# LOCAL DUAL AND POLY-SCALE REFINABILITY

QIYU SUN

Abstract. For a compactly supported function $f$, let $S_n(f), n \geq 0$, be the space of all finite linear combinations of $f(M^n \cdot -k), k \in \mathbf{Z}$. In this paper, we consider the explicit construction of local duals of $f$ and the poly-scale refinability of functions in $S_0(f)$ when $f$ is an $M$-refinable function. We show that for any $M$-refinable function $f$, there exists a local dual of $f$ in $S_N(f)$ for some $N \geq 0$ (Theorem 1.1), and that any function in $S_0(f)$ is poly-scale refinable (Theorem 1.2).

## 1. Introduction

Let $L^2 := L^2(\mathbf{R})$ be the space of all square integrable functions on $\mathbf{R}$ and $\langle \cdot, \cdot \rangle$ be the usual inner product on $L^2$. For compactly supported functions $f$ and $g$ in $L^2$, we say that $g$ is a *local dual* of $f$ if

$$\langle f(\cdot - k),\ g(\cdot - k') \rangle = \delta_{kk'} \quad \forall\ k, k' \in \mathbf{Z}, \tag{1.1}$$

where $\delta$ is the usual Kronecker symbol. Given a compactly supported function $f$, we define the corresponding *semi-convolution* $f*'$ on $\ell$, the space of all sequences on $\mathbf{Z}$, by

$$f*' : \ell \ni \{c(k)\} \longmapsto \sum_{k \in \mathbf{Z}} c(k) f(\cdot - k).$$

We see that for any $k \in \mathbf{Z}$, the $k$th component of a sequence $c$ contributes only to $f *' c$ at the neighborhood $k + \mathrm{supp} f$ of the location $k$, and hence the semi-convolution is locally defined. If there exists a local dual $g$ of $f$, then for any $k \in \mathbf{Z}$, the $k$-th component $c(k)$ of the sequence $c := \{c(k)\}$ can be recovered from the restriction of the semi-convolution $f *' c$ to a neighborhood of the location $k$ in a stable way because

$$c(k) = \langle f *' c, g(\cdot - k) \rangle \quad \forall\ k \in \mathbf{Z}, \tag{1.2}$$

and hence the semi-convolution $f*'$ has a local inverse. From (1.2), we see that the semi-convolution $f*'$ is one-to-one, which is known as *linear independent shifts* of $f$ ([10, 13]). Conversely if $f$ has linear independent shifts, then a local dual $g$ of $f$

*Date*: December 8, 2003.

*Key words and phrases.* local dual, linear independent shifts, refinability, poly-scale refinability.

Partial results of this paper were announced in the 2002 Fall Southeastern Section Meeting of AMS, Orlando, November 9–10, 2002.

can be found ([1, 9, 14]). Thus for any given compactly supported $L^2$ function $f$, the existence of its local dual in $L^2$ is equivalent to its linear independent shifts.

We say that a compactly supported function $f$ is *$M$-refinable*, or *refinable* for short, if

$$f=\sum_{k\in\mathbf{Z}} a(k) f(M\cdot -k) \quad \text{and} \quad \hat f(0)=1, \tag{1.3}$$

where the dilation $M$ is a fixed integer larger than 2, and $\{c(k)\}$ is a finitely supported sequence on $\mathbf{Z}$ and satisfies $\sum_{k\in\mathbf{Z}} a(k)=M$ ([2, 5, 12]). Here the Fourier transform $\hat f$ of an integrable function $f$ is defined by $\hat f(\xi)=\int_{\mathbf{R}} f(x)e^{-ix\xi}dx$. The sequence $\{a(k)\}$ and the function $H(\xi)=\frac{1}{M}\sum_{k\in\mathbf{Z}} a(k)e^{-ik\xi}$ are called the *mask* and the *symbol* of the refinable function $f$ respectively. The first topic considered in this paper is the explicit construction of a local dual of a given refinable function. Our motivation is based on the following three easy observations: the first observation is that if $f$ has orthonormal shifts, that is, (1.1) holds for $g$ being replaced by $f$, then $f$ itself is the local dual of $f$, and hence there is a local dual in the *shift-invariant space* $S_0(f)$ generated by $f$,

$$S_0(f)=\{f *' c:\ c\in \ell_0\},$$

where $\ell_0$ is the space of all finitely supported sequences on $\mathbf{Z}$; the second observation is that if $f$ is $M$-refinable, then the spaces

$$S_n(f)=\{h(M^n\cdot):\ h\in S_0(f)\},\ n\ge 0,$$

satisfy the following nestedness condition,

$$S_0(f)\subset S_1(f)\subset\cdots\subset S_n(f)\subset\cdots\to L^2;$$

and the third observation is that for the hat function $h(x)=\max(1-|x|,0)$, the function $-\frac{1}{2}(h(2x-1)+h(2x+1))+3h(2x)$ in $S_1(h)$ is a local dual of $h$, where we set $M=2$. The above three observations inspire us to consider the problem whether for any refinable function $f$ we can find its local dual in $S_N(f)$ for some $N\ge 0$. An affirmative answer to the above problem for $M=2$ follows from the intertwining multiresolution analysis in [8]. In this paper, we give an affirmative answer to the above problem for any dilation $M$, and prove the following result where the local dual can be chosen from the dilation of the shifts of another refinable function.

