Sobolev Exponent Estimate and Asymptotic Regularity of M Band Daubechies' Scaling Functions

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Abstract

In this paper, direct estimate of Sobolev exponent of refinable distributions and its application to the asymptotic estimate of Sobolev exponent of M band Daubechies' scaling functions are considered.

1 Introduction

Fix integer $M \geq 2$. We say that a tempered distribution ϕ is *refinable* if it satisfies such a refinement equation

$$\phi(x) = \sum_{k\in\mathbb{Z}} c_k \phi(Mx - k), \tag{1}$$

where the coefficients c_k are summable and satisfy $\sum_{k\in\mathbb{Z}} c_k = M$. Define the *symbol* H of the refinement equation (1) by

$$H(\xi) = \frac{1}{M} \sum_{k\in\mathbb{Z}} c_k e^{-ik\xi}. \tag{2}$$

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Generally the symbol H can be put into the factorized form

$$H(\xi) = \Big(\frac{1-e^{-iM\xi}}{M-Me^{-i\xi}}\Big)^N \mathcal{L}(\xi), \tag{3}$$

where $N \geq 1$ and $\mathcal{L}$ is bounded.

Define the Fourier transform of an integrable function f by

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)dx.$$

The Fourier transform of a tempered distribution is understood as usual. For a distribution f with measurable Fourier transform, the Sobolev exponent $s_p(f)$ is defined by

$$s_p(f) = \sup\{\gamma; \int_{\mathbb{R}} |\hat{f}(\xi)|^p(1+|\xi|)^{p\gamma}d\xi < \infty\}, \quad 0 < p < \infty$$

and

$$s_\infty(f) = \sup\{\gamma; |\hat{f}(\xi)|(1+|\xi|)^\gamma \text{ is bounded}\}.$$

The Hölder exponent $\alpha(f)$ of a continuous function f is defined by

$$\alpha(f) = \sup\{\gamma; f \in C^\gamma\},$$

where C^γ denotes the usual Hölder class. Then

$$s_p(f) + \frac{1}{p} \geq s_q(f) + \frac{1}{q} \tag{4}$$

for any compactly supported distribution f and $0 < p \leq q \leq \infty$, and

$$s_\infty(f) - 1 \leq s_1(f) \leq \alpha(f) \leq s_\infty(f)$$

for any compactly supported continuous function f.

There are considerable literature devoted to estimate the Sobolev exponent and Hölder exponent of the refinable distribution ϕ, for instance [E], [HW], [Vi] for $s_2(\phi)$, [CD] for $s_1(\phi)$, [Her] and [FL] for $s_p(\phi)$. The following are two elementary estimates of Sobolev exponent ([D, Lemmas 7.1.5 and 7.1.6]).

Theorem 1.1 *Let $M = 2$, ϕ be the refinable distribution in (1), H be the symbol with the factorized form (3) and $\{\xi_0, \xi_1, \cdots, \xi_{P-1}\} \subset [-\pi, \pi]$ be any non-trivial invariant cycle for the map $\tau\xi = 2\xi$ (modulo 2π), which means $\xi_m = \tau\xi_{m-1}$, $m = 1, \cdots, P-1, \tau\xi_{P-1} = \xi_0$ and $\xi_0 \neq 0$. If $\hat{\phi}(\xi_0) \neq 0$, then for all integer $k \geq 1$ there exists a constant $C > 0$ independent of k such that*

$$|\hat{\phi}(2^{kP+1}\xi_0)| \geq C(1 + |2^{kP+1}\xi_0|)^{-N+\tilde{\mathcal{K}}},$$

where $\tilde{\mathcal{K}} = \sum_{m=0}^{P-1} \ln |\mathcal{L}(\xi_m)|/(P \ln 2)$.

Theorem 1.2 *Let $M = 2$, ϕ be the refinable distribution in (1) and H be the symbol with the factorized form (3). Suppose that $[-\pi, \pi] = D_1 \cup D_2 \cup \cdots \cup D_Q$ and there exists $q > 0$ so that*

$$\begin{cases} |\mathcal{L}(\xi)| \leq q, & \xi \in D_1, \\ |\mathcal{L}(\xi)\mathcal{L}(2\xi)| \leq q^2, & \xi \in D_2, \\ \quad \vdots & \quad \vdots \\ |\mathcal{L}(\xi)\mathcal{L}(2\xi) \cdots \mathcal{L}(2^{Q-1}\xi)| \leq q^Q, & \xi \in D_Q. \end{cases}$$

Then $|\hat{\phi}(\xi)| \leq C(1 + |\xi|)^{-N+\mathcal{K}}$, where $\mathcal{K} = \ln q / \ln 2$.

Combining (4) and the estimates in Theorems 1.1 and 1.2, we obtain the following estimate of Sobolev exponent

$$N - \mathcal{K} \leq s_\infty(\phi) \leq N - \tilde{\mathcal{K}} \tag{5}$$

and

$$s_p(\phi) \geq N - \mathcal{K} - \frac{1}{p}. \tag{6}$$

Define

$$a_{M,N}(s) = \sum_{s_1 + \cdots + s_{M-1} = s} \prod_{j=1}^{M-1} \begin{pmatrix} N - 1 + s_j \\ s_j \end{pmatrix} (\sin \frac{j\pi}{M})^{-2s_j}, \quad 0 \leq s \leq N-1 \tag{7}$$

and

$$\mathcal{P}_{M,N}(\xi) = \sum_{s=0}^{N-1} a_{M,N}(s) \sin^{2s} \frac{\xi}{2}. \tag{8}$$

Let $\mathcal{L}_{M,N}(\xi)$ be a trigonometric polynomial with real coefficients satisfying

$$|\mathcal{L}_{M,N}(\xi)|^2 = \mathcal{P}_{M,N}(\xi).$$

and let $\phi_{M,N}$ be the refinable distribution in (1) with corresponding symbol $(\frac{1-e^{-iM\xi}}{M-Me^{-i\xi}})^N \mathcal{L}_{M,N}(\xi)$. The functions $\phi_{M,N}$ above are the well-known Daubechies' scaling functions when $M = 2$ ([D]). So we call the functions $\phi_{M,N}$ as M band Daubechies' scaling functions. The functions $\phi_{M,N}$ were introduced by Heller in [H] and independently by Bi, Dai and Sun in [BDS].

There are a much large literature devoted to estimate the regularity of Daubechies' scaling functions (see [D], [CD], [LS] and references therein). For $M = 2$, Volker ([V]), independently Cohen and Conze ([CC]), proved that

$$s_\infty(\phi_{2,N}) = (1 - \ln 3/\ln 4)N + o(N).$$

In [BDS], Bi, Dai and Sun improved the asymptotic estimate above as

$$s_\infty(\phi_{2,N}) = (1 - \frac{\ln 3}{\ln 4})N + \frac{\ln N}{4\ln 2} + O(1)$$

by using the estimate (5) and precise estimates of $\mathcal{L}_{2,N}$. Recently in [LS], Lau and Sun gave more precise asymptotic estimate

$$-C/N \le s_p(\phi_{2,N}) - N + \ln|\mathcal{L}_{2,N}(2\pi/3)|/\ln 2 \le 0 \qquad (9)$$

when $0 < p < \infty$ and

$$s_\infty(\phi_{2,N}) = N - \ln|\mathcal{L}_{2,N}(2\pi/3)|/\ln 2,$$

where C is a constant independent of N. This affirms the phenomenon

$$\lim_{N\to\infty} s_p(\phi_{2,N}) - s_q(\phi_{2,N}) = 0, \quad \forall\ 0 < p, q \le \infty$$

observed by Cohen and Daubechies in [CD]. By the method used in [FS], C/N in the lower bound estimate in (9) can be improved by Cr^N for some constants C and $0 < r < 1$ independent of N.

For $M \ge 3$, Bi, Dai and Sun ([BDS]) proved that

$$s_\infty(\phi_{M,N}) = \frac{4N\ln(\sin M\pi/(2M+2))^{-1} + \ln N}{4\ln M} + O(1)$$

when M is even and

$$s_\infty(\phi_{M,N}) = \frac{\ln N}{4 \ln M} + O(1)$$

when M is odd. Independently Soardi ([So]) proved

$$\alpha(\phi_{4,N}) = -\frac{N \ln \sin 2\pi/5}{2 \ln 2} + o(N)$$

and

$$\alpha(\phi_{M,N}) = \frac{\ln N}{4 \ln M} + o(\ln N)$$

when $M = 3, 5$. Heller and Wells ([HW]) gave similar estimate of $s_2(\phi_{M,N})$ for $M = 3, 4$.

The purpose of this paper are to establish a direct estimate of Sobolev exponent of refinable distributions and to apply the direct estimate above to the asymptotic estimate of Sobolev exponent of the M band Daubechies' scaling functions.

