A Class of M-Dilation Scaling Functions with Regularity Growing Proportionally to Filter Support Width

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September 1995

ABSTRACT In this paper, a class of M-dilation scaling functions with regularity growing proportionally to filter support width are constructed. This answers a question proposed by Daubechies in p.338 of her book *Ten Lectures on Wavelets(1992).*

AMS Subject Classification: 42C15.

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[2]This author is partilally supported by the National Natural Science Foundation of China and the Zhejiang Provincial Science Foundation of China.

1. Introduction

Let $M \geq 2$ be a fixed integer. A multiresolution analysis for dilation M consists of a sequence of closed subspaces V_j of $L^2(R)$ that satisfy the following conditions (see [C], [D], [M]):

i) $V_j \subset V_{j+1}, \forall j \in Z$;

ii) $\overline{\cup_{j \in Z} V_j} = L^2(R)$;

iii) $\cap_{j \in Z} V_j = \{0\}$;

iv) $f \in V_j \iff f(2^{-j} \cdot) \in V_0$;

v) there exists a function ϕ in V_0 such that $\{\phi(\cdot - n); n \in Z\}$ is an orthonormal basis of V_0.

The function ϕ is called an M-dilation scaling function. It is easy to see that ϕ satisfies the refinement equation

$$\phi(x) = \sum_{n \in Z} c_n \phi(Mx - n), \tag{1}$$

where the sequence $\{c_n\}$ satisfies

$$\sum_{n \in Z} c_n = M.$$

In this paper we shall only deal with compactly supported M-dilation scaling functions. In this case the sequence $\{c_n\}$ must have finite length. The function

$$H(\xi) = \frac{1}{M} \sum_{n \in Z} c_n e^{in\xi} \tag{2}$$

is called a symbol corresponding to the refinement equation (1).

The filter support width $W(\phi)$ of an M-dilation scaling function ϕ is defined as the difference of the largest and the smallest indices of the nonzero c_n. The regularity $R(\phi)$ of ϕ is defined as the supremum of α such that $\phi \in C^\alpha$, where C^α denotes the Hölder class of index α.

In her book [D, p.338], Daubechies remarks that:

At present, I know of no explicit scheme that provides an infinite family of m_0 (i.e., symbols H), *for dilation* 3(i.e., $M = 3$), *with regularity growing proportionally to the filter support width.*

To our knowledge, this question is still open. The purpose of this paper is to construct a class of M-dilation scaling functions ϕ_N for which there exists a constant λ_M independent of N such that

$$R(\phi_N) \geq \lambda_M W(\phi_N), \tag{3}$$

where $M \geq 3$. On the other hand it is already known that (see [DL])

$$W(\phi_N) \geq R(\phi_N).$$

These facts give an affirmative answer to Daubechies' question.

The regularity of ϕ has been studied in several papers, see for example [BDS], [D], and [HW]. In general, to study the regularity of ϕ we need to consider the symbol (2) first. By the Fourier transform, we see that all the symbols H satisfy

$$\sum_{l=0}^{M-1} |H(\xi + 2l\pi/M)|^2 = 1. \tag{4}$$

The solutions H of the equation (4) are determined by (see [BDS], [H])

$$|H(\xi)|^2 = \left(\frac{\sin^2(M\xi/2)}{M^2 \sin^2 \xi/2}\right)^N \sum_{s=0}^{N-1} {}_N^M a(s) \sin^{2s}\frac{\xi}{2} + \left(\sin \frac{M\xi}{2}\right)^{2N} R(\xi), \tag{5}$$

where

$${}_N^M a(s) = \sum_{s_1+\cdots+s_{M-1}=s} \prod_{j=1}^{M-1} \binom{N-1+s_j}{s_j} \frac{1}{\sin^{2s_j} j\pi/M}$$

and R is a real-valued trigonometric polynomial such that

i) $\sum_{l=0}^{M-1} R(\xi + 2l\pi/M) = 0$

and

ii) the right hand side of (5) is nonnegative.

