Nonhomogeneous Refinement Equation: Existence, Regularity and Biorthogonality*

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Abstract

In this paper, the existence, regularity and biorthogonality of the solution of the nonhomogeneous refinement equation

$$\Phi(x) = \sum_{k \in \mathbf{Z}^{d'}} c_k \Phi(2x-k) + G(x), \quad x \in \mathbb{R}^d$$

are considered. Also new class of biorthogonal wavelet basis on a non-uniform grid is constructed.

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1 Introduction

1.1 Overview

Fix d' and d with $d' \leq d$ as the dimensions of spaces and N as the length of vectors. Let $g_j,\ 1 \leq j \leq N$ be compactly supported distributions and $c_k, k \in \mathbb{Z}^{d'}$ be a family of $N \times N$ matrices such that $c_k \neq 0$ for only finite indices $k \in \mathbb{Z}^{d'}$. Set $\Phi(x) = (\phi_1(x), \cdots, \phi_N(x))^T$ and $G(x) = (g_1(x), \cdots, g_N(x))^T$. In this paper, we shall consider the existence, regularity and biorthogonality of the solution of such a nonhomogeneous refinement equation

$$\Phi(x) = \sum_{k \in \mathbb{Z}^{d'}} c_k \Phi(2x - k) + G(x), \quad x \in \mathbb{R}^d. \tag{1}$$

Hereafter we identify $k' \in \mathbb{Z}^{d'}$ (resp. $x' \in \mathbb{R}^{d'}$) to $(k', 0) \in \mathbb{Z}^d$ (resp. $(x', 0) \in \mathbb{R}^d$). Obviously any solution of the equation (1) can be written as the sum of a fixed solution of the equation (1) and a solution of the corresponding refinement equation

$$\Phi(x) = \sum_{k \in \mathbb{Z}^{d'}} c_k \Phi(2x - k), \quad x \in \mathbb{R}^d. \tag{2}$$

So we call the equation (1) as the nonhomogeneous refinement equation. For the refinement equation, there is a much large literature (for instance [CDP], [D], [GHM], [HC], [H], [HSW], [J], [Ji], [JiS], [LCY], [MS], [RS]).

1.2 Motivation

The nonhomogeneous refinement equation appeared in the constructions of wavelets on bounded domain, multiwavelets and biorthogonal wavelets at non-uniform grid (see for instance [Me], [CDD], [CDV], [Ma], [GHM], [HM], [HSW], [CMS]).

In the multiresolution approximation on the unit interval $[0, 1]$, the approximation space of scale n is spanned by interior functions, left edge functions and right edge functions. Generally the interior functions are the scaling functions on the line with their support contained in $[0, 1]$, which satisfy refinement equations, and the left and right edge functions are modified from restriction of scaling functions on the line, which satisfy nonhomogeneous refinement equations.

Let ϕ be an orthonormal scaling functions supported on $[0, N]$. In the multiresolution approximation proposed by Meyer([Me]), $\phi(2^n x - k), 0 \le k \le 2^n - N$ are chosen as its interior functions, $\phi(2^n x + k)\chi_{[0,1]}(x),\ 1 \le k \le N-1$ as its left edge functions and $\phi(2^n x - 2^n + k)\chi_{[0,1]}(x), 1 \le k \le N-1$ as its right edge functions, where $\chi_{[0,1]}$ is the characteristic function on $[0, 1]$. Set

$$\Phi_n^L(x) = (\phi(2^n x + 1), \cdots, \phi(2^n x + N - 1))^T \chi_{[0,1]}(x)$$

and

$$\Phi_n^R(x) = (\phi(2^n x - 2^n + N - 1), \cdots, \phi(2^n x - 2^n + 1))^T \chi_{[0,1]}(x).$$

Then for sufficiently large scale n, Φ_n^L and Φ_n^R satisfy the nonhomogeneous refinement equation

$$\begin{aligned} \Phi_n^L(x) &= A_1 \Phi_{n+1}^L(x) + G_n^L(x) = A_1 \Phi_n^L(2x) + G_n^L(x) \\ \Phi_n^R(x) &= A_2 \Phi_{n+1}^R(x) + G_n^R(x) = A_2 \Phi_n^R(2x) + G_n^R(x) \end{aligned}$$

where A_1, A_2 are constant matrices and G_n^L, G_n^R are vectors with linear combination of interior functions as their components.

In the multiresolution approximation on the interval proposed by Cohen, Daubechies and Vial([CDV]), $\phi(2^n x - k), a \le k \le 2^n - N - b$ are chosen as its interior functions, $2^{nl} x^l - \sum_{k \ge 0} P_l(k) \phi(2^n x - k), 0 \le l \le N_0$ as its left edge functions and $2^{nl} x^l - \sum_{k \le 2^n - N} P_l(k) \phi(2^n x - k), 0 \le l \le N_0$ as its right edge functions, where a, b are nonnegative integers and $P_l, 0 \le l \le N_0$ are some polynomials with degree at most l such that

$$\sum_{k \in \boldsymbol{Z}} P_l(k) \phi(x - k) = x^l, \quad 0 \le l \le N_0.$$

Similarly the vector with left edge functions as its components and the one with right edge functions as its components satisfy the following type of nonhomogeneous refinement equation

$$\Phi(x) = A\Phi(2x) + G(x). \tag{3}$$

In [Ma], Madych studied multiresolution approximation on $[0, 1]$ through discrete orthogonal transform. After appropriate choose of matrix A and function G in (3), the solution of the nonhomogeneous refinement equation (3) together with interior functions span the approximation space of scale n.

The multiresolution approximation on the interval in [CDV] is generalized to the one on bounded domain in high dimensions by Cohen, Dahmen and Devore([CDD]). The edge functions also satisfy a nonhomogeneous refinement equation of the form (1) with G as linear combination of interior functions. In the wavelet construction on bounded domain, the regularity and biorthogonality of edge functions are known.

Nonhomogeneous refinement equation also occurs in the multiwavelet construction. For example, let h be the hat function defined by

$$h(x) = \begin{cases} 1-|x|, & x \in [-1,1], \\ 0, & otherwise, \end{cases}$$

and w_c satisfy a nonhomogeneous refinement equation of the form

$$w_c(x) = h(2x-1) + c(w_c(2x) + w_c(2x-1)).$$

Then (h, w_c) leads to the symmetric orthogonal continuous scaling vector when $c = -\frac{1}{5}$, and the pair (h, w_c) and $(h, w_{\tilde{c}})$ with $\tilde{c} = \frac{2c+1}{5c-1}$ and $-1 < c < 1/7$ leads to a family of symmetric biorthogonal scaling vector (see [GHM], [HM]).

The perturbation of Daubechies' orthonormal scaling functions and wavelets in [HSW] is another example to use a solution of a nonhomogeneous refinement equation and a scaling function as two components of a new orthonormal scaling vector. Let ϕ_D and ψ_D be Daubechies' orthonormal scaling functions and wavelets respectively. In [HSW], Huang, Sun and Wang considered solutions $(\phi_1(x), \phi_2(x))^T$ of the refinement equation

$$\begin{cases} \phi_1(x) = \sum_{k\in \boldsymbol{Z}} c_{1,k}\phi_1(2x-k) + \sum_{k\in \boldsymbol{Z}} c_{2,k}\phi_2(2x-k), \\ \phi_2(x) = \sum_{k\in \boldsymbol{Z}} c_{3,k}\phi_2(2x-k) \end{cases}$$

in the neighborhood of $(\psi_D(x), \phi_D(x))^T$. It leads to nontrivial orthonormal scaling vector with arbitrary regularity. In [A], Ayache use the perturbation of tensor-product of Daubechies orthonormal scaling functions to construct compactly supported orthonormal wavelets of non-tensor product type with arbitrary regularity on the plane.

The nonhomogeneous refinement equation is one of the cornerstone in the construction of biorthogonal wavelet basis on arbitrary triangulation of a polygon by lifting scheme in [CES] from hierarchical basis. The dual refinable function $\phi(x_1, x_2)$ at the intersection of different type of grid satisfies a nonhomogeneous refinement equation of the form

$$\phi(x_1, x_2) = \sum_{k \in \boldsymbol{Z}} c_k \phi(2x_1 - k, 2x_2) + G(x_1, x_2)$$

and the one at exceptional node satisfies a nonhomogeneous refinement equation of the form

$$\phi(x_1, x_2) = \phi(2x_1, 2x_2) + G(x_1, x_2)$$

(see [CES] for detail). In the construction of biorthogonal wavelets in one dimension with non-uniform grid at last section, the solution of a nonhomogeneous refinement equation of the form

$$\phi(x) = \phi(2x) + G(x)$$

is used as the dual refinable function at the intersection node of different type of grid.

In the construction of multiwavelets and biorthogonal wavelets on non-uniform grid, the existence, regularity and biorthogonality of the solution of a nonhomogeneous refinement equation are considered as the case may be. At least for constructing more practical and efficient multiwavelets and biorthogonal wavelets on non-uniform grid, it is necessary to study systematical about the existence, regularity, biorthogonality and other properties of solutions of nonhomogeneous refinement equations.

To our knowledge, there are several authors started to working on the nonhomogeneous refinement equations (for instance [DH] and [SZ]). They use different technique to consider the existence and uniqueness of compactly supported distributional solution, convergences of corresponding cascade algorithm and other properties of nonhomogeneous refinement equation.

1.3 Main Results

In Section 2, we consider the existence and explicit expression of compactly supported distributional solution of the nonhomogeneous refinement equation (1) and its general setting which includes nonstationary refinement equations and continuous refinement equation as special cases. The main results are

Theorems 2.1 and 2.8, where necessary and sufficient conditions are given to the existence of compactly supported distributional solutions of nonhomogeneous refinement equations. We use the vector F of compactly supported distributions in Theorem 2.1 for more freedom and for better convergences of the explicit expression. When F is chosen as vector of distributions supported on the origin, we give explicit necessary and sufficient condition on the mask of the corresponding refinement equation for the existence of the solution of nonhomogeneous refinement equation in Theorem 2.3. Similar results are obtained by Dinsenbacher and Hardin([DH]), and Strang and Zhou([SZ]). For scale case, i.e., $N = 1$, we can choose $P(D)\phi_0$ as the function F in Theorem 2.1 where $P(D)$ is a differential operator and ϕ_0 is a solution of a refinement equation (Theorem 2.4). We apply the explicit expression in Theorem 2.1 to solve the corresponding refinement equation (2) and obtain the result about the existence of compactly supported distributional solutions of refinement equations in [JS] and [Z]. At last, we give explicit necessary and sufficient condition to the existence of compactly supported distributional solution of the equation (3), simple form of the nonhomogeneous refinement equation (1) (Theorem 2.7).

In Section 3, we use the behavior of symbol H and regularity of G to estimate the regularity of the solution Φ of the nonhomogeneous refinement equation (1) in Bessel potential space $L^{p,\gamma}$ by complex interpolation method and to estimate Sobolev exponent $s_p(\Phi), 1 < p < \infty$ (Theorems 3.1 and 3.5). Some examples are included to show that these estimates cannot be improved in general. The number α in (22) and (24) is very important to our estimate. We study the relationship between the number α and the spectral radius of the symbol $H(0)$ in Theorems 3.4, 3.8 and 3.9.