The paper is organized as follows. In Section 2, we shall establish the same upper bound estimate of Sobolev exponent $s_p(\phi)$, $0 < p < \infty$ as the one of $s_\infty(\phi)$ (Theorem 2.1). Obviously it is impossible to obtain precise estimate of $s_p(\phi), 0 < p < \infty$ by combining (4) and estimate of $s_\infty(\phi)$. So we estimate the lower bound of $s_p(\phi), 0 < p \leq \infty$ directly under natural assumptions (12) and (13). The lower bound estimate in Theorem 2.2 seems complicated, but precise. In Section 3, we shall consider the application of the lower and upper bound estimates above of $s_p(\phi)$ to the M band Daubechies' scaling functions, and hence generalize the corresponding result in [LS], where only 2 band Daubechies' scaling functions are considered (Theorem 3.1). By the relationship between Hölder exponent and Sobolev exponent, we obtain similar asymptotic estimate of Hölder exponent of the M band Daubechies' scaling functions (Corollary 2.2). From Theorem 3.1, we see that the phenomenon in [CD] for the 2 band Daubechies' scaling functions appears for the M band Daubechies' scaling functions too when $M \geq 3$ (Corollary 3.3). The technical estimates of $\mathcal{L}_{M,N}$ and $a_{M,N}(s)$ are another important part of Section 3 (Theorems 3.6-3.8, 3.8$'$, 3.11 and 3.11$'$). They are used to obtain the asymptotic estimate of $s_p(\phi_{M,N})$ and are interesting themselves. The corresponding estimates for $\mathcal{L}_{2,N}$ can be found in [D], [CS] and [LS].

2 Direct Estimate of Sobolev Exponent

In this section, we shall discuss the upper and lower bounds of Sobolev exponent of refinable distributions with corresponding symbol having the factorized form (3).

2.1 Upper Bounds

Theorem 2.1 *Let ϕ be the refinable function in (1), H be the corresponding symbol with the factorized form (3), and $\{\xi_0, \xi_1, \cdots, \xi_{P-1}\} \subset [-\pi, \pi]$ be any non-trivial invariant cycle for the map $\tau\xi = M\xi$ (modulo 2π), which means $\xi_m = \tau\xi_{m-1}$, $m = 1, \cdots, P-1, \tau\xi_{P-1} = \xi_0$ and $\xi_0 \neq 0$. Assume that H and $\hat{\phi}$ are continuous and $\hat{\phi}(\xi_0) \neq 0$. Then*

$$s_p(\phi) \leq N - \tilde{\mathcal{K}}, \quad 0 < p \leq \infty$$

where $\tilde{\mathcal{K}} = \sum_{m=0}^{P-1} \ln |\mathcal{L}(\xi_m)|/(P \ln M)$.

Theorem 2.1 is proved by the method in [CDR] and its modification in [LS]. For the perfection we include the proof here.

Proof. By taking Fourier transform at both sides of (1), we obtain

$$\hat{\phi}(\xi) = H(\xi/M)\hat{\phi}(\xi/M). \tag{10}$$

Thus by using (10) for k times, we get

$$\hat{\phi}(\xi) = \prod_{j=1}^{k} H(\xi/M^j)\hat{\phi}(\xi/M^k). \tag{11}$$

Without loss of generality, we assume that $\mathcal{L}(\xi_m) \neq 0, 0 \leq m \leq P-1$. Recall that $\{\xi_0, \cdots, \xi_{P-1}\}$ is a non-trivial cycle. Then $M\xi_m \notin 2\pi\mathbb{Z}$ and $\mathcal{L}$ is continuous at $\xi_m, m = 0, 1, \cdots, P-1$ by the factorized form (3) and the continuity of H. Thus there exists $0 < \delta < 1$ for any $\epsilon > 0$ such that

$$|\mathcal{L}(\xi_m + \xi)| \geq (1-\epsilon)|\mathcal{L}(\xi_m)|, \ \forall \ \xi \in [-\delta, \delta], \ m = 0, 1, \cdots, P-1$$

and

$$|\hat{\phi}(\xi_0 + \xi)| \geq (1-\epsilon)|\hat{\phi}(\xi_0)| > 0, \quad \xi \in [-\delta, \delta]$$

by the continuity of $\hat{\phi}$ at ξ_0 and $\mathcal{L}$ at ξ_m. By computation, we have

$$M^j \xi_0 = \xi_{j'} \quad \text{modulo } 2\pi$$

if $j' = j \quad \text{modulo} \quad P$. Thus it follows from (11) that

$$|\hat{\phi}(M^{Pk}\xi_0 + \xi)| \geq CM^{-PkN}(1-\epsilon)^{Pk} |\prod_{m=0}^{P-1} \mathcal{L}(\xi_m)|^k, \quad \forall\, \xi \in [-\delta, \delta].$$

Hence

$$\begin{aligned} \int_{M^{P(k-1)}|\xi_0|+1}^{M^{Pk}|\xi_0|+1} |\hat{\phi}(\xi)|^p d\xi \quad &\geq \quad C \int_{-\delta}^{\delta} |\hat{\phi}(M^{pk}\xi_0 + \xi)|^p d\xi \\ &\geq \quad CM^{-pPkN}(1-\epsilon)^{Pkp} \Big(\prod_{m=0}^{P-1} |\mathcal{L}(\xi_m)| \Big)^{kp} \delta \end{aligned}$$

and Theorem 2.1 follows. $\square$

2.2 Lower Bounds

We say that $D_m, 1 \leq m \leq Q$ be an *partition* of $[-\pi, \pi]$ if D_m are mutually disjoint and $[-\pi, \pi] = \cup_{m=1}^{Q} D_m$.

Theorem 2.2 *Let $q > 0$, ϕ be the refinable distribution in (1) and H be the corresponding symbol with the factorized form (3). Suppose that $D_m, 1 \leq m \leq Q$ is an partition of $[-\pi, \pi]$, and $\mathcal{L}(\xi)$ satisfies*

$$\begin{cases} |\mathcal{L}(\xi)| \leq q, & \xi \in D_1, \\ |\mathcal{L}(\xi)\mathcal{L}(M\xi)| \leq q^2, & \xi \in D_2, \\ \quad \vdots & \quad \vdots \\ |\mathcal{L}(\xi)\mathcal{L}(M\xi)\cdots\mathcal{L}(M^{Q-1}\xi)| \leq q^Q, & \xi \in D_Q, \end{cases} \tag{12}$$

and

$$\begin{cases} |\mathcal{L}(\xi)| \leq rq, & \xi \in \mathcal{D}_1, \\ |\mathcal{L}(\xi)\mathcal{L}(M\xi)| \leq (rq)^2, & \xi \in \mathcal{D}_2, \\ \quad \vdots & \quad \vdots \\ |\mathcal{L}(\xi)\mathcal{L}(M\xi)\cdots\mathcal{L}(M^{Q-1}\xi)| \leq (rq)^Q, & \xi \in \mathcal{D}_Q, \end{cases} \tag{13}$$

where $0 < r \le 1$, $D_0 \subset [-\pi, \pi]$ *and*

$\mathcal{D}_m = \{\xi \in D_m; M^j\xi \in D_0 + 2\pi\mathbb{Z} \quad \text{for some} \ \ 0 \le j \le m-1\}, \quad 1 \le m \le Q.$

Then for any integer $R \ge 1$ *and* $0 < p < \infty$, *we have*

$$s_p(\phi) \ge N - \mathcal{K} - \frac{\ln \sum_{(\epsilon_1,\cdots,\epsilon_R)\in\{0,1,\cdots,M-1\}^R} r^{p\delta(\epsilon_1,\cdots,\epsilon_R)}}{Rp\ln M}$$

and

$$s_\infty(\phi) \ge N - \mathcal{K},$$

where $\mathcal{K} = \ln q/\ln M$ *and* $\delta(\epsilon_1, \cdots, \epsilon_R)$ *denotes the cardinality of the set* $J(\epsilon_1, \cdots, \epsilon_R)$ *defined by*

$$J(\epsilon_1, \cdots, \epsilon_R) = \{0 \le j \le R-1;\ 2\pi M^j(\sum_{i=1}^{R} \epsilon_i M^{-i} + [0, M^{-R}]) \subset D_0 + 2\pi\mathbb{Z}\}.$$

Obviously the lower bound estimate in Theorem 2.2 reduces to (6) when $r = 1$. For $0 < r < 1$, the lower bound estimate above of Sobolev exponent is better than the one in (6) when D_0 contains a small interval.

To prove Theorem 2.2, we need a lemma. Define

$$I_k(\xi) = \{j;\ 0 \le j \le k-1,\ M^j\xi \in D_0 + 2\pi\mathbb{Z}\}$$

and $i_k(\xi)$ as the cardinality of the set $I_k(\xi)$. Then we have

Lemma 2.3 *Let* $\mathcal{L}$ *be as in Theorem 2.2. Then there exists a constant* C *independent of* k *such that*

$$\prod_{j=0}^{k-1} |\mathcal{L}(M^j\xi)| \le C r^{i_k(\xi)} q^k. \tag{14}$$

Proof. We prove (14) by induction. It is easy to see that (14) holds for all $k \le Q$ by letting the constant C chosen large enough. Inductively we assume that (14) holds for $k \le k_0$.

