k \geq 1$, *B-spline* B_k is the refinable function with its symbol G_k being

$$G_k(z) = \Big(\frac{2 - z^M - z^{-M}}{M^2(2 - z - z^{-1})}\Big)^k.$$

The B-spline B_k can also be defined as the convolution of the characteristic function on $[0, 1]$ for k times,

$$B_k = \chi_{[0,1]} * \chi_{[0,1]} * \cdots * \chi_{[0,1]} \quad (k \text{ times}).$$

A compactly supported function f is said to be *piecewise smooth* if there exist $a_1 < a_2 < \cdots < a_K$ such that supp $f \subset [a_1, a_K]$ and f coincides with some function $f_j \in C^\infty(\mathbb{R})$ on (a_j, a_{j+1}) for any $1 \leq j \leq K - 1$. In [9], the second author proved

Lemma 7 *Let ϕ be the refinable function in (1). Assume that ϕ is piecewise smooth and that its integer translates are globally linearly independent. Then ϕ is a B-spline.*

Proof of Theorem 6. Let $\mathcal{Q}_{\zeta_0}$ be the family of restriction of all functions in $C^\infty(\mathbb{R})$ on $(0, 1)$. Then dim $\mathcal{Q}_{\zeta_0}(\phi) = \zeta_0$ or N' by Theorem 1.

If dim $\mathcal{Q}_{\zeta_0}(\phi) = N'$, then $\phi(\cdot + j)$ is the restriction on $(0, 1)$ of some function in $C^\infty(\mathbb{R})$ for any $0 \leq j \leq N' - 1$. Thus ϕ is piecewise smooth and a B-spline by Lemma 7. Hence the assertion holds when dim$\mathcal{Q}_{\zeta_0}(\phi) = N'$.

If dim $\mathcal{Q}_{\zeta_0}(\phi) = \zeta_0$, then $\mathcal{Q}_{\zeta_0}(\phi) = \Pi^*_{\zeta_0}$ by (6). Let $f \in C^\infty(a, b)$ be any element in $V_0(\phi)$. Without loss of generality we assume that $(a, b) \subset (0, 1)$. Write

$$\begin{aligned} f(x) &= \sum_{j<0} d_j\phi(x + j) + \sum_{j \geq N'} d_j\phi(x + j) + \sum_{j=0}^{N'-1} d_j\phi(x + j) \\ &= f_1(x) + f_2(x) + f_3(x). \end{aligned}$$

Then by (6) it suffices to prove that there exists a polynomial $Q \in \Pi_{\zeta_0}$ such that

$$d_j = Q(j), \quad \forall\ 0 \leq j \leq N' - 1.$$

Let $\mathcal{W}(\phi)$ be the linear space of all vectors $w \in \mathbb{R}^{N'}$ such that $w^T\Phi \in \mathcal{Q}_{\zeta_0}(\phi)$. Then it follows from the definition of $\mathcal{Q}_{\zeta_0}$ and nonsingularity of $B_l, l = 0, 1, \ldots, M-1$ that

$$B_l^T \mathcal{W}(\phi) = \mathcal{W}(\phi) \quad \forall\, l = 0, 1, \ldots, M-1. \tag{11}$$

Let m and n be integers such that $[2^{-n}m, 2^{-n}(m+1)] \subset (a,b)$ and let $\epsilon_1, \cdots, \epsilon_n \in \{0, 1, \cdots, M-1\}$ be chosen that $m = M^{n-1}\epsilon_1 + M^{n-2}\epsilon_2 + \cdots + \epsilon_n$. By using (5) for n times, we obtain

$$B_{\epsilon_1} \cdots B_{\epsilon_n} \Phi(x) = \Phi(2^{-n}m + 2^{-n}x), \quad x \in (0,1). \tag{12}$$

