

**Title:** Stability of the Shifts of Global Supported Distributions

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# Stability of the Shifts of Global Supported Distributions

Qiyu Sun

For a tempered distributions with  $\ell^1$  decay, we characterize its stable shifts via its Fourier transform and via a shift-invariant space of summable sequences. Also we show that if the tempered distribution with  $\ell^1$  decay has stable shifts, then we can recover all distributions in  $V_\infty$ , the space of all linear combinations of its shifts using bounded sequences, in a stable way via using  $C^\infty$  dual functions with  $\ell^1$  decay at infinity. If, additionally, that tempered distribution has compactly supported, then the above  $C^\infty$  dual functions can be chosen to have exponential decay at infinity.

## 1. INTRODUCTION

Let  $\ell, \ell^p, \ell^\infty$  and  $\ell_0$  be the spaces of all sequences on  $\mathbf{Z}^d$ , all  $p$ -summable sequences, all bounded sequences, and all sequences with finite support, and denote their  $N$  copies by  $(\ell)^N, (\ell^p)^N, (\ell^\infty)^N$  and  $(\ell_0)^N$  respectively, where  $1 \leq p < \infty$ . The space  $\ell^p, 1 \leq p \leq \infty$ , is imbedded by the usual  $\ell^p$  norm  $\|\cdot\|_p$ . It is obvious that

$$(\ell_0)^N \subset (\ell^\infty)^N \subset (\ell^p)^N \subset (\ell)^N, \quad 1 \leq p < \infty.$$

For any  $W = (w(j))_{j \in \mathbf{Z}^d} \in (\ell)^N$  and  $D = (d(j))_{j \in \mathbf{Z}^d} \in (\ell_0)^N$ , define the action  $(W, D) \mapsto D(W) = W(D)$  by

$$D(W) := \sum_{j \in \mathbf{Z}^d} d(j)^T w(-j) = \sum_{j \in \mathbf{Z}^d} w(j)^T d(-j) = W(D).$$

Here and hereafter  $A^T$  is the transpose of a vector (matrix)  $A$ .

Denote the space of all Schwartz functions and of all compactly supported  $C^\infty$  functions by  $\mathcal{S}$  and  $\mathcal{D}$  respectively. Let  $\ll \cdot, \cdot \gg$  be the action between a tempered distribution and a Schwartz function, or between a global supported distribution and a function in  $\mathcal{D}$ . We say that a tempered distribution  $F$  has  $\ell^1$  decay if

$$\ll F(\cdot + x), h \gg \text{ is continuous about } x \quad \forall h \in \mathcal{S} \quad (1)$$

and if there exist positive constants  $C$  and  $k_0$  independent of  $h$  such that

$$\| \ll \mathcal{L}(F), h \gg \|_1 \leq C \sum_{|\alpha| \leq k_0} \sup_{x \in \mathbf{R}^d} |D^\alpha h(x)(1 + |x|)^{k_0}| \quad \forall h \in \mathcal{S}. \quad (2)$$

Let  $\mathcal{L}^p$ ,  $1 \leq p \leq \infty$ , be the space of all functions  $f$  such that  $\sum_{j \in \mathbf{Z}^d} |f(\cdot + j)|$  is  $p$ -integrable on  $[-\pi, \pi]^d$  ([9]). It is easy to check that compactly supported distributions,  $\mathcal{L}^p$  functions and integrable functions on  $\mathbf{R}^d$  are tempered distributions with  $\ell^1$  decay. Refinable distributions with smooth symbol are shown in this paper to be tempered distributions with  $\ell^1$  decay too (see Proposition A.7 for detail).

For any tempered distribution  $F = (f_1, \dots, f_N)^T$  with  $\ell^1$  decay, let

$$i(F) := \{\ll \mathcal{L}(F), h \gg : h \in \mathcal{D}\}$$

and

$$\mathcal{K}_\infty(F) := \{W \in (\ell^\infty)^N : W(\mathcal{L}(F)) = 0\},$$

where  $\mathcal{L}(F)$  is an  $(\ell^1)^N$ -valued distribution defined by  $\mathcal{L}(F) := (F(\cdot + j))_{j \in \mathbf{Z}^d}$ . We remark that the space  $i(F)$  is spanned by  $\mathcal{L}(F)(x)$ ,  $x \in \mathbf{R}^d$ , for the case that  $F$  is continuous and compactly supported (see [18] for more properties of  $i(F)$  for a compactly supported distribution  $F$ ).

We say that a linear subspace of  $(\ell^1)^N$  is *shift-invariant* if it is invariant under the *shift operators*  $\tau_k$ ,  $k \in \mathbf{Z}^d$ , where  $\tau_k : (w(j))_{j \in \mathbf{Z}^d} \mapsto (w(j-k))_{j \in \mathbf{Z}^d}$ . Note that  $\tau_k(\ll \mathcal{L}(F), h \gg) = \ll \mathcal{L}(F), h(\cdot + k) \gg$  and that  $(\tau_k W)(\mathcal{L}(F)) = W(\mathcal{L}(F))(\cdot + k)$  for any  $k \in \mathbf{Z}^d$ ,  $h \in \mathcal{D}$  and  $W \in (\ell^\infty)^N$ . Then  $i(F)$  and  $\mathcal{K}_\infty(F)$  are shift-invariant. Let

$$\mathcal{I}(F) := \ell^1 \text{ closure of } i(F).$$

Note that  $\ll W(\mathcal{L}(F)), h \gg = W(\ll \mathcal{L}(F), h \gg)$  for any  $W \in (\ell^\infty)^N$  and  $h \in \mathcal{D}$ . Then  $\mathcal{K}_\infty(F)$  is the annihilator of  $\mathcal{I}(F)$  in  $(\ell^\infty)^N$ , i.e.,

$$\mathcal{K}_\infty(F) = \{W \in (\ell^\infty)^N : W(D) = 0 \quad \forall D \in \mathcal{I}(F)\}. \quad (3)$$