For $\xi \in \mathcal{D}_m, 1 \le m \le Q$, we have

$$\begin{aligned}\prod_{j=0}^{k_0} |\mathcal{L}(M^j\xi)| &= \prod_{j=0}^{m-1} |\mathcal{L}(M^j\xi)| \times \prod_{j=m}^{k_0} |\mathcal{L}(M^j\xi)| \\ &\le (qr)^m \times \left(C r^{i_{k_0+1-m}(M^m\xi)} q^{k_0+1-m}\right) \le C r^{i_{k_0+1}(\xi)} q^{k_0+1},\end{aligned}$$

where the first inequality follows from (13) and the induction assumption, and the last one holds because of $i_{k_0+1}(\xi) \le m + i_{k_0+1-m}(M^m\xi)$.

For $\xi \in D_m \backslash \mathcal{D}_m, 1 \le m \le Q$, we have $i_{k_0+1}(\xi) = i_{k_0+1-m}(M^m\xi)$ and

$$\begin{aligned}\prod_{j=0}^{k_0} |\mathcal{L}(M^j\xi)| &= \prod_{j=0}^{m-1} |\mathcal{L}(M^j\xi)| \times \prod_{j=m}^{k_0} |\mathcal{L}(M^j\xi)| \\ &\le q^m \times \left(Cr^{i_{k_0+1-m}(M^m\xi)} q^{k_0+1-m}\right) = Cr^{i_{k_0+1}(\xi)} q^{k_0+1}.\end{aligned}$$

Hence (14) holds for $k = k_0 + 1$ by the assumption on $D_m, 1 \le m \le Q$. □

Proof of Theorem 2.2. For $2M^{k-1}\pi \le |\xi| \le 2M^k\pi$, it follows from (3), (11) and Lemma 2.3 that

$$\begin{aligned}|\hat{\phi}(\xi)| &\le C|\xi|^{-N} \prod_{j=0}^{k-1} |\mathcal{L}(M^{-k+j}\xi)| \\ &\le CM^{-kN} q^k r^{i_k(M^{-k}\xi)}.\end{aligned}$$

Therefore

$$s_\infty(\phi) \ge N - \mathcal{K}.$$

For any integer $R \ge 1$, it is easy to prove that

$$i_{kR}(2\pi(\sum_{j=1} \epsilon_j M^{-j} + M^{-kR}\eta)) \ge \sum_{j=1}^{k} \delta(\epsilon_{(j-1)R}, \cdots, \epsilon_{jR}), \quad \eta \in [0,1].$$

Thus for $0 < p < \infty$ and $k \ge 1$, we get

$$\begin{aligned}&\int_{2M^{kR-1}\pi \le |\xi| \le 2M^{(k+1)R-1}\pi} |\hat{\phi}(\xi)|^p d\xi \le C \int_{2M^{kR-1}\pi}^{2M^{kR}\pi} |\hat{\phi}(\xi)|^p d\xi \\ \le\ & CM^{-kNRp} q^{kRp} \int_0^{2M^{kR}\pi} r^{p i_{kR}(M^{-kR}\xi)} d\xi \\ \le\ & CM^{-kNRp} q^{kRp} \sum_{0 \le \epsilon_j \le M-1, 1 \le j \le R} \cdots \sum_{0 \le \epsilon_j \le M-1, (k-1)R+1 \le j \le kR} \\ & \qquad \int_0^1 r^{p i_{kR}(2\pi(\sum_{j=1}^{kR} \epsilon_j M^{-j} + M^{-kR}\eta))} d\eta \\ \le\ & CM^{-kNRp} q^{kRp} \sum_{0 \le \epsilon_j \le M-1, 1 \le j \le R} \cdots \sum_{0 \le \epsilon_j \le M-1, (k-1)R+1 \le j \le kR} \prod_{i=1}^{k} r^{p\delta(\epsilon_{(i-1)R+1}, \cdots, \epsilon_{iR})} \\ \le\ & CM^{-kNRp} q^{kRp} \Big(\sum_{(\epsilon_1, \cdots, \epsilon_R) \in \{0,1,\cdots,M-1\}^R} r^{p\delta(\epsilon_1, \cdots, \epsilon_R)} \Big)^k.\end{aligned}$$

Hence

$$s_p(\phi) \geq N - \mathcal{K} - \frac{\ln \sum_{(\epsilon_1,\cdots,\epsilon_R)\in\{0,1,\cdots,M-1\}^R} r^{p\delta(\epsilon_1,\cdots,\epsilon_R)}}{Rp\ln M}.$$

□

3 Asymptotic Estimate

In this section, we shall prove an asymptotic estimate of Sobolev exponent $s_p(\phi_{M,N})$ of the M band Daubechies' scaling functions and some estimates of $a_{M,N}(s)$ and $\mathcal{L}_{M,N}$.

3.1 Asymptotic Estimate of Sobolev Exponent

In this subsection, we shall apply Theorems 2.1 and 2.2 and some estimates of $\mathcal{L}_{M,N}$ (Theorems 3.6 and 3.7) to the following asymptotic estimate of $s_p(\phi_{M,N}), 0 < p \leq \infty$.

Theorem 3.1 *Let $\phi_{M,N}$ and $\mathcal{L}_{M,N}$ be defined as above. Then there exist constants C and $0 < r_0 < 1$ independent of N such that*

$$-Cr_0^N \leq s_p(\phi_{M,N}) - N + \frac{\ln|\mathcal{L}_{M,N}(M\pi/(M+1))|}{\ln M} \leq 0,\ \ 0 < p \leq \infty$$

when M is even, and

$$-Cr_0^N \leq s_p(\phi_{M,N}) - N + \frac{\ln|\mathcal{L}_{M,N}(\pi)|}{\ln M} \leq 0,\ \ 0 < p < \infty$$

and

$$s_\infty(\phi_{M,N}) = N - \frac{\ln|\mathcal{L}_{M,N}(\pi)|}{\ln M}$$

when M is odd.

For the terms $|\mathcal{L}_{M,N}(\pi)|$ and $|\mathcal{L}_{M,N}(M\pi/(M+1))|$ in Theorem 3.1, we can use $\mathcal{P}_{M,N}$ in (8) to compute them directly. By comparing the estimate in [BDS] and the one above, we also have the following asymptotic estimate

$$\ln|\mathcal{L}_{M,N}(\pi)| = N\ln M - \ln N/4 + O(1)$$

for odd M and

$$\ln|\mathcal{L}_{M,N}\Big(\frac{M\pi}{M+1}\Big)| = N\Big(\ln M - \ln(\sin\frac{M\pi}{2M+2})^{-1}\Big) - \frac{\ln N}{4} + O(1)$$

for even M (see also Theorem 3.11′).

By Theorem 3.1, we have

Corollary 3.2 *Let $\phi_{M,N}$ and $\mathcal{L}_{M,N}$ be defined as above. Then there exist constants C and $0<r<1$ independent of N such that*

$$-Cr^N \le \alpha(\phi_{M,N}) - N + \frac{\ln|\mathcal{L}_{M,N}(M\pi/(M+1))|}{\ln M} \le 0$$

when M is even and

$$-Cr^N \le \alpha(\phi_{M,N}) - N + \frac{\ln|\mathcal{L}_{M,N}(\pi)|}{\ln M} \le 0$$

when M is odd.

By Theorem 3.1, we also have

Corollary 3.3 *Let $\phi_{M,N}$ be defined as above. Then*

$$\lim_{N\to\infty} s_p(\phi_{M,N}) - s_q(\phi_{M,N}) = 0, \quad \forall\ 0<p,q\le\infty.$$

The phenomenon above was observed by Cohen and Daubechies in [CD] and affirmed by Lau and Sun in [LS] when $M=2$.

Because Theorem 3.1 is proved in [LS] when $M=2$ after little modification, we assume that $M\ge 3$ from now on. To prove Theorem 3.1, we need some estimates of $\sum_{(\epsilon_1,\cdots,\epsilon_R)\in\{0,1,\cdots,M-1\}^R} r^{\delta(\epsilon_1,\cdots,\epsilon_R)}$ and $\mathcal{L}_{M,N}$.