In the construction of biorthogonal wavelets on bounded domain or nonuniform grid, solutions of nonhomogeneous refinement equations are used as functions near boundary or near intersection of different type of grid when $d' < d$. For edge functions in the wavelet construction on the interval, we only consider the biorthogonality between edge functions and interior functions or between themselves instead of the one between their integer translates. So the biorthogonality of solutions of nonhomogeneous refinement equations should be different with the one of refinement equations when $d' < d$. In Section 4, we consider the restricted biorthogonality between solutions of nonhomogeneous refinement equations and give a practical condition (Theorems 4.1 and 4.2). In Example 4.3, the restricted biorthogonality of solutions of nonhomogeneous refinement equations and the biorthogonality of solutions of refinement equations are compared in some sense.

In some applications such as construction of curves and surfaces, and finite element method, the user provides data, sampled on a closely spaced but irregular grid. Resampling onto regular grid is typically costly and may generate unwanted artifacts. In [SS2], it is shown how to build a multiresolution analysis and an associated transform on the original non-uniform grid. In [W], Warren shows how spectral analysis can be used to analysis the limit function of an interpolating subdivision on non-uniform grid of semi-regular case (see Remark 5.9 for its definition). In [DGS] Daubechies, Guskov and Swelden study the smoothness of the limit function of unequally spaced interpolating subdivision schemes by the commutation formula. For the spline context, global subdivision scheme for non-uniform spline was introduced in [QG]. In last section, we discuss the construction of biorthogonal wavelets on one-dimsional non-uniform grids of semi-regular case, consider its unit decomposition and regularity(Propositions 5.6 and 5.7), and use corresponding wavelets to characterize certain Sobolev space(Theorem 5.8). To our surprise, the regularity of dual refinable functions is same as the one on non-uniform grid. Thus we may construct refinable functions with arbitrary regularity on any non-uniform grids of semi-regular case (Remark 5.10). In the construction of biorthogonal wavelets, we find that the dual refinable function at the intersection node of different type of grid satisfies a nonhomogeneous refinement equation. This construction is also the one-dimensional model of biorthogonal wavelets on triangulation of a polygon in [CES].

1.4 Notations

$\|f\|_p := (\int_{\mathbb{R}^d} |f(x)|^p dx)^{1/p}, 1 \le p < \infty$;
$\|F\|_p := (\int_{\mathbb{R}^d} \sum_{k=1}^N |f_k(x)|^p dx)^{1/p}, 1 \le p < \infty$ when $F(x) = (f_1(x), \cdots, f_N(x))^T$;
$\delta_{kk'}$: the Kronecker symbol defined by $\delta_{kk'} = \begin{cases} 1, & k = k' \\ 0, & k \ne k' \end{cases}$;
$\rho(A)$: the spectral radius of a matrix A;
$r(A)$: the rank of a matrix A;
$\mathbb{Z}_+^d$: the set of integers in $\mathbb{Z}^d$ with nonnegative components;
$Re\ z$: real part of a complex number z;
$Im\ z$: imaginary part of a complex number z;

$Re\ (z_1, \cdots, z_d) =: (Re\ z_1, \cdots, Re\ z_d)$;
$Im\ (z_1, \cdots, z_d) =: (Im\ z_1, \cdots, Im\ z_d)$;
$s! =: s_1! \cdots s_d!$ for $s = (s_1, \cdots, s_d) \in \mathbb{Z}_+^d$;
$\xi^s =: \xi_1^{s_1} \cdots \xi_d^{s_d}$ for $\xi = (\xi_1, \cdots, \xi_d) \in \mathbb{C}^d$ and $s = (s_1, \cdots, s_d) \in \mathbb{Z}_+^d$;
χ_K: the characteristic function on the set K.

2 Existence

In this section, we will consider the existences of compactly supported distributional solutions of the nonhomogeneous refinement equation (1) and its general setting (19). In one dimension, Dinsenbacher and Hardin, Strang and Zhou established in [DH] and [SZ] some characterization to the existence of compactly supported distributional solutions of nonhomogeneous refinement equations.

Define the Fourier transform $\hat{f}$ of an integrable function f by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi} f(x) dx.$$

The Fourier transform of a compactly supported distribution is understood as usual. For $F(x) = (f_1(x), \cdots, f_N(x))^T$, $\hat{F}(\xi)$ is interprated as $(\hat{f}_1(\xi), \cdots, \hat{f}_N(\xi))^T$.

Set the symbol $H(\xi)$ of the nonhomogeneous refinement equation (1) by

$$H(\xi) = \frac{1}{2^d} \sum_{k \in \boldsymbol{Z}^d} c_k e^{-ik\xi}, \quad \xi \in \mathbb{R}^d. \tag{4}$$

Then the nonhomogeneous refinement equation (1) can be written as

$$\hat{\Phi}(\xi) = H(\frac{\xi}{2})\hat{\Phi}(\frac{\xi}{2}) + \hat{G}(\xi). \tag{5}$$

By using (5) for n times, we obtain

$$\hat{\Phi}(\xi) = H(2^{-1}\xi) \cdots H(2^{-n}\xi)\hat{\Phi}(2^{-n}\xi) + \sum_{j=0}^{n-1} H(2^{-1}\xi) \cdots H(2^{-j}\xi) G(2^{-j}\xi). \tag{6}$$

Theorem 2.1 *Let $G(x)$ be a vector of compactly supported distributions, c_k be $N \times N$ matrices such that $c_k \neq 0$ for only finite indices and $H(\xi)$ be defined by (4). Then a necessary and sufficient condition such that there are compactly supported distributional solutions $\phi_j, 1 \leq j \leq N$ of the nonhomogeneous refinement equation* (1) *is that there exist compactly supported distributions $f_j(x), 1 \leq j \leq N$ such that*

$$\hat{G}(\xi) - \hat{F}(\xi) + H(\frac{\xi}{2})\hat{F}(\frac{\xi}{2}) = O(|\xi|^\tau) \quad as \quad \xi \to 0, \tag{7}$$

where τ is the minimal nonnegative integer with $\rho(H(0)) < 2^\tau$ and $\hat{F}(\xi) = (\hat{f}_1(\xi), \cdots, \hat{f}_N(\xi))^T$.

Let $\Phi_1(x)$ be defined with the help of Fourier transform by

$$\begin{aligned}\hat{\Phi}_1(\xi) = \quad & \hat{F}(\xi) + \textstyle\sum_{n=0}^{\infty} H(2^{-1}\xi)\cdots H(2^{-n}\xi) \\ & (\hat{G}(2^{-n}\xi) - \hat{F}(2^{-n}\xi) + H(2^{-n-1}\xi)\hat{F}(2^{-n-1}\xi))\end{aligned} \tag{8}$$

Then any solution Φ of the nonhomogeneous refinement equation (1) can be written as

$$\Phi(x) = \Phi_1(x) + \Phi_0(x),$$

where $\Phi_0(x)$ is a solution of the corresponding refinement equation (2).

To prove Theorem 2.1, we need a simple lemma.

Lemma 2.2 *Let $H(\xi)$ be a matrix with trigonometrical polynomial entries. Then for any $\delta > 0$ there exists a constant C such that*

$$\|H(\frac{\xi}{2})\cdots H(\frac{\xi}{2^n})\| \le Ce^{C|\xi|}(\rho(H(0)) + \delta)^n, \quad \forall\ n \ge 1,\ \xi \in C^d.$$

Proof. For any $\delta > 0$ there exists a norm $\|\cdot\|_*$ such that

$$\|H(0)\|_* \le \rho(H(0)) + \delta/2.$$

Recall that $H(\xi)$ is a matrix with trigonometrical polynomial entries. Then there exists a constant C such that

$$\|H(\xi) - H(0)\|_* \le Ce^{C|\xi|}|\xi|, \quad \forall\ \ \xi \in \mathbb{C}^d.$$

Therefore by the equivalence of different norms we have

$$\begin{aligned} & \|H(\frac{\xi}{2})\cdots H(\frac{\xi}{2^n})\| \le C\|H(\frac{\xi}{2})\cdots H(\frac{\xi}{2^n})\|_* \\ \le\ \ & C\prod_{j=1}^{n}(\|H(0)\|_* + Ce^{C2^{-j}|\xi|}2^{-j}|\xi|) \le Ce^{C|\xi|}(\rho(H(0)) + \delta)^n. \end{aligned}$$

Proof of Theorem 2.1 By (5), the left hand side of (7) equals zero when $\hat{F}(\xi)$ is replaced by $\hat{\Phi}(\xi)$. Hence the necessity follows.

Obviously the sufficiency reduces to right hand side of (8) being well-defined, analytic and Fourier transform of compactly supported distributions, and satisfying (5).

By (7) and Lemma 2.2 with $0 < \delta < 2^\tau - \rho(H(0))$, we obtain

$$\begin{aligned} & \|H(2^{-1}\xi)\cdots H(2^{-n}\xi)[\hat{G}(2^{-n}\xi) - \hat{F}(2^{-n}\xi) + H(2^{-n-1}\xi)\hat{F}(2^{-n-1}\xi)]\| \\ \leq\ & \|H(2^{-1}\xi)\cdots H(2^{-n}\xi)\| \times |\hat{G}(2^{-n}\xi) - \hat{F}(2^{-n}\xi) + H(2^{-n-1}\xi)\hat{F}(2^{-n-1}\xi)| \\ \leq\ & Ce^{C|\xi|}(\rho(H(0)) + \delta)^n 2^{-n\tau}, \quad \forall\, n \geq 1,\ \xi \in \mathbb{C}^d. \end{aligned}$$

Therefore the sum in the right hand side of (8) converges absolutely and uniformly on any bounded domain of $\mathbb{C}^d$. Hence the function $\hat{\Phi}_1(\xi)$ in (8) is well-defined.

Denote

$$a_n(\xi) = H(2^{-1}\xi)\cdots H(2^{-n}\xi)[\hat{G}(2^{-n}\xi) - \hat{F}(2^{-n}\xi) + H(2^{-n-1}\xi)\hat{F}(2^{-n-1}\xi)]. \tag{9}$$

Then $a_n(\xi),\ n \geq 0$ are analytic functions. Thus $\hat{\Phi}_1(\xi)$ is analytic by uniform convergences on any bounded domain of $\mathbb{C}^d$.