We say that a vector-valued tempered distribution  $F$  with  $\ell^1$  decay has *stable shifts* if  $\mathcal{K}_\infty(F) = \{0\}$ . For the stable shifts, there is a long list of

publications on the characterizations and applications especially for compactly supported distributions and refinable distributions (see for instance [7, 10, 14] for compactly supported distributions, [2, 5, 6, 13, 16, 19] for compactly supported refinable distributions, and [9] for global supported functions in  $\mathcal{L}^p$ ). From (3) it follows that

$$\text{If } \mathcal{I}(F) = (\ell^1)^N, \text{ then } F \text{ has stable shifts.} \quad (4)$$

The converse is shown in this paper to be true too.

Define the Fourier transform  $\hat{f}$  by  $\hat{f}(\xi) := \int_{\mathbf{R}^d} f(x)e^{-ix\xi} dx$  for an integrable function  $f$  and interpret the one as usual for a (vector-valued) tempered distribution. Set

$$[\hat{f}, \hat{g}](\xi) := \sum_{j \in \mathbf{Z}^d} \hat{f}(\xi + 2j\pi) \overline{\hat{g}(\xi + 2j\pi)}$$

for any measurable functions  $f$  and  $g$  with  $\hat{f}\hat{g}$  being integrable. In this paper, we give the following characterization of the stable shifts of a vector-valued distribution with  $\ell^1$  decay.

**THEOREM 1.1.** *Let  $F = (f_1, \dots, f_N)^T$  be a tempered distribution with  $\ell^1$  decay. Then the following statements are equivalent to each other.*

- (i)  *$F$  has stable shifts.*
- (ii) *The matrix  $(\hat{F}(\xi + 2j\pi))_{j \in \mathbf{Z}^d}$  is of full rank for any  $\xi \in \mathbf{R}^d$ .*
- (iii) *There exist  $h_1, \dots, h_N \in \mathcal{D}$  such that  $([\hat{f}_i, \hat{h}_i](\xi))_{1 \leq i, i' \leq N}$  is invertible for any  $\xi \in \mathbf{R}^d$ .*

(iv)  $\mathcal{I}(F) = (\ell^1)^N$ .

For a tempered distribution  $F = (f_1, \dots, f_N)^T$  with  $\ell^1$  decay, let

$$V_\infty(F) := \{D(\mathcal{L}(F)) : D \in (\ell^\infty)^N\},$$

and the *semi-convolution*  $F *' : (\ell^\infty)^N \mapsto V_\infty(F)$  be defined by  $F *' D = D(\mathcal{L}(F))$ . Then the semi-convolution  $F *'$  is a one-to-one correspondence between the sequence space  $(\ell^\infty)^N$  and the space  $V_\infty(F)$  of distributions provided that  $F$  has stable shifts. This arises naturally the problem how to find the inverse of the semi-convolution  $F *'$ . For any tempered distribution  $F$  with  $\ell^1$  decay, define  $\mathcal{L}^*(F) = (F(\cdot - j))_{j \in \mathbb{Z}^d}$ . In this paper, we show that the inverse of the semi-convolution  $F *'$  can be writing explicitly via a vector-valued  $C^\infty$  function with  $\ell^1$  decay.

**THEOREM 1.2.** *Let  $F = (f_1, \dots, f_N)^T$  be a tempered distribution with  $\ell^1$  decay. If the semi-convolution  $F *' : (\ell^\infty)^N \mapsto V_\infty(F)$  is one-to-one, then its inverse is bounded, and there is a vector-valued  $C^\infty$  function  $G = (G_1, \dots, G_N)^T$  with  $\ell^1$  decay at infinity such that*

$$(F *')^{-1} f = \ll f, \mathcal{L}^*(G) \gg \quad \forall f \in V_\infty(F).$$

Furthermore,  $G$  can be chosen to be linear combination of the shifts of some functions  $g_1, \dots, g_N \in \mathcal{D}$  using  $\ell^1$  coefficient, i.e.,

$$G_i = \sum_{i'=1}^N \sum_{j' \in \mathbf{Z}^d} r_{ii'}(j') g_{i'}(\cdot + j'), \quad 1 \leq i \leq N$$

for some sequences  $(r_{ii'}(j))_{j \in \mathbf{Z}^d} \in \ell^1, 1 \leq i, i' \leq N$ .

From Theorem 1.2, when  $F$  has stable shifts, we can reconstruct all functions  $f \in V_\infty(F)$  in a stable way from the average sampling  $\ll f, G_i(\cdot - j) \gg, 1 \leq i \leq N, j \in \mathbf{Z}^d$  via  $C^\infty$  functions  $G_i$  with  $\ell^1$  decay. If, additionally,  $F$  in Theorem 1.2 is compactly supported, then the  $C^\infty$  function  $G$  in Theorem 1.2 can be chosen to have exponential decay at infinity (see Theorem 3.4 for detail).