Lemma 3.4 *Let $M\ge 4$ be even, $0<r\le 1$ and $D_0=[-(M-1)\pi/M,(M-1)\pi/M]$. Then*

$$\sum_{(\epsilon_1,\cdots,\epsilon_R)\in\{0,1,\cdots,M-1\}^R} r^{\delta(\epsilon_1,\cdots,\epsilon_R)} \le (2+(M-2)r)(1+(M-1)r)^{R-1}.$$

Proof. By computation, we have

$$2\pi(\frac{\epsilon_1}{M} + [0, \frac{1}{M}]) \subset D_0 + 2\pi\mathbb{Z}$$

when $\epsilon_1 \in \{0, 1, \cdots, M-1\}$ and $\epsilon_1 \neq M/2, M/2-1$, and

$$2\pi(\frac{\epsilon_1}{M} + \frac{\epsilon_2}{M^2} + [0, \frac{1}{M^2}]) \subset D_0 + 2\pi\mathbb{Z}$$

when $\epsilon_1 = M/2$ and $\epsilon_2 \geq M/2$. Hence

$$\delta(\epsilon_1, \epsilon_2, \cdots, \epsilon_R) = 1 + \delta(\epsilon_2, \cdots, \epsilon_R)$$

when $\epsilon_1 \neq M/2, M/2-1$, or $\epsilon_1 = M/2$ and $\epsilon_2 \geq M/2$. Recall that

$$\delta(\epsilon_1, \cdots, \epsilon_R) = \delta(M-1-\epsilon_1, \cdots, M-1-\epsilon_R).$$

Then we get

$$\begin{aligned}
& \sum_{(\epsilon_1,\cdots,\epsilon_R)\in\{0,1,\cdots,M-1\}^R} r^{\delta(\epsilon_1,\cdots,\epsilon_R)} \\
= & \sum_{(\epsilon_1,\cdots,\epsilon_R)\in\{0,1,\cdots,M-1\}^R, \epsilon_1\neq M/2, M/2-1} r^{\delta(\epsilon_1,\cdots,\epsilon_R)} \\
& +2 \sum_{(\epsilon_1,\cdots,\epsilon_R)\in\{0,1,\cdots,M-1\}^R, \epsilon_1=M/2} r^{\delta(\epsilon_1,\cdots,\epsilon_R)} \\
\leq & (M-2)r \sum_{(\epsilon_2,\cdots,\epsilon_R)\in\{0,1,\cdots,M-1\}^{R-1}} r^{\delta(\epsilon_2,\cdots,\epsilon_R)} \\
& +2r \sum_{(\epsilon_2,\cdots,\epsilon_R)\in\{0,1,\cdots,M-1\}^{R-1}, \epsilon_2\geq M/2} r^{\delta(\epsilon_2,\cdots,\epsilon_R)} \\
& +2 \sum_{(\epsilon_2,\cdots,\epsilon_R)\in\{0,1,\cdots,M-1\}^{R-1}, \epsilon_2\leq M/2-1} r^{\delta(\epsilon_2,\cdots,\epsilon_R)} \\
= & (1+(M-1)r) \sum_{(\epsilon_2,\cdots,\epsilon_R)\in\{0,1,\cdots,M-1\}^{R-1}} r^{\delta(\epsilon_2,\cdots,\epsilon_R)} \\
\leq & \cdots \\
\leq & (1+(M-1)r)^{R-1} \sum_{\epsilon_R\in\{0,1,\cdots,M-1\}} r^{\delta(\epsilon_R)} \\
= & (2+(M-2)r)(1+(M-1)r)^{R-1}.
\end{aligned}$$

□

Lemma 3.5 *Let $M \geq 3$ be odd, $0 < r \leq 1$ and $D_0 = [-(M-1)\pi/M, (M-1)\pi/M]$. Then*

$$\sum_{(\epsilon_1,\cdots,\epsilon_R)\in\{0,1,\cdots,M-1\}^R} r^{\delta(\epsilon_1,\cdots,\epsilon_R)} \leq (1+(M-1)r)^R.$$

Proof. By computation, we have

$$2\pi(\frac{\epsilon_1}{M} + [0, \frac{1}{M}]) \subset D_0 + 2\pi\mathbb{Z}$$

when $\epsilon_1 \neq (M-1)/2$. Thus

$$\delta(\epsilon_1, \cdots, \epsilon_R) = 1 + \delta(\epsilon_2, \cdots, \epsilon_R)$$

when $\epsilon_1 \neq (M-1)/2$. Hence we obtain

$$\begin{aligned}
& \sum_{(\epsilon_1,\cdots,\epsilon_R)\in\{0,1,\cdots,M-1\}^R} r^{\delta(\epsilon_1,\cdots,\epsilon_R)} \\
= & \sum_{(\epsilon_1,\cdots,\epsilon_R)\in\{0,1,\cdots,M-1\}^R,\epsilon_1\neq(M-1)/2} r^{\delta(\epsilon_1,\cdots,\epsilon_R)} \\
& + \sum_{(\epsilon_1,\cdots,\epsilon_R)\in\{0,1,\cdots,M-1\}^R,\epsilon_1=(M-1)/2} r^{\delta(\epsilon_1,\cdots,\epsilon_R)} \\
\leq & (1+(M-1)r) \sum_{(\epsilon_2,\cdots,\epsilon_R)\in\{0,1,\cdots,M-1\}^{R-1}} r^{\delta(\epsilon_2,\cdots,\epsilon_R)} \\
\leq & \cdots \\
\leq & (1+(M-1)r)^{R-1} \sum_{\epsilon_R\in\{0,1,\cdots,M-1\}} r^{\delta(\epsilon_R)} = (1+(M-1)r)^R.
\end{aligned}$$

□

Theorem 3.6 *Let $M \geq 4$ be even and $\mathcal{L}_{M,N}$ be defined as above. Then there exist constants $0 < r_1, r_2 < 1$ and C such that*

$$|\mathcal{L}_{M,N}(\xi)| \leq |\mathcal{L}_{M,N}(\frac{M\pi}{M+1})|, \quad \xi \in [-\frac{M\pi}{M+1}, \frac{M\pi}{M+1}] \tag{15}$$

and

$$|\mathcal{L}_{M,N}(\xi)\mathcal{L}_{M,N}(M\xi)| \leq (1+Cr_1^N)^2|\mathcal{L}_{M,N}(\frac{M\pi}{M+1})|^2, \quad |\xi| \in [\frac{M\pi}{M+1}, \pi]. \tag{16}$$

Furthermore

$$|\mathcal{L}_{M,N}(\xi)| \le Cr_2^N |\mathcal{L}_{M,N}(\frac{M\pi}{M+1})|, \quad \xi \in [-\frac{(M-1)\pi}{M}, \frac{(M-1)\pi}{M}] \tag{17}$$

and

$$|\mathcal{L}_{M,N}(\xi)\mathcal{L}_{M,N}(M\xi)| \le C^2 r_2^{2N} |\mathcal{L}_{M,N}(\frac{M\pi}{M+1})|^2, \quad |\xi| \in [\pi - \frac{(M-1)\pi}{M^2}, \pi]. \tag{18}$$

Theorem 3.7 *Let $M \ge 3$ be odd and $\mathcal{L}_{M,N}$ be defined as above. Then there exist constants $0 < r_3 < 1$ and C such that*

$$|\mathcal{L}_{M,N}(\xi)| \le |\mathcal{L}_{M,N}(\pi)|, \quad \xi \in [-\pi, \pi] \tag{19}$$

and

$$|\mathcal{L}_{M,N}(\xi)| \le Cr_3^N |\mathcal{L}_{M,N}(\pi)|, \quad \xi \in [-\frac{(M-1)\pi}{M}, \frac{(M-1)\pi}{M}]. \tag{20}$$

We postpone the proof of Theorems 3.6 and 3.7 to next subsection. For a moment, we assume that the estimates of $\mathcal{L}_{M,N}$ in Theorems 3.6 and 3.7 hold and start to prove Theorem 3.1 by using the estimates of $\mathcal{L}_{M,N}$ above and Theorems 2.1 and 2.2.

Proof of Theorem 3.1. The upper bound estimate follows from Theorem 2.1 and the facts that $\{-M\pi/(M+1), M\pi/(M+1)\}$ is a non-trivial invariant cycle when M is even and that $\{\pi\}$ is a non-trivial invariant cycle when M is odd.

We divide two cases to prove the lower bound estimate of $s_p(\phi_{M,N})$.

Case 1. *M is even*

By Lemma 3.4, Theorem 3.6, and by letting $D_0 = [-(M-1)\pi/M, (M-1)\pi/M]$, $D_1 = [-M\pi/(M+1), M\pi/(M+1)]$, $D_2 = [-\pi, -M\pi/(M+1)] \cup [M\pi/(M+1), \pi]$, $q = (1 + Cr_1^N)|\mathcal{L}_{M,N}(M\pi/(M+1))|$ and $r = Cr_2^N$ in Theorem 2.2, we obtain

$$s_\infty(\phi_{M,N}) \ge N - \frac{\ln|\mathcal{L}_{M,N}(M\pi/(M+1))| + \ln(1 + Cr_1^N)}{\ln M}$$

and

$$s_p(\phi_{M,N}) \geq N - \frac{\ln|\mathcal{L}_{M,N}(M\pi/(M+1))| + \ln(1+Cr_1^N)}{\ln M} - \frac{\ln(2+C^p(M-1)r_2^{Np}) + R\ln(1+C^p(M-1)r_2^{Np})}{pR\ln M}. \quad (21)$$

Hence the assertion for even M is proved by letting R tend to infinity in (21) and $1 > r_0 > \max(r_1, r_2)$.

Case 2 *M is odd.*

By Lemma 3.5, Theorem 3.7 and by letting $D_0 = [-(M-1)\pi/M, (M-1)\pi/M]$, $D_1 = [-\pi, \pi]$, $q = |\mathcal{L}_{M,N}(\pi)|$ and $r = Cr_3^N$ in Theorem 2.2, we obtain

$$s_\infty(\phi_{M,N}) \geq N - \frac{\ln|\mathcal{L}_{M,N}(\pi)|}{\ln M}$$

and

$$s_p(\phi_{M,N}) \geq N - \frac{\ln|\mathcal{L}_{M,N}(\pi)|}{\ln M} - \frac{\ln(1+C^p(M-1)r_3^{Np})}{p\ln M}. \quad (22)$$

Hence the assertion for odd M is proved by letting $1 > r_0 > r_3$. □

3.2 Estimates of $\mathcal{L}_{M,N}$

In this subsection, we shall prove Theorems 3.6 and 3.7 and give some elementary estimates of $a_{M,N}(s)$ and $\mathcal{L}_{M,N}$ (Theorems 3.8, 3.8′, 3.11 and 3.11′).

