M-BAND SCALING FUNCTION WITH FILTER HAVING VANISHING MOMENTS TWO AND MINIMAL LENGTH

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Abstract. In this paper, we consider the Hölder continuity, local linearity, linear independence and interpolation problem of the M-band scaling function with its filter having vanishing moments two and minimal length, and explicit construction of wavelets. Especially we find some new properties which is not true when $M = 2$, such as local linearity, local linear dependence, differentiability at adjoint M-adic points and interpolation problem at integer knots.

1. Introduction

Let $M \geq 2$ be a fixed positive integer. A compactly supported and square integrable function ϕ is called a *scaling function* if it satisfies $\int_R \phi(x)dx = 1$, $\int_R \phi(x)\phi(x-k) = \delta_k$ and a refinement equation

$$\phi(x) = \sum_{k\in Z} c_k \phi(Mx - k), \tag{1}$$

where $\{c_k\}$ is a sequence with finite length and satisfying $\sum_{k\in Z} c_k = M$, and δ_k is the Kronecker symbol defined by $\delta_k = 1$ when $k = 0$ and $\delta_k = 0$ when $k \neq 0$. For a sequence $\{c_k\}$ with finite length, define

$$H(z) = \frac{1}{M}\sum_{k\in Z} c_k z^k.$$

Then $H(z)$ is uniquely determined by the scaling function ϕ and conversely for an $H(z)$ there exists a unique solution of (1) with $\int_R \phi(x)dx = 1$. So we say that $H(z)$ is the *filter* of the refinement equation (1) or the filter corresponding to the scaling function ϕ. For

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an integer $N \geq 1$, we say that $H(z)$ has *vanishing moments* N if there exists a Laurent polynomial $\tilde{H}(z)$ such that

$$H(z) = \left(\frac{1 - z^M}{M(1-z)}\right)^N \tilde{H}(z).$$

For a scaling function ϕ, let closed subspaces $V_j,\ j \in Z,$ of square integrable function space L^2 be spanned by $\{M^{j/2}\phi(M^j \cdot -k); k \in Z\}$. Then $\{V_j\}_{j\in Z}$ is a *multiresolution* by elementary wavelet theory (see [BDS] for example), i.e., it satisfies the following conditions:

(i) $V_j \subset V_{j+1}$ for all $j \in Z$, and $f \in V_j$ if and only if $f(M\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(ii) $\cup_{j\in\mathbb{Z}} V_j$ is dense in the square integrable function space L^2 and $\cap_{j\in\mathbb{Z}} V_j = \{0\}$;

(iii) there exists a function ϕ in V_0 such that $\{\phi(\cdot - k); k \in Z\}$ is an orthonormal basis of V_0.

Define the wavelet space $W_j,\ j \in Z,$ by the complement spaces of V_j in V_{j+1}. Then we have the following wavelet decompostion

$$L^2 = \oplus_{j\in\mathbb{Z}} W_j = V_k + \oplus_{j\geq k} W_j.$$

The wavelet theory when $M = 2$ can be found in many books on wavelets(for example [D], [M]). The wavelet theory for general $M \geq 3$ is also developed early. Auscher([A]) considered the construction of M-band wavelets in his thesis. Welland and Lundberg ([WL]) constructed M-band wavelets with arbitrary high index of smoothness. Heller ([H]), independently Bi, Dai and the first author ([BDS]), considered the design of filter with vanishing moments N and finite length. Bi, Dai and the first author ([BDS]) studied the asympotic behaviour of regularity of the scaling functions with their filter having vanishing moments N and minimal length as N tends to infinity for all $M \geq 3$. Similar problem was also considered by Heller and Well in [HW] for $M = 3, 4$. From the asympotic behaviour, we get that the regularity of that class of scaling functions is about $C \ln N$ when M is odd. It inspired Shi and the first author ([SS]) to construct another class of scaling functions with their filter having vanishing moments N and with their regularity at least CN for some positive number C independent of N. For $M = 3$, Dai, Huang and the first author considered in [DHS] local and global linear independence of solutions of the five-coefficient refinement equation (1), and find some examples of solutions of refinement equations which are globally linearly independent but locally linearly dependent. In the engineer literature, the publications on the design of M-band linear phase filter bank by Vaidyanathan and his group are important contribution to the M-band wavelet theory(see [SVN] and the references therein).

Let

$$_2H(z) = \left(\frac{1+z}{2}\right)^2 \left(\frac{1 \pm \sqrt{3}}{2} + \frac{1 \mp \sqrt{3}}{2} z\right).$$

Then it is proved ([D]) that a filter G with vanishing moments two and minimal length can be written as $G(z) = {}_2H(z)z^k$ for some integer k when $M = 2$. The scaling function ϕ_2 corresponding to the filter $\left(\frac{1+z}{2}\right)^2 \left(\frac{1+\sqrt{3}}{2} + \frac{1-\sqrt{3}}{2} z\right)$ is a very important example of scaling function when $M = 2$ (see [D]). The other scaling function ϕ_2' with its filter $\left(\frac{1+z}{2}\right)^2\left(\frac{1-\sqrt{3}}{2}+\frac{1+\sqrt{3}}{2}z\right)$ is related to ϕ_2. In particular $\phi_2'(x)=\phi_2(3-x)$. Daubechies and Lagarais ([DL]) considered the Hölder continuity of ϕ_2. Pollen ([P]) studied the differentiability of ϕ_2 at dyadic points.

The scaling function ϕ_M with its corresponding filter

$$H(z)=\left(\frac{1-z^M}{M-Mz}\right)^2\left(\frac{1+\sqrt{\frac{2M^2+1}{3}}}{2}+\frac{1-\sqrt{\frac{2M^2+1}{3}}}{2}z\right),$$

which has vanishing moments two and minimal length, is also important in M-band wavelet theory for general $M\ge 3$. Inspired by [DL] and [P], we consider in this paper the Hölder continuity, differentiability at M-adic and adjoint M-adic points, linear independence, interpolation problem of ϕ_M, and explicit construction of wavelets. Through this paper, it is found some new properties of ϕ_M when $M\ge 3$ which does not holds for ϕ_2. In particular, ϕ_M is locally linear on an open set with full measure and locally linearly dependent when $M\ge 3$, $\tilde{\phi}_M$ is differentiable at M-adic points when $M=3$, and $\tilde{\phi}_M$ is not interpolatable at Z when $M=11$, where $\tilde{\phi}_M(x)=\phi(2+\frac{1}{M-1}-x)$.

In this paper, we will assume that $M\ge 3$.