**Theorem 1.1.** *Let $f$ and $g$ be compactly supported refinable functions in $L^2$. If $f$ and $g$ have linear independent shifts, then there is $h\in S_N(g)$ for some $N\ge 0$ so that $h$ is a local dual of $f$.*

From Theorem 1.1, we see that for any refinable function $f\in L^2$ with linear independent shifts, a spline of order $L$ with knots on $\mathbf{Z}/M^N$ can be a local dual of $f$ when $N$ is sufficiently large. For the case that both $f$ and $g$ are B-splines, a function in $S_1(g)$ can be chosen to be a local dual of $f$ from the proof of Theorem

1.1. For instance, $3\chi_{[0,1/2]} - \chi_{[1/2,1)}$ is a local dual of the hat function $h$, where we set $M = 2$ and we denote the characteristic function on a measurable set $E$ by $\chi_E$.

Given any compactly supported refinable function $f \in L^2$ with linear independent shifts, the functions in $S_N(f)$ have been used in the construction of orthonormal wavelets ([2, 5, 12]), tight affine frames ([3, 4, 6]), and also the local dual (Theorem 1.1). The second topic of this paper is the refinability of functions in $S_0(f)$. Given a refinable function $f$, a function $g \in S_0(f)$ is not refinable in general. In particular, one may easily verify that $g = \sum_{k\in\mathbf{Z}} d(k) f(\cdot - k) \in S_0(f)$ is refinable if and only if $D(z^M)H(z)/D(z)$ is a Laurent polynomial, where $D(z) = \sum_{k\in\mathbf{Z}} d(k)z^k$ and $H(z) = \sum_{k\in\mathbf{Z}} a(k)z^k$ with $\{a(k)\}$ being the mask of the refinable function $f$. Recently, poly-scale refinability, a weak concept of refinability, was introduced by Dekel and Dyn ([7]). For a compactly supported function $f$, we say that $f$ is *poly-scale refinable* if

$$f = \sum_{n=1}^{N} \sum_{k\in\mathbf{Z}} c_n(k) f(M^n \cdot - k) \tag{1.4}$$

for some $N \geq 1$, where $\{c_n(k)\} \in \ell_0, 1 \leq n \leq N$. Clearly, a refinable function is poly-scale refinable, and a compactly supported function $f$ is poly-scale refinable if and only if

$$S_0(f) \subset S_1(f) + \cdots + S_N(f)$$

for some $N \geq 1$. In this paper, we show that given a refinable function $f$ having linear independent shifts, any function $g \in S_0(f)$ is poly-scale refinable.

**Theorem 1.2.** *Let $f$ be a compactly supported refinable function. Then any function in $S_0(f)$ is poly-scale refinable.*

Given a compactly supported refinable function $f$ and a function $g \in S_0(f)$, we see from Theorem 1.2 that $S_0(g) \subset S_1(g) + S_2(g) + \cdots + S_N(g)$ for some $N \geq 1$. Also we notice that $S_0(g) \subset S_0(f)$. The above two observations inspire us to consider the following problem whether

$$S_0(f) \subset S_1(g) + \cdots + S_N(g) \tag{1.5}$$

holds for some $N \geq 1$.

**Theorem 1.3.** *Let $f$ be a compactly supported refinable distribution having linear independent shifts, and let $g = \sum_{k\in\mathbf{Z}} d(k) f(\cdot - k) \in S_0(f)$ for some nonzero sequence $\{d(k)\} \in \ell_0$. Assume that there does not exist $\xi_0 \in \mathbf{R}$ so that $d(M^n\xi_0) = 0$ for all $n \geq 0$, where we set $d(\xi) := \sum_{k\in\mathbf{Z}} d(k)e^{-ik\xi}$. Then (1.5) holds for some positive integer $N$.*

The condition on $d$ in Theorem 1.3 cannot be dropped out in general. For instance, if $g = f - f(\cdot - 1)$, then any function $h$ in $S_N(g), N \geq 1$, satisfies $\int_{\mathbf{R}} h(x)dx = 0$, which implies that (1.5) does not hold for any $N \geq 1$.

## 2. Local Dual

In this section, we give a constructive proof to the following slight generalization of Theorem 1.1.