Thus

$$B_{\epsilon_n}^T \cdots B_{\epsilon_1}^T (d_0, \cdots, d_{N'-1})^T \in \mathcal{W}(\phi)$$

by (12) and the assumption that $f \in C^\infty(a,b)$, and

$$(d_0, \cdots, d_{N'-1})^T \in \mathcal{W}(\phi)$$

by (11). Hence the assertion follows from (6) and $\mathcal{Q}_{\zeta_0}(\phi) = \Pi_{\zeta_0}^*$. $\square$

Remark 2 (Local Linear Independence) We say that the integer translates of a compactly supported distribution f are *locally linearly independent* if for any open set A

$$\sum_{j \in \mathbf{Z}} d_j f(\cdot + j) \equiv 0 \quad \text{on} \quad A \quad \text{implies} \quad d_j = 0 \quad \forall\, j \in K_f(A),$$

where $j \in K_f(A)$ means that the restriction of $f(\cdot + j)$ on A is not identically zero. Obviously local linear independence of integer translates of a compactly supported distribution implies its global linear independence, and the converse is not true. By the definition, we see that the integer translates of ϕ are locally linearly independent if $V_0(\phi)$ has the separation property of Malgouyres. Hence by Theorem 6, we have the following result about local and global linear independence, which is proved by the second author ([8]) for $M = 2$, and by Wang ([12]) and the second author ([10]) for $M \geq 2$.

Corollary 8 *Let ϕ and $B_l, l = 0, 1, \cdots, M-1$ be as in (1) and (4). Assume that $B_l, l = 0, 1, \cdots, M-1$ are nonsingular. Then global and local linear independence of the integer translates of ϕ are equivalent to each other.*

Remark 3 (B-Spline) There are close relationship between refinable function and B-spline. In [5] , it was proved that a refinable function which is piecewise polynomial must be finitely linear combination of integer translates of a B-spline. In [9], the second author proved that a refinable function which is piecewise smooth must be linear combination of integer translates of a B-spline. By using Theorem 1, we shall prove that a refinable function which is smooth on some small interval must be B-spline under some additional assumption. Precisely we have

Theorem 9 *Let ϕ and $B_l, l = 0, 1, \cdots, M-1$ be as in (1) and (4). Assume that the integer translates of ϕ are globally linearly independent and that $B_l, l = 0, 1, \cdots, M-1$ are nonsingular. If there exists an open interval $(a, b) \subset [0, N/(M-1)]$ such that $\phi \in C^\infty(a, b)$, then ϕ is a B-spline.*

If we left out global linear independence of integer translates of ϕ, the conclusion of Theorem 9 is not true in general. For example, let ϕ_1 be the solution of the refinement equation

$$\phi_1(x) = \sum_{j=0}^{N_1} \tilde{c}_j \phi_1(2x - j).$$

Then $\phi(x) = \phi_1(x) + \phi_1(x - 2N_1) + \phi_1(x - 4N_1)$ satisfies the refinement equation

$$\phi(x) = \sum_{j=0}^{N_1} \tilde{c}_j (\phi(2x - j) - \phi(2x - j - 2N_1) + \phi(2x - j - 4N_1))$$

and $\phi(x) \equiv 0$ on $(N_1, 2N_1)$.

In [1, 11], Bi, Debnath, Zhang and the second author showed that for $K < M$ there exists an open set $A \subset (0, 1)$ with Lebesgue measure one for M-band Daubechies' scaling functions $\phi_{M,K}$ such that $\phi_{M,K}$ are polynomials on $(a, b)+j$ for any $(a, b) \subset A$ and $0 \le j \le N' - 1$ (see [1, 11] for precise statement). Therefore the nonsingularity of $B_l, l = 0, 1, \cdots, M-1$ in Theorems 6 and 9 can not be left out in general.

Proof of Theorem 9. Let $\mathcal{Q}_{\zeta_0}$ be the family of restriction of all functions in $C^\infty(\mathbb{R})$ on $(0, 1)$. Then $\dim \mathcal{Q}_{\zeta_0}(\phi) = \zeta_0$ or N' by Theorem 1.