Recall that $H(\xi)$ is a matrix with trigonomatrical polynomial entries and that G is a vector with compactly supported distribution components. Then there exist constants C and A such that

$$\|H(\xi)\| \leq Ce^{A|Im\ \xi|}, \quad \forall\, \xi \in \mathbb{C}^d$$

and

$$|\hat{G}(\xi)| \leq C(1 + |\xi|)^C e^{A|Im\ \xi|}, \quad \forall\, \xi \in \mathbb{C}^d.$$

By (8) and uniform convergence, there exists a constant C such that

$$|\hat{\Phi}_1(\xi)| \leq C, \quad \forall\, |\xi| \leq 2.$$

Thus for $2^{n-1}\pi \leq |\xi| \leq 2^n\pi$, we have

$$\begin{aligned} & \|H(2^{-1}\xi)\cdots H(2^{-n}\xi)\hat{\Phi}_1(2^{-n}\xi)\| \\ \leq\ & C\prod_{j=0}^{n}\|H(2^{-j}\xi)\| \leq C\prod_{j=1}^{n}(Ce^{A|Im\ 2^{-j}\xi|}) \leq C^n e^{A|Im\ \xi|} \end{aligned} \tag{10}$$

and

$$\begin{aligned}
&\| \sum_{j=0}^{n} H(2^{-1}\xi)\cdots H(2^{-j}\xi)G(2^{-j}\xi)\| \\
\leq\ & C\sum_{j=0}^{n}\Big[\Big(\prod_{l=1}^{j} Ce^{A|Im\ 2^{-l}\xi|}\Big)\times (C(1+|\xi|)^{C}e^{A|Im\ 2^{-j}\xi|})\Big] \qquad (11)\\
\leq\ & nC^{n+2}(1+|\xi|)^{C}e^{A|Im\ \xi|}.
\end{aligned}$$

Combining (6), (10) and (11), we get

$$|\hat{\Phi}_1(\xi)| \leq C(1+|\xi|)^{C}e^{A|Im\ \xi|}.$$

This proves that all components of $\hat{\Phi}_1$ are Fourier transform of compactly supported distributions supported in $[-A, A]^d$ by Paley-Wiener Theorem.

Hence it remains to prove that $\Phi_1(x)$ satisfies the nonhomogeneous refinement equation (1). By (9), we have

$$a_{n+1}(\xi) = H(\frac{\xi}{2})a_n(\frac{\xi}{2}), \quad n \geq 0.$$

Hence

$$\begin{aligned}
&\hat{\Phi}_1(\xi) - H(\frac{\xi}{2})\hat{\Phi}_1(\frac{\xi}{2}) \\
=\ & \hat{F}(\xi) + \sum_{n=0}^{\infty} a_n(\xi) - H(\frac{\xi}{2})\hat{F}(\frac{\xi}{2}) - \sum_{n=0}^{\infty} H(\frac{\xi}{2})a_n(\frac{\xi}{2}) \\
=\ & \hat{F}(\xi) - H(\frac{\xi}{2})\hat{F}(\frac{\xi}{2}) + a_0(\xi) = \hat{G}(\xi).
\end{aligned}$$

This proves that $\hat{\Phi}_1$ satisfies (5) and completes the proof.

By Theorem 2.1, the problem to solve the nonhomogeneous refinement equation (1) reduces to finding compactly supported distributions $f_j, 1 \leq j \leq N$ of the equation (7). Now we give a simple method to construct $f_j, 1 \leq j \leq N$ with their supports at the origin.

Write

$$\hat{G}(\xi) = \sum_{|s|\leq \tau-1, s\in \boldsymbol{Z}_+^d} G_s \xi^s + O(|\xi|^{\tau}) \qquad (12)$$

and

$$H(\xi) = \sum_{|s|\le\tau-1, s\in\boldsymbol{Z}_+^d} H_s\xi^s + O(|\xi|^\tau). \tag{13}$$

Set

$$S = I - (2^{-|s|}H_{s-t})_{s,t\in\boldsymbol{Z}_+^d, 0\le|s|,|t|\le\tau-1} \tag{14}$$

and

$$\tilde{G} = (G_s^T)^T_{s\in\boldsymbol{Z}_+^d, 0\le|s|\le\tau-1}, \tag{15}$$

where we set $H_s = 0$ when $s \notin \mathbb{Z}_+^d$ or $|s| > \tau - 1$. In one dimension,

$$S = \begin{pmatrix} I - H_0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{2}H_1 & I - \frac{1}{2}H_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2^{-\tau+2}H_{\tau-2} & -2^{-\tau+2}H_{\tau-3} & \cdots & I - 2^{-\tau+2}H_0 & 0 \\ -2^{-\tau+1}H_{\tau-1} & -2^{-\tau+1}H_{\tau-2} & \cdots & -2^{-\tau+1}H_1 & I - 2^{-\tau+1}H_0 \end{pmatrix}$$

and

$$\tilde{G} = (G_0^T, \cdots, G_{\tau-1}^T)^T.$$

Theorem 2.3 *Let G_s, H_s, S and $\tilde{G}$ be defined by* (12)-(15). *Then the necessary and sufficient condition such that there exist some compactly supported distributional solutions $f_j, 1 \le j \le N$ of the equation (7) is*

$$r(S) = r(S^*)$$

where $S^* = (S, \tilde{G})$ is the augmented matrix.

Proof. Write

$$\hat{F}(\xi) = \sum_{s\in\boldsymbol{Z}_+^d, 0\le|s|\le\tau-1} F_s\xi^s + O(|\xi|^\tau). \tag{16}$$

Then the equation (7) for $\hat{F}(\xi)$ is equivalent to

$$G_s + 2^{-|s|} \sum_{t\in\boldsymbol{Z}_+^d, |t|\le\tau-1} H_{s-t}F_t - F_s = 0, \quad s \in \mathbb{Z}_+^d, 0 \le |s| \le \tau - 1$$

or

$$S\tilde{F} = \tilde{G}, \tag{17}$$

where $\tilde{F} = (F_s^T)^T_{s \in \mathbf{Z}_+^d, 0 \le |s| \le \tau - 1}$. Observe that the equation (17) is solvable if and only if $r(S) = r(S^*)$. Thus the necessity follows from (17) and the sufficiency follows by letting $\hat{F}(\xi) = \sum_{s \in \mathbf{Z}_+^d, 0 \le |s| \le \tau - 1} F_s \xi^s$ and $(F_s^T)^T_{s \in \mathbf{Z}_+^d, 0 \le |s| \le \tau - 1}$ be a solution of the equation (17).

Obviously the solution of the equation (7) is not unique. The functions $f_j, 1 \le j \le N$ chosen in Theorem 2.3 are distributions supported on the origin.

For $N = 1$ and $d = 1$, we can simplify the conditions in Theorem 2.1 and construct the functions F in (7) explicitly. Let $\phi_0(x)$ is the solution of refinement equation

$$\phi_0(x) = H(0)^{-1} \sum_{k \in \mathbf{Z}} c_k \phi_0(2x - k).$$

Theorem 2.4 *If $H(0) \neq 2^l$ for all nonnegative integers l, then there exists compactly supported distributional solution of the nonhomogeneous refinement equation (1) and the function $F(x)$ in (7) can be chosen that*

$$\hat{F}(\xi) = \hat{\phi}_0(\xi) \sum_{s=0}^{\tau-1} \frac{(\hat{\phi}_0^{-1} \hat{G})^{(s)}(0)(2\xi)^s}{s!(2^s - H(0))}.$$

If $H(0) = 2^l$ for some nonnegative integer l, then the sufficient and necessary condition such that there exists compactly supported distributional solution of the nonhomogeneous refinement equation (1) is

$$(\hat{G}(\xi)\phi_0^{-1}(\xi))^{(l)}(0) = 0,$$

where $f^{(l)}$ denotes l-derivatives of f. In this case, $F(x)$ in (7) can be chosen that

$$\hat{F}(\xi) = \hat{\phi}_0(\xi) \sum_{s=0}^{l-1} \frac{(\hat{\phi}_0^{-1} \hat{g})^{(s)}(0)\xi^s}{s!(2^s - H(0))}.$$

Theorem 2.4 follows from Theorems 2.1, 2.3 and the facts that $r(S) = \tau$ when $H(0) \neq 2^l$ for all nonnegative integer l and that $r(S) = \tau - 1 = l$ when $H(0) = 2^l$ for some nonnegative integer l.

In Theorem 2.4, the function F is chosen that $P(D)\phi_0$ for some differential operator $P(D)$ and is more regular than the distribution supported at the origin when ϕ_0 is sufficiently smooth.

Theorems 2.1 and 2.3 can be used to study the existence and explicit expression of nonzero solutions of the refinement equation (2).

Theorem 2.5 *Let $\hat{F}(\xi)$ be a solution of the equation (7) with $\hat{G}(\xi) \equiv 0$. Assume that $F_s \neq 0$ in (16) for some $s \in \mathbb{Z}_+^d, |s| \leq \tau - 1$. Then*

$$\hat{\Phi}_1(\xi) = \lim_{n\to\infty} H(2^{-1}\xi)\cdots H(2^{-n}\xi)\hat{F}(2^{-n}\xi)$$

is a nonzero solution of the refinement equation (2). The dimension of compactly supported distributional solutions of the refinement equation (2) is $NA(\tau, d) - r(S)$, where $A(\tau, d)$ denotes the dimension of the space of all polynomials in $\mathbb{R}^d$ with degree at most $\tau - 1$.

Proof. For any solution Φ of the refinement equation (2),

$$\hat{\Phi}(\xi) = H(\frac{\xi}{2})\cdots H(\frac{\xi}{2^n})\hat{\Phi}(\frac{\xi}{2^n}) \to 0$$

as $n \to \infty$ when

$$\hat{\Phi}(\xi) = O(|\xi|^\tau) \quad as \quad |\xi| \to 0.$$

Thus the solution $\Phi(x)$ of the refinement equation (2) isn't identically zero if and only if $\hat{\Phi}^{(s)}(0) \neq 0$ for some $s \in \mathbb{Z}_+^d, 0 \leq |s| \leq \tau - 1$. By (8), we have $\hat{\Phi}_1^{(s)}(0) = s!F_s$ for all $s \in \mathbb{Z}_+^d, 0 \leq |s| \leq \tau - 1$, where F_s is defined by (16). Then Theorem 2.5 follows from Theorem 2.3.

Set

$$S_{l_0} = I - (2^{-|s|}H_{s-t})_{s,t\in\mathbb{Z}_+^d, l_0\leq|s|,|t|\leq\tau-1}, \quad 1 \leq l_0 \leq \tau - 1$$

and

$$T_{l_0} = I - (2^{-|s|}H_{s-t})_{s,t\in\mathbb{Z}_+^d, 0\leq|s|,|t|\leq l_0-1}, \quad 1 \leq l_0 \leq \tau - 1.$$

Then the matrices S_{l_0} and T_{l_0} are block lower triangular matrices with diagonal block $I - 2^{-|s|}H_0$ and

$$S = \begin{pmatrix} T_{l_0} & 0 \\ B & S_{l_0} \end{pmatrix},$$

where B is a matrix. By (14), Let l_0 be the minimal nonnegative integer such that 2^{l_0} is an eigenvalue of $H(0)$. Then T_{l_0} is nonsingular and there exists a vector α such that $\alpha H_0 = 2^{l_0}\alpha$. Let $\tilde{\alpha} = \alpha\delta_{s_0}$ be the vector with the s_0-th block component α and other block components zero where $|s_0| = l_0$. Then the s_0-th block component of $\tilde{\alpha}S_{l_0}$ is $\alpha(I - 2^{-|l_0|}H_0) = 0$ and other block components zero from the fact that S_{l_0} is the block lower triangular matrix. Hence S_{l_0} is singular and S is singular too. This proves that there exists nonzero compactly supported solution of the refinement equation (2). Obviously the assertion above doesn't mean that there exists compactly supported solution Φ of the refinement equation (2) with $\hat{\Phi}(0) \neq 0$ even when one is an eigenvalue of $H(0)$. For example, $(0, C\delta'(x))^T$ are all compactly supported solutions of the refinement equation

$$\Phi(x) = \begin{pmatrix} 2 & 0 \\ -2 & 4 \end{pmatrix} \Phi(2x) + \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \Phi(2x-1),$$

where $\delta'(x)$ denotes the derivative of the delta distribution. It is easy to see that any solution with $S_1F = 0$ satisfies $S(0, F^T)^T = 0$. Hence there are not any compactly supported solution Φ of the refinement equation (2) with $\hat{\Phi}(0) \neq 0$ when $r(S) = r(S_1) + N$ because $S(\Phi_s)_{0\le|s|\le\tau-1, s\in\boldsymbol{Z}_+^d} = 0$, where Φ_s is defined by (16) for $\hat{\Phi}(\xi)$. Conversely the dimension of solutions of the linear equation $SF = 0$ with the first block zero is $N(A(\tau, d) - 1) - r(S_1) < NA(\tau, d) - r(S)$ when $r(S) \le r(S_1) + N - 1$. This proves that there exists a solution $\tilde{F} = (\tilde{F}_s)$ of the linear $SF = 0$ with the first block nonzero. Let Φ be defined by (8) with $F(\xi) = \sum_{s\in\boldsymbol{Z}_+^d} \tilde{F}_s\xi^s$. Then Φ satisfies the refinement equation (2) and $\hat{\Phi}(0) = F_0 \neq 0$. This proves that