For a vector-valued compactly supported distribution  $F$ , let

$$\mathcal{K}(F) := \{W \in (\ell)^N : W(\mathcal{L}(F)) = 0\}.$$

We remark that  $W(\mathcal{L}(F))$  is well defined for any  $W \in (\ell)^N$  since  $\mathcal{L}(F)$  is  $(\ell_0)^N$ -valued distribution at this time. We say that  $F$  has *linear independent shifts* if  $\mathcal{K}(F) = \{0\}$ . Obviously if a compactly supported distribution  $F$  has linearly independent shifts, then  $F$  has stable shifts. It was proved in [1, 20] that for a compactly supported distribution  $F$ ,  $F$  has linear independent shifts if and only if there exist  $h_1, \dots, h_N \in \mathcal{D}$  such that the matrix  $([\widehat{f}_i, \widehat{h}_{i'}](\xi))_{1 \leq i, i' \leq N}$  is the identity matrix for all  $\xi \in \mathbf{C}^d$ . So when  $F$  has

linear independent shifts, we can find functions  $g_1, \dots, g_N \in \mathcal{D}$  such that  $(F^*)^{-1}(f) = \ll f, \mathcal{L}^*(G) \gg$  for all  $f \in V_\infty(F)$ , where  $G = (g_1, \dots, g_N)^T$ . In other words, under the linear independence assumption to  $F$ , the  $C^\infty$  function  $G$  in Theorem 1.2 can be chosen to be compactly supported.

The paper is organized as follows. The stable shifts of a tempered distribution with  $\ell^1$  decay is discussed in Sections 2. In particular, the proofs of Theorems 1.1 and 1.2 are gathered there. Some remarks are given in Section 3, especially on the stable shifts of compactly supported distributions and the computation of the space  $\mathcal{I}(F)$  of a refinable distribution  $F$ . In the appendix, we give some basic properties of tempered distributions with  $\ell^1$  decay, and show that a refinable distribution with smooth symbol has  $\ell^1$  decay.

## 2. PROOFS

Let  $\mathcal{F}(D) := \sum_{j \in \mathbf{Z}^d} d(j)e^{-ij \cdot}$  be the Fourier series of a sequence  $D \in \ell^1$ , and  $\mathcal{F}^{-1}(f)$  be the Fourier sequence of a  $2\pi$  periodic distribution  $f$ . Denote the space of all  $2\pi$  periodic functions  $f$  with Fourier series in  $\ell^1$  by  $\mathcal{W}$ , and define  $\|f\|_{\mathcal{W}} := \|\mathcal{F}^{-1}(f)\|_1$ . Any function in the space  $\mathcal{W}$  is said to belong to the Wiener class. For any function  $R \in \mathcal{W}$  such that  $R(\xi) \neq 0$  for all  $\xi \in \mathbf{R}^d$ , Wiener theorem says that  $R^{-1} \in \mathcal{W}$  too.



*Proof of Theorem 1.1.* We divide the proof into the following steps:

(i)  $\implies$  (ii)  $\implies$  (iii)  $\implies$  (iv)  $\implies$  (i). Note that it follows from (4) that (iv) implies (i). Then it remains to prove the rest of the implications.

(i)  $\implies$  (ii) On the contrary, suppose that there exist  $\alpha \in \mathbf{C}^N \setminus \{0\}$  and  $\xi_0 \in \mathbf{R}^d$  such that  $\alpha^T \widehat{F}(\xi_0 + 2j\pi) = 0$  for all  $j \in \mathbf{Z}^d$ . This together with (A.1) and Proposition A.6 implies that

$$\begin{aligned} \ll \sum_{j \in \mathbf{Z}^d} e^{ij\xi_0} \alpha^T F(\cdot - j), h \gg &= \sum_{j \in \mathbf{Z}^d} e^{ij\xi_0} \ll \alpha^T F(\cdot - j), h \gg \\ &= [\alpha^T \widehat{F}, \widehat{h}](\xi_0) = 0 \quad \forall h \in \mathcal{S}. \end{aligned}$$

Therefore  $(e^{ij\xi_0} \alpha)_{j \in \mathbf{Z}^d} \in \mathcal{K}_\infty(F)$ , which is a contradiction.

(ii)  $\implies$  (iii) By the assumption, there exists a finite set  $K(\xi) \subset \mathbf{Z}^d$  for any  $\xi \in [-\pi, \pi]^d$  such that  $\text{rank}(\widehat{F}(\xi + 2k\pi))_{k \in K(\xi)} = N$ . Let  $\varphi \in \mathcal{D}$  be supported in  $[-\pi/2, \pi/2]^d$  and take value one at the origin, and define  $h_{\xi, k} \in \mathcal{S}, k \in K(\xi)$ , by  $\widehat{h}_{\xi, k}(\eta) = \varphi(\eta - \xi - k)$ . Noting that  $[\widehat{F}, \widehat{h}_{\xi_0, k}](\xi_0) = F(\xi_0 + 2k\pi)$  for any  $\xi_0 \in [-\pi, \pi]^d$  and  $k \in K(\xi_0)$ , then by the compactness of  $[-\pi, \pi]^d$  and continuity of  $\widehat{F}$  by Proposition A.6, there exist a finite set  $\xi_\lambda, \lambda \in \Lambda$ , and  $h_{\lambda, k_\lambda} \in \mathcal{S}, k_\lambda \in K(\xi_\lambda)$ , such that  $([\widehat{F}, \widehat{h}_{\lambda, k_\lambda}](\xi))_{k_\lambda \in K(\xi_\lambda), \lambda \in \Lambda}$  has full rank  $N$  for any  $\xi \in [-\pi, \pi]^d$ . This, together with (2) and the density of functions in  $\mathcal{D}$  in  $\mathcal{S}$ , leads to

$$\text{rank}([\widehat{F}, \widehat{h}_i](\xi))_{1 \leq i \leq r} = N \quad \forall \xi \in [-\pi, \pi]^d \quad (5)$$

for some functions  $h_1, \dots, h_r \in \mathcal{D}$ . Note that  $[\widehat{F}, h_i], 1 \leq i \leq r$ , belong to the Wiener class. Then by (5), there exists a  $r \times N$  matrix  $B(\xi)$  with entries in the Wiener class such that

$$([\widehat{F}, \widehat{h}_i](\xi))_{1 \leq i \leq r} B(\xi) = I_N \quad \forall \xi \in [-\pi, \pi]^d. \quad (6)$$