2. Preliminary

From the design of filters in [H], or [BDS], we see that the filter $H(z)$ of a scaling function with vanishing moments two and minimal length must be

$$H(z)=\left(\frac{1-z^M}{M-Mz}\right)^2(\alpha+\beta z)z^k$$

and

$$(\alpha+\beta z)(\alpha+\beta z^{-1})=1+\left(\sum_{s=1}^{M-1}\theta_s^{-1}\right)(2-z-z^{-1}),$$

where k is an integer and $\theta_s=4\sin^2 s\pi/M, 1\le s\le M-1$. By the indentity in [BDS]

$$\frac{\sin^2 M\xi/2}{M^2\sin^2\xi/2}=\prod_{s=1}^{M}(1-4\theta_s^{-1}\sin^2\xi/2),$$

and by

$$\frac{\sin M\xi/2}{M\sin\xi/2}=1-\frac{M^2-1}{24}\xi^2+O(\xi^4),$$

we get

$$\sum_{s=1}^{M-1} \theta_s^{-1} = \frac{M^2-1}{12},$$

where $O(\xi^4)$ means the term bounded by $C\xi^4$ for some constant C independent of ξ when ξ is sufficiently small. Therefore α and β are two roots of the equation

$$x^2 - x - \frac{M^2-1}{6} = 0,$$

i.e.,

$$\alpha = \frac{1\pm\theta}{2}, \beta = \frac{1\mp\theta}{2},$$

where $\theta = \sqrt{\frac{2M^2+1}{3}}$. Denote the scaling function with its filter

$$H(z) = \left(\frac{1-z^M}{M-Mz}\right)^2 (\frac{1+\theta}{2} + \frac{1-\theta}{2} z)$$

by ϕ_M. Then the scaling function with its filter $H(z) = \left(\frac{1-z^M}{M-Mz}\right)^2 (\frac{1+\theta}{2} + \frac{1-\theta}{2} z) z^k$ is $\phi_M(-\frac{k}{M-1} + x)$ and the scaling function with its filter $H(z) = \left(\frac{1-z^M}{M-Mz}\right)^2 (\frac{1-\theta}{2} + \frac{1+\theta}{2} z) z^k$ is $\phi_M(\frac{2M+k-1}{M-1} - x)$ by elementary computation. Hereafter we assume that the filter with vanishing moments two and minimal length is

$$\begin{aligned} H(z) &= \left(\frac{1-z^M}{M-Mz}\right)^2 (\frac{1+\theta}{2} + \frac{1-\theta}{2} z) \\ &= \frac{1}{M} \sum_{k=0}^{2M-1} \frac{(1+\theta)(M-|k-M+1|) + (1-\theta)(M-|k-M|)}{2M} z^k. \end{aligned} \tag{2}$$

Observe that $\theta = 9$ when $M = 11$. Hence the filter with vanishing moments two and minimal length has rational coefficients. For $M = 2$, we have not seen examples of compactly supported orthonormal scaling function except Haar scaling function such that its filter has rational coefficients to our knowledge.

Obviously ϕ_M is supported in $[0, 2+\frac{1}{M-1}]$ by the refinement equation (1). Observe that

$$|\frac{1+\theta}{2} + \frac{1-\theta}{2} e^{i\xi}| \le \theta < M$$

Then ϕ_M is Hölder continuous and its Hölder index is at least $1 - \ln\theta / \ln M$ by the criterion in [D]. In particular, we will show in section 5 that the Hölder index of ϕ_M is exactly $1 - \ln\frac{1+\theta}{2} / \ln M$.

3. Basic Properties

In this section, we will give some elementary properties of ϕ_M.

Proposition 1. *For $x \in R$, we have*

$$\begin{cases} \sum_{k\in Z} \phi(x-k) = 1, \\ \sum_{k\in Z} (1 + \frac{1-\theta}{2(M-1)} + k)\phi(x-k) = x. \end{cases}$$

Proof. Recall that the filter of ϕ_M has the factor $(\frac{1-z^M}{1-z})^2$. Then ϕ_M satisfies the Strang-Fix condition of order two, i.e., $\hat{\phi}_M(2k\pi) = 0$ and $(\hat{\phi}_M)'(2k\pi) = 0$ holds for all nonzero integers k, where the Fourier transform $\hat{f}$ of an integrable function f is defined by

$$\hat{f}(\xi) = \int_R e^{-ix\xi} f(x)dx.$$

By the definition of ϕ_M, we get

$$\hat{\phi}_M(0) = \int_R \phi_M(x)dx = 1.$$

Hence there exists a constant c such that

$$\sum_{k\in Z} \phi(x-k) = 1 \tag{3}$$

and

$$\sum_{k\in Z} (c+k)\phi(x-k) = x. \tag{4}$$

Therefore the proof of Proposition 1 reduces to

$$c = 1 + \frac{1-\theta}{2M-2}.$$

¿From (1)-(3), we get

$$\phi_M(1) = \frac{2M-1+\theta}{2M}\phi_M(1) + \frac{2M-3+\theta}{2M}\phi_M(2).$$

and

$$\phi_M(1) + \phi_M(2) = 1.$$

By the above two equations, we have

$$\phi_M(1) = \frac{2M-3+\theta}{2M-2}$$

and

$$\phi_M(2) = \frac{1-\theta}{2M-2}.$$

Hence

$$c = \phi_M(1) + 2\phi_M(2) = 1 + \frac{1-\theta}{2M-2}$$

by (4) and Proposition 1 follows. ■

Recall that ϕ_M is supported in $[0, 2+\frac{1}{M-1}]$. So Proposition 1 can be rewritten as the following result.

Proposition 2. *For $0 \le x \le 1$, we have*

$$\begin{cases} \phi_M(x) - \phi_M(x+2) = x - \frac{1-\theta}{2M-2}, \\ \phi_M(x+1) + 2\phi_M(x+2) = -x + 1 + \frac{1-\theta}{2M-2}, \\ 2\phi_M(x) + \phi_M(x+1) = x + 1 - \frac{1-\theta}{2M-2}. \end{cases}$$

Proposition 3. *For $0 \le x \le 1$, we have*

$$\begin{cases} \phi_M(\frac{x}{M}) = \frac{1+\theta}{2M}\phi_M(x), \\ \phi_M(\frac{x+1}{M}) = \frac{1-\theta}{2M}\phi_M(x) + \frac{1+\theta}{2M}x + \frac{4+M+3\theta}{6M}, \\ \phi_M(1+\frac{x}{M}) = \frac{1+\theta}{2M}\phi_M(1+x) + \frac{1-\theta}{2M}x + \frac{(2M-3+\theta)(2M-1-\theta)}{4M(M-1)}, \\ \phi_M(1+\frac{x+1}{M}) = \frac{1-\theta}{2M}\phi_M(1+x) - \frac{1+\theta}{2M}x + \frac{5M-7}{6M}, \\ \phi_M(2+\frac{x}{M}) = \frac{1+\theta}{2M}\phi_M(2+x) - \frac{1-\theta}{2M}x + \frac{(1-\theta)(2M-1-\theta)}{4M(M-1)}, \\ \phi_M(2+\frac{x+1}{M}) = \frac{1-\theta}{2M}\phi_M(2+x), \end{cases} \tag{5}$$

and for $x \in [\frac{1}{M-1}, 1]$, we have

$$\begin{cases} \phi(x) = x - \frac{1-\theta}{2M-2}, \\ \phi(x+1) = -x + \frac{1-\theta}{2M-2} + 1, \\ \phi(x+2) = 0. \end{cases} \tag{6}$$