**Theorem 2.1.** *Let $f$ be a compactly supported refinable function in $L^2$ and have linear independent shifts, and let $g$ be a compactly supported function in $L^2$, have linear independent shifts and satisfy $\int_{\mathbf{R}} g(x)dx \neq 0$. Then there exists a function $h$ in $S_N(g)$ for some $N \geq 0$ so that $h$ is a local dual of $f$.*

We remark that the condition $\int_{\mathbf{R}} g(x)dx \neq 0$ in Theorem 2.1 cannot be dropped out in general. For instance, let $f = \chi_{[0,1)}$ and $g = \chi_{[-1/2,0)} - \chi_{[0,1/2]}$. If we can find $h \in S_N(g), N \geq 0$, so that $h$ is a local dual of $f$, then

$$\langle f(\cdot - k), h\rangle = \delta_k,\ \ k \in \mathbf{Z}. \tag{2.1}$$

Write $h = \sum_{k\in\mathbf{Z}} a_N(k)g(2^N \cdot -k)$ for some $\{a_N(k)\} \in \ell_0$. Then direct calculation yields

$$\langle f(\cdot - k), h\rangle = 2^{-N}a_N(2^N k) - 2^{-N}a_N(2^N(k+1)),\ \ k \in \mathbf{Z}. \tag{2.2}$$

By (2.1) and (2.2), the sequence $\{a_N(k)\}$ is not finitely supported, which is a contradiction.

We say that $\xi_0$ is an *$m$-symmetric root* of a trigonometric polynomial $a$ if $a(\xi_0 + 2k\pi/m) = 0$ for all $k \in \mathbf{Z}$. To prove Theorem 2.1, we need a lemma about $m$-symmetric root of a trigonometric polynomial.

**Lemma 2.2.** *Let $f$ be a compactly supported refinable function having linear independent shifts, and $A$ be a nonzero trigonometric polynomial. Let $H$ be the symbol of $f$ and define*

$$H_n(\xi) = H(M^{n-1}\xi)\cdots H(\xi) \tag{2.3}$$

*for $n \geq 1$. Denote the set of all $M^n$-symmetric roots of $H_nA$ by $K_n$ with multiplicity being not counted, where we say that $a$ and $b$ are two different $M^n$-symmetric roots if $a - b \notin 2M^{-n}\pi\mathbf{Z}$. Then $K_n \subset 2M^{-n}\pi\mathbf{Z}$ for sufficiently large $n$.*

*Proof.* Let $f$ satisfy the refinement equation

$$f = \sum_{k\in\mathbf{Z}} a(k)f(M \cdot -k), \tag{2.4}$$

where $\{a(k)\} \in \ell_0$ satisfies $\sum_{k\in\mathbf{Z}} a(k) = M$. Taking Fourier transform of both sides of the equation (2.4), we get

$$\hat{f}(\xi) = H(\xi/M)\hat{f}(\xi/M), \tag{2.5}$$

and using (2.5) iteratively for $n$ times, we obtain

$$\hat{f}(\xi) = H_n(\xi/M^n)\hat{f}(\xi/M^n). \tag{2.6}$$

Recall that $f$ has linear independent shifts, then for any complex number $\xi_0$ there exists an integer $k := k(\xi_0)$ so that $\hat{f}(\xi_0 + 2k\pi) \neq 0$ ([10, 13]). This together with (2.6) concludes that

$$H_n(\xi) \text{ does not have any } M^n\text{-symmetric root for any } n \geq 1. \tag{2.7}$$

Let $Z(A)$ be the set of all roots of $A(\xi)$ with multiplicity being not counted. By (2.7), we have

$$K_n \subset Z(A) + 2M^{-n}\pi\mathbf{Z}, \tag{2.8}$$

and hence

$$\#K_n < \infty \tag{2.9}$$

for all $n \geq 1$. Here $\#E$ is the cardinality of a finite set $E$.

For any $\xi_0 \in K_n$, let $k_1$ be an integer so chosen that $H(M^{n-1}\xi_0 + 2k_1\pi/M) \neq 0$, and set $\xi_0' = \xi_0 + 2M^{-n}k_1\pi$. Then $\xi_0' \in K_{n-1}$ since $H_n(\xi_0 + 2M^{-n}k\pi)A(\xi_0 + 2M^{-n}k\pi) = 0$ for all $k \in \mathbf{Z}$ and $H_n(\xi) = H(M^{n-1}\xi)H_{n-1}(\xi)$. Also one may verify that if $\xi_0, \xi_0' \in K_n$ are two different $M^n$-symmetric roots of $H_nA$, then $\xi_0 + 2M^{-n}k_1\pi$ and $\xi_0' + 2M^{-n}k_1'\pi$ are two different $M^{n-1}$-symmetric roots of $H_{n-1}A$, where $k_1, k_1' \in \mathbf{Z}$. Therefore we conclude that