Set

$$h_1(\xi) = \frac{\sin(\xi/2)\cos(M\xi/2)}{\cos(\xi/2)\sin(M\xi/2)}.$$

Then $h_1(0) = 1/M$, $h_1(\pi/M) = 0$ and

$$\frac{d}{d\xi}h_1(\xi) = \frac{\sin M\xi - M\sin\xi}{4\cos^2(\xi/2)\sin^2(M\xi/2)} < 0.$$

Thus h_1 decreases strictly on $[0, \pi/M]$ and there exists unique $\xi(x) \in [0, \pi/M]$ for $0 \leq x \leq 1$ such that $h_1(\xi(x)) = (1-x)/M$. Furthermore there exists a constant such that

$$C^{-1}x^{1/2} \leq \xi(x) \leq Cx^{1/2}.$$

For $0 \le x \le 1$, define

$$x_j^0(x) = \frac{\sin^2(\xi(x)/2)}{\sin^2 j\pi/M - \sin^2(\xi(x)/2)}, \quad 1 \le j \le M-1 \tag{23}$$

and

$$D_x = \{(x_1, \cdots, x_{M-1});\ 0 \le x_j \le x, \sum_{j=1}^{M-1} x_j = x\}.$$

To estimate $a_{M,N}(s)$, we introduce an auxiliary function F on D_x,

$$F(x_1, \cdots, x_{M-1}) = \sum_{j=1}^{M-1} (1+x_j)\ln(1+x_j) - x_j \ln x_j - 2x_j \ln\sin\frac{j\pi}{M}. \tag{24}$$

Theorem 3.8 *Let $0 \le s \le N-1$ and $a_{M,N}(s)$ be defined by (7). Then there exists a constant C independent of N and s such that*

$$C^{-1}N^{-C} \le a_{M,N}(s)\exp(-(N-1)F(x_1^0(\frac{s}{N-1}), \cdots, x_{M-1}^0(\frac{s}{N-1})) \le CN^C.$$

To prove Theorem 3.8, we need two lemmas.

Lemma 3.9 *Let F be defined by (24) and $0 \le x \le 1$. Then F takes its maximum at $(x_1^0(x), \cdots, x_{M-1}^0(x))$ and its maximum is*

$$2\ln M + (1-x)\ln\sin^2\frac{\xi(x)}{2} - \ln\sin^2\frac{M\xi(x)}{2}.$$

Lemma 3.9 was proved in [BDS] for $x = 1$.

Proof. First we prove $(x_1^0(x), \cdots, x_{M-1}^0(x)) \in D_x, 0 \le x \le 1$. Set

$$h_2(t) = 4^{M-1}\prod_{j=1}^{M-1}(\sin^2\frac{j\pi}{M} - t).$$

Then for $t = \sin^2\xi/2$

$$h_2(t)t = \frac{1-\cos\xi}{2} \times 2^{M-1}\prod_{j=1}^{M-1}(\cos\xi - \cos\frac{2j\pi}{M}) = \frac{1-\cos M\xi}{2} = \sin^2\frac{M\xi}{2}.$$

and

$$\frac{d}{dt}h_2(t) = \frac{M\sin(M\xi)\sin^2(\xi/2) - \sin\xi\sin^2(M\xi/2)}{\sin\xi\sin^4(\xi/2)}.$$

Obviously

$$\sum_{j=1}^{M-1} x_j^0(x) = -t\frac{d}{dt}\ln h_2(t) = -t\frac{h_2'(t)}{h_2(t)}$$

where $t = \sin^2\xi(x)/2$. Hence

$$\sum_{j=1}^{M-1} x_j^0(x) = -\frac{M\cos(M\xi(x)/2)\sin(\xi(x)/2)}{\cos(\xi(x)/2)\sin(M\xi(x)/2)} + 1 = x.$$

Observe that

$$\frac{\partial^2}{\partial x_i \partial x_j}F(x_1, \cdots, x_{M-1}) = \begin{cases} -(x_j(1+x_j))^{-1}, & j = i \\ 0, & j \neq i. \end{cases}$$

Then F is strictly convex. Set

$$D_x^{\text{inner}} = \{(x_1, \cdots, x_{M-1}) \in D_x; 0 < x_j < x, 1 \le j \le M-1\}$$

Then D_x^{inner} is open and convex, $(x_1^0(x), \cdots, x_{M-1}^0(x)) \in D_x^{\text{inner}}$ and D_x is the closure of D_x^{inner}. By computation, we have

$$\frac{\partial}{\partial x_j}F(x_1, \cdots, x_{M-1}) = \ln\frac{1+x_j}{x_j\sin^2(j\pi/M)}.$$

Thus

$$\frac{\partial}{\partial x_j}F(x_1^0(x), \cdots, x_{M-1}^0(x)) = -\ln\sin^2\frac{\xi(x)}{2}$$

is independent of $1 \le j \le M-1$. Hence the maximum of F on D_x is taken at $(x_1^0(x), \cdots, x_{M-1}^0(x))$.

By (23) and (24), we get

$$\begin{aligned} F(x_1^0(x), \cdots, x_{M-1}^0(x)) &= \sum_{j=1}^{M-1} \ln(1+x_j^0(x)) - x_j^0(x)\ln\sin^2\frac{\xi(x)}{2} \\ &= \sum_{j=1}^{M-1} \ln\sin^2\frac{j\pi}{M} - \ln\left|\sin^2\frac{j\pi}{M} - \sin^2\frac{\xi(x)}{2}\right| - x_j^0(x)\ln\sin^2\frac{\xi(x)}{2} \\ &= -\ln h_2(\sin^2\frac{\xi(x)}{2}) + \ln h_2(0) - x\ln\sin^2\frac{\xi(x)}{2} \\ &= (1-x)\ln\sin^2\frac{\xi(x)}{2} - \ln\sin^2\frac{M\xi(x)}{2} + 2\ln M. \end{aligned}$$

□

Lemma 3.10 *Let F be defined by (24). If $x_j \geq 0$ and*

$$|x_j - x_j^0(x)| \leq C_1/N, \; 1 \leq j \leq M-1,$$

then

$$|F(x_1, \cdots, x_{M-1}) - F(x_1^0(x), \cdots, x_{M-1}^0(x))| \leq C\frac{\ln N}{N}.$$

Proof. By (24), we have

$$\frac{\partial}{\partial x_j} F(x_1, \cdots, x_{M-1}) = \ln \frac{1 + x_j}{x_j \sin^2(j\pi/M)} = \ln(1 + x_j^{-1}) - \ln \sin^2 \frac{j\pi}{M}.$$

For $|\xi(x)| \geq C_0/\sqrt{N}$ with some sufficiently large constant C_0, we have $|x_j^0(x)| \geq 2C_1/N$ and $|x_j| \geq C_1/N$ because $|x_j - x_j^0(x)| \leq C_1/N$. So we obtain

$$|\frac{\partial}{\partial x_j} F(x_1, \cdots, x_{M-1})| \leq C \ln N$$

and

$$\begin{aligned} & |F(x_1, \cdots, x_{M-1}) - F(x_1^0(x), \cdots, x_{M-1}^0(x)| \\ \leq \; & C \ln N \max_{1 \leq j \leq M-1} |x_j - x_j^0(x)| \leq C\ln N/N. \end{aligned} \tag{25}$$

For $|\xi(x)| \leq C_0/\sqrt{N}$, we have $|x_j^0(x)| \leq C/N$ and $|x_j| \leq C/N$. So we get

$$\begin{aligned} & |F(x_1, \cdots, x_{M-1}) - F(x_1^0(x), \cdots, x_{M-1}^0(x))| \\ \leq \; & |F(x_1, \cdots, x_{M-1})| + |F(x_1^0(x), \cdots, x_{M-1}^0(x))| \\ \leq \; & CN^{-1} + C \max_{0 \leq x \leq C/N} |x \ln x| \leq C \ln N/N. \end{aligned} \tag{26}$$

Hence Lemma 3.10 follows from (25) and (26). □

Proof of Theorem 3.8. By the Stirling formula

$$k! = k^k e^{-k} \sqrt{2\pi(k+1)}(1 + o(1)),$$

there exists a constant C independent of s and N such that

$$\begin{aligned} & C^{-1}N^{-C}\exp\Big((N-1)F(\frac{s_1}{N-1},\cdots,\frac{s_{M-1}}{N-1})\Big) \\ \leq\ & \prod_{j=1}^{M-1}\begin{pmatrix} N-1+s_j \\ s_j \end{pmatrix}(\sin\frac{j\pi}{M})^{-2s_j} \\ \leq\ & CN^C\exp\Big((N-1)F(\frac{s_1}{N-1},\cdots,\frac{s_{M-1}}{N-1})\Big). \end{aligned} \tag{27}$$