Proof. The first and second formulas in (5) follows easily from the refinement equation (1) and the third formula in Proposition 2. The other formulas in (5) follow from the first and second formulas and Proposition 2. Recall that ϕ_M is supported in $[0, 2+\frac{1}{M-1}]$. By the first formula in Propositon 2, we get the first and third formula in (6). The second formula in (6) follows from the first one and Proposition 2. ■

Using the first two formulas in (5) for k times, we get

Propsotion 4. *Let $\epsilon_i \in \{0,1\}$ for all $1 \le i \le k$. Then we have*

$$\phi_M(\sum_{i=1}^{k} \frac{\epsilon_i}{M^i} + \frac{x}{M^k}) = \alpha(\epsilon_1, \cdots, \epsilon_k)\phi(x) + \beta(\epsilon_1, \cdots, \epsilon_k)x + \phi_M(\sum_{i=1}^{k} \frac{\epsilon_i}{M^i}),$$

where

$$\begin{cases} \alpha(\epsilon_1, \cdots, \epsilon_k) = M^{-k}(\dfrac{1-\theta}{2})^{\sum_{j=1}^{k}\epsilon_j}(\dfrac{1+\theta}{2})^{k-\sum_{j=1}^{k}\epsilon_j}, \\ \beta(\epsilon_1, \cdots, \epsilon_k) = M^{-k+1}\beta(\epsilon_1) + M^{-k+1}\displaystyle\sum_{i=2}^{k}\beta(\epsilon_i)(\dfrac{1-\theta}{2})^{\sum_{j=1}^{i-1}\epsilon_j}(\dfrac{1+\theta}{2})^{i-1-\sum_{j=1}^{i-1}\epsilon_j}, \end{cases}$$

and $\beta(0) = 0$, $\beta(1) = \frac{1+\theta}{2M}$.

4. Local Linearity

We say that a function is *locally linear* on an open set if it is a linear function on its every connected components.

Theorem 1. *Let $M \ge 3$. Then there exists an open set $A \subset (0, 2+\frac{1}{M-1})$ with Lebesgue measure $2+\frac{1}{M-1}$ such that ϕ_M is locally linear on A.*

Proof. For $(\epsilon_1, \cdots, \epsilon_k) \in \{0,1\}^k$, define

$$A(\epsilon_1, \cdots, \epsilon_k) = (\sum_{i=1}^{k} \frac{\epsilon_i}{M^i} + \frac{1}{(M-1)M^{k+1}}, \sum_{i=1}^{k} \frac{\epsilon_i}{M^i} + \frac{1}{M^{k+1}}) \subset (0,1).$$

Then any two elements of the set

$$\{A(\epsilon_1, \cdots, \epsilon_k); \epsilon_i \in \{0,1\}, 1 \le i \le k, k = 1, 2, \cdots\}$$

of intervals have empty intersection. Also it is easy to check that $A(\epsilon_1, \cdots, \epsilon_k)$ has empty intersection with $(\frac{1}{M(M-1)}, \frac{1}{M})$ and $(\frac{1}{M-1}, 1)$. Define

$$A_1 = (\frac{1}{M-1}, 1) \cup (\frac{1}{M(M-1)}, \frac{1}{M}) \cup \big(\cup_{k=1}^{\infty} \cup_{(\epsilon_1, \cdots, \epsilon_k) \in \{0,1\}^k} A(\epsilon_1, \cdots, \epsilon_k)\big).$$

Then $A_1 \subset (0,1)$ and

$$|A_1| = \frac{M-2}{M-1} + \frac{M-2}{M(M-1)} + \frac{M-2}{M(M-1)} \sum_{k=1}^{\infty} (\frac{2}{M})^k = 1.$$

Furthermore ϕ_M is a linear function on $(\frac{1}{M(M-1)}, \frac{1}{M})$ and $(\frac{1}{M-1}, 1)$ by the first formula in (5) and (6), and is also a linear function on $A(\epsilon_1, \cdots, \epsilon_k)$ when $\epsilon_i \in \{0,1\}, 1 \le i \le k$ since

$$\begin{aligned} \phi_M(x) = & \frac{1+\theta}{2M} \times \alpha(\epsilon_1, \cdots, \epsilon_k) \phi_M\big(M^{k+1}(x - \sum_{i=1}^{k} \frac{\epsilon_i}{M^i})\big) \\ & + M^k \times \beta(\epsilon_1, \cdots, \epsilon_k) \times (x - \sum_{i=1}^{k} \frac{\epsilon_i}{M^i}) + \phi_M(\sum_{i=1}^{k} \frac{\epsilon_i}{M^i}) \end{aligned}$$

when $x \in A(\epsilon_1 \cdots, \epsilon_k)$ by Proposition 4 and (6). Then ϕ_M is locally linear on A_1.

Define

$$A = A_1 \cup (A_1 + 1) \cup ((A_1 \cap (0, \frac{1}{M-1})) + 2).$$

Then $A \subset (0, 2 + \frac{1}{M-1})$ and $|A| = 2 + \frac{1}{M-1}$. Moreover ϕ_M is locally linear by Proposition 2. ■

5. Hölder Continuity

Denote the set of all M-*adic points* in $[0, 2 + \frac{1}{M-1}]$ by D_M, i.e.,

$$D_M = \big\{x = \sum_{i=0}^{k} \frac{\epsilon_i}{M^i} \in [0, 2 + \frac{1}{M-1}]; \epsilon_i \in \{0, 1 \cdots, M-1\}, k = 0, 1, 2, \cdots\big\}.$$

We say that $x \in [0, 2 + \frac{1}{M-1}]$ is *adjoint* M-*adic* if $2 + \frac{1}{M-1} - x$ is M-adic. Denote the set of all adjoint M-adic points in $[0, 2 + \frac{1}{M-1}]$ by D'_M. For D_M, we further decompose it as the union of D_M^1 and D_M^2, where

$$D_M^1 = \big\{x = l + \sum_{i=1}^{k} \frac{\epsilon_i}{M^i} \in [0, 2 + \frac{1}{M-1}]; l \in 0, 1, 2, \epsilon_i \in \{0,1\}, k = 1, 2, \cdots\big\}$$

and

$$D_M^2 = D_M \backslash D_M^1.$$

Denote the set of all adjoint M-adic points of D_M^j by $D_M^{j\prime}, j = 1, 2$. Then

$$D'_M = D_M^{1\prime} \cup D_M^{2\prime}.$$

We define the *left(right) local Hölder index* of a continuous function ϕ at x by the supremum of all α such that

$$|\phi(y) - \phi(x)| \leq C|y - x|^\alpha$$

holds for all y less(larger) than x and a constant C independent of y, where $|y - x|$ is sufficiently small. We denote the left(right) local Hölder index of ϕ_M by $\alpha_L(x)(\alpha_R(x))$.