Corollary 2.6 *Let S and S_{l_0} be defined as above. The necessary and sufficient condition such that there exist compactly supported solutions of the refinement equation (2) is that 2^l is an eigenvalue of $H(0)$ for some nonnegative integer l. Furthermore there exist compactly supported solutions Φ of the refinement equation (2) with $\hat{\Phi}(0) \neq 0$ if and only if $r(S) \le r(S_1) + N - 1$.*

It is easy to see that S_{l_0} is of full rank when $2^l, l \ge l_0$ are not eigenvalues of $H(0)$. Thus $r(S) \le NA(\tau, d) - 1 \le r(S_1) + N - 1$ when $2^l, l \ge 1$ are not eigenvalues of $H(0)$ and one is an eigenvalue of $H(0)$. Hence under the assumption $2^l, l \ge 1$ are not eigenvalues of $H(0)$, the necessary and sufficient condition such that there exist compactly supported solution Φ of the refinement equation (2) with $\hat{\Phi}(0) \neq 0$ is that one is an eigenvalue of $H(0)$. The assertion above was proved by Jiang and Shen ([JS]) for $N \geq 2$ and Zhou ([Z]) for $N = 2$. In particular, there are considerable literature concerning the existence and uniqueness of compactly supported solutions of refinement equations (see for instance [HC], [CDP], [H], [HSW], [LCY]).

The nonhomogeneous refinement equation in one dimension below is an important class of nonhomogeneous refinement equation

$$\Phi(x) = H\Phi(2x) + G(x), \tag{18}$$

where H is an $N \times N$ matrix. By elementary property of a matrix, it can be written as $U^{-1}TU$, where U is nonsingular, $T = diag(E(\lambda_1), \cdots, E(\lambda_{l_0}))$ and

$$E(\lambda_l) = \begin{pmatrix} \lambda_l & & & 0 \\ 1 & \lambda_l & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_l \end{pmatrix}, \quad 1 \leq l \leq l_0.$$

Write $U\Phi(x) = (\tilde{\Phi}_1(x)^T, \cdots, \tilde{\Phi}_{l_0}(x)^T)^T$ and $UG(x) = (\tilde{G}_1(x)^T, \cdots, \tilde{G}_{l_0}(x)^T)^T$. Then

$$\tilde{\Phi}_l(x) = E(\lambda_l)\tilde{\Phi}_l(2x) + \tilde{G}_l(x), \quad 1 \leq l \leq l_0.$$

Therefore the nonhomogeneous refinement equation (18) is essentially the combination of the following type of nonhomogeneous refinement equations

Type I: $\Phi(x) = \lambda\Phi(2x) + G(x), \quad N = 1.$

Type II: $\Phi(x) = E(\lambda)\Phi(2x) + G(x).$

By Theorems 2.1 and 2.3, we have

Theorem 2.7 *The nonhomogeneous refinement equation of type I is solvable if and only if $\lambda \neq 2^l$ for all nonnegative integers l or $\lambda = 2^l$ and $G^{(l)}(0) = 0$. The nonhomogeneous refinement equation of type II is solvable if and only if $\lambda \neq 2^l$ for all nonnegative integer l or $\lambda = 2^l$ and $G_1^{(l)}(0) = 0$, where $G_1(x)$ denotes the first component of $G(x)$.*

The nonhomogeneous refinement equation can be formulated in general setting. Fix a matrix A with the norms of all eigenvalues strictly larger than one. Let $\mathbb{Z}_0$ be $\mathbb{Z}$ or $\mathbb{Z}_+$. For integrable functions f and g, the convolution $f * g$ between f and g is defined by

$$f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy.$$

The convolution of two compactly supported distributions is understood as usual. Denote the space of compactly supported distributions by $\mathcal{D}'$. Consider the generalized nonhomogeneous refinement equation

$$\Phi_n(x) = H_n * \Phi_{n+1}(Ax) + G_n(x), \quad n \in \mathbb{Z}_0, \tag{19}$$

where all entries of H_n and all components of Φ_n and $G_n, n \in \mathbb{Z}_0$ are in a bounded set of $\mathcal{D}'$. Obviously nonstationary refinement equation and continuous refinement equation are special cases of the generalized nonhomogeneous refinement equation (19) (see [CD], [ChS], [DR], [DM] and references therein).

Theorem 2.8 *Let $G_n(x), H_n(x), n \in \mathbb{Z}_0$ be in a bounded set of $\mathcal{D}'$. Then the following are equivalent:*

1) *There exists a solution Φ_n of the generalized nonhomogeneous refinement equation (19) with its components in a bounded set of $\mathcal{D}'$,*

2) *There exists F_n with its components in a bounded set of $\mathcal{D}'$ such that*

$$|\hat{G}_n(\xi) - \hat{F}_n(\xi) + H_{n+1}(B\xi)\hat{F}_{n+1}(B\xi)| \le C|\xi|^\tau, \quad \forall\ |\xi| \le 1, \quad n \in \mathbb{Z}_0,$$

where C is a constant independent of $n \in \mathbb{Z}_0$, B denotes the inverse of the transfer of A, and τ is the minimal nonnegative integer such that $\|H_n(0)\|\rho(B)^\tau \le r < 1$ holds for some constant $0 < r < 1$ and some norm $\|\cdot\|$ independent of $n \in \mathbb{Z}_0$.

Set

$$\begin{aligned}\hat{\Phi}_{n,1}(\xi) \quad &= \hat{F}_n(\xi) + \textstyle\sum_{j=0}^\infty H_{n+1}(B\xi)\cdots H_{n+j}(B^j\xi) \\ &(\hat{G}_{n+j}(B^j\xi) - \hat{F}_{n+j}(B^j\xi) + H_{n+j+1}(B^{j+1}\xi)\hat{F}_{n+j+1}(B^{j+1}\xi))\end{aligned}$$

Then any solution $\Phi_n, n \in \mathbb{Z}_0$ of the generalized nonhomogeneous refinement equation (19) can be written as

$$\Phi_n(x) = \Phi_{n,1}(x) + \Phi_{n,0}(x),$$

where $\Phi_{n,1}$ is the inverse Fourier transform of $\hat{\Phi}_{n,1}$ and $\Phi_{n,0}$ is a solution of the corresponding refinement equation

$$\Phi_{n,0}(x) = H_n * \Phi_{n+1}(Ax).$$

Theorem 2.8 can be proved by the same procedure used in the proof of Theorem 2.1, we omit the detail here. Similarly we can establish the corresponding results of Theorems 2.3 and 2.5 to the generalized nonhomogeneous refinement equation (19).

3 Regularity

The analysis of the smoothness of compactly supported solution of refinement equations is widely studied (for instance see [CDP], [CGV], [J], [Ji], [MaS], [MS], [RS] and references therein). In this section, we divide two subsections to discuss the regularity in Bessel potential spaces and estimate of Sobolev exponent of the solution of the nonhomogeneous refinement equation (1). These estimates can used directly to the regularity estimate of dual refinable function at the intersection node of different type of triangles of a triangulation in [CES].

3.1 Time Domain Estimate

For nonnegative integer γ, $f \in L^{p,\gamma}, 1 \leq p < \infty$ means that f and $f^{(s)}$ are p-integrable for all $s \in \mathbb{Z}_+^d$ with $|s| \leq \gamma$. For noninteger $\gamma > 0$ and $1 < p < \infty$, $L^{p,\gamma}$ is defined as the complex interpolation space of L^{p,γ_1} and L^{p,γ_2}, where γ_1, γ_2 are nonnegative integers with $\gamma_1 < \gamma < \gamma_2$ (see [BL]). For $F(x) = (f_1(x), \cdots, f_N(x))^T$, $F \in L^{p,\gamma}$ is interprated as $f_1, \cdots, f_N \in L^{p,\gamma}$. For $1 < p < \infty$, the norm $\|f\|_{p,\gamma}$ on $L^{p,\gamma}$ can be defined by the L^p norm of $(\hat{f}(\xi)(1+|\xi|^2)^{\gamma/2})^\vee$, where $f^\vee$ denotes the inverse Fourier transform of f. For $p = 2$, the Bessel potential space $L^{2,\gamma}$ is just the usual Sobolev space.

Let

$$TF(x) = \sum_{k \in \mathbf{Z}^d} c_k F(2x - k). \tag{20}$$

Then the nonhomogeneous refinement equation (1) can be written as

$$TF(x) = F(x) - G(x).$$

Write

$$H(2^{n-1}\xi) \cdots H(\xi) = 2^{-nd} \sum_{k \in \mathbf{Z}^d} c_{n,k} e^{-ik\xi}. \tag{21}$$

Theorem 3.1 *Let $1 < p < \infty$ and $\gamma \geq 0$. Let Φ be the solution of the nonhomogeneous refinement equation (1), and let G, H and $c_{n,k}$ be as in (1), (4) and (21) respectively. If $G \in L^{p,\gamma}$ and*

$$2^{-nd} \sum_{k \in \mathbf{Z}^d} |c_{n,k}|^p \leq C 2^{-\alpha np} \tag{22}$$

with $\alpha > \gamma$, *then* $\Phi \in L^{p,\gamma}$.

To prove Theorem 3.1, we need a lemma.