Hence it follows from (27) and Lemma 3.9 that

$$\begin{aligned} a_{M,N}(s) &= \sum_{s_1+\cdots+s_{M-1}=s}\prod_{j=1}^{M-1}\begin{pmatrix} N-1+s_j \\ s_j \end{pmatrix}(\sin\frac{j\pi}{M})^{-2s_j} \\ &\leq CN^C\sum_{s_1+\cdots+s_{M-1}=s}\exp\Big((N-1)F(\frac{s_1}{N-1},\cdots,\frac{s_{M-1}}{N-1})\Big) \\ &\leq CN^C\exp\Big((N-1)F(x_1^0(\frac{s}{N-1}),\cdots,x_{M-1}^0(\frac{s}{N-1}))\Big). \end{aligned}$$

Conversely let $s_{j,1}, 1\leq j\leq M-1$ be integers such that $\sum_{j=1}^{M-1}s_{j,1}=s$ and

$$|\frac{s_{j,1}}{N-1}-x_{M-1}^0(\frac{s}{N-1})|\leq\frac{C}{N},\quad 1\leq j\leq M-1.$$

Then we get

$$\begin{aligned} a_{M,N}(s) &\geq \prod_{j=1}^{M-1}\begin{pmatrix} N-1+s_{j,1} \\ s_{j,1} \end{pmatrix}(\sin\frac{j\pi}{M})^{-2s_{j,1}} \\ &\geq C^{-1}N^{-C}\exp\Big((N-1)F(\frac{s_{1,1}}{N-1},\cdots,\frac{s_{M-1,1}}{N-1})\Big) \\ &\geq C^{-1}N^{-C}\exp\Big((N-1)F(x_1^0(\frac{s}{N-1}),\cdots,x_{M-1}^0(\frac{s}{N-1}))-C\ln N\Big) \\ &\geq C^{-1}N^{-C}\exp\Big((N-1)F(x_1^0(\frac{s}{N-1}),\cdots,x_{M-1}^0(\frac{s}{N-1}))\Big), \end{aligned}$$

where the third inequality follows from Lemma 3.10. □

By the proof of Theorem 3.8 and the method used in [BDS], we have

Theorem 3.8′ *Let $a_{M,N}(s)$ be defined by (7) and $0<\delta<1$. Then there exists a constant C independent of N and s such that*

$$C^{-1}N^{-1/2}\leq a_{M,N}(s)\exp(-(N-1)F(x_1^0(\frac{s}{N-1}),\cdots,x_{M-1}^0(\frac{s}{N-1}))\leq CN^{-1/2}$$

hold for all $\delta(N-1) \leq s \leq N-1$ *and* $N \geq 2$.

To estimate $|\mathcal{L}_{M,N}(\xi)|$, we introduce auxiliary functions

$$g(\xi) = \begin{cases} |M\sin(\xi/2)/\sin(M\xi/2)|, & |\xi| \leq \pi/M, \\ M|\sin(\xi/2)|, & \pi/M \leq |\xi| \leq \pi, \\ g(x-2k\pi), & \xi \in 2k\pi + [-\pi,\pi]. \end{cases} \tag{28}$$

and

$$\begin{aligned} \tilde{F}(x,\xi) &= F(x_1^0(x), \cdots, x_{M-1}^0(x)) + 2x\ln|\sin\frac{\xi}{2}| \\ &= 2\ln M + (1-x)\ln\sin^2\frac{\xi(x)}{2} - \ln\sin^2\frac{M\xi(x)}{2}, \quad x \in [0,1]. \end{aligned} \tag{29}$$

Theorem 3.11 *Let* g *and* $\mathcal{L}_{M,N}$ *be as above. Then there exists a constant* C *independent of* ξ *such that*

$$C^{-1}N^{-C}g(\xi)^N \leq |\mathcal{L}_{M,N}(\xi)| \leq CN^C g(\xi)^N.$$

For $M=2$, Cohen and Sere ([CS]) proved $|\mathcal{L}_{2,N}(\xi)| \leq g(\xi)^N$ and Lau and Sun ([LS]) showed $C^{-1}N^{-C}g(\xi)^N \leq |\mathcal{L}_{2,N}(\xi)|$. To prove Theorem 3.11, we need two lemmas.

Lemma 3.12 *Let* $F(x_1,\cdots,x_{M-1})$ *and* $g(\xi)$ *be defined by (24) and (28) respectively. Then*

$$\max_{(x_1,\cdots,x_{M-1})\in D_x, 0\leq x\leq 1} F(x_1,\cdots,x_{M-1}) + 2x\ln|\sin\frac{\xi}{2}| = 2\ln g(\xi).$$

Proof. By Lemma 3.9 and the fact that $\tilde{F}(x,\xi)$ and $g(\xi)$ are 2π periodic even functions, it suffices to prove

$$\max_{0\leq x\leq 1} \tilde{F}(x,\xi) = 2\ln g(\xi), \quad 0 \leq \xi \leq \pi.$$

By (23) and (24), we get

$$\begin{aligned}
&\frac{d}{dx}F(x_1^0(x),\cdots,x_{M-1}^0(x)) \\
= &\sum_{j=1}^{M-1}\frac{\partial}{\partial x_j}F(x_1^0(x),\cdots,x_{M-1}^0(x))\times\frac{d}{dx}x_j^0(x) \\
= &\ln\frac{1}{\sin^2(\xi(x)/2)}\sum_{j=1}^{M-1}\frac{d}{dx}x_j^0(x)=\ln\frac{1}{\sin^2(\xi(x)/2)},
\end{aligned}$$

where the last equality follows from $\sum_{j=1}^{M-1}x_j^0(x)=x$. Therefore for $0\le\xi\le\pi/M$, $\tilde{F}(x,\xi)$ takes its maximum when x satisfies $\xi(x)=\xi$, and for $\pi/M\le\xi\le\pi$, $\tilde{F}(x,\xi)$ takes its maximum at $x=1$. Hence Lemma 3.12 follows from Lemma 3.9. □

Lemma 3.13 *Let $\tilde{F}(x,\xi)$ be defined by (29) and let $z_0(\xi)$ be 2π periodic function with its restriction on $[0,\pi]$ satisfying $\xi(z_0(\xi))=\xi,\xi\in[0,\pi/M]$ and $z_0(\xi)=1$ when $\xi\in[\pi/M,\pi]$. If $|x-z_0(\xi)|\le C_1/N$, then there exists a constant C independent of N and ξ such that*

$$|\tilde{F}(x,\xi)-2\ln g(\xi)|\le C\ln N/N.$$

Proof. Obviously it suffices to prove the assertion for $\xi\in[0,\pi]$. By the proof of Lemma 3.12, we obtain

$$\frac{d}{dx}\tilde{F}(x,\xi)=\ln\frac{\sin^2\xi/2}{\sin^2\xi(x)/2}.$$

For $\xi\in[C_0/\sqrt{N},\pi]$ with some sufficiently large constant C_0, we have $|z_0(\xi)|\ge 2C_1/N$ and $|x|\ge C_1/N$ for all $|x-z_0(\xi)|\le C_1/N$. Thus $|\xi(x)|\ge C/\sqrt{N}$ and

$$|\frac{d}{dx}\tilde{F}(x,\xi)|\le C\ln N.$$

Hence

$$|\tilde{F}(x,\xi)-2\ln g(\xi)|=|\tilde{F}(x,\xi)-\tilde{F}(z_0(\xi),\xi)|\le C\ln N/N. \tag{30}$$

For $\xi \in [0, C_0/\sqrt{N}]$, we have $|z_0(\xi)| \le C/N$ and $|x| \le C/N$. Recall that there exists a constant C such that $C^{-1}x^{1/2} \le \xi(x) \le Cx^{1/2}$. Then

$$\begin{aligned}|\tilde{F}(x,\xi)| &\le 2|\ln \frac{M\sin\xi(x)/2}{\sin(M\xi(x)/2)}| + 2|x\ln\sin\frac{\xi(x)}{2}| \\ &\le C|\xi(x)|^2 + C|x\ln x| + C|x| \le C\ln N/N\end{aligned}$$

and

$$2\ln g(\xi) = 2\Big|\ln\frac{M\sin\xi/2}{\sin(M\xi/2)}\Big| \le C|\xi|^2 \le \frac{C}{N}.$$

Thus we have

$$|\tilde{F}(x,\xi) - 2\ln g(\xi)| \le C\ln N/N. \tag{31}$$

Hence Lemma 3.13 follows from (30) and (31). □

Proof of Theorem 3.11. Obviously it suffices to prove

$$C^{-1}N^{-C}g(\xi)^{2N} \le |\mathcal{L}_{M,N}(\xi)|^2 \le CN^C g(\xi)^{2N}.$$

By Theorem 3.8 and Lemma 3.12, there exists a constant C independent of N and ξ such that

$$\begin{aligned}|\mathcal{L}_{M,N}(\xi)|^2 &\le CN^C\sum_{s=0}^{N-1}\exp((N-1)\tilde{F}(\frac{s}{N-1},\xi)) \\ &\le CN^C\exp(2(N-1)\ln g(\xi)) \le CN^C g(\xi)^{2N}.\end{aligned} \tag{32}$$