By the Riesz Lemma (see [D], p.172), such symbol H exists. Let ${}_N H$ be a solution of (5) with $R = 0$, and let ${}_N\phi$ be the solution of (1) corresponding to the symbol ${}_N H$. In [BDS], Bi, Dai and Sun prove the following estimates on the regularity of ${}_N\phi$,

$$|R({}_N\phi) - \frac{\ln N}{4 \ln M}| \le C,$$

when M is odd, and

$$|R({}_N\phi) - \frac{4N \ln\left(\sin \frac{M\pi}{2M+2}\right)^{-1} + \ln N}{4 \ln M}| \le C,$$

when M is even. For the special cases $M = 3, 4, 5$, similar results are obtained by Heller and Wells in [HW]. This result shows that for these special ${}_N\phi$ the regularity does not grow proportionally to the filter support width when M is odd. To construct M-dilation scaling functions with regularity growing proportionally to the filter support width we use the symbol H_N determined by

$$|H_N(\xi)|^2 = \sum_{k_0+\cdots+k_{M-1}=MN-M+1} \alpha_N(k_0, \cdots, k_{M-1}) \frac{(MN-M+1)!}{k_0! \cdots k_{M-1}!} \times \prod_{l=0}^{M-1} \left(\frac{\sin M\xi/2}{M \sin(\xi/2 + l\pi/M)}\right)^{2k_l},$$

where $N \ge 1$, and $\alpha_N(k_0, \cdots, k_{M-1})$ is defined by

$$\alpha_N(k_0, \cdots, k_{M-1}) = \begin{cases} 0, & \text{if } k_0 \le N-1, \\ \frac{1}{\#(E)}, & \text{if } k_0 \ge N, \end{cases}$$

where $E = \{j : k_j \geq N\}$ and $\#(E)$ is the cadinality of E. Let ϕ_N be the solution of (1) corresponding to a symbol H_N. Then we have the following

Theorem. *Let $M \geq 3$ and $N \geq 2$ be any natural numbers. Then ϕ_N is a M-dilation scaling function and there exists a constant C independnet of N such that*

$$(\frac{1}{2} - \frac{(M-1)\ln(1+\frac{1}{M-1})}{2\ln M})N - \frac{\ln N}{4\ln M} - C \leq$$
$$R(\phi_N) \leq (\frac{1}{2} - \frac{(M-1)\ln(1+\frac{1}{M})}{2\ln M})N - \frac{\ln N}{4\ln M} + C.$$

Remark 1. Observe that $W(\phi_N) \leq 2(M-1)MN$. Therefore the regularity $R(\phi_N)$ of ϕ_N grows proportionally to the filter support width $W(\phi_N)$, i.e., (3) holds.

Remark 2. Let $D(\phi) = R(\phi)/W(\phi)$ be the rate of regularity and filter support width of a scaling function ϕ. Then

$$D(\phi_N) \geq \frac{1}{4M(M-1)}(1 - \frac{(M-1)\ln(1+\frac{1}{M-1})}{\ln M}) - C\frac{\ln N}{N},$$

and

$$D({}_N\phi) \leq \frac{\ln N}{4NM\ln M} + \frac{C}{N}$$

when M is odd, and

$$D({}_N\phi) \leq \frac{\ln\big(\sin M\pi/(2M+2)\big)^{-1}}{M\ln M} + C\frac{\ln N}{N}$$

when M is even. Therefore we get

$$D(\phi_N)/D({}_N\phi) \geq \frac{N}{\ln N}(\frac{\ln M}{M-1} - \ln(1+\frac{1}{M-1})) - C$$

when M is odd, and

$$D(\phi_N)/D({}_N\phi) \geq \frac{\ln M - (M-1)\ln(1+\frac{1}{M-1})}{4(M-1)\ln(\sin M\pi/(2M+2))^{-1}} - C\ln N/N$$

when M is even. This shows that $D(\phi_N)$ of the M-dilation scaling function ϕ_N is larger than the one of ${}_N\phi$ even when M is an even integer larger than 4.

2. Proof of the Theorem

To prove the theorem, we estimate $H_N(\xi)$ first. Let

$$h(\xi) = \frac{\sin^2 M\xi/2}{M^2\sin^2\xi/2}$$

and

$$B_N(\xi) = \big(h(\xi)\big)^{-N} |H_N(\xi)|^2.$$

Then for all real valued ξ we have

$$\begin{aligned} B_N(-\xi) = & \sum_{k_0+\cdots+k_{M-1}=MN-M+1} \alpha_N(k_0, k_{M-1}, \cdots, k_1) \frac{(MN-M+1)!}{k_0!k_1!\cdots k_{M-1}!} \times \\ & \qquad (h(\xi))^{k_0-N} \prod_{l=1}^{M-1} (h(\xi+2l\pi/M))^{k_l} \\ =& B_N(\xi) \end{aligned}$$

and

$$B_N(\xi) \geq 0.$$

Therefore by the Riesz Lemma ([D], p.172) we obtain the existence of $H_N(\xi)$ with