By the density of trigonometric polynomials in the Wiener class, for some sufficiently small constant  $\delta > 0$  chosen later, there exists a  $r \times N$  matrix  $\tilde{B}(\xi)$  with trigonometric polynomial entries  $b_{s i'}(\xi), 1 \leq s \leq r, 1 \leq i' \leq N$ , such that

$$\|B(\xi) - \tilde{B}(\xi)\|_{\mathcal{W}} \leq \delta. \quad (7)$$

Then it follows from (6) and (7) that

$$\left\| \left( \left[ \widehat{f}_i, \sum_{s=1}^r \overline{b_{s i'}} \widehat{h}_s \right] (\xi) \right)_{1 \leq i, i' \leq N} - I_N \right\|_{\mathcal{W}} \leq C\delta \quad (8)$$

for some positive constant  $C$  independent of  $\delta$ . Then  $([\widehat{f}_i, \widehat{H}_{i'}](\xi))_{1 \leq i, i' \leq N}$  is invertible for any  $\xi \in [-\pi, \pi]^d$  by defining  $H_1, \dots, H_N$  by  $\widehat{H}_{i'}(\xi) = \sum_{s=1}^r \overline{b_{s i'}}(\xi) \widehat{h}_s(\xi), 1 \leq i' \leq N$ , and choosing  $\delta$  sufficient small.

**(iii)  $\implies$  (iv)** Let  $v_i, 1 \leq i \leq N$ , be vectors with  $i$ -th component one and other components zero. Then it suffices to prove that for any  $\epsilon > 0$ , there exists  $H_1^\epsilon, \dots, H_N^\epsilon \in \mathcal{D}$  such that

$$\|[\widehat{F}, \widehat{H}_i^\epsilon](\xi) - v_i\|_{\mathcal{W}} \leq \epsilon, \quad 1 \leq i \leq N. \quad (9)$$

Let  $h_1, \dots, h_N \in \mathcal{D}$  be as in (iii). Then  $([\widehat{f}_i, \widehat{h}_{i'}](\xi))$  is invertible for any  $\xi \in \mathbf{R}^d$ . By the density of trigonometric polynomials in the Wiener space, there exists an  $N \times N$  matrix  $B^\epsilon(\xi) = (b_{i'i'}^\epsilon(\xi))_{1 \leq i, i' \leq N}$  with trigonometric polynomial entries such that

$$\|([\widehat{f}_i, \widehat{h}_{i'}](\xi))_{1 \leq i, i' \leq N} B^\epsilon(\xi) - I_N\|_{\mathcal{W}} \leq \epsilon. \quad (10)$$

Define  $H_1^\epsilon, \dots, H_N^\epsilon$  by  $\widehat{H}_i^\epsilon(\xi) = \sum_{i'=1}^N \widehat{h}_{i'}(\xi) \overline{b_{i'i}^\epsilon(\xi)}$ ,  $1 \leq i \leq N$ . Then (9) follows from (10).  $\blacksquare$

*Proof of Theorem 1.2.* Let  $h_1, \dots, h_N \in \mathcal{D}$  be as (iii) of Theorem 1.1. Then  $B(\xi) = ([\widehat{f}_i, \widehat{h}_{i'}](\xi))_{1 \leq i, i' \leq N}$  is invertible for any  $\xi \in \mathbf{R}^d$ . Recall that all entries of  $B(\xi)$  are in the Wiener class by (2) and (A.1). Then  $(B(\xi))^{-1} = (\mathcal{F}(R_{i'i'})(\xi))_{1 \leq i, i' \leq N}$  for some sequences  $R_{i'i'} = (r_{i'i'}(j))_{j \in \mathbf{Z}^d} \in \ell^1$  by Wiener theorem. Therefore we obtain

$$\sum_{i''=1}^N \sum_{j' \in \mathbf{Z}^d} \ll f_i(\cdot + j), h_{i''}(\cdot + j') \gg r_{i''i'}(j') = \delta_{i'i'} \delta_{j0}. \quad (11)$$

Let  $G_i = \sum_{i''=1}^N \sum_{j' \in \mathbf{Z}^d} r_{i''i}(j') h_{i''}(\cdot + j')$ ,  $1 \leq i \leq N$ , and  $G = (G_1, \dots, G_N)^T$ .

Then it follows from (11) that

$$D = \ll D(\mathcal{L}(F)), \mathcal{L}^*(G) \gg \quad \forall D \in (\ell^\infty)^N.$$

This proves that  $(F_{*'})^{-1}(f) = \ll f, \mathcal{L}^*(G) \gg$  for all  $f \in V_\infty(F)$ .  $\blacksquare$

### 3. SOME REMARKS

In this section, we discuss the stable shifts of a compactly supported distribution, and the computation of the space  $\mathcal{I}(F)$  for a refinable distribution  $F$ .