Let $0 \le u \le N-1$ be an integer such that

$$|\frac{u}{N-1} - z_0(\xi)| \le \frac{1}{N-1}.$$

Then by Theorem 3.8 and Lemma 3.13, there exists a constant C independent of N and ξ such that

$$\begin{aligned}|\mathcal{L}_{M,N}(\xi)|^2 &\ge C^{-1}N^{-C}\exp((N-1)\tilde{F}(\frac{u}{N-1},\xi)) \\ &\ge C^{-1}N^{-C}\exp((N-1)\tilde{F}(z_0(\xi),\xi) - C\ln N) \\ &\ge C^{-1}N^{-C}g(\xi)^{2N}.\end{aligned} \tag{33}$$

Hence Theorem 3.11 follows from (32) and (33). □

For $|\mathcal{L}_{M,N}(\xi)|$, we also have

Theorem 3.11′ *Let g and $\mathcal{L}_{M,N}$ be as above and $0 < \delta < \pi/M$. Then there exists a constant C independent of ξ and N such that*

$$\begin{aligned} C^{-1}\min(1, N^{-1/4}||\xi| - \frac{\pi}{M}|^{-1/2})g(\xi)^N &\leq |\mathcal{L}_{M,N}(\xi)| \\ &\leq C\min(1, N^{-1/4}||\xi| - \frac{\pi}{M}|^{-1/2})g(\xi)^N \end{aligned}$$

for $\xi \in [-\pi, -\pi/M] \cup [\pi/M, \pi]$, and

$$C^{-1}g(\xi)^N \leq |\mathcal{L}_{M,N}(\xi)| \leq Cg(\xi)^N$$

for $\xi \in [-\pi/M, -\delta] \cup [\delta, \pi/M]$.

Proof. Obviously it suffices to prove the assertion for $\xi \in [\delta, \pi]$. Recall that

$$\frac{d}{dx}\tilde{F}(x, \xi) = \ln \frac{\sin^2 \xi/2}{\sin^2 \xi(x)/2}.$$

Then $\frac{d}{dx}\tilde{F}(x, \xi) > 0$ when $0 < x \leq z_0(\xi) \leq 1$ and $\frac{d}{dx}\tilde{F}(x, \xi) < 0$ when $z_0(\xi) \leq x \leq 1$. By computation, we have

$$\frac{d^2}{dx^2}\tilde{F}(x, \xi) = -\frac{\cos \xi(x)/2}{\sin \xi(x)/2} \times \frac{d\xi(x)}{dx} < 0.$$

Then there exist positive constants $\delta_1 < z_0(\xi)/2$, θ_1 and θ_2 such that

$$\begin{aligned} -\theta_1(x - z_0(\xi))^2 &\leq \tilde{F}(x, \xi) - \tilde{F}(z_0(\xi), \xi) - \ln \frac{\sin^2 \xi/2}{\sin^2 \xi(z_0(\xi))/2}(x - z_0(\xi)) \\ &\leq -\theta_2(x - z_0(\xi))^2 \end{aligned} \tag{34}$$

holds for all $z_0(\xi) - \delta_1 \leq x \leq \min(1, z_0(\xi) - \delta_1)$ and $\xi \in [\delta, \pi]$. Hence by Theorems 3.8 and 3.8′ we obtain

$$\begin{aligned} &\sum_{0 \leq s \leq (z_0(\xi) - \delta_1)(N-1)} a_{M,N}(s) \sin^{2s} \frac{\xi}{2} \\ \leq\ & CN^C \sum_{0 \leq s \leq (z_0(\xi) - \delta_1)(N-1)} \exp\Big((N-1)\tilde{F}(\frac{s}{N-1}, \xi)\Big) \\ \leq\ & CN^C \exp\Big((N-1)\tilde{F}(z_0(\xi) - \delta_1, \xi)\Big), \end{aligned} \tag{35}$$

and

$$\begin{aligned}
&\sum_{(z_0(\xi)+\delta_1)(N-1)\le s\le N-1} a_{M,N}(s)\sin^{2s}\frac{\xi}{2}\\
\le\ & CN^C \sum_{(z_0(\xi)+\delta_1)(N-1)\le s\le N-1} \exp\Big((N-1)\tilde F(\frac{s}{N-1},\xi)\Big)\\
\le\ & CN^C \exp\Big((N-1)\tilde F(z_0(\xi)+\delta_1,\xi)\Big) \qquad (36)
\end{aligned}$$

if $z_0(\xi)+\delta_1<1$, and

$$\begin{aligned}
&\sum_{(z_0(\xi)-\delta_1)(N-1)\le s\le \min(N-1,(z_0(\xi)+\delta_1)(N-1))} a_{M,N}(s)\sin^{2s}\frac{\xi}{2}\\
\approx\ & N^{-1/2}\sum_{(z_0(\xi)-\delta_1)(N-1)\le s\le \min(N-1,(z_0(\xi)+\delta_1)(N-1))} \exp\Big((N-1)\tilde F(\frac{s}{N-1},\xi)\Big)\\
\approx\ & N^{1/2}\int_{z_0(\xi)-\delta_1}^{\min(1,z_0(\xi)+\delta_1)} \exp((N-1)\tilde F(x,\xi))dx. \qquad (37)
\end{aligned}$$

Hereafter $A\approx B$ means $C^{-1}A\le B\le CA$ for some absolute constant C independent of parameters in the terms A and B.

For $\xi\in[\pi/M,\pi]$, we have $z_0(\xi)=1$. Thus by (34) we obtain

$$\begin{aligned}
&\int_{z_0(\xi)-\delta_1}^{1} \exp((N-1)\tilde F(x,\xi))dx\\
\le\ & g(\xi)^{2N}\int_{z_0(\xi)-\delta_1}^{1} \exp\Big((N-1)(\ln\frac{\sin^2\xi/2}{\sin^2\pi/(2M)}(x-1)-\theta_2(x-1)^2)\Big)dx\\
\le\ & C_1 g(\xi)^{2N}\min(N^{-1/2},N^{-1}|\xi-\frac{\pi}{M}|^{-1})
\end{aligned}$$

and similarly

$$\int_{z_0(\xi)-\delta_1}^{1} \exp((N-1)\tilde F(x,\xi))dx \ge C_2 g(\xi)^{2N}\min(N^{-1/2},N^{-1}|\xi-\frac{\pi}{M}|^{-1}).$$

This shows that

$$\int_{z_0(\xi)-\delta_1}^{1} \exp((N-1)\tilde F(x,\xi))dx \approx g(\xi)^{2N}\min(N^{-1/2},N^{-1}|\xi-\frac{\pi}{M}|). \qquad (38)$$

when $\xi\in[\pi/M,\pi]$.

For $\xi \in [\delta, \pi/M]$, we have $z_0(\xi) < 1$. Similarly by (34) we obtain

$$\begin{aligned}
&\int_{z_0(\xi)-\delta_1}^{\min(1,z_0(\xi)+\delta_1)} \exp((N-1)\tilde{F}(x,\xi))dx \\
\leq\ & g(\xi)^{2N} \int_{z_0(\xi)-\delta_1}^{\min(1,z_0(\xi)+\delta_1)} \exp\Big(-\theta_2(N-1)(x-z_0(\xi))^2\Big)dx \leq C_3 g(\xi)^{2N} N^{-1/2}
\end{aligned}$$

and

$$\int_{z_0(\xi)-\delta_1}^{\min(1,z_0(\xi)+\delta_1)} \exp((N-1)\tilde{F}(x,\xi))dx \geq C_3 g(\xi)^{2N} N^{-1/2}.$$

Therefore

$$\int_{z_0(\xi)-\delta_1}^{\min(1,z_0(\xi)+\delta_1)} \exp((N-1)\tilde{F}(x,\xi))dx \approx N^{-1/2} g(\xi)^{2N}. \qquad (39)$$

when $\xi \in [\delta, \pi/M]$. Hence Theorem 3.11′ follows from (35)-(39) and $|\mathcal{L}_{M,N}(\xi)|^2 = \sum_{s=0}^{N-1} a_{M,N}(s) \sin^{2s}\frac{\xi}{2}$. □

To prove Theorems 3.6 and 3.7, we need the following property of $g(\xi)$.

Lemma 3.14 *Let g be defined by (28). Then $g(\xi)$ increases strictly on $[0,\pi]$. Furthermore*

$$0 \leq g(\xi)g(M\xi) \leq |g(\frac{M\pi}{M+1})|^2,\ \ |\xi| \in [\frac{M\pi}{M+1}, \pi], \qquad (40)$$

and there exists $0 < r_4 < 1$ such that

$$g(\xi)g(M\xi) \leq r_4^2 |g(\frac{M\pi}{M+1})|^2,\ \ |\xi| \in [\pi - \frac{(M-1)\pi}{M^2}, \pi] \qquad (41)$$

when M is even.

Proof. By computation, we have

$$\frac{d}{d\xi}\Big(\frac{\sin M\xi}{\sin\xi}\Big) = \frac{\cos\xi\cos M\xi}{\sin^2\xi}(M\tan\xi - \tan M\xi) < 0,\ \ \xi \in (0, \frac{\pi}{2M}).$$

Hence $\sin M\xi / \sin\xi$ decreases strictly on $(0, \pi/(2M))$ and g increases strictly on $[0,\pi]$ by (28).