Theorem 2. *Let $M \geq 3$. Then ϕ_M is right differentiable at all points in D_M^2 and D'_M, and not right differentiable at all points in D_M^1. Furthermore the right local Hölder index $\alpha_R(x)$ is $1 - \ln \frac{1+\theta}{2} / \ln M$ when $x \in D_M^1$.*

Proof. First we show that ϕ_M is right differentiable at any point in D_M^2. By Proposition 2, it suffices to prove that ϕ_M is right differentiable at $\sum_{i=1}^k \frac{\epsilon_i}{M^i}$, when $2 \leq \epsilon_i \leq M - 1$ holds for some $1 \leq i \leq k$. In particular, we will prove that $\phi_M(\sum_{i=1}^k \frac{\epsilon_i}{M^i} + y)$ is a linear function when $0 \leq y \leq M^{-k-1}$. Let i_0 be the minimal number such that $2 \leq \epsilon_{i_0} \leq M - 1$. Then

$$\phi_M(\sum_{i=1}^k \frac{\epsilon_i}{M^i} + y) - \alpha(\epsilon_1, \cdots, \epsilon_{i_0-1})\phi_M(\sum_{i=i_0}^k \frac{\epsilon_i}{M^{i-i_0+1}} + M^{i_0-1}y)$$

is a linear function of y by Proposition 4. Recall that ϕ_M is linear on $(\frac{1}{M-1}, 1)$ and

$$\frac{1}{M-1} < \frac{2}{M} \leq \sum_{i=i_0}^k \frac{\epsilon_i}{M^{i-i_0+1}} + M^{i_0-1}y < 1$$

when $0 \leq y \leq M^{-k-1}$. Then $\phi_M(\sum_{i=i_0}^k \frac{\epsilon_i}{M^{i-i_0+1}} + M^{i_0-1}y)$ is a linear function of y when $0 \leq y \leq M^{-k-1}$ and ϕ_M is right differentiable at all points in D_M^2.

Secondly we show that ϕ_M is right differentiable at all points in D'_M. For $x_0 = l + \frac{1}{M-1} - \sum_{i=1}^k \frac{\epsilon_i}{M^i} \in D'_M$, we can rewrite it as

$$x_0 = l + \sum_{i=1}^k \frac{1-\epsilon_i}{M^i} + \frac{1}{M^k(M-1)} = l' + \sum_{i=1}^k \frac{\epsilon'_i}{M^i} + \frac{1}{M^k(M-1)}, \tag{7}$$

where $l, l' \in \{0, 1, 2\}, \epsilon_i, \epsilon'_i \in \{0, 1, \cdots, M - 1\}$.

When $2 \leq \epsilon'_i \leq M - 1$ holds for some $1 \leq i \leq k$, we deduce that $\phi_M(x_0 + y)$ is a linear function of y when $0 \leq y \leq M^{-k-2}$ by the same procedure as to prove that ϕ_M is right differential at points in D_M^2.

Therefore it suffices to show that $\phi_M(x_0 + y)$ is a linear function of y when $0 \leq y \leq M^{-k-2}$ and ϵ_i in (7) satisfies $\epsilon'_i \in \{0, 1\}$ for all $1 \leq i \leq k$. When $\epsilon'_i = 1$ for all $1 \leq i \leq k$ and $0 \leq y \leq \frac{M-2}{M-1}$, $\phi(x_0 + y)$ is a linear function of y since $x_0 = l' + \frac{1}{M-1}$. Therefore we assume that $e_i', 1 \le i \le k$ is not identically 1. Let $1 \le i_0 \le k$ be the maximal index such that $\epsilon_{i_0}' = 0$. Then $\epsilon_i' = 1$ when $i_0 + 1 \le i \le k$ and

$$x_0 = l' + \sum_{i=1}^{i_0-1} \frac{\epsilon_i'}{M^i} + \frac{1}{M^{i_0}(M-1)}$$

is the left endpoint of the interval $l' + A(\epsilon_1', \cdots, \epsilon_{i_0-1}')$. Therefore $\phi_M(x_0 + y)$ is a linear function of y when $0 \le y \le M^{-k-2}$ by the proof of Theorem 1. Thus ϕ_M is right differentiable at all points in D_M'.

Finally we prove that ϕ_M is not right differentiable at any point in D_M^1. Observe that the right local Hölder index of ϕ_M is at least one at points where ϕ_M is right differentiable. Therefore it suffices to prove that the right local Hölder index of ϕ_M at any point in D_M^1 is less than 1.

By the first formula in (5), we get

$$|\phi_M(x) - \phi_M(0)| = |(\frac{1+\theta}{2M})^{k-1}\phi_M(M^{k-1}x)| \le C|x|^{1-\ln\frac{1+\theta}{2}/\ln M}$$

when $x \in (M^{-k}, M^{-k+1})$ and

$$|\phi_M(\frac{1}{M^k}) - \phi_M(0)| = \big(\frac{1+\theta}{2M}\big)^k |\phi_M(1)| \ge C\big(\frac{1}{M^k}\big)^{1-\ln\frac{1+\theta}{2}/\ln M},$$

where C is a constant independent of x and k. Hence $1 - \frac{\ln\frac{1+\theta}{2}}{\ln M} < 1$ is the right local Hölder index of ϕ_M at zero point. By Proposition 2 and 4, we know that the right local Hölder index of ϕ_M at any point in D_M^1 is the same as the one at zero point. ■

Theorem 3. *Let $M \ge 3$. Then ϕ_M is left differentiable at $D_M^{2\prime}$ and D_M, and ϕ_M is left differentiable at $D_M^{1\prime}$ when $M = 3$ and is not when $M \ge 4$. Furthermore the left local Hölder index of ϕ_M at $D_M^{1\prime}$ is $1 - \ln\frac{\theta-1}{2}/\ln M$ when $M \ge 4$.*

Proof. Let $\tilde{\phi}_M(x) = \phi_M(2 + \frac{1}{M-1} - x)$. Then $\tilde{\phi}_M$ is also a scaling function with its filter $\big(\frac{1-z^M}{M-Mz}\big)^2(\frac{1-\theta}{2} + \frac{1+\theta}{2}z)$. For $\tilde{\phi}_M$, we can establish corresponding results of Proposition 2-4. In particular, the results of Proposition 2-4 hold when ϕ_M is replaced by $\tilde{\phi}_M$ and θ by $-\theta$ in Proposition 2-4. Then by the procedure used in the proof of Theorem 2, we deduce that $\tilde{\phi}_M$ is right differentiable at any point in D_M' and D_M^2., i.e., ϕ_M is left differentiable at all points in D_M and $D_M^{2\prime}$.