If dim $\mathcal{Q}_{\zeta_0}(\phi) = N'$, then $\phi(\cdot + j)$ is the restriction on $(0,1)$ of some function in $C^\infty(\mathbb{R})$ for any $0 \le j \le N'-1$. Thus ϕ is piecewise smooth and a B-spline by Lemma 7.

If dim $\mathcal{Q}_{\zeta_0}(\phi) = \zeta_0$, then

$$\mathcal{Q}_{\zeta_0}(\phi) = \Big\{ \sum_{j=0}^{N'-1} Q(j)\phi(\cdot + j);\ Q \in \Pi_{\zeta_0} \Big\}$$

by (6). Let

$$\mathcal{W}(\phi) = \Big\{ d \in \mathbb{R}^{N'};\quad d^T\Phi \in \mathcal{Q}_{\zeta_0}(\phi) \Big\}.$$

Thus any vector $d \in \mathcal{W}(\phi)$ can be written as

$$d = (Q(1), \cdots, Q(N'))^T \tag{13}$$

for some polynomial $Q \in \Pi_{\zeta_0}$. By the assumption on ϕ and the procedure used in the proof of Theorem 6, there exist $\epsilon_1, \cdots, \epsilon_n \in \{0, 1, \cdots, M-1\}$ and $1 \le i_0 \le N'$ such that

$$B_{\epsilon_n}^T \cdots B_{\epsilon_1}^T e_{i_0} \in \mathcal{W}(\phi),$$

where $e_j, 1 \le j \le N'$ is the vector with j-th component one and other components zero. Hence

$$e_{i_0} \in \mathcal{W}(\phi) \tag{14}$$

by (7) and nonsingularity of B_l for all $0 \le l \le M-1$. Let Q_{i_0} be the Langrage interpolation polynomial with degree $N'-1$ which takes value zero at the integer knots $1 \le j \le N$ except one at the integer knot $j = i_0$. In fact,

$$Q_{i_0}(x) = \Big(\prod_{1 \le j \le N', j \ne i_0} (i_0 - j) \Big)^{-1} \prod_{1 \le j \le N', j \ne i_0} (x - j).$$

By (13) and (14), we have

$$Q_{i_0} \in \Pi_{\zeta_0}.$$

Hence $\zeta_0 = N'$ and ϕ is a B-spline. □

Remark 4 (Hölder Continuity and Integrability) By taking $\mathcal{Q}_{\zeta_0}$ in Theorem 1 as the family of all Hölder continuous functions on $(0,1)$ or all p-integrable functions on $(0,1)$ with $p \ge 1$, and using the same procedure as the one used in the proof of Theorem 9, we have

Theorem 10 *Let ϕ and $B_l, l = 0, 1, \cdots, M-1$ be as in (1) and (4). Assume that the integer translates of ϕ are globally linearly independent and that $B_l, l = 0, 1, \cdots, M-1$ are nonsingular. If ϕ is Hölder continuous on an open interval $(a, b) \subset [0, N/(M-1)]$, then ϕ is Hölder continuous on $(j, j+1) \subset [0, N']$ for any $0 \leq j \leq N'-1$.*

and

Theorem 11 *Let $1 \leq p \leq \infty$, ϕ and $B_l, l = 0, 1, \cdots, M-1$ be as in (1) and (4). Assume that the integer translates of ϕ are globally linearly independent and that $B_l, l = 0, 1, \cdots, M-1$ are nonsingular. If ϕ is p-integrable on an open interval $(a, b) \subset [0, N/(M-1)]$, then ϕ is p-integrable.*

The refinable function ϕ in Theorem 10 is not Hölder continuous at integer knots $j, 0 \leq j \leq N'$ in general. The characteristic function on $[0, 1]$ is such an example.

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