Observe that

$$\frac{d}{d\xi}(\sin\frac{\xi}{2}\sin\frac{M\xi}{2}) = \frac{1}{2}\cos\frac{\xi}{2}\cos\frac{M\xi}{2}(\tan\frac{M\xi}{2} - \tan\frac{\xi}{2}) \neq 0,\ \xi \in (\frac{M\pi}{M+1}, \pi)$$

and

$$\sin\frac{\xi}{2}\sin\frac{M\xi}{2}\Big|_{\xi=\pi} = 0$$

when M is even. Then $|\sin\frac{\xi}{2}\sin\frac{M\xi}{2}|$ decreases strictly on $[\frac{M\pi}{M+1}, \pi]$. Recall that

$$g(\xi)g(M\xi) = M^2|\sin\frac{\xi}{2}\sin\frac{M\xi}{2}|,\ \xi \in [\frac{M\pi}{M+1}, \frac{(M^2-1)\pi}{M^2}].$$

Then $g(\xi)g(M\xi)$ decreases strictly on $[\frac{M\pi}{M+1}, \frac{(M^2-1)\pi}{M^2}]$.

For $\xi \in [\frac{(M^2-1)\pi}{M^2}, \pi]$, we have

$$\begin{aligned} 0 \leq g(\xi)g(M\xi) &\leq g(\pi)g(\frac{\pi}{M}) = M^2\sin^2\frac{\pi}{2M} \\ &< M^2\sin^2\frac{3\pi}{8} \leq M^2\sin^2\frac{M\pi}{2(M+1)} = g(\frac{M\pi}{M+1})^2. \end{aligned}$$

This proves (40).

From the proof of (40), we see that (41) holds for

$$r_4 = \Big(\frac{\cos((M-1)\pi/(2M^2))\sin((M-1)\pi/(2M))}{\sin^2(M\pi/(2M+2))}\Big)^{1/2} < 1$$

□

Now we start to prove Theorems 3.6 and 3.7.

Proof of Theorem 3.6. Recall that

$$|\mathcal{L}_{M,N}(\xi)|^2 = \sum_{s=0}^{N-1} a_{M,N}(s)(\sin\frac{\xi}{2})^{2s}.$$

Then

$$|\mathcal{L}_{M,N}(\xi)| \leq |\mathcal{L}_{M,N}(\frac{M\pi}{M+1})|, \quad \xi \in [-\frac{M\pi}{M+1}, \frac{M\pi}{M+1}]$$

and (15) holds.

By Theorem 3.11 and Lemma 3.14, we have

$$\begin{aligned}|\mathcal{L}_{M,N}(\xi)| &\leq CN^C g(\xi)^N \leq CN^C r_5^N g(\frac{M\pi}{M+1})^N \\ &\leq Cr_1^N |\mathcal{L}_{M,N}(\frac{M\pi}{M+1})|, \quad \forall\, \xi \in [-\frac{(M-1)\pi}{M}, \frac{(M-1)\pi}{M}],\end{aligned}$$

and

$$\begin{aligned}|\mathcal{L}_{M,N}(\xi)\mathcal{L}_{M,N}(M\xi)| &\leq CN^C (g(\xi)g(M\xi))^N \leq Cr_4^{2N} |g(\frac{M\pi}{M+1})|^{2N} \\ &\leq C^2 r_2^{2N} |\mathcal{L}_{M,N}(\frac{M\pi}{M+1})|^2, \quad \forall\, |\xi| \in [\pi - \frac{(M-1)\pi}{M^2}, \pi],\end{aligned}$$

where $r_5 = g(\frac{(M-1)\pi}{M})/g(\frac{M\pi}{M+1})$, $r_5 < r_1 < 1$ and $r_4 < r_2 < 1$. This proves (17) and (18).

By (18), it suffices to prove (16) for $\xi \in [\frac{M\pi}{M+1}, \pi - \frac{(M-1)\pi}{M^2}]$. Recall that $\max_{0\leq x\leq 1} F(x,\xi) = \tilde{F}(1,\xi)$ and $\tilde{F}(x,\xi)$ increases strictly about $0 \leq x \leq 1$. Then there exist constants C and $0 < r_2 < 1$ by Theorems 3.8 and 3.11 such that

$$\begin{aligned}\sum_{s\leq\beta(N-1)} a_{M,N}(s)(\sin\xi/2)^{2s} &\leq CN^C \exp((N-1)\tilde{F}(\beta,\xi)) \\ &\leq Cr_2^N \sum_{s=0}^{N-1} a_{M,N}(s)(\sin\frac{\xi}{2})^{2s},\end{aligned}$$

when $\xi \in [-\pi + (M-1)\pi/M^2, -(M-1)\pi/M] \cup [(M-1)\pi/M, \pi - (M-1)\pi/M^2]$, where $\beta = 3/(2M)$. By computation, we have $M\xi \in M\pi - [(M-1)\pi/M, M\pi/(M+1)]$. Hence we get

$$\sum_{s=0}^{N-1} a_{M,N}(s)(\sin\frac{\xi}{2})^{2s} \leq (1 + Cr_2^N) \sum_{s\geq\beta(N-1)} a_{M,N}(s)(\sin\frac{\xi}{2})^{2s}$$

and

$$|\mathcal{L}_{M,N}(\xi)\mathcal{L}_{M,N}(M\xi)| \leq (1 + Cr_2^N)^2 \sum_{k,l\geq\beta(N-1)} a_{M,N}(k)a_{M,N}(l)\sin^{2k}\frac{\xi}{2}\sin^{2l}\frac{M\xi}{2}.$$

Set

$$F_{k,l}(\xi) = \sin^{2k}\frac{\xi}{2}\sin^{2l}\frac{M\xi}{2}, \quad \beta(N-1) \leq k, l \leq N-1.$$

Then

$$\frac{d}{d\xi}F_{k,l}(\xi) = \frac{1}{4}(k\tan\frac{M\xi}{2} + Ml\tan\frac{\xi}{2})\sin^{2k-2}\frac{\xi}{2}\sin^{2l-2}\frac{M\xi}{2}\sin(M\xi)\sin\xi.$$

For $\xi \in [\frac{M\pi}{M+1}, \pi - \frac{(M-1)\pi}{M^2}]$, we have $\sin\xi\sin(M\xi) < 0$,

$$\begin{aligned}(k\tan\frac{M\xi}{2} + Ml\tan\frac{\xi}{2})\Big|_{\xi=M\pi/(M+1)} &= ((-1)^{M/2+1}k + Ml)\tan\frac{M\pi}{2(M+1)} \\ &\geq (N-1)(M\beta-1)\tan\frac{M\pi}{2(M+1)} > 0,\end{aligned}$$

and

$$\frac{d}{d\xi}(k\tan\frac{M\xi}{2} + Ml\tan\frac{\xi}{2}) > 0.$$

Therefore $k\tan\frac{M\xi}{2} + Ml\tan\frac{\xi}{2} > 0$ and $\frac{d}{dx}F_{k,l}(\xi) < 0$ on $[\frac{M\pi}{M+1}, \pi - \frac{(M-1)\pi}{M^2}]$. So $F_{k,l}(\xi)$ decreases on $[\frac{M\pi}{M+1}, \pi - \frac{(M-1)\pi}{M^2}]$. Hence for $\xi \in [\frac{M\pi}{M+1}, \pi - \frac{(M-1)\pi}{M^2}]$, we obtain

$$\begin{aligned}|\mathcal{L}_{M,N}(\xi)\mathcal{L}_{M,N}(M\xi)| &\leq (1+Cr_2^N)^2 \sum_{k,l\geq\beta(N-1)} a_{M,N}(k)a_{M,N}(l)F_{k,l}(\frac{M\pi}{M+1}) \\ &\leq (1+Cr_2^N)^2|\mathcal{L}_{M,N}(\frac{M\pi}{M+1})|^2.\end{aligned}$$

□

Proof of Theorem 3.7. Recall that

$$|\mathcal{L}_{M,N}(\xi)|^2 = \sum_{s=0}^{N-1} a_{M,N}(s)(\sin\frac{\xi}{2})^{2s}.$$

Then we have $|\mathcal{L}_{M,N}(\xi)| \leq |\mathcal{L}_{M,N}(\pi)|$.

By Theorem 3.11 and Lemma 3.14, there exists a constant C such that

$$\begin{aligned}|\mathcal{L}_{M,N}(\xi)| &\leq CN^C g(\xi)^N \leq CN^C r_5^N g(\pi)^N \leq CN^C r_5^N |\mathcal{L}_{M,N}(\pi)| \\ &\leq Cr_3^N|\mathcal{L}_{M,N}(\pi)|, \quad \xi \in [-\frac{(M-1)\pi}{M}, \frac{(M-1)\pi}{M}],\end{aligned}$$

where $r_5 = g(\frac{(M-1)\pi}{M})/g(\pi) < 1$ and the last inequality holds by letting $r_5 < r_3 < 1$. □

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