Observe that $\tilde{\phi}_M(0) = 0$, $\tilde{\phi}_M(1) \ne 0$ and $\tilde{\phi}_M(\frac{x}{M}) = \frac{1-\theta}{2M}\tilde{\phi}_M(x)$ for $0 \le x \le 1$. Then we get

$$|\tilde{\phi}_M(x) - \tilde{\phi}_M(0)| \le C|x|^{1-\ln\frac{\theta-1}{2}/\ln M}$$

when $0 \le x \le 1/M$ and

$$|\tilde{\phi}_M(\frac{1}{M^k}) - \tilde{\phi}_M(0)| \ge C(\frac{1}{M^k})^{1-\ln\frac{\theta-1}{2}/\ln M},$$

where C is a constant independent of x and $k \geq 1$. Then the right local Hölder index of $\tilde{\phi}_M$ at zero point is $1 - \frac{\ln \frac{\theta-1}{2}}{\ln M}$. Observe that $(\theta-1)/2 < 1$ when $M = 3$. Hence $\tilde{\phi}_M$ is right differentiable at zero point and consequently at any point in D_M^1. Hence ϕ_M is left differentiable at any point in $D_M^{1\prime}$. Also observe that $(\theta-1)/2 > 1$ when $M \geq 4$. Then $\tilde{\phi}_M$ is not right differentiable at zero point and any point in D_M^1. Hence ϕ_M is not left differentiable at $D_M^{1\prime}$, and the left local Hölder index of ϕ_M at all points in $D_M^{1\prime}$ is $1 - \ln \frac{\theta-1}{2} / \ln M$. ■

From Theorem 2 and 3, we get

Corollary. *ϕ_M is differentiable at any adjoint M-adic point when $M = 3$.*

Theorem 4. *Write $x_0 = \sum_{i=1}^{\infty} \frac{\epsilon_i}{M^i} + l \in [0, 2 + \frac{1}{M-1}]$, where $\epsilon_i \in \{0, 1 \cdots, M-1\}$, $l \in \{0, 1, 2\}$ and ϵ_i is not identically $M-1$ for sufficiently large i. Then we have*

(i) If there exists an index i such that $2 \leq \epsilon_i \leq M-1$, then ϕ_M is differentiable at x_0;

(ii) If $\epsilon_i \in \{0, 1\}$ for all $i \geq 1$, then the right and left local Hölder index of ϕ_M at x_0 is at least $1 - \max\big(0, \big(\alpha(x_0) \ln \frac{\theta-1}{2} + (1-\alpha(x_0)) \ln \frac{\theta+1}{2}\big)/\ln M\big)$, where $\alpha(x_0) = \lim_{k\to\infty} \inf_{l \geq k} \left(\sum_{i=1}^{l} \epsilon_i\right)/l$.

Proof. At first we prove the first part. In particular we will prove that $\phi_M(x_0 + y)$ is a linear function of y when y is sufficient small. By Proposition 2 and 3, we can assume that $l = 0, \epsilon_1 \in \{0, 1\}$ without loss of generality. Let $i_0 \geq 2$ be the minimal index such that $\epsilon_{i_0} \geq 2$. Rewrite x_0 as

$$x_0 = \sum_{i=1}^{i_0-1} \frac{\epsilon_i}{M^i} + \frac{x'}{M^{i_0-1}}$$

with $\frac{1}{M-1} < x' < 1$. By Proposition 4, we deduce that $\phi_M(x_0+y) - \alpha(\epsilon_1, \cdots, \epsilon_{i_0-1})\phi_M(x' + M^{i_0-1}y)$ is a linear function of y when $0 < x' + M^{i_0-1}y < 1$. Let y be chosen small enough that

$$\frac{1}{M-1} < \frac{2}{M} + M^{i_0-1}y < x' + M^{i_0-1}y < 1.$$

Then $\phi_M(x_0 + y)$ is a linear function of y when $\frac{1}{M-1} < x' + M^{i_0-1}y < 1$ since ϕ_M in a linear function on $(\frac{1}{M-1}, 1)$. This proves that ϕ_M is differentiable at x_0.

Now we prove the second part. Let $y \in (M^{-k-1}, M^{-k})$ with $k \geq 2$. Then by Proposition 2, we get

$$\begin{aligned}
&|\phi_M(x_0 + y) - \phi_M(x_0)| \\
\leq &|\alpha(\epsilon_1, \cdots, \epsilon_{k-1})| \Big|\phi_M(\sum_{i=k}^{\infty} \frac{\epsilon_i}{M^{i-k+1}} + M^{k-1}y) - \phi_M(\sum_{i=k}^{\infty} \frac{\epsilon_i}{M^{i-k+1}})\Big|, \\
&+ |\beta(\epsilon_1, \cdots, \epsilon_{k-1})| M^{k-1}|y| \\
\leq &C\big(|\alpha(\epsilon_1, \cdots, \epsilon_{k-1})| + |\beta(\epsilon_1, \cdots, \epsilon_{k-1})|\big)
\end{aligned}$$

where C is a constant independent of $y \in (M^{-k-1}, M^{-k})$.

By Proposition 4 and the definition of $\alpha(x_0)$ there exists an integer N for any $\epsilon > 0$ such that $\frac{\sum_{i=1}^{k}\epsilon_i}{k} \geq \alpha(x_0) - \epsilon$ for all $k \geq N$. Then there exists a constant C for any $\epsilon > 0$ such that

$$\begin{aligned}&|\alpha(\epsilon_1, \cdots, \epsilon_{k-1})|\\ \leq&|(\frac{1-\theta}{2M})^{\sum_{i=1}^{k-1}\epsilon_i} \times (\frac{1+\theta}{2M})^{k-1-\sum_{i=1}^{k-1}\epsilon_i}|\\ \leq&CM^{-k} \times (\frac{\theta-1}{2})^{k(\alpha(x_0)-\epsilon)} \times (\frac{\theta+1}{2})^{k(1-\alpha(x_0)+\epsilon)},\end{aligned}$$

and

$$\begin{aligned}&|\beta(\epsilon_1, \cdots, \epsilon_{k-1})|\\ \leq&CM^{-k}\sum_{j=N}^{k-1}(\frac{\theta-1}{2})^{j(\alpha(x_0)-\epsilon)} \times (\frac{\theta+1}{2})^{j(1-\alpha(x_0)+\epsilon)} + CM^{-k}.\end{aligned}$$