$$\#K_n \leq \#K_{n-1} \quad \forall\, n \geq 1. \tag{2.10}$$

Moreover, we see that $\#K_n = \#K_{n-1}$ if and only if

$$K_{n-1} = \{\xi_0 + 2M^{-n}k(\xi_0)\pi :\ \xi_0 \in K_n\}, \tag{2.11}$$

where $k(\xi_0) \in \mathbf{Z}$ is chosen so that

$$H(M^{n-1}\xi_0 + 2k\pi/M) = 0 \quad \forall\, k \notin k(\xi_0) + M\mathbf{Z}, \tag{2.12}$$

and

$$H(M^{n-1}\xi_0 + 2k\pi/M) \neq 0 \quad \forall\, k \in k(\xi_0) + M\mathbf{Z}. \tag{2.13}$$

By (2.10) and (2.11), there exist an integer $N_0$ and a finite set $K$ such that

$$K_n = K \quad \text{modulo } 2M^{-n}\pi\mathbf{Z} \tag{2.14}$$

for all $n \geq N_0$.

For any $\xi_0 \in K$ and $n \geq N_0$, $H(M^n\xi_0 + 2k\pi/M) = 0$ for $k \in \mathbf{Z}\backslash M\mathbf{Z}$ and $H(M^n\xi_0 + 2k\pi/M) \neq 0$ for $k \in M\mathbf{Z}$ by (2.12) and (2.13). Then for any $1 \leq k \leq M-1$, $\{M^n\xi_0 + 2k\pi/M : 1 \leq k \leq M-1, n \geq N_0\}$ is a finite set modulo $2\pi\mathbf{Z}$, which implies that $M^{n_1}\xi_0 + 2k\pi/M = M^{n_2}\xi_0 + 2k\pi/M \bmod 2\pi\mathbf{Z}$ for some integers $N_0 \leq n_1 < n_2$. Therefore $\eta_0 := M^{n_1}\xi_0$ is rational and we may assume that

$$\eta_0 = 2j_0\pi/(m-1) \tag{2.15}$$

for some $0 \leq j_0 \leq m-2$, where $m = M^{n_2 - n_1}$, and

$$H(M^n\eta_0 + 2k\pi/M) = 0 \quad \forall\, k \in \mathbf{Z}\backslash M\mathbf{Z} \quad \text{and} \quad n \geq 0. \tag{2.16}$$

From (2.16), it follows that

(2.17)
$$H_{n_2-n_1}(\eta_0+2k\pi/m)=H(M^{n_2-n_1-1}(\eta_0+2k\pi/m))\cdots H(\eta_0+2k\pi/m)=0$$

for all $k\in\mathbf{Z}\backslash m\mathbf{Z}$, which implies that for any $k\in\mathbf{Z}$ with $k=j_0+\cdots+m^{l-1}j_0+m^lk'$ for some $l\ge 0$ and $k'-j_0\in m\mathbf{Z}$, we have

$$\begin{aligned}\hat f(\eta_0+2k\pi) &= \hat f(m\eta_0+2(k-j_0)\pi)\\ &= H_{n_2-n_1}(\eta_0+2(k-j_0)\pi/m)\hat f(\xi_0+2(k-j_0)\pi/m)\\ &= H_{n_2-n_1}(\eta_0)\hat f(\xi_0+2(k-j_0)\pi/m)\\ &= \cdots = H_{n_2-n_1}(\eta_0)^l\hat f(\xi_0+2k'\pi)\\ &= H_{n_2-n_1}(\eta_0)^lH_{n_2-n_1}(\eta_0+2(k'-j_0)\pi/m)\hat f(\eta_0+2(k'-j_0)\pi/m)\\ &= 0. \end{aligned} \tag{2.18}$$

On the other hand, we see that if $0<j_0\le m-2$, then the union of the sets $U_l(\eta_0):=j_0+\cdots+m^lj_0+m^l\mathbf{Z}\backslash m^{l+1}\mathbf{Z}$, $l\ge 0$, is the whole integer set $\mathbf{Z}$, because $\mathbf{Z}\backslash(\cup_{l=0}^LU_l)=j_0+\cdots+m^Lj_0+m^{L+1}\mathbf{Z}$, and $\min\{|k|:\ k\in\mathbf{Z}\backslash(\cup_{l=1}^LU_l)\}\ge\min(m^{L+1}-(j_0+\cdots+m^lj_0),(j_0+\cdots+m^Lj_0))\to\infty$ as $L\to\infty$. Therefore $j_0=0$ by (2.18) and the linear independent assumption on $f$. This proves that for any $\xi_0\in K$ there exists an integer $n$ so that $M^n\xi_0\in 2\pi\mathbf{Z}$. Hence $K\subset 2M^{-N_1}\pi\mathbf{Z}$ for some sufficiently large integer $N_1$. This together with (2.14) proves that $K_n\subset 2M^{-n}\mathbf{Z}$ for all $n\ge\max(N_0,N_1)$ and hence the result follows. □

Now we reach the stage of the proof of Theorem 2.1.