Lemma 3.2 *Let* T *be defined by (20),* $c_{n,k}$ *by (21) and let* F *be the vector of compactly supported p-integrable functions. Then there exists a constant* C *dependent only of the support of* F *such that*

$$\|T^n F\|_p \le C(2^{-nd} \sum_{k \in \boldsymbol{Z}^d} |c_{n,k}|^p)^{1/p} \|F\|_p.$$

Proof. By (20) and (21), we have

$$T^n F(x) = \sum_{k \in \boldsymbol{Z}^d} c_{n,k} F(2^n x - k).$$

Hence by the assumption on F, we obtain

$$|T^n F(x)|^p \le C \sum_{k \in \boldsymbol{Z}^d} |c_{n,k}|^p |F(2^n x - k)|^p.$$

Thus

$$\|T^n F\|_p \le C(2^{-nd} \sum_{k \in \boldsymbol{Z}^d} |c_{n,k}|^p)^{1/p} \|F\|_p.$$

Proof of Theorem 3.1. By taking $\xi = 0$ in (21), we obtain

$$H(0)^n = 2^{-n} \sum_{k \in \boldsymbol{Z}^d} c_{n,k}.$$

Hence

$$\|H(0)^n\| \le 2^{-n} \sum_{k \in \boldsymbol{Z}^d} |c_{n,k}| \le C(\sum_{k \in \boldsymbol{Z}^d} |c_{n,k}|^p)^{1/p} 2^{-dn/p} \le C 2^{-\alpha n}.$$

Thus

$$\rho(H(0)) \le 2^{-\alpha} < 1$$

and $\tau = 0$ in Theorem 2.1. Therefore we may use zero vector as a solution of the equation (7) and we obtain by (8)

$$\hat{\Phi}(\xi) = \sum_{n=0}^{\infty} H(2^{-1}\xi) \cdots H(2^{-n}\xi)\hat{G}(2^{-n}\xi).$$

Hence

$$\Phi(x) = \sum_{n=0}^{\infty} T^n G(x). \tag{23}$$

By (23) and Lemma 3.2, we have

$$\begin{aligned} \|\Phi\|_p &\leq \sum_{n=0}^{\infty} \|T^n G\|_p \\ &\leq \sum_{n=0}^{\infty} (2^{-nd} \sum_{k\in \boldsymbol{Z}^d} |c_{n,k}|^p)^{1/p} \|G\|_p \\ &\leq C \sum_{n=0}^{\infty} 2^{-\alpha n} \|G\|_p \leq C\|G\|_p. \end{aligned}$$

Observe that

$$(TG)'(x) = 2TG'(x).$$

Therefore

$$\Phi'(x) = \sum_{n=0}^{\infty} 2^n T^n G'(x).$$

Hence for $\alpha > 1$,

$$\|\Phi'\|_p \leq \sum_{n=0}^{\infty} 2^n \|T^n G'\|_p \leq C \sum_{n=0}^{\infty} 2^{-(\alpha-1)n} \|G'\|_p \leq C\|G\|_{p,1}.$$

Inductively we can prove that

$$\|\Phi\|_{p,\gamma} \leq C\|G\|_{p,\gamma}$$

for all nonnegative integers $\gamma < \alpha$. This proves the assertion when γ is an integer.

Now we prove the assertion for all $\gamma > 0$. Recall that $L^{p,\gamma}$ is the complex interpolation space between L^{p,γ_1} and L^{p,γ_2}, where γ_1 and γ_2 are integers and $\gamma_1 < \gamma < \gamma_2$. Therefore for every $G(x) \in L^{p,\gamma}$, there exists an analytic function $G(x,z)$ about z on $\Omega = \{z \in \mathbb{C}; 0 < Re\ z < 1\}$ such that

- $G(x,z)$ is continuous about z on $\{z \in \mathbb{C}; 0 \le Re\ z \le 1\}$,
- $G(x,\theta) = G(x)$, where $\theta = (\gamma - \gamma_1)/(\gamma_2 - \gamma_1)$,
- $\|G(x,it)\|_{p,\gamma_1} \le Ce^{C|t|}\|G\|_{p,\gamma}, \quad \forall\ t \in \mathbb{R}$,
- $\|G(x,1+it)\|_{p,\gamma_2} \le Ce^{C|t|}\|G\|_{p,\gamma}, \quad \forall\ t \in \mathbb{R}$.

Without loss of generality, we assume that $G(x,z)$ is supported in a fixed compact set independent of $z \in \Omega$, otherwise by multiplying a compactly supported smooth function which takes values 1 on the support of G.

For $F(x) = (f_1(x), \cdots, f_N(x))$, define

$$T_z F(x) = \sum_{n=0}^{\infty} 2^{-n(\gamma_1 - \gamma + (\gamma_2 - \gamma_1)z)} T^n F(x),\ \ z \in \Omega.$$

Then

$$T_\theta G = \sum_{n=0}^{\infty} T^n G$$

satisfies the nonhomogeneous refinement equation (1) by (23). Furthermore there exists a constant C such that

$$\|T_{it}F\|_{p,\gamma_1} \le Ce^{C|t|}\|F\|_{p,\gamma_1}, \quad \forall\ t \in \mathbb{R}$$

and

$$\|T_{1+it}F\|_{p,\gamma_2} \le Ce^{C|t|}\|F\|_{p,\gamma_2}, \quad \forall\ t \in \mathbb{R}.$$

Set

$$R_z G(x) = \Big((T_z G(\cdot, z))^{\wedge}\ (\cdot)(1 + |\cdot|^2)^{(\gamma_1 + (\gamma_2 - \gamma_1)z)/2} \Big)^{\vee}(x).$$

Then $R_z G(x)$ is an analytic function about z on Ω and continuous on $\{z; 0 \le Re\ z \le 1\}$. Furthermore for $t \in \mathbb{R}$ we have

$$\|R_{it}G\|_p \le C\|T_{it}G(\cdot, it)\|_{p,\gamma_1} \le Ce^{C|t|}\|G(\cdot, it)\|_{p,\gamma_1} \le Ce^{C|t|}\|G\|_{p,\gamma},$$

and

$$\|R_{1+it}G\|_p \le C\|T_{1+it}G(\cdot, 1+it)\|_{p,\gamma_2} \le Ce^{C|t|}\|G(\cdot, 1+it)\|_{p,\gamma_2} \le Ce^{C|t|}\|G\|_{p,\gamma}.$$

Thus

$$\|R_\theta G\|_p \le C\|G\|_{p,\gamma}$$

by complex interpolation theorem ([BL]). Hence Theorem 2.1 follows from

$$\|R_\theta G\|_p = \|T_\theta G\|_{p,\gamma} = \|\Phi\|_{p,\gamma}.$$

The condition $\alpha > \gamma$ in Theorem 3.1 cannot be improved in general.

Example 3.3 *Let ψ be a compactly supported orthonormal wavelet of a multiresolution and belongs to C^{γ_0} for some large integer γ_0 (see [D]). Then ψ belongs to L^{p,γ_0} for $1 < p < \infty$. Let ϕ_1 be the solution of the nonhomogeneous refinement equation*

$$\phi_1(x) = 2^{-\alpha}(\phi_1(2x) + \phi_1(2x-1)) + \psi(x).$$

Then

$$2^{-n}\sum_{k\in\boldsymbol{Z}} |c_{n,k}|^p = 2^{-\alpha pn}$$

and

$$\phi_1(x) = \sum_{j=0}^{\infty} 2^{-\alpha j} \sum_{k=0}^{2^j-1} \psi(2^j x - k).$$

For $\gamma_0 > \gamma$, by the characterization of $L^{p,\gamma}$(see [D]), $\|\phi_1\|_{p,\gamma}$ is equivalent to

$$\|(\sum_{j=0}^{\infty} 2^{2(\gamma-\alpha)j} \sum_{k=0}^{2^j-1} \chi_{[0,1]}(2^j\cdot - k))^{1/2}\|_p = \|\Big(\sum_{j=0}^{\infty} 2^{2(\gamma-\alpha)j}\chi_{[0,1]}(\cdot)\Big)^{1/2}\|_p.$$

Hence $\phi_1 \in L^{p,\gamma}$ if and only if $\alpha > \gamma$.

We say that $\Phi(x) = (\phi_1(x), \cdots, \phi_N(x))^T \in L^p$ is L^p *stable* if there exists constant C such that

$$C^{-1}(\sum_{k\in\boldsymbol{Z}^d} |e_k|^p)^{1/p} \le \|\sum_{k\in\boldsymbol{Z}^d} e_k^T\Phi(x)\|_p \le C(\sum_{k\in\boldsymbol{Z}^d} |e_k|^p)^{1/p}$$

holds for all sequence of vectors $\{e_k, k\in \mathbb{Z}^d\}$ (see [JM]).

In the proof of Theorem 3.1, we see that $\rho(H(0)) \le 2^{-\alpha}$ when the symbol satisfies (22). Conversely we have

Theorem 3.4 *Let $H(\xi)$ be the symbol of the nonhomogeneous refinement equation* (1)*, and let $\tilde{\Phi}(x) \in L^p$ be compactly supported solutions of the refinement equation (2) with symbol $\lambda H(\xi)$ where $|\lambda| = \rho(H(0))^{-1}$. If $\tilde{\Phi}$ is L^p stable, then* (22) *holds for $\alpha = -\frac{log|\rho(H(0))|}{log2}$.*

Proof. By the L^p stability of $\tilde{\Phi}(x)$, there exists a constant C such that

$$(2^{nd}\int_{\mathbb{R}^d} |\sum_{k\in\boldsymbol{Z}^d} c_{n,k}\tilde{\Phi}(2^n x-k)|^p dx)^{1/p} \geq C(\sum_{k\in\boldsymbol{Z}^d} \|c_{n,k}\|^p)^{1/p}.$$

Observe that

$$\begin{aligned}&(\sum_{k\in\boldsymbol{Z}^d} c_{n,k}\tilde{\Phi}(2^n\cdot -j))\hat{}\ (\xi)\\ =\ & H(2^{-1}\xi)\cdots H(2^{-n}\xi)\hat{\tilde{\Phi}}(2^{-n}\xi) = \lambda^{-n}\hat{\tilde{\Phi}}(\xi).\end{aligned}$$

Thus

$$(2^{-nd}\sum_{k\in\boldsymbol{Z}^d} \|c_{n,k}\|^p)^{1/p} \leq C[\rho(H(0))]^n.$$

3.2 Frequency Domain Estimate

For a compactly supported function f, the Sobolev exponent $s_p(f)$ is defined by

$$s_p(f) = \sup\{\gamma; \int_{\mathbb{R}^d} |\hat{f}(\xi)|^p(1+|\xi|)^{p\gamma}d\xi < \infty\}$$

for $1 \leq p < \infty$ and

$$s_\infty(f) = \sup\{\gamma; \hat{f}(\xi)(1+|\xi|)^\gamma \quad is\ \ a\ \ bounded\ \ function\}.$$

For a vector $F(x) = (f_1(x), \cdots, f_N(x))$, $s_p(F)$ is interprated as

$$s_p(F) = \min\{s_p(f_j), 1 \leq j \leq N\}.$$

Theorem 3.5 *Let $1 \le p \le \infty, 1 \le r \le \infty$ and $\alpha \ge 0$. Let Φ be the solution of the nonhomogeneous refinement equation (1), $G(x)$ be as in (1) and H be the symbol of (1). Assume that*

$$s_{pr}(G) \ge \alpha + \frac{d(r-1)}{rp}$$

and there exists a constant C such that for all $n \ge 1$,

$$\Big(\int_{|\xi|\le 2^n\pi} \|H(\xi)\cdots H(2^{-n}\xi)\|^{pr/(r-1)} d\xi\Big)^{(r-1)/r} \le C 2^{-n(\alpha p + d/r)}. \tag{24}$$

Then $s_p(\Phi) \ge \alpha$.