Then the right local Hölder index of ϕ_M is at least

$$1 - \max\big(0, \big(\alpha(x_0)\ln\frac{\theta-1}{2} + (1-\alpha(x_0))\ln\frac{\theta+1}{2}\big)/\ln M\big).$$

To estimate the left local Hölder index of ϕ_M, we suffice to estimate the right local Hölder index of $\tilde{\phi}_M$, where $\tilde{\phi}_M$ is defined as in the proof of Theorem 3. Observe that

$$2 + \frac{1}{M-1} - l - \sum_{i=1}^{\infty}\frac{\epsilon_i}{M^i} = (2-l) + \sum_{i=1}^{\infty}\frac{1-\epsilon_i}{M^i}$$

when $\epsilon_i \in \{0,1\}$. Therefore by the same procedure as the one to estimate the right local Hölder index of ϕ_M, we get that the right local Hölder index of $\tilde{\phi}_M$ at $2 + \frac{1}{M-1} - x_0$ is at least $1 - \max\big(0, \big(\alpha(x_0)\ln\frac{\theta-1}{2} + (1-\alpha(x_0))\ln\frac{\theta+1}{2}\big)/\ln M\big)$, when $x_0 = \sum_{i=1}^{\infty}\frac{\epsilon_i}{M^i}$ and $\epsilon_i \in \{0,1\}$ for all $i \geq 1$. ■

The *global Hölder index*, or simply *Hölder index*, of a continuous function ϕ is defined as the supremum of α such that

$$|\phi(x+y) - \phi(x)| \leq C|y|^{\alpha}$$

holds for all x, y and a constant C independent of x and y.

Theorem 5. *Let $M \geq 3$. Then the Hölder index of ϕ_M is $1 - \frac{\ln\frac{1+\theta}{2}}{\ln M}$.*

Proof. By Theorem 2, Proposition 2-3, it suffices to prove that

$$|\phi_M(x+y) - \phi_M(x)| \leq C|y|^{1-\ln\frac{1+\theta}{2}/\ln M}$$

holds for all $0 < x < 2/M$ and $0 < y \leq 1/M^3$. Let k be the integer such that $y \in (M^{-k-1}, M^{-k})$ and let $\epsilon_i \in \{0, 1, \cdots, M-1\}, 1 \leq i \leq k-1$, be chosen that

$$\sum_{i=1}^{k-1} \frac{\epsilon_i}{M^i} \leq x < \sum_{i=1}^{k-1} \frac{\epsilon_i}{M^i} + \frac{1}{M^{k-1}}.$$

Then by Proposition 4 and (6) we get

$$|\phi_M(x) - \phi_M(\sum_{i=1}^{k-1} \frac{\epsilon_i}{M^i})| \leq CM^{-k+s-1}(|\alpha(\epsilon_1, \cdots, \epsilon_{s-1})| + |\beta(\epsilon_1, \cdots, \epsilon_{s-1})|) \leq 2C\big(\frac{1+\theta}{2M}\big)^k, \tag{8}$$

where s is defined as the minimal index such that $\epsilon_s \geq 2$ if such an s exists, otherwise as k, and C is a constant independent of x.

Let $\epsilon'_i \in \{0, 1, \cdots, M-1\}, 1 \leq i \leq k-1$, be chosen that

$$\sum_{i=1}^{k-1} \frac{\epsilon'_i}{M^i} \leq x + y < \sum_{i=1}^{k-1} \frac{\epsilon'_i}{M^i} + \frac{1}{M^{k-1}}.$$

Similarly there exists a constant C independent of x and y such that

$$|\phi_M(x+y) - \phi_M(\sum_{i=1}^{k-1} \frac{\epsilon'_i}{M^i})| \leq C(\frac{1+\theta}{2M})^k. \tag{9}$$

If $\epsilon_i = \epsilon'_i$ holds for all $1 \leq i \leq k-1$, then the result follows from the inequality

$$|\phi_M(x+y) - \phi_M(x)| \leq |\phi_M(x+y) - \phi_M(\sum_{i=1}^{k-1} \frac{\epsilon'_i}{M^i})| + |\phi_M(x) - \phi_M(\sum_{i=1}^{k-1} \frac{\epsilon_i}{M^i})|.$$

Then the matter reduces to the case $\epsilon_i \neq \epsilon'_i$ for some $1 \leq i \leq k-1$. In this case, we have

$$\sum_{i=1}^{k-1} \frac{\epsilon_i}{M^i} + \frac{1}{M^{k-1}} = \sum_{i=1}^{k-1} \frac{\epsilon'_i}{M^i}.$$

When $\epsilon_1 \geq 2$, we have

$$|\phi_M(\sum_{i=1}^{k-1} \frac{\epsilon'_i}{M^i}) - \phi_M(\sum_{i=1}^{k-1} \frac{\epsilon_i}{M^i})| \leq M^{-k-1}.$$

Thus it suffices to prove the result when $\epsilon_1 = 0$ or 1. By Proposition 4, we get

$$\begin{aligned} &|\phi_M(\sum_{i=1}^{k-1} \frac{\epsilon'_i}{M^i}) - \phi_M(\sum_{i=1}^{k-1} \frac{\epsilon_i}{M^i})| \\ \leq &CM^{-k+i_0-1}\big(|\alpha(\epsilon_1, \cdots, \epsilon_{i_0-1})| + |\beta(\epsilon_1, \cdots, \epsilon_{i_0-1})|\big) \leq C\big(\frac{1+\theta}{2M}\big)^k, \end{aligned} \tag{10}$$

where i_0 is the minimal index such that $\epsilon_{i_0-1} \geq 2$ if it exists, otherwise $k-1$. Then Theorem 5 follows from (8)-(10). ■

6. Linear Independence

We say that a nonzero compactly supported distribution ϕ is *locally linearly independent* if for every open set O the statement $\sum_{k\in Z} d_k\phi(x-k)=0$ holds on the open set O implies $d_k=0$ for all indices k which satisfy that $\phi(\cdot-k)$ is not indentically zero on O. Otherwise we say that ϕ is *locally linearly dependent.* If the statement $\sum_{k\in Z} d_k\phi(x-k)=0$ holds for all real x implies $d_k=0$ for all $k\in Z$, then we say that ϕ is *globally linearly independent.*

In [DHS], we constructed examples of solutions of the refinement equations (1) when $M=3$, which are globally linearly independent, but locally linearly dependent. To our surprise, we will show that ϕ_M is locally linearly dependent when $M\geq 3$. When $M=2$, it is proved that local and global linear independence of compactly supported solutions of refinement equations are equivalent to each other.