*Proof of Theorem 2.1.* Define the *correlation* of two compactly supported $L^2$ functions $f$ and $g$ by

$$A_{f,g}(\xi):=\sum_{k\in\mathbf{Z}}\langle f,g(\cdot-k)\rangle e^{-ik\xi}. \tag{2.19}$$

Recall that $f$ is a refinable function in $L^2$, which implies the unit partition property $\sum_{k\in\mathbf{Z}}f(\cdot-k)\equiv 1$. Thus

$$A_{f,g}(0)=\int_{\mathbf{R}}g(x)dx\ne 0. \tag{2.20}$$

Denote the set of all $M^n$-symmetric roots of $H_nA_{f,g}$ by $K_n$ with multiplicity being not counted, where as before we say $a$ and $b$ are two different $M^n$-symmetric roots if $a-b\notin 2M^{-n}\pi\mathbf{Z}$. By (2.20) and Lemma 2.2, $K_N\subset 2M^{-N}\pi\mathbf{Z}$ for some integer $N$. Thus $K_N$ is an empty set, since otherwise $0\in K_N$, and then $A_{f,g}(0)=H_N(0)A_{f,g}(0)=0$, which contradicts (2.20). From the definition of $K_N$ and the assertion $K_N=\emptyset$, it follows that $H_N(\xi)A_{f,g}(\xi)$ does not have any $M^N$-symmetric root. By Bezout identity, there exists a trigonometric polynomial $B(\xi)$ so that

$$\sum_{k=0}^{M^N-1} H_N\Big(\frac{\xi+2k\pi}{M^N}\Big)A_{f,g}\Big(\frac{\xi+2k\pi}{M^N}\Big)\overline{B\Big(\frac{\xi+2k\pi}{M^N}\Big)} = 1. \tag{2.21}$$

Define $h$ by $\hat{h}(\xi) = B(\xi/M^N)\hat{g}(\xi/M^N)$. Obviously $h \in S_N(g)$. By (2.6) and (2.21), we have

$$\begin{aligned} A_{f,h}(\xi) &= \sum_{k\in\mathbf{Z}} \hat{f}(\xi+2k\pi)\overline{\hat{h}(\xi+2k\pi)} \\ &= \sum_{k\in\mathbf{Z}} H_N\Big(\frac{\xi+2k\pi}{M^N}\Big)\overline{B\Big(\frac{\xi+2k\pi}{M^N}\Big)}\hat{f}\Big(\frac{\xi+2k\pi}{M^N}\Big)\overline{\hat{g}\Big(\frac{\xi+2k\pi}{M^N}\Big)} \\ &= \sum_{k=0}^{M^N-1} H_N\Big(\frac{\xi+2k\pi}{M^N}\Big)A_{f,g}\Big(\frac{\xi+2k\pi}{M^N}\Big)\overline{B\Big(\frac{\xi+2k\pi}{M^N}\Big)} \\ &= 1, \end{aligned}$$

and hence $h$ is a local dual of $f$. □

## 3. Poly-scale Refinability

In this section, we give the proofs of Theorems 1.2 and 1.3. To prove Theorems 1.2 and 1.3, we need a lemma about trigonometric polynomials.

**Lemma 3.1.** *For any given nonzero trigonometric polynomial $d(\xi)$, there exist an integer $N$ and trigonometric polynomials $c_l(\xi), 1 \le l \le N$, so that*

$$d(M^N\xi) = c_1(M^{N-1}\xi)d(M^{N-1}\xi) + c_2(M^{N-2}\xi)d(M^{N-2}\xi) + \cdots + c_N(\xi)d(\xi). \tag{3.1}$$

*Furthermore, if there does not exist $\xi_0 \in \mathbf{C}$ so that $d(M^n\xi_0) = 0$ for all $n \ge 0$, then there exist an integer $N$ and trigonometric polynomials $c_l^*(\xi), 1 \le l \le N$, so that*

$$1 = c_1^*(M^{N-1}\xi)d(M^{N-1}\xi) + c_2^*(M^{N-2}\xi)d(M^{N-2}\xi) + \cdots + c_N^*(\xi)d(\xi). \tag{3.2}$$