Proof. By letting n tend to infinity in (6) and combining $\rho(H(0)) < 1$ (see Theorem 3.8 below), we obtain

$$\hat{\Phi}(\xi) = \sum_{n=0}^{\infty} H(2^{-1}\xi)\cdots H(2^{-n}\xi)\hat{G}(2^{-n}\xi).$$

Hence it suffices to prove that for any $\beta < \alpha$ there exist constants C and $\delta > 0$ such that

$$\int_{\mathbb{R}^d} |H(\xi)\cdots H(2^{-n}\xi)\hat{G}(2^{-n}\xi)|^p (1+|\xi|)^{p\beta} d\xi \le C 2^{-n\delta}.$$

Write

$$\tilde{G}_{\beta,n}(\xi) = \sum_{k\in\mathbf{Z}^d} |\hat{G}(\xi + 2k\pi)|^p (1 + 2^n|\xi + 2k\pi|)^{\beta p}.$$

Then

$$\begin{aligned}
& \int_{\mathbb{R}^d} |H(\xi)\cdots H(2^{-n}\xi)\hat{G}(2^{-n}\xi)|^p (1+|\xi|)^{\beta p} d\xi \\
\le\ & \int_{|\xi|\le 2^n\pi} \|H(\xi)\cdots H(2^{-n}\xi)\|^p |\tilde{G}_{\beta,n}(2^{-n}\xi)| d\xi \\
\le\ & C 2^{-n(\alpha p + d/r)} \Big(\int_{|\xi|\le 2^n\pi} |\widetilde{G}_{\beta,n}(2^{-n}\xi)|^r d\xi\Big)^{1/r} \\
\le\ & C 2^{-\alpha n p} \Big(\int_{|\xi|\le\pi} |\widetilde{G}_{\beta,n}(\xi)|^r d\xi\Big)^{1/r} \\
\le\ & C_\delta 2^{-n\alpha p} \Big(\int_{|\xi|\le\pi} \Big|\sum_{k\in\mathbf{Z}^d} |\widehat{G}(\xi + 2k\pi)|^{pr} (1 + 2^n|\xi + 2k\pi|)^{\beta pr}
\end{aligned}$$

$$(1+|\xi+2k\pi|)^{d(r-1)+\delta}d\xi\Big)^{1/r}$$
$$\leq C_\delta 2^{-(\alpha-\beta)np}\Big(\int_{\mathbb{R}^d}|\hat{G}(\xi)|^{rp}(1+|\xi|)^{rp\beta+d(r-1)+\delta}d\xi\Big)^{1/r}$$
$$\leq C_\delta 2^{-n(\alpha-\beta)p},$$

where $\delta<(\alpha-\beta)r$, the second inequality follows from the assumption on the symbol H and Holder inequality, the fourth one holds because of

$$\begin{aligned}\sum_{k\in\boldsymbol{Z}^d}|e_k| &\leq \Big(\sum_{k\in\boldsymbol{Z}^d}|e_k|^r(1+|\xi+2k\pi|)^{d(r-1)+\delta}\Big)^{1/r}\\ &\qquad\times\Big(\sum_{k\in\boldsymbol{Z}^d}(1+|\xi+2k\pi|)^{-d-\delta/(r-1)}\Big)^{(r-1)/r}\\ &\leq C_\delta\Big(\sum_{k\in\boldsymbol{Z}^d}|e_k|^r(1+|\xi+2k\pi|)^{d(r-1)+\delta}\Big)^{1/r},\end{aligned}$$

and the last one follows from the assumption on $s_{pr}(G)$.

The assumptions on $s_{pr}(G)$ and the symbol $H(\xi)$ in Theorem 3.5 cannot be improved in general.

Example 3.6 *Let B_{2k} be the spline of order $2k$ defined by*

$$\hat{B}_{2k}(\xi)=\left(\frac{\sin\xi}{\xi}\right)^{2k}.$$

Then $\Phi(x)=\frac{1}{1-2^{-\gamma}}B_{2k}$ satisfies the nonhomogeneous refinement equation

$$\Phi(x)=2^{-2k+1-\gamma}\sum_{j=-k}^{k}\binom{2k}{k-|j|}\Phi(2x-j)+B_{2k}(x).$$

Observe that $s_p(B_{2k})=s_\infty(B_{2k})-1/p=2k-1/p$. Thus

$$s_p(\Phi)=2k-\frac{1}{p}=s_{pr}(B_{2k})-\frac{r-1}{pr}.$$

On the other hand,

$$\Big(\int_{|\xi|\leq 2^n\pi}|H(2^{-1}\xi)\cdots H(2^{-n}\xi)|^{pr/(r-1)}d\xi\Big)^{(r-1)/r}$$
$$\leq 2^{-\gamma pn}\Big(\int_{|\xi|\leq 2^n\pi}|\cos(2^{-2}\xi)\cdots\cos(2^{-n-1}\xi)|^{2kpr/(r-1)}d\xi\Big)^{(r-1)/r}\leq C2^{-\gamma pn}.$$

Therefore the equation (24) holds when $\gamma>2k-1/p+d/r$.

Example 3.7 *Let $\gamma > 0$ and $\Phi(x)$ be the solution of the nonhomogeneous refinement equation*

$$\Phi(x) = 2^{-2k+1-\gamma} \sum_{j=-k}^{k} \binom{2k}{k-|j|} \Phi(2x-j) + B_{2k}(8x).$$

By the same procedure as in Example 3.6, we have

$$\Big(\int_{|\xi|\le 2^n\pi} |H(\xi)\cdots H(2^{-n}\xi)|^{pr/(r-1)}\Big)^{(r-1)/r} d\xi \le C2^{-n\gamma p}.$$

For $|\xi| \le 2^{n_0+1}\pi$, we have

$$\begin{aligned}
\hat\Phi(\xi) &= \sum_{n=0}^{\infty} 2^{-\gamma n} \left(\frac{\sin\xi}{2^n \sin 2^{-n}\xi}\right)^{2k} \hat B_{2k}(2^{-n-3}\xi) \\
&\ge 2^{-(n_0-1)\gamma} \left(\frac{\sin\xi}{2^{n_0-1}\sin 2^{-n_0+1}\xi}\right)^{2k} \hat B_{2k}(2^{-n_0-2}\xi) \\
&\ge (\frac{2}{\pi})^{2k} 2^{-(n_0-1)\gamma} \left(\frac{\sin\xi}{2^{n_0-1}\sin 2^{-n_0+1}\xi}\right)^{2k} \ge 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{2^{n_0}\pi\le|\xi|\le 2^{n_0+1}\pi} |\hat\Phi(\xi)|^{pr} d\xi \\
\ge\;& (\frac{2}{\pi})^{2kpr} 2^{-\gamma(n_0-1)pr} \int_{|\xi|\le 2^{n_0-1}\pi} \left(\frac{\sin\xi}{2^{n_0-1}\sin 2^{-n_0+1}\xi}\right)^{2kpr} d\xi \\
\ge\;& (\frac{2}{\pi})^{4kpr} 2^{-\gamma(n_0-1)pr} \int_{|\xi|\le 2^{n_0-2}\pi} \left(\frac{\sin\xi}{\xi}\right)^{2kpr} d\xi \\
\ge\;& C2^{-\gamma n_0 pr}, \quad \forall\ n_0 \ge 1.
\end{aligned}$$

Hence

$$s_p(\Phi) \le \gamma.$$

Obviously

$$s_{pr}(B_{2k}) = 2k - \frac{1}{p} > \gamma + \frac{d(r-1)}{rp}$$

as k is chosen large enough.

The condition (24) is important to the estimate of Sobolev exponent of the solution of the nonhomogeneous refinement equation (1). From the proof of Theorem 3.5, we see that the condition (24) can be replaced by the following weaker condition: for any $\delta > 0$ there exists C_δ such that

$$\Big(\int_{|\xi|\le 2^n\pi} \|H(\xi)\cdots H(2^{-n}\xi)\|^{pr/(r-1)}d\xi\Big)^{(r-1)/r} \le C_\delta 2^{-n(\alpha p+d/r-\delta)} \tag{25}$$

and that the condition $s_{pr}(G) \ge \alpha + \frac{d(r-1)}{rp}$ by

$$\sum_{k\in\boldsymbol{Z}} |\hat{G}(\xi+2k\pi)|^p(1+|\xi+2k\pi|)^{\beta p} \in L^r([-\pi,\pi])$$

for all $\beta < \alpha$. There is close relationship between the number α in (24) or (25) and the spectral radius of $H(0)$. In particular we have

Theorem 3.8 *If $H(\xi)$ satisfies (24) or (25), then the spectral radius $\rho(H(0))$ satisfies*

$$\rho(H(0)) \le 2^{-\alpha-d/(rp)}.$$

Proof. Let $\tilde{H}(\xi) = \rho(H(0))^{-1}H(\xi)$. Then $\rho(\tilde{H}(0)) = 1$. By Lemma 2.2, there exists a constant C such that

$$\|\tilde{H}(\xi)\cdots\tilde{H}(2^{-n}\xi)\| \le Ce^{C|\xi|}(1+\delta)^n,\ n\ge 0$$

where $0<\delta<1/2$. Hence for $|\xi|\le 1$, we have

$$\begin{aligned}
&\|\tilde{H}(\xi)\cdots\tilde{H}(2^{-n}\xi) - \tilde{H}(0)^{n+1}\| \\
\le\ & \sum_{j=1}^{n} \|\tilde{H}(\xi)\cdots\tilde{H}(2^{-j+1}\xi)\|\|\tilde{H}(2^{-j}\xi)-\tilde{H}(0)\|\|\tilde{H}(0)^{n-j}\| \\
\le\ & C\sum_{j=1}^{n}(1+\delta)^j e^{C|\xi|}(2^{-j}|\xi|)(n-j)^C \\
\le\ & Cn^C|\xi|.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|H(\xi)\cdots H(2^{-n}\xi)\| \ge \|H(0)^{n+1}\| - Cn^C\rho(H(0))^{n+1}|\xi| \\
\ge\ & \rho(H(0))^{n+1}(1-Cn^C|\xi|) \ge \frac{1}{2}[\rho(H(0))]^{n+1}
\end{aligned}$$

when $|\xi| \leq \frac{1}{2}C^{-1}n^{-C}$. Hence

$$\int_{|\xi|\leq \frac{1}{2}C^{-1}n^{-C}} \|H(\xi)\cdots H(2^{-n}\xi)\|^{pr/(r-1)}d\xi \geq C^{-1}n^{-C}[\rho(H(0))]^{npr/(r-1)}$$

and $\rho(H(0)) \leq 2^{-(\alpha+d/(rp))}$ by (24) or (25).

We say that compactly supported distributions $f_j, 1 \leq j \leq N$ is *weakly stable* if the rank of $N\times\infty$ matrix $(\hat{f}_j(\xi+2k\pi))_{1\leq j\leq N, k\in \mathbf{Z}^d}$ equals N for all $\xi \in \mathbb{R}^d$ (see [R]). For compactly supported functions $f_j \in L^p, 1 \leq p < \infty$, weak stability and L^p stability of $f_j, 1 \leq j \leq N$ are equivalent to each other (see [JM]).