Theorem 6. *Let $M\geq 3$. Then the scaling function ϕ_M is globally linearly independent but locally linearly dependent.*

Proof. The global linear independence of ϕ_M follows easily form the orthonomality of $\{\phi_M(\cdot-k); k\in Z\}$, i.e., $\int_R \phi_M(x)\phi_M(x-k)dx=\delta_k$.

To prove local linear dependence of ϕ_M, it suffices to find an open set $O\subset(0,1)$ nonzero numbers a,b and c such that $\phi_M(x+j)$ is not identically zero on O when $j=0,1,2$, and

$$a\phi(x)+b\phi(x+1)+c\phi(x+2)=0$$

holds on the open set O. Let $O=(\frac{1}{M(M-1)},\frac{1}{M}), a=(1-\theta)M/2+(M^2-1)/6\neq 0, b=(M^2-1)/6\neq 0, c=(1+\theta)M/2+(M^2-1)/6\neq 0$. By the refinement equation (1) and the fact that ϕ_M is supported on $[0,2+\frac{1}{M-1}]$, we get

$$\begin{cases} \phi(x)=\frac{1+\theta}{2M}\phi_M(Mx), \\ \phi(x+1)=\frac{2M-1+\theta}{2M}\phi_M(Mx+1)+\frac{2M-1-\theta}{2M}\phi_M(Mx), \\ \phi(x+2)=\frac{1-\theta}{2M}\phi_M(Mx+1), \end{cases}$$

when $x\in(\frac{1}{M(M-1)},\frac{1}{M})$. Hence

$$a\phi_M(x)+b\phi_M(x+1)+c\phi_M(x+2)=0, x\in O.$$

Now the matter reduces to prove that $\phi_M(x+j)$ is not indentically zero on O when $j=0,1,2$. Observe that $\phi_M(x+j), j=0,1,2$, is a linear function on O with slope $(1+\theta)/2, -\theta, (\theta-1)/2$ respectively. Hence $\phi(x+j)$ is not indentically zero on O for all $j=0,1,2$. ■

7. Interpolation

For an L^2 closed subspace G of continuous functions on R and a point $x_0 \in [0,1)$, we say that interpolation problem at $x_0 + Z$ is well-posed if there exists a constant C such that

$$C^{-1}\|f\|_2 \le \big(\sum_{k\in Z} |f(x_0+k)|^2\big)^{1/2} \le C\|f\|_2$$

holds for all $f \in G$. We say that a continuous function ϕ is *interpolatable* at $x_0 + Z$ if the interpolation problem at $x_0 + Z$ is well-posed for the L^2 closed space spanned by $\{\phi(\cdot - k), k \in Z\}$. It is proved that a compactly supported and continuous scaling function ϕ is interpolatable at $x_0 + Z$ if and only if there does not exist $\xi \in R$ such that

$$\sum_{k\in Z} \phi(x_0+k)e^{ik\xi} = 0. \tag{11}$$

(see [W] for example)

For M-band scaling function ϕ_M, we have

Theorem 7. *Let $M \ge 2$. Then ϕ_M is interpolatable at $x_0 + Z$ if and only if $\phi_M(x_0 + 1) \ne \frac{1}{2}$, where $x_0 \in [0,1)$. Moreover there exists $x_0 \in [0,1)$ such that ϕ_M is not interpolatable at $x_0 + Z$ and such x_0 can be chosen as $\frac{3}{2} - \frac{\theta-1}{2M-2}$ when $M \ge 11$.*

Proof. Let $x_0 \in [0,1)$ be any point such that ϕ_M is not interpolatable at $x_0 + Z$. Then by (11) there exists $\xi \in R$ such that

$$e^{-i\xi}\phi(x_0) + \phi(x_0+1) + e^{i\xi}\phi(x_0+2) = 0. \tag{12}$$

By Proposition 1, we write (12) as

$$\begin{cases} (1-\cos\xi)\phi(x_0+1) + \cos\xi = 0, \\ (\phi(x_0) - \phi(x_0+2))\sin\xi = 0. \end{cases}$$

It is easy to see that $\phi(x_0+1) = \frac{1}{2}$ when $\sin\xi = 0$. Therefore the proof of the first part reduces to prove that $\sin\xi = 0$. Contradictly we assume $\sin\xi \ne 0$. Then $\phi(x_0) = \phi(x_0+2)$. By Proposition 1, we get

$$\begin{cases} 2\phi(x_0) + \phi(x_0+1) = 1 \\ \frac{1-\theta}{2(M-1)}(2\phi(x_0) + \phi(x_0+1)) = x_0 \end{cases}$$

and hence

$$x_0 = \frac{1-\theta}{2M-2},$$

which is a contradiction since $x_0 \in [0,1)$ and $\frac{1-\theta}{2M-2} < 0$. This proves $\sin\xi = 0$ and the first part of Theorem 7.

By the proof of Proposition 1, we get

$$\phi_M(1) = \frac{2M-3+\theta}{2M-2} > 1$$

and

$$\phi_M(2) = \frac{1-\theta}{2M-2} < 0.$$

Recall that ϕ_M is continuous on $[1,2]$. Hence there exists $x_0 \in [0,1)$ such that $\phi_M(x_0 + 1) = 1/2$ by mean-valued theorem for continuous function. This proves that ϕ_M is not interpolatable at $x_0 + Z$ by the first part.

By the second formula in Proposition 2, ϕ_M is linear function with slope -1 on $[1 + \frac{1}{M-1}, 2]$. Observe that

$$\frac{3}{2} - \frac{\theta-1}{2M-2} \geq 1 + \frac{1}{M-1}$$

when $M \geq 11$. Hence we get

$$\phi_M(\frac{3}{2} - \frac{\theta-1}{2M-2}) = \phi_M(2) + 2 - \frac{3}{2} + \frac{\theta-1}{2M-2} = \frac{1}{2}$$

and ϕ_M is not interpolatable at $\frac{3}{2} - \frac{\theta-1}{2M-2} + Z$. ■

Observe that

$$\frac{3}{2} - \frac{\theta-1}{2M-2} = \frac{11}{10}$$

when $M = 11$. By Theorem 7, we get

Corollary. *Let*

$$\tilde{\phi}_{11}(x) = \phi_{11}(2 + \frac{1}{10} - x).$$

Then the scaling function $\tilde{\phi}_{11}$ is a scaling function with its filter

$$\left(\frac{1-z^{11}}{11(1-z)}\right)^2 (-4+5z)$$

and is not interpolatable at Z.

The existence of x_0 in Theorem 7 for which $\sum_{k\in Z} \phi_M(x_0+k)e^{-ik\xi} = 0$ for some $\xi \in R$ can be extended to general compactly supported continuous function.