*Proof.* We let $d_0(\xi) = d(\xi)$, and inductively for $n \ge 1$, we let $d_n(\xi)$ be the trigonometric polynomial with minimal degree so that any common factor between $d(\xi)$ and $d_{n-1}(\xi)$ is a factor of $d_n(M\xi)$. Here the degree of a trigonometric polynomial $p(\xi) := \sum_{k=k_1}^{k_2} p(k)e^{-ik\xi}$ with $p(k_2)p(k_1) \ne 0$ is defined by $\deg p = k_2 - k_1$. Note that $d_n(\xi)$ is a factor of $d_{n-1}(\xi/M)\cdots d_{n-1}(\xi/M + 2(M-1)\pi/M)$ since $d_{n-1}(\xi)$ is a factor of the trigonometric polynomial $d_{n-1}(\xi)\cdots d_{n-1}(\xi+2(M-1)\pi/M)$. Then the degree of $d_n$ is less than the one of $d_{n-1}$, which implies that there exists an integer $N$ so that the degree of $d_n$ is the same as the one of $d_N$ for all $n \ge N$. Recall that $d_{n+1}(M\xi)$ is a factor of $d_n(\xi)\cdots d_n(\xi+2(M-1)\pi/M)$. Then comparing the degrees of the above two trigonometric polynomials, we get

$$d_{n+1}(M\xi) = c_n(\xi)d_n(\xi)\cdots d_n(\xi+2(M-1)\pi/M), \tag{3.3}$$

where $c_n(\xi), n \geq N$, are nonzero monomials. This together with the construction of $d_{n+1}$ implies that

$$d_n(\xi) \text{ is a factor of } d(\xi) \text{ for all } n \geq N. \tag{3.4}$$

Therefore there exists a trigonometric polynomial $c_0$ so that

$$d(\xi) = c_0(\xi) d_N(\xi). \tag{3.5}$$

From the construction of $d_n$, there exist trigonometric polynomials $a_n$ and $b_n$ so that

$$d_n(M\xi) = a_n(\xi) d(\xi) + b_n(\xi) d_{n-1}(\xi) \tag{3.6}$$

for all $1 \leq n \leq N$. The assertion (3.1) follows from (3.5) and (3.6).

Now we prove (3.2). Let $d_n, n \geq 0$, and $N$ be as above. By (3.6), it suffices to show that $d_N$ is a monomial when there does not exist $\xi_0 \in \mathbf{C}$ so that $M^n\xi_0, n \geq 0$, are roots of $d(\xi)$. Suppose, on the contrary, that $d_N$ is not a monomial. Then $d_N(\eta_1) = 0$ for some $\eta_1 \in \mathbf{C}$. Iteratively using (3.3), we obtain $d_{n+N}(M^n\eta_1) = 0$ for all $n \geq 0$. This together with (3.4) implies that $d(M^n\eta_1) = 0$ for all $n \geq 0$, which is a contradiction. □

Now we start to prove Theorems 1.2 and 1.3.

*Proof of Theorem 1.2.* Write $g = \sum_{k\in\mathbf{Z}} d(k) f(\cdot - k)$ for some $\{d(k)\} \in \ell_0$, and assume that $f$ satisfies the refinement equation $f = \sum_{k\in\mathbf{Z}} a(k) f(M \cdot -k)$ for some $\{a(k)\} \in \ell_0$. Then setting $H(\xi) = \frac{1}{M}\sum_{k\in\mathbf{Z}} a(k) e^{-ik\xi}$ and $d(\xi) = \sum_{k\in\mathbf{Z}} d(k) e^{-ik\xi}$ yields

$$\hat{f}(M\xi) = H(\xi)\hat{f}(\xi) \quad \text{and} \quad \hat{g}(\xi) = d(\xi)\hat{f}(\xi). \tag{3.7}$$

By Lemma 3.1, there exist a positive integer $N$ and trigonometric polynomials $c_n(\xi), 1 \leq n \leq N$, so that

$$d(M^N\xi) = \sum_{n=1}^{N} c_n(M^{N-n}\xi) d(M^{N-n}\xi). \tag{3.8}$$

Multiplying $\hat{f}(M^N\xi)$ at both sides of (3.8) and using (3.7), we obtain

$$\begin{aligned} \hat{g}(M^N\xi) &= \sum_{n=1}^{N} c_n(M^{N-n}\xi) d(M^{N-n}\xi) \hat{f}(M^N\xi) \\ &= \sum_{n=1}^{N} c_n(M^{N-n}\xi) H(M^{N-1}\xi) \cdots H(M^{N-n}\xi) \hat{g}(M^{N-n}\xi). \end{aligned} \tag{3.9}$$

Hence taking inverse Fourier transform at both sides of (3.9) proves that $g$ is polyscale refinable. □

*Proof of Theorem 1.3.* We use the same argument as in the proof of Theorem 1.2 except the equation (3.8) being replaced by (3.2). We omit the details of the proof here. □

## 4. Generalization, Applications and Problems

In this section, we give few remarks about generalization, applications and open problems related to Theorems 1.1, 1.2 and 1.3.