#### 3.1. The Stable Shifts of Compactly Supported Distributions

Define the convolution  $D * W$  of two summable sequences  $D = (d(j))_{j \in \mathbf{Z}^d}$  and  $W = (w(j))_{j \in \mathbf{Z}^d}$  by  $D * W := (\sum_{j' \in \mathbf{Z}^d} d(j')w(j - j'))_{j \in \mathbf{Z}^d}$ . Then  $\mathcal{F}(D * W) = \mathcal{F}(D)\mathcal{F}(W)$  for any  $D$  and  $W \in \ell^1$ . For any compactly supported distribution  $F = (f_1, \dots, f_N)^T$ , let

$$i_S(F) = \{D \in (\ell_0)^N : D * R \in i(F) \text{ for some } R \in \ell_0 \\ \text{with } \mathcal{F}(R)(\xi) \neq 0 \quad \forall \xi \in \mathbf{R}^d\}.$$

It is easy to check that  $i_S(F)$  is shift-invariant, and  $i(F) \subset i_S(F) \subset \mathcal{I}(F) \cap (\ell_0)^N$ . For the stable shifts of a compactly supported distribution, we have the following result corresponding to Theorem 1.1, except the subspace  $\mathcal{I}(F)$  of  $(\ell^1)^N$  replaced by the subspace  $i_S(F)$  of  $(\ell_0)^N$ .

**THEOREM 3.3.** *Let  $F = (f_1, \dots, f_N)^T$  be a compactly supported distribution on  $\mathbf{R}^d$ . Then the following statements are equivalent to each other.*

(i)  $F$  has stable shifts.

(ii) The matrix  $(\widehat{F}(\xi + 2j\pi))_{j \in \mathbf{Z}^d}$  is of full rank for any  $\xi \in \mathbf{R}^d$ .

(iii) There exist  $h_1, \dots, h_N \in \mathcal{D}$ , and a trigonometric polynomial  $\alpha(\xi)$  with  $\alpha(\xi) \neq 0$  for any  $\mathbf{R}^d$  such that  $([\widehat{f}_i, \widehat{h}_{i'}](\xi))_{1 \leq i, i' \leq N} = \alpha(\xi)I_N$  for any  $\xi \in \mathbf{R}^d$ , where  $I_N$  is the  $N \times N$  identity matrix.

(iv)  $i_S(F) = (\ell_0)^N$ .

We remark that the equivalence between (i) and (ii) in Theorem 3.3 was proved in [14] for  $N = 1$  and [10] for  $N \geq 1$ .

**THEOREM 3.4.** *Let  $F = (f_1, \dots, f_N)^T$  be a compact supported distribution. If the semi-convolution  $F *' : (\ell^\infty)^N \mapsto V_\infty(F)$  is one-to-one, then its inverse is bounded, and there exists a vector-valued  $C^\infty$  function  $G = (G_1, \dots, G_N)^T$  with exponential decay at infinity such that  $(F *')^{-1} f = \ll f, \mathcal{L}^*(G) \gg$  for all  $f \in V_\infty(F)$ . Furthermore,  $G$  can be chosen to be linear combination of the shifts of some functions  $g_1, \dots, g_N \in \mathcal{D}$  using sequences with exponential decay, i.e.,*

$$G_i = \sum_{j \in \mathbf{Z}^d} r(j) g_i(\cdot + j), \quad 1 \leq i \leq N,$$

and there exist positive constants  $C$  and  $\lambda_0$  independent of  $j$  such that  $|r(j)| \leq C e^{-\lambda_0 |j|}$  for all  $j \in \mathbf{Z}^d$ .

We omit the detail of the proofs of Theorems 3.3 and 3.4 since we can use the same procedure to prove Theorems 1.1 and 1.2 except using the

fact that the inverse of a trigonometric polynomial  $R(\xi)$  with  $R(\xi) \neq 0$  for all  $\xi \in \mathbf{R}^d$  has exponential decay instead of using Wiener theorem.

### 3.2. Refinable Distributions

A tempered distribution  $F = (f_1, \dots, f_N)^T$  is said to be *refinable* if it has continuous Fourier transform, and satisfies a *refinement equation*

$$F = \sum_{j \in \mathbf{Z}^d} c(j) F(2 \cdot -j), \quad (12)$$

where  $c(j), j \in \mathbf{Z}^d$ , are  $N \times N$  matrices such that  $\sum_{j \in \mathbf{Z}^d} \|c(j)\| < \infty$ . The existence of distributional solutions of the refinement equation (2) is well studied especially for compactly supported solutions (see for instance [3, 4, 8, 12, 17, 21]). A square matrix is said to satisfy *Condition E* if one is its spectral radius, its only eigenvalue on the unit circle, and its simple eigenvalue. Set  $H(\xi) = 2^{-d} \sum_{j \in \mathbf{Z}^d} c(j) e^{-ij\xi}$ , which is known as the *symbol*  $H(\xi)$  of the refinement equation (12), or of the refinable distribution  $F$ . In the appendix of this paper, we show that if  $H(\xi) \in C^{d+1}$  and  $H(0)$  satisfies Condition E, then the nonzero distributional solution of the refinement equation (12) has  $\ell^1$  decay (see Proposition A.7 for detail).