Theorem 8. *Let ϕ be a compactly supported continuous function. Then there exist $x_0 \in [0,1)$ and $\xi \in R$ such that*

$$\sum_{k\in Z} \phi(x_0+k)e^{-ik\xi} = 0.$$

If ϕ is further assumed that $\{\phi(\cdot - k); k \in Z\}$ is a Riesz basis of the space V_0 spanned by $\{\phi(\cdot - k); k \in Z\}$, then there exists $x_0 \in [0,1)$ such that the interpolation problem at $x_0 + Z$ is not well-posed for V_0.

Proof. Without loss of generality, we assume that ϕ is supported in $[0,N]$ for some integer N. Define

$$G(x,z) = \sum_{k\in Z} \phi(x+k)z^{-k}.$$

Then $G(x,z)$ is a polynomial of z for all $x \in [0,1]$. Contradictly if

$$\sum_{k\in Z} \phi(x_0+k)e^{-ik\xi} \neq 0$$

hold for all $x \in [0,1]$ and $\xi \in R$. Then $G(x,z) \neq 0$ when $|z| = 1$. Denote the cardinality of zero of $G(x,z)$ including its multiplicity by $k(x)$. Then $k(x)$ is a integer and a continuous function of x by Rouché theorem. Therefore we get $k(x) = k(0)$ for all $x \in [0,1]$. On the other hand we have $G(1,z) = zG(0,z)$ and $k(1) = k(0)+1$, which contradicts to $k(1) = k(0)$. This proves the first part.

The second part follows from (11).■

8. Wavelets

Let $V_j, j \in Z$, be the closed subspace of L^2 spanned by $\{\phi_M(M^j \cdot -k); k \in Z\}$. Then $\{V_j\}$ is a multiresolution of L^2 (see [BDS]). Denote the orthogonal complement space of V_j in V_{j+1} by $W_j, j \in Z$.

Denote $e_0 = (1,1,\cdots,1)$ and $e_1 = (0,1,2,\cdots,M-1)$.

Theorem 9. *Let U be the complement space of the one spanned by e_0 and e_1 in R^M, and let $e_i = (e_{i0},\cdots,e_{i(M-1)}), 2 \le i \le M-1$ be orthonormal basis of U. Define*

$$\begin{cases} \psi_{1,M}(x) = \sqrt{\frac{M-\theta}{M+\theta}} \sum_{j=0}^{M-1} \frac{1+2j+\theta}{2M}\phi(Mx-j) \\ \qquad\qquad -\sqrt{\frac{M+\theta}{M-\theta}} \sum_{j=0}^{M-1} \frac{2M-1-2j-\theta}{2M}\phi(Mx-j-M), \\ \psi_{i,M}(x) = M^{1/2}\sum_{j=0}^{M-1} e_{ij}\phi(Mx-j), 2 \le i \le M-1. \end{cases}$$

Then $\{M^{j/2}\psi_{i,M}(M^jx-k); 1 \le i \le M-1, k \in Z\}$ is an orthonormal basis of W_j for all $j \in Z$, and $\{M^{j/2}\psi_{i,M}(M^jx-k); 1 \le i \le M-1, j,k \in Z\}$ is an orthonormal basis of the space L^2 of square integrable functions.

Proof. Write $H(z) = \frac{1}{M}\sum_{j=0}^{M-1} z^j H_j(z^M)$. Then we get

$$(H_0(z), \cdots, H_{M-1}(z)) = (\frac{1+\theta}{2M}e_0 + \frac{1}{M}e_1) + (\frac{2M-1-\theta}{2M}e_0 - \frac{1}{M}e_1)z.$$

Define

$$U(z) = A + Bz,$$

where

$$A = \begin{pmatrix} \frac{1+\theta}{2M}e_0 + \frac{1}{M}e_1 \\ \sqrt{\frac{M-\theta}{M+\theta}}(\frac{1+\theta}{2M}e_0 + \frac{1}{M}e_1) \\ e_2 \\ \vdots \\ e_{M-1} \end{pmatrix}$$

and

$$B = \begin{pmatrix} \frac{2M-1-\theta}{2M}e_0 - \frac{1}{M}e_1 \\ -\sqrt{\frac{M+\theta}{M-\theta}}(\frac{2M-1-\theta}{2M}e_0 - \frac{1}{M}e_1) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then the first row of $U(z)$ is $(H_0(z), \cdots, H_{M-1}(z))$. Furthermore it is easy to check that $U(z^{-1})U(z)^T = I$, where $U(z)^T$ denotes the transpose of $U(z)$ and I is the unit matrix. Then Theorem 9 follows from the fact that $\{\phi_M(\cdot - k); k \in Z\}$ is an orthonormal basis of V_0 by complicated but usual computation. ■

References

[A] P. Auscher, Ondelette et espaces de Hardy, Thesis, Paris, 1989.

[BDS] N. Bi, X. Dai and Q. Sun, Construction of compactly supported M-band wavelets, Research Report 9518, Center for Mathematical Sciences, Zhejiang University, 1995.

[DHS] X. Dai, D. Huang and Q. Sun, Some properties of five-coefficient refinement equation, *Arch. Math.*, 66(1996), 299-309.

[D] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics 61, Philadephia, 1992.

[DL] I. Daubechies and J. L. Lagarais, Two-scale difference equation:II. Local regularity, infinity products of matrices and fractrals, *SIAM J. Math. Anal.*, 23(1992), 1031-1079.

[H] P. N. Heller, Rank M wavelets with N vanishing moments, *SIAM J. Matrix Anal. Appl.*, 16(1995), 502-519.

[HW] P. N. Heller and R. O. Wells, Jr., Sobolev regularity for rank m wavelet systems, preprint.

[M] Y. Meyer, *Ondelettes et Opérateurs, I: Ondelettes*, Hermann, Paris, 1990.

[P] D. Pollen, Daubechies' scaling function on [0,3], *Wavelet-A Tutorial in Theory and Application*, C. K. Chui ed., Academic Press, Inc. 3-13, 1992.

[SS] X. Shi and Q. Sun, A class of M-dilation scaling functions with regularity growing proportionally to filter support width, Research Report 9606, Center for Mathematical Sciences, Zhejiang University, 1996.

[SVN] A. K. Soman, P. P. Vaidyanathan and T. Q. Nguyen, Linear phase paraunitary filter banks: theory, factorizations and designs, *IEEE Trans. Signal Processing*, 41(1993), 3480-3496.

[WL] G. V. Welland and M. Lundberg, Construction of compact p-wavelets, *Constr. Approx.*, 9(1993), 347-370.

[W] G. Walter, A sampling theorem for wavelet subspaces, *IEEE Trans. on Information Theory*, 38(1992), 881-884.