For a vector-valued compactly supported function $F = (f_1, \dots, f_r)^T$ on $\mathbf{R}^d$, we define the *semi-convolution* $F*'$ by

$$F*' : \ (\ell)^r \ni \{c(k)\} \longmapsto \sum_{k\in\mathbf{Z}^d} c(k)^T F(\cdot - k),$$

where $(\ell)^r$ is the $r$ copies of the space of all sequences on $\mathbf{Z}^d$, and the *shift-invariant spaces* $S_n(F)$ by

$$S_n(F) = \{F *' c(M^n \cdot) : \ c \in (\ell_0)^r\}.$$

We say that $F$ is *$M$-refinable* if

$$S_0(F) \subset S_1(F),$$

and *poly-scale refinable* if

$$S_0(F) \subset S_1(F) + \cdots + S_N(F)$$

for some $N \geq 1$. For two vector-valued compactly supported functions $F = (f_1, \dots, f_r)^T \in L^2$ and $G = (g_1, \dots, g_r)^T \in L^2$, we say that $G$ is a *local dual* of $F$ if $\langle f_i(\cdot - k), g_{i'}(\cdot - k')\rangle = \delta_{ii'}\delta_{kk'}$ for all $1 \leq i, i' \leq r$ and $k, k' \in \mathbf{Z}^d$.

First we consider the generalization of Theorem 1.1 to the vector-valued case.

**Theorem 4.1.** *Let $F = (f_1, \dots, f_r)^T$ and $G = (g_1, \dots, g_r)^T$ be vector-valued compactly supported functions in $L^2$. Assume that $F$ is $M$-refinable and has linear independent shifts, the symbol $H(\xi)$ of the refinable function $F$ satisfies $\det H(\xi) \not\equiv 0$, $G$ satisfies $\int_{\mathbf{R}} G(x)dx \neq 0$, and the correlation matrix $\sum_{k\in\mathbf{Z}} \widehat{F}(\xi + 2k\pi)\overline{\widehat{G}(\xi + 2k\pi)^T}$ is nonsingular for some $\xi_0 \in \mathbf{C}$. Then there exist functions $h_1, \dots, h_r \in S_N(g)$ for some $N \geq 0$ so that $(h_1, \dots, h_r)^T$ is a local dual of $F$.*

As an easy consequence of Theorem 4.1, we see that for a vector-valued refinable function $F \in L^2$ having linear independent shifts, if the corresponding symbol has nonzero determinant, then a local dual to $F$ can be found in $S_N(F)$ for some $N \geq 1$. We do not include the proof of Theorem 4.1 here since the proof is similar to the one of Theorem 2.1, except more delicate analysis about symmetric roots of a matrix-valued trigonometric polynomial.

Next we consider the compactly supported distributional solution of a poly-scale refinement equation. As mentioned in [7], for a poly-scale refinement equation, there is a corresponding vector-valued refinement equation. In particular, let $F = (f_1, \dots, f_r)^T$ satisfy the following *poly-scale refinement equation*,

$$F = \sum_{n=1}^{N} \sum_{k \in \mathbf{Z}} c_n(k) F(M^n \cdot -k), \tag{4.1}$$

where $\{c_n(k)\} \in \ell_0^{r\times r}, 1 \le n \le N$, set

$$H_n(\xi) = M^{-n} \sum_{k\in\mathbf{Z}} c_n(k) e^{-ik\xi},\ 1 \le n \le N,$$

and define

$$\tilde{H}(\xi) = \begin{pmatrix} H_1(M^{N-1}\xi) & H_2(M^{N-2}\xi) & \cdots & H_{N-1}(M\xi) & H_N(\xi) \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & I & 0 \end{pmatrix}.$$

Then the vector-valued function $\tilde{F}$ defined by

$$\widehat{\tilde{F}}(\xi) = \begin{pmatrix} \widehat{F}(M^{N-1}\xi) \\ \vdots \\ \widehat{F}(M\xi) \\ \widehat{F}(\xi) \end{pmatrix}$$

satisfies the refinement equation

$$\widehat{\tilde{F}}(\xi) = H(\xi/M)\widehat{\tilde{F}}(\xi/M).$$

As an application of Theorem 2.1 in [11], we have the following result about the existence of nonzero compactly supported distributional solution of a poly-scale refinement equation.