For the  $N \times N$  matrices  $c(j), j \in \mathbf{Z}^d$ , with  $\sum_{j \in \mathbf{Z}^d} \|c(j)\| < \infty$ , define

$$B_k = (c(2i - i' + k))_{i, i' \in \mathbf{Z}^d}, \quad k \in \mathbf{Z}^d. \quad (13)$$

It is easy to check that  $B_k, k \in \mathbf{Z}^d$ , are bounded operators on  $(\ell^1)^N$ . For the shift-invariant space  $\mathcal{I}(F)$ , we have

THEOREM 3.5. *Let  $F$  be a nonzero distributional solution of the refinement equation (12) with continuous Fourier transform,  $c(j), j \in \mathbf{Z}^d$ , be the family of  $N \times N$  matrices in (12) such that  $\sum_{j \in \mathbf{Z}^d} \|c(j)\| < \infty$ , and let  $\Phi$  be a compactly supported  $C^{k_0(F)}$  function such that*

$$\lim_{n \rightarrow \infty} \sum_{|\alpha| \leq k_0(F)} \sup_{x \in \mathbf{R}^d} |D^\alpha(h - h_n)(x)(1 + |x|)^{k_0}| = 0 \quad \forall h \in \mathcal{S}, \quad (14)$$

where  $k_0(F)$  is the minimal integer  $k_0$  such that (2) holds for  $F$ ,  $h_n, n \geq 1$ , are finite linear combinations of  $\Phi(2^n \cdot -j), j \in \mathbf{Z}^d$ . Assume that  $H(\xi) = 2^{-d} \sum_{j \in \mathbf{Z}^d} c(j)e^{-ij\xi} \in C^{d+1}$  and that  $H(0)$  satisfies Condition E. Then  $\mathcal{I}(F)$  is the minimal closed subspace of  $(\ell^1)^N$  which is invariant under  $B_k, k \in \mathbf{Z}^d$ , and which contains the initial sequence  $\ll \mathcal{L}(F), \Phi \gg$ .

Similar invariant spaces of refinable distributions can be found in [7, 18].

*Proof of Theorem 3.5.* By Proposition A.7,  $F$  is a tempered distribution with  $\ell^1$  decay. Hence  $\mathcal{I}(F)$  is well defined and shift-invariant. Let  $V$  be the minimal closed linear subspace of  $(\ell^1)^N$  which is invariant under  $B_k, k \in \mathbf{Z}^d$ , and which contains the sequence  $\ll \mathcal{L}(F), \Phi \gg$ . Note that for any  $h \in \mathcal{D}$  and  $k \in \mathbf{Z}^d$ ,  $B_k D = 2^d \ll \mathcal{L}(F), h(2 \cdot -k) \gg$  by (12). Then it suffices to prove

$$\ll \mathcal{L}(F), \Phi \gg \in \mathcal{I}(F), \quad (15)$$

and for any  $h \in \mathcal{D}$ ,

$$\ll \mathcal{L}(F), h \gg \in V. \quad (16)$$

Let  $g \in \mathcal{D}$  with  $\hat{g}(0) = 1$ , and let  $g_n * \Phi, n \geq 1$ , be the convolution between  $g_n$  and  $\Phi$ , where  $g_n(x) = n^d g(x/n)$ . Then  $g_n * \Phi \in \mathcal{D}$  and are supported in a compact set independent of  $n \geq 1$ . By  $\Phi \in C^{k_0(F)}$ ,

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbf{R}^d} |D^\alpha (g_n * \Phi - \Phi)(x)| = \lim_{n \rightarrow \infty} \|g_n * (D^\alpha \Phi) - D^\alpha \Phi\|_\infty = 0$$

for any  $|\alpha| \leq k_0(F)$ . Therefore  $\ll \mathcal{L}(F), g_n * \Phi \gg$  converges to  $\ll \mathcal{L}(F), \Phi \gg$  in  $(\ell^1)^N$  by (2). This, together with  $\ll \mathcal{L}(F), g_n * \Phi \gg \in i(F)$  and closedness of  $\mathcal{I}(F)$  in  $(\ell^1)^N$  topology, implies that  $\ll \mathcal{L}(F), \Phi \gg \in \mathcal{I}(F)$ . This proves (15).

By the assumption on  $h_n$ ,  $\ll \mathcal{L}(F), h_n \gg \in V$  since for any  $k \in \mathbf{Z}^d$ ,

$$\ll \mathcal{L}(F), \Phi(2^n \cdot -k) \gg = 2^{-nd} B_k B_0^{n-1} \ll \mathcal{L}(F), \Phi \gg \in V.$$

Hence  $\ll \mathcal{L}(F), h \gg \in V$  by (2), (14) and closedness of  $V$ .  $\blacksquare$

Let us make a remark to compute the initial sequence  $\ll \mathcal{L}(F), \Phi \gg$  when  $\Phi$  is chosen to be refinable too. Assume that  $\Phi$  is a compactly supported  $C^{k_0(F)}$  function, and satisfies (14) and the following refinement equation

$$\Phi = \sum_{j \in \mathbf{Z}^d} d(j) \Phi(2 \cdot -j), \quad (17)$$



where the sequence  $(d(j))_{j \in \mathbf{Z}^d}$  has finite support. Set  $\Psi(x) = \ll F(\cdot + x), \Phi \gg$ . Then  $\Psi$  is continuous by (2), and

$$\begin{aligned} \Psi(x) &= 2^{-d} \sum_{j, j' \in \mathbf{Z}^d} c(j)d(j') \ll F(2x - j + \cdot), \Phi(\cdot - j') \gg \\ &= \sum_{j \in \mathbf{Z}^d} \left( 2^{-d} \sum_{j' \in \mathbf{Z}^d} c(j + j')d(j') \right) \Psi(2x - j) \end{aligned} \quad (18)$$

by (12) and (17). Define an operator  $B$  on  $(\ell^1)^N$  by

$$B = \left( 2^{-d} \sum_{j' \in \mathbf{Z}^d} c(2i - i' + j')d(j') \right)_{i, i' \in \mathbf{Z}^d}.$$

Then  $(\Psi(j))_{j \in \mathbf{Z}^d}$  is an eigenvalue of  $B$  with eigenvalue one by (17) and the continuity of  $\Phi$ . Hence  $\ll \mathcal{L}(F), \Phi \gg = \mathcal{L}(\Psi)(0)$  can be computed if the operator  $B$  on  $(\ell^1)^N$  has one as its simple eigenvalue. This gives a way to compute  $\ll \mathcal{L}(F), G \gg$  through finding eigenvector of the operator  $B$  with eigenvalue one.