$$H_N(\xi) = \Big(\frac{1-e^{iM\xi}}{M(1-e^{i\xi})}\Big)^N \tilde{H}_N(\xi)$$

and

$$|\tilde{H}_N(\xi)|^2 = B_N(\xi).$$

From the definition of $\alpha_N(k_0, \cdots, k_{M-1})$ and from

$$\sum_{l=0}^{M-1} h(\xi + 2l\pi/M) = 1,$$

we get

$$\alpha_N(k_0, k_1, \cdots, k_{M-1}) + \alpha_N(k_1, k_2, \cdots, k_{M-1}, k_0) + \cdots + \alpha_N(k_{M-1}, k_0, \cdots, k_{M-2}) = 1$$

and

$$\begin{aligned} & \sum_{l=0}^{M-1} |H_N(\xi+2l\pi/M)|^2 \\ =& \sum_{k_0+\cdots+k_{M-1}=MN-M+1} \big(\alpha_N(k_0, k_1, \cdots, k_{M-1}) + \alpha_N(k_1, k_2, \cdots, k_{M-1}, k_0) \\ & \quad + \cdots + \alpha_N(k_{M-1}, k_0, \cdots, k_{M-2})\big) \times \frac{(MN-M+1)!}{k_0!k_1!\cdots k_{M-1}!} \prod_{l=0}^{M-1} \big(h(\xi+2l\pi/M)\big)^{k_l} \\ =& \Big(\sum_{l=0}^{M-1} h(\xi+2l\pi/M)\Big)^{MN-M+1} \\ =& 1. \end{aligned}$$

Therefore (4) holds for $H_N(\xi)$. Recall that $H_N(\xi) \neq 0$ when $|\xi| \leq \pi/M$. Hence the solution ϕ_N of (1) corresponding to the symbol $H_N(\xi)$ is an M-dilation scaling function by an elementary argument ([D], p.182, Theorem 6.3.1 with $K = [-\pi, \pi]$).

To estimate the regularity of ϕ_N, we need some estimates on $B_N(\xi)$. From the Stirling formula, which says that $n!$ is equivalent to $n^n e^{-n}\sqrt{n}$, from $1/(M-1) \le \alpha_N(k_0, \cdots, k_{M-1}) \le 1$ when $k_0 \ge N$ and $h(2\pi/(M-1)) = 1/M^2$, we get

$$\begin{aligned} B_N(\xi) \le & \sum_{k_0+\cdots+k_{M-1}=MN-M+1, k_0\ge N} \frac{(MN-M+1)!}{k_0!\cdots k_{M-1}!}\times \\ & (h(\xi))^{k_0-N}\prod_{l=1}^{M-1}(h(\xi+2l\pi/M))^{k_l} \\ = & \sum_{0\le k_0\le (N-1)(M-1)} \frac{(MN-M+1)!}{(k_0+N)!((N-1)(M-1)-k_0)!}\times \\ & (h(\xi))^{k_0}(1-h(\xi))^{(N-1)(M-1)-k_0} \\ \le & \frac{(MN-M+1)!}{N!((N-1)(M-1))!} \le CM^N(1+1/(M-1))^{(M-1)N}N^{-1/2} \end{aligned}$$

and

$$\begin{aligned} B_N(2\pi/(M-1)) \ge & \frac{1}{M-1}\sum_{k_1+\cdots+k_{M-1}=(M-1)(N-1)} \frac{(MN-M+1)!}{N!k_1!\cdots k_{M-1}!}\times \\ & \prod_{l=1}^{M-1}(h(2\pi/(M-1)+2l\pi/M))^{k_l} \\ \ge & \frac{1}{M-1}\times\frac{(MN-M+1)!}{N!((N-1)(M-1))!}(1-\frac{1}{M^2})^{(N-1)(M-1)} \\ \ge & CM^N(1+1/M)^{(M-1)N}N^{-1/2}. \end{aligned}$$

Therefore we get

$$R(\phi_N) \ge (\frac{1}{2} - \frac{(M-1)\ln(1+1/(M-1))}{2\ln M})N - \frac{\ln N}{4\ln M} - C$$

by an argument as in [D, p.217]. Observe that $2M\pi/(M-1) = 2\pi/(M-1) + 2\pi$. By an argument similar that in p.220 of [D] we obtain

$$R(\phi_N) \le (\frac{1}{2} - \frac{(M-1)\ln(1+1/M)}{2\ln M})N - \frac{\ln N}{4\ln M} - C.$$

This completes the proof of the Theorem.

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