Theorem 3.9 *Let H be the symbol of the nonhomogeneous refinement equation* (1)*, and let $\tilde{\Phi}(x) = (\tilde{\phi}_1, \cdots, \tilde{\phi}_N)$ be compactly supported solution of the refinement equation (2) with symbol $(\rho(H(0)))^{-1}H(\xi)$. If $\tilde{\Phi}$ is weakly stable and $s_{pr/(r-1)}(\tilde{\Phi}) \geq 0$, then* (25) *holds for $H(\xi)$ with*

$$\alpha = -\frac{log|\rho(H(0))|}{log2} - \frac{d}{rp}.$$

Proof. Obviously it suffices to prove that for any $\delta > 0$ there exists a constant C_δ such that

$$\int_{|\xi|\leq 2^n\pi} \|\tilde{H}(\xi)\cdots\tilde{H}(2^{-n}\xi)\|^{pr/(r-1)}d\xi \leq C_\delta 2^{n\delta},$$

where $\tilde{H}(\xi) = \rho(H(0))^{-1}H(\xi)$. By the weak stability of $\tilde{\Phi}$ and continuity of $\hat{\tilde{\Phi}}(\xi)$, there exists a bounded set K of $\mathbb{Z}^d$ such that the rank of the matrix $(\hat{\tilde{\Phi}}(\xi+2k\pi))_{k\in K}$ equals N for $\xi \in [-\pi,\pi]^d$. Therefore

$$(\sum_{k\in K} |A\hat{\tilde{\Phi}}(\xi+2k\pi)|^{pr/(r-1)})^{(r-1)/(rp)} \geq C\|A\|$$

holds for all $N\times N$ matrix A. Hence

$$\begin{aligned} & \int_{|\xi|\leq 2^n\pi} \|\tilde{H}(2^{-1}\xi)\cdots\tilde{H}(2^{-n}\xi)\|^{pr/(r-1)}d\xi \\ \leq\ & C\int_{|\xi|\leq 2^n\pi}\sum_{k\in K} |\tilde{H}(2^{-1}\xi)\cdots\tilde{H}(2^{-n}\xi)\hat{\tilde{\Phi}}(2^{-n}\xi+2k\pi)|^{pr/(r-1)}d\xi \\ \leq\ & C\int_{|\xi|\leq 2^n\pi}\sum_{k\in K} |\hat{\tilde{\Phi}}(\xi+2^nk\pi)|^{pr/(r-1)}d\xi \\ \leq\ & C\int_{|\xi|\leq C2^n\pi} |\hat{\tilde{\Phi}}(\xi)|^{pr/(r-1)}d\xi \leq C_\delta 2^{n\delta}, \end{aligned}$$

where the last inequality follows from $s_{pr/(r-1)}(\tilde{\Phi}) \geq 0$.

4 Biorthogonality

In this section, we shall discuss the restricted biorthogonality of the solutions Φ and $\widetilde{\Phi}$ of the nonhomogeneous refinement equations

$$\begin{cases} \Phi(x) = \sum_{k\in\mathbf{Z}^{n'}} c_k \Phi(2x-k) + G(x), \\ \widetilde{\Phi}(x) = \sum_{k\in\mathbf{Z}^{n'}} \tilde{c}_k \widetilde{\Phi}(2x-k) + \widetilde{G}(x), \end{cases} \tag{26}$$

where $\Phi(x) = (\phi_1(x), \cdots, \phi_N(x))^T$ and $\widetilde{\Phi}(x) = (\tilde{\phi}_1(x), \cdots, \tilde{\phi}_N(x))^T$.

We say that the solutions Φ and $\widetilde{\Phi}$ of the nonhomogeneous refinement equation (26) are *restricted biorthogonality* or *biorthogonal on* $\mathbb{Z}^{d'}$ if

$$\int_{\mathbb{R}^d} \Phi(x-k)\widetilde{\Phi}(x-k')^T dx = \delta_{kk'} I, \quad \forall\ k, k' \in \mathbb{Z}^{d'}. \tag{27}$$

The masks $\{c_k\}$ and $\{\tilde{c}_k\}$ of the nonhomogeneous refinement equation (26) are said to be *discrete biorthogonal* if

$$2^{-d} \sum_{k\in\mathbf{Z}^{d'}} c_k \tilde{c}_{k+2l}^T = \delta_{l0} I - G_l, \quad \forall\ l \in \mathbb{Z}^{d'}, \tag{28}$$

where $G_l = \int_{\mathbb{R}^d} G(x-l)\widetilde{G}(x)^T dx$.

Theorem 4.1 *Let Φ and $\widetilde{\Phi}$ be compactly supported solutions of (26). Assume that Φ and $\widetilde{\Phi}$ are square integrable, and satisfy*

$$\int_{\mathbb{R}^d} \Phi(2x-k)\widetilde{G}(x)^T dx = \int_{\mathbb{R}^d} \widetilde{\Phi}(2x-k)G(x)^T dx = 0, \quad \forall\ \ k \in \mathbb{Z}^{d'}. \tag{29}$$

Then the mask $\{c_k\}$ and $\{\tilde{c}_k\}$ are discrete biorthogonal when Φ and $\widetilde{\Phi}$ are biorthogonal on $\mathbb{Z}^{d'}$. Conversely if

$$\Big(\int_{|\xi|\le 2^n\pi} \|H(\xi)\cdots H(2^{-n}\xi)\|^2 d\xi\Big) \times \Big(\int_{|\xi|\le 2^n\pi} \|\tilde{H}(\xi)\cdots \tilde{H}(2^{-n}\xi)\|^2 d\xi\Big) \to 0$$

as n tends to infinity and the masks $\{c_k\}$ and $\{\tilde{c}_k\}$ are discrete biorthogonal, then Φ and $\widetilde{\Phi}$ are biorthogonal on $\mathbb{Z}^{d'}$.

Proof. The assertion of the first part follows easily from the nonhomogeneous refinement equations (26) and the assumptions on Φ and $\widetilde{\Phi}$.

Now we start to prove the assertion of the second part. For $n \geq 1$ and $k \in \mathbb{Z}^{d'}$, write

$$\Phi(x-k) = \sum_{l\in \boldsymbol{Z}^{d'}} c_{n,l}^k \Phi(2^n x - l) + \sum_{j=0}^{n-1} \sum_{l\in \boldsymbol{Z}^{d'}} d_{j,l}^k G(2^j x - l)$$

and

$$\widetilde{\Phi}(x-k) = \sum_{l\in \boldsymbol{Z}^{d'}} \tilde{c}_{n,l}^k \widetilde{\Phi}(2^n x - l) + \sum_{j=0}^{n-1} \sum_{l\in \boldsymbol{Z}^{d'}} \tilde{d}_{j,l}^k \widetilde{G}(2^j x - l).$$

Then $c_{1,l}^k = c_{l-2k}$ and $c_{n,l}^k = \sum_{l'\in \boldsymbol{Z}^{d'}} c_{n-1,l'}^k c_{l-2l'}$ by induction. Thus

$$\sum_{l\in \boldsymbol{Z}^{d'}} c_{n,l}^k e^{-il\xi} = 2^{nd} H(2^{n-1}\xi)\cdots H(\xi) e^{-i2^n k\xi}.$$

By the discrete orthogonality of the masks $\{c_k\}$ and $\{\tilde{c}_k\}$ and by the assumptions on Φ and $\widetilde{\Phi}$, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^d} \Phi(x-k)\widetilde{\Phi}(x-k')^T dx - \delta_{kk'} I \\ = & \sum_{l,l'\in \boldsymbol{Z}^{d'}} c_{n,l}^k \Big(\int_{\mathbb{R}^d} \Phi(2^n x - l)\widetilde{\Phi}(2^n x - l')^T dx - 2^{-nd}\delta_{ll'} I\Big)(\tilde{c}_{n,l'}^{k'})^T. \end{aligned}$$

Therefore

$$\begin{aligned} & \Big\| \int_{\mathbb{R}^d} \Phi(x-k)\widetilde{\Phi}(x-k')^T dx - \delta_{kk'} I \Big\| \\ \leq & \sum_{l,l'\in \boldsymbol{Z}^{d'}} \|c_{n,l}^k\| \Big\| \int_{\mathbb{R}^d} \Phi(2^n x - l)\widetilde{\Phi}(2^n x - l')^T dx - 2^{-nd}\delta_{ll'} I\Big\| \|\tilde{c}_{n,l'}^{k'}\| \\ \leq & \Big(\sum_{l,l'\in \boldsymbol{Z}^{d'}} \|c_{n,l}^k\|^2 \Big\| \int_{\mathbb{R}^d} \Phi(2^n x - l)\widetilde{\Phi}(2^n x - l')^T dx - 2^{-nd}\delta_{ll'} I\Big\|\Big)^{1/2} \times \\ & \Big(\sum_{l,l'\in \boldsymbol{Z}^{d'}} \|\tilde{c}_{n,l'}^{k'}\|^2 \Big\| \int_{\mathbb{R}^d} \Phi(2^n x - l)\widetilde{\Phi}(2^n x - l')^T dx - 2^{-nd}\delta_{ll'} I\Big\|\Big)^{1/2} \\ \leq & C 2^{-nd} \Big(\sum_{l\in \boldsymbol{Z}^{d'}} \|c_{n,l}^k\|^2 \Big)^{1/2} \times \Big(\sum_{l'\in \boldsymbol{Z}^{d'}} \|\tilde{c}_{n,l'}^{k'}\|^2 \Big)^{1/2} \\ \leq & C 2^{-nd} \Big(\int_{|\xi|\leq\pi} \Big\| \sum_{l\in \boldsymbol{Z}^{d'}} c_{n,l}^k e^{-il\xi} \Big\|^2 d\xi \Big)^{1/2} \times \Big(\int_{|\xi|\leq\pi} \Big\| \sum_{l'\in \boldsymbol{Z}^{d'}} \tilde{c}_{n,l'}^{k'} e^{-il'\xi} \Big\|^2 d\xi \Big)^{1/2} \\ = & C \Big(\int_{|\xi|\leq 2^{n-1}\pi} \|H(\xi)\cdots H(2^{-n+1}\xi)\|^2 d\xi \Big)^{1/2} \Big(\int_{|\xi|\leq 2^{n-1}\pi} \|\tilde{H}(\xi)\cdots \tilde{H}(2^{-n+1}\xi)\|^2 d\xi \Big)^{1/2} \\ \to & 0, \quad as \quad n\to\infty, \end{aligned}$$

where the third inequality follows

$$\| \int_{\mathbb{R}^d} \Phi(x-l)\widetilde{\Phi}(x-l')^T dx - \delta_{ll'} I \| \le \begin{cases} C, & \text{when} \quad |l-l'| \le C, \\ 0, & \text{otherwise}, \end{cases}$$

where C is a constant independent of l and $l' \in \mathbb{Z}^{d'}$.

By the proof of Theorem 4.1, we have

Theorem 4.2 *Let $\Phi, \tilde{\Phi}, G$ and $\tilde{G}$ as in (26). If*

$$\int_{\mathbb{R}^d} G(2^n x - k)\tilde{G}(x)^T dx = \int_{\mathbb{R}^d} G(x)\tilde{G}(2^n x - k)^T dx = 0, \quad \forall\, k \in \mathbb{Z}^{d'},\ n \ge 1. \tag{30}$$

and

$$\Big(\int_{|\xi| \le 2^n \pi} \|H(\xi)\cdots H(2^{-n}\xi)\|^2 d\xi\Big) + \Big(\int_{|\xi| \le 2^n \pi} \|\tilde{H}(\xi)\cdots \tilde{H}(2^{-n}\xi)\|^2 d\xi\Big) \to 0$$

as n tends to infinity, then (29) holds.

The conditions (29) and (30) on G and $\tilde{G}$ seems not easy to check in general, but practical in some applications such as the construction of biorthogonal wavelet basis on non-uniform grid(see [CES]).