## APPENDIX: DISTRIBUTIONS WITH $\ell^1$ DECAY

In this appendix, we give some basic properties of tempered distributions with  $\ell^1$  decay, and show the distributional solution of a refinement equation with smooth symbol has  $\ell^1$  decay.

**PROPOSITION A.6.** *Let  $F$  be a vector-valued tempered distribution on  $\mathbf{R}^d$  with  $\ell^1$  decay. Then*

(i)  $\widehat{F}$  is continuous.

(ii)  $\widehat{F}$  has polynomial increase at infinity, i.e., there exists a polynomial  $Q(\xi)$  such that  $|\widehat{F}(\xi)| \leq Q(\xi)$  for all  $\xi \in \mathbf{R}^d$ .

(iii)  $\sum_{j \in \mathbf{Z}^d} d(j)^T F(\cdot - j)$  converges in distributional sense for any sequence  $(d(j))_{j \in \mathbf{Z}^d} \in (\ell^\infty)^N$ .

(iv) For any  $g \in \mathcal{D}$ ,  $[\widehat{F}, \widehat{g}]$  is well defined and continuous, and

$$[\widehat{F}, \widehat{g}](\xi) = \sum_{j \in \mathbf{Z}^d} \ll F(\cdot + j), g \gg e^{-ij\xi}. \quad (\text{A.1})$$

*Proof.* For any Schwartz function  $h$ , set  $\|h\|_{\mathcal{D}^{k_0}} = \sum_{|\alpha| \leq k_0} \|D^\alpha h(x)(1 + |x|)^{k_0}\|_\infty$  and  $h_t = h(\cdot - t)$  for any  $t \in \mathbf{R}^d$ . Then there exists a positive constant  $C$  independent of  $h$  and  $t$  such that  $\|h_t\|_{\mathcal{D}^{k_0}} \leq C\|h\|_{\mathcal{D}^{k_0}}$  for all  $|t| \leq 1$  and  $h \in \mathcal{S}$ . This together with (2) leads to

$$\sum_{j \in \mathbf{Z}^d} |h * F(t + j)| = \sum_{j \in \mathbf{Z}^d} |h_t * F(j)| \leq C\|h\|_{\mathcal{D}^{k_0}}.$$

Then by integrating over  $[0, 1]^d$  at the both sides of the above inequality, we get  $h * F \in L^1$  and

$$\|h * F\|_1 \leq C\|h\|_{\mathcal{D}^{k_0}}. \quad (\text{A.2})$$

Therefore  $\widehat{h\widehat{F}}$  is bounded and continuous by Riemann Lemma, and the continuity of  $\widehat{F}$  follows.

Let  $\varphi \in \mathcal{D}$  with  $\varphi(\xi) = 1$  on the unit disk, and define  $h_N, N \geq 1$ , by  $\widehat{h}_N(\xi) = (1 + |\xi|)^{-N_1} \varphi(\xi/N)$ , where  $N_1 \geq k_0 + d + 2$ . Then there exist

positive constants  $C_1$  and  $C_2$  independent of  $N \geq 1$  such that

$$\|h_N\|_{\mathcal{D}^{k_0}} \leq C_1 \sum_{|\beta|, |\alpha| \leq k_0} \|D^\beta (\widehat{h}_N(\xi) \xi^\alpha)\|_1 \leq C_2 \quad \forall N \geq 1.$$

This together with (A.2) leads to

$$\sup_{|\xi| \leq N} (1 + |\xi|)^{-N_1} |\widehat{F}(\xi)| \leq \|\widehat{h}_N \widehat{F}\|_\infty \leq \|h_N * F\|_1 \leq C,$$

where  $C$  is a positive constant independent of  $N$ . This proves (ii).

For any  $D = (d(j))_{j \in \mathbf{Z}^d} \in (\ell^\infty)^N$  and Schwartz function  $h$ , define

$$D_F(h) = \lim_{K \rightarrow \infty} \sum_{|j| \leq K} d(j) \ll F(\cdot - j), h \gg.$$

Then for any Schwartz function  $h$ ,  $D_F(h)$  is well-defined and

$$|D_F(h)| \leq \|D\|_\infty \|\ll \mathcal{L}(F), h \gg\|_1 \leq C \|D\|_\infty \|h\|_{\mathcal{D}^{k_0}}$$

by (2). This shows that  $h \mapsto D_F(h)$  is a continuous linear form on the space of all Schwartz functions. Denote that continuous form by  $\sum_{j \in \mathbf{Z}^d} d(j)^T F(\cdot - j) = D(\mathcal{L}(F))$ . By the above procedure,  $\sum_{|j| \leq K} d(j)^T F(\cdot - j)$  converges to  $D(\mathcal{L}(F))$  in distributional sense.