**Theorem 4.2.** *Let $\{c_l(k)\} \in \ell_0^{r\times r}, 1 \le l \le N$, and set*

$$H(z, w) = \sum_{l=1}^{N} \sum_{k\in\mathbf{Z}} c_l(k) z^k w^l.$$

*Then there exists a nonzero compactly supported distributional solution $F$ of the poly-scale refinement equation*

$$F = \sum_{n=1}^{N} \sum_{k \in \mathbf{Z}} c_n(k) F(M^n \cdot -k), \tag{4.2}$$

*if and only if $H(1, M^{-l}) - I$ is singular for some nonnegative integer $l$.*

Now we consider the possible application of Theorem 1.3. Let $f$ be a compactly supported refinable function on the real line, and let $g \in S_0(f)$. By Theorem 1.3, there exist a positive integer $N$ and sequences $\{c_n(k)\} \in \ell_0, 1 \le n \le N$, so that $f = \sum_{n=1}^N \sum_{k\in\mathbf{Z}} c_n(k) g(M^n \cdot -k)$. Then taking inner product with an $L^2$ function $h$ leads to

$$\langle h, f(\cdot - k)\rangle = \sum_{n=1}^N \sum_{l\in\mathbf{Z}} c_n(l)\langle h, g(M^n \cdot - M^n k - l)\rangle \quad \text{for all } k \in \mathbf{Z},$$

which implies that the local information of the sequence $\{\langle h, f(\cdot - k)\rangle\}$ can be obtained from the local information of $\{\langle h, g(M^n \cdot - k')\rangle\}$ at the finer level $n$ between 1 and $N$. The above formula is useful in the reconstruction of the original function or signal in the shift-invariant space generated by $f$, especially when it is costly for designing a good average sampler $g$ to obtain the data at initial level than for using an easy average sampler $g$ to obtain the data at finer level.

Finally, we propose some problems about local dual and poly-scale refinability:

**Problem 1**. Let $F$ and $G \in L^2$ be vector-valued compactly supported refinable functions on $\mathbf{R}^d, d \ge 2$, and have linear independent shifts. Can we find local dual of $F$ in $S_N(G)$ for some $N \ge 1$.

**Problem 2**. Let $F = (f_1, \dots, f_r)^T$ be a vector-valued compactly supported (poly-scale) refinable functions having linear independent shifts. Characterize all functions $g_1, \dots, g_r \in S_0(F)$ which are poly-scale refinable.

**Problem 3**. Let $F = (f_1, \dots, f_r)^T$ be a vector-valued compactly supported refinable functions having linear independent shifts. Characterize all functions $g_1, \dots, g_r \in S_0(F)$ so that $S_0(F) \subset S_1(G) + \cdots + S_N(G)$ for some $N \ge 1$, where $G = (g_1, \dots, g_r)^T$.

## References


[1] A. Ben-Artzi and A. Ron, On the integer translates of a compactly supported function: dual bases and linear projectors, *SIAM J. Math. Anal.*, 21(1990), 1550–1562.

[2] C. K. Chui, "An Introduction to Wavelets", Academic Press, 1992.

[3] C. K. Chui, W. He and J. Stöckler, Compactly supported tight and sibling frames with maximal vanishing moments, *Appl. Comput. Harmon. Anal.*, 13(2002), 224–262

[4] C. K. Chui, W. He, J. Stöckler and Q. Sun, Compactly supported tight affine frames with integer dilations and maximum vanishing moments, *Adv. Comput. Math.*, 18(2003), 159–187.

[5] I. Daubechies, "Ten Lectures on Wavelets", CBMF Conference Series in Applied Mathematics, 61, SIAM, Philadelphia, 1992.

[6] I. Daubechies, B. Han, A. Ron and Z. Shen, Framelets: MRA-based constructions of wavelet frames, *Appl. Comput. Harmon. Anal.*, 14(2003), 1–46.

[7] S. Dekel and N. Dyn, Poly-scale refinability and subdivision, *Appl. Comput. Harmon. Anal.*, 13(2002), 35–62.

[8] G. C. Donovan, J. S. Geronimo, and D. P. Hardin, Intertwining multiresolution analyses and the construction of piecewise-polynomial wavelets, *SIAM J. Math. Anal.*, 27(1996), 1791–1815.

[9] P. G. Lemarié-Rieusset, On the existence of compactly supported dual wavelets, *Appl. Comput. Harmon. Anal.*, 4(1997), 117–118

[10] R.-Q. Jia and C. A. Micchelli, On linear independence for integer translates of a finite number of functions, *Proc. Edinburgh Math. Soc.*, 36(1993), 69–85.

[11] Q. Sun, Compactly supported distributional solutions of nonstationary nonhomogeneous refinement equations, *Acta Math. Sinica*, 17(2001), 1–14.

[12] Q. Sun, N. Bi and D. Huang, "An Introduction to Multiband Wavelets", Zhejiang University Press, China, 2001.

[13] A. Ron, A necessary and sufficient condition for the linear independence of the integer translates of a compactly supported distribution, *Constr. Approx.*, 5(1989), 297–308.

[14] K. Zhao, Global linear independence and finitely supported dual basis, *SIAM J. Math. Anal.*, 23(1992), 1352–1355.

(Qiyu Sun) Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA

*E-mail address*: qsun@mail.ucf.edu