When G and $\tilde{G}$ are linear combinations of integer translates of solutions of refinement equations, the condition (30) and the restricted biorthogonality can be formulated in simple form.

Example 4.3 *Let $\Psi(x) = (\psi_1(x), \cdots, \psi_{N_1}(x))^T$ and $\widetilde{\Psi}(x) = (\tilde{\psi}_1(x), \cdots, \tilde{\psi}_{N_1}(x))^T$ be the compactly supported solutions of the refinement equation*

$$\Psi(x) = \sum_{k \in \mathbf{Z}^d} e_k \Psi(2x-k) \tag{31}$$

and

$$\widetilde{\Psi}(x) = \sum_{k \in \mathbf{Z}^d} \tilde{e}_k \widetilde{\Psi}(2x-k). \tag{32}$$

Assume that Ψ and $\widetilde{\Psi}$ are biorthogonal on $\mathbb{Z}^d$, i.e.,

$$\int_{\mathbb{R}^d} \Psi(x-k)\widetilde{\Psi}(x-k')^T dx = \delta_{kk'}, \quad \forall\, k, k' \in \mathbb{Z}^d. \tag{33}$$

Then

$$2^{-d} \sum_{k \in \mathbf{Z}^d} e_k \tilde{e}^T_{k+2l} = \delta_{l0} I. \tag{34}$$

Let $E(\xi)$ and $\widetilde{E}(\xi)$ be the symbol of the refinement equation (31) and (32) respectively. Then we can write (34) as

$$\sum_{\epsilon \in \{0,1\}^d} E(\xi + \epsilon\pi) \overline{\widetilde{E}(\xi + \epsilon\pi)}^T = I. \tag{35}$$

Assume that G and $\tilde{G}$ in the nonhomogeneous refinement equations (26) are linear combinations of $\Psi(2x-k)$ and $\widetilde{\Psi}(2x-k)$ respectively. In other words there exists a family of matrices d_k and $\tilde{d}_k$ such that

$$G(x) = \sum_{k \in \mathbf{Z}^d} d_k \Psi(2x-k) \tag{36}$$

and

$$\tilde{G}(x) = \sum_{k \in \mathbf{Z}^d} \tilde{d}_k \widetilde{\Psi}(2x-k). \tag{37}$$

By computation and (33), we obtain

$$\int_{\mathbb{R}^d} G(x-l)\tilde{G}(x-l')dx = \sum_{k \in \mathbf{Z}^d} d_{k-2l} \tilde{d}^T_{k-2l'}.$$

Thus the discrete biorthogonality (27) of the mask $\{c_k\}$ and $\{\tilde{c}_k\}$ of the nonhomogeneous refinement equation (26) with G and $\widetilde{G}$ in (36) and (37) reduces to

$$2^{-d} \sum_{k \in \mathbf{Z}^{d'}} c_k \tilde{c}^T_{k+2l} = \delta_{l0} - 2^{-d} \sum_{k \in \mathbf{Z}^d} d_k \tilde{d}^T_{k+2l}, \quad \forall\ l \in \mathbb{Z}^{d'}. \tag{38}$$

Set

$$D(\xi) = 2^{-d} \sum_{k \in \mathbf{Z}^d} d_k e^{-ik\xi}$$

and

$$\widetilde{D}(\xi) = 2^{-d} \sum_{k \in \mathbf{Z}^d} \tilde{d}_k e^{-ik\xi}.$$

Then the equation (38), or the discrete biorthogonality (27), can be written as

$$\begin{aligned} & I - \pi^{-d+d'} \int_{[-\pi,\pi]^{d-d'}} \sum_{\epsilon\in\{0,1\}^{d'}} D(\xi'+\epsilon\pi,\eta)\overline{\widetilde{D}(\xi'+\epsilon\pi,\eta)}^T d\eta \\ = \; & 2^{d-d'} \sum_{\epsilon\in\{0,1\}^{d'}} H(\xi'+\epsilon\pi)\overline{\widetilde{H}(\xi'+\epsilon\pi)}^T, \quad \forall\ \xi' \in I\!R^{d'}. \end{aligned} \tag{39}$$

Observe that

$$\begin{aligned} & \int_{I\!R^d} G(2^n x - k)\widetilde{G}(x)^T dx \\ = \; & \sum_{k',l\in \mathbf{Z}^d} d_{k'-2k} \int_{I\!R^d} \Psi(2^{n+1}x-k')\widetilde{\Psi}(2x-l)^T \tilde{d}_l^T dx \\ = \; & \sum_{k',l,l'\in \mathbf{Z}^d} d_{k'-2k} \int_{I\!R^d} \Psi(2^{n+1}x-k')\widetilde{\Psi}(2^{n+1}x-l')^T (c_{n,l'}^l)^T \tilde{d}_l^T dx, \end{aligned}$$

where $\widetilde{\Psi}(x-l) = \sum_{l'\in \mathbf{Z}^d} c_{n,l'}^l \widetilde{\Psi}(2^n x - l')$. *Then the equation (30) can be rewritten as*

$$\begin{aligned} \int_{[-\pi,\pi]^{d-d'}} & \sum_{\epsilon\in\{0,1\}^{d'}} D(\xi'+\epsilon\pi,\eta)\overline{\tilde{E}(\xi'+\epsilon\pi,\eta)}^T \\ & \overline{\tilde{E}(2\xi',2\eta)}^T \cdots \overline{\tilde{E}(2^{n-1}\xi',2^{n-1}\eta)}^T \overline{\tilde{D}(2^n\xi',2^n\eta)}^T d\eta = 0 \end{aligned} \tag{40}$$

and

$$\begin{aligned} \int_{[-\pi,\pi]^{d-d'}} & \sum_{\epsilon\in\{0,1\}^{d'}} D(2^n\xi',2^n\eta)E(2^{n-1}\xi',2^{n-1}\eta)\cdots E(2\xi',2\eta) \\ & E(\xi'+\epsilon\pi,\eta)\overline{\tilde{D}(\xi'+\epsilon\pi,\eta)}^T d\eta = 0, \quad \forall\ n \geq 1. \end{aligned} \tag{41}$$

Obviously the equations (40) and (41) hold when

$$\sum_{\epsilon\in\{0,1\}^{d'},\epsilon'\in\{0,1\}^{d-d'}} D(\xi'+\epsilon\pi,\eta+\epsilon'\pi)\overline{\tilde{E}(\xi'+\epsilon\pi,\eta+\epsilon'\pi)}^T = 0 \tag{42}$$

and

$$\sum_{\epsilon\in\{0,1\}^{d'},\epsilon'\in\{0,1\}^{d-d'}} E(\xi'+\epsilon\pi,\eta+\epsilon'\pi)\overline{\tilde{D}(\xi'+\epsilon\pi,\eta+\epsilon'\eta)}^T = 0. \tag{43}$$

When $d = d'$, the equation (35), (39), (42) and (43) is equivalent to

$$\sum_{\epsilon \in \{0,1\}^d} \begin{pmatrix} H(\xi + \epsilon\pi) & D(\xi + \epsilon\pi) \\ 0 & E(\xi + \epsilon\pi) \end{pmatrix} \overline{\begin{pmatrix} \widetilde{H}(\xi + \epsilon\pi) & \widetilde{D}(\xi + \epsilon\pi) \\ 0 & \widetilde{E}(\xi + \epsilon\pi) \end{pmatrix}}^T = I. \quad (44)$$

This is the necessary condition of biorthogonality of refinable vectors $(\Phi^T, \Psi^T)^T$ and $(\tilde{\Phi}^T, \tilde{\Psi}^T)^T$ (see [CDP], [HSW], [LCY] and references therein). Obviously the conditions (39)-(41) is strictly weaker than the necessary condition (44) to the biorthogonality of refinable vectors $(\Phi^T, \Psi^T)^T$ and $(\tilde{\Phi}^T, \tilde{\Psi}^T)^T$ when $d' < d$.

5 Examples

In this section, we shall construct a new class of biorthogonal wavelets on a non-uniform grid on the line by modifying cardinal refinable functions on uniform grid, consider its unit decomposition, regularity and characterization of Sobolev spaces. Also we construct refinable functions with arbitrary regularity on any non-uniform grids of semi-regular case.

First we recall the construction of biorthogonal wavelet basis on the uniform grid (see [CDF] and references therein). Let ϕ be the solution the refinement equation

$$\phi(x) = \sum_{k=-2N_2-1}^{2N_2+1} c_k \phi(2x-k) \tag{45}$$

with $c_{2k} = \delta_{k0}$ and $c_k = c_{-k}, |k| \le 2N_2+1$. Assume that ϕ be square integrable and cardinal, which means

$$\phi(k) = \delta_{k0}, \quad \forall k \in \mathbb{Z},$$

and satisfy $\int_{\mathbb{R}} \phi(x)dx = 1$. Then ϕ is symmetric about the origin and the mask $c_k, -2N_2-1 \le k \le 2N_2+1$ satisfies $\sum_{k=0}^{N_2} c_{2k+1} = 1/2$.

Set $\phi_{n,k}(x) = \phi(2^n x - k)$. Then $\phi_{n,k}, k \in \mathbb{Z}$ is linear combination of $\phi_{n+1,l}, l \in \mathbb{Z}$,

$$\phi_{n,k} = \sum_{l\in\mathbb{Z}} c_{l-2k}\phi_{n+1,l}. \tag{46}$$

Let V_n be the closed subspace of L^2 spanned by $\phi_{n,k}, k \in \mathbb{Z}$. Then

$$V_n \subset V_{n+1}, \ \forall \ n \in \mathbb{Z}$$

by (46). Define operators $P_{n,1}$ by

$$P_{n,1}(f)(x) = \sum_{k\in\mathbb{Z}} f(2^{-n}k)\phi_{n,k}(x).$$

Recall that ϕ is cardinal. Then

$$P_{n,1}f(x) = f(x), \quad \forall \ f \in V_n.$$

Define $Q_{n,1}f(x) = P_{n+1,1}f(x) - P_{n,1}f(x)$. Then

$$Q_{n,1}f(x) = \sum_{k\in\mathbb{Z}} \Big(f(2^{-n-1}(2k+1)) - \sum_{l\in\mathbb{Z}} c_{2k+1-2l} f(2^{-n}l)\Big)\phi_{n+1,2k+1}(x).$$

Obviously $Q_{n,1}f$ is not well-defined for square integrable function f. So $Q_{n,1}$ is not appropriately used to characterize L^2 or Sobolev spaces.

Let $\tilde{\phi}$ be the solution of another refinement equation

$$\tilde{\phi}(x) = \sum_{k=-N_3}^{N_3} \tilde{c}_k \tilde{\phi}(2x-k), \tag{47}$$

with $\tilde{c}_k = \tilde{c}_{-k}$. Furthermore we assume that $\tilde{\phi}$ is square integrable and satisfies $\int_{\mathbb{R}} \tilde{\phi}(x)dx = 1$, and that ϕ and $\tilde{\phi}$ are biorthogonal, i.e.,

$$\int_{\mathbb{R}} \phi(x-k)\tilde{\phi}(x-k')dx = \delta_{kk'}, \quad \forall\ k, k' \in \mathbb{Z}.$$

Then

$$\sum_{k\in\boldsymbol{