By the assumptions on  $F$  and  $\widehat{g}$ ,  $\widehat{F} \widehat{g}$  is continuous and integrable. Thus  $[\widehat{F}, \widehat{g}]$  is well defined and continuous. By direct computation, we have

$$\begin{aligned} \ll F(\cdot + j), g \gg &= (2\pi)^{-d} \int_{\mathbf{R}^d} e^{ij\xi} \widehat{F}(\xi) \overline{\widehat{g}(\xi)} d\xi \\ &= (2\pi)^{-d} \int_{[-\pi, \pi]^d} e^{ij\xi} [\widehat{F}, \widehat{g}](\xi) d\xi \quad \forall j \in \mathbf{Z}^d. \end{aligned}$$

Hence (A.1) follows from the uniqueness theorem about the Fourier sequences of  $2\pi$  periodic integrable functions.  $\blacksquare$

PROPOSITION A.7. *Let  $F = (f_1, \dots, f_N)^T$  be a tempered distribution, have continuous Fourier transform, and satisfy the refinement equation  $F = \sum_{j \in \mathbf{Z}^d} c(j)F(2 \cdot -j)$ , where  $\sum_{j \in \mathbf{Z}^d} \|c(j)\| < \infty$ . Define  $H(\xi) = 2^{-d} \sum_{j \in \mathbf{Z}^d} c(j)e^{-ij\xi}$ . Assume that  $H(\xi) \in C^{d+1}$ , and that  $H(0)$  satisfies Condition E. Then  $F$  is a tempered distribution with  $\ell^1$  decay.*

To prove Proposition A.7, we need the following estimate about the limit of  $H(\xi/2)H(\xi/4) \cdots H(\xi/2^n)$ . We omit the detail of the proof here since it can be proved by the same way as in [3, 8].

LEMMA A.8. *Let  $H(\xi)$  be a matrix-valued  $2\pi$  periodic function on  $\mathbf{R}^d$  and set  $\widehat{F}_n(\xi) = H(\xi/2) \cdots H(\xi/2^n)$ ,  $n \geq 0$ . Assume that  $H(\xi) \in C^{d+1}$  and that  $H(0)$  satisfies Condition E. Then for any  $|\alpha| \leq d+1$ ,  $D^\alpha F_n(\xi)$  converges uniformly to some continuous function on any compact set. Furthermore there exist positive constants  $C$  and  $B$  independent of  $n$  and  $\xi$  such that*

$$\|D^\alpha \widehat{F}_n(\xi)\| \leq C(1 + |\xi|)^B \tag{A.3}$$

for all  $|\alpha| \leq r$ ,  $n \geq 1$  and  $\xi \in \mathbf{R}^d$ .

*Proof of Proposition A.7.* Let  $\widehat{F}_n(\xi) = H(\xi/2)H(\xi/4)\cdots H(\xi/2^n)$ ,  $n \geq 1$  be as in Lemma A.8. Then  $F_n(\xi)$  converges, and its limit  $M(\xi)$  is continuous and dominated by a polynomial, which implies that  $M(\xi)$  is a tempered distribution. Denote the inverse Fourier transform of  $M(\xi)\widehat{F}(0)$  by  $G$ . Then  $G$  is a tempered distribution too. Now it remains to prove that  $G = F$  and that  $G$  is a tempered distribution with  $\ell^1$  decay.

Define  $H_K(\xi) = \sum_{|j| \leq K} c(j)e^{-ij\xi}$ . Then from the definition of  $F$ ,  $H_K\widehat{F}$  converges to  $\widehat{F}(2\cdot)$  in distributional sense, since  $\sum_{|j| \leq K} c(j)F(2\cdot - j)$  converges to  $F$  in distributional sense. On the other hand,  $H_N(\xi)\widehat{F}(\xi)$  converges to  $H(\xi)\widehat{F}(\xi)$  uniformly on any compact set, and hence converges in distributional sense. Therefore  $\widehat{F}(\xi) = H(\xi/2)\widehat{F}(\xi/2)$ . Using the above formula for  $n$  times leads to  $\widehat{F}(\xi) = H(\xi/2)\cdots H(\xi/2^n)\widehat{F}(\xi/2^n)$ . Then letting  $n$  tend to infinity and using the continuity of  $\widehat{F}$  and Lemma A.8, we obtain  $\widehat{F}(\xi) = M(\xi)\widehat{F}(0)$ . This proves  $F = G$ .

By (A.3),  $\widehat{G}\widehat{h}$  is integrable for any Schwartz function  $h$ , hence  $\ll G(\cdot + x), h \gg$  is continuous about  $x$ . This prove (1) for  $G$ . Still by (A.3), there exist positive constants  $C$  and  $B$  independent of  $\xi$  and  $\alpha$  such that

$$|D^\alpha \widehat{G}(\xi)| \leq C(1 + |\xi|)^B \quad (\text{A.4})$$

for all  $|\alpha| \leq d + 1$  and  $\xi \in \mathbf{R}^d$ . Therefore for any Schwartz function  $h$ ,

$$\sum_{j \in \mathbf{Z}^d} \left| \int_{\mathbf{R}^d} \widehat{G}(\xi)\widehat{h}(\xi)e^{-ij\xi} d\xi \right|$$

$$\begin{aligned}
&\leq C_1 \sum_{j \in \mathbf{Z}^d} (1 + |j|)^{-d-1} \sup_{\xi \in \mathbf{R}^d} \sum_{|\gamma| \leq d+1} |D^\gamma(\widehat{G}\widehat{h}(\xi))|(1 + |\xi|)^{d+1} \\
&\leq C_2 \sup_{\xi \in \mathbf{R}^d} \sum_{|\gamma| \leq d+1} |D^\gamma\widehat{h}(\xi)|(1 + |\xi|)^{B+d+1} < \infty,
\end{aligned}$$

where  $C_1, C_2$  are positive constants independent of  $h$ , and  $B$  is the constant in (A.4). This proves (2) for  $G$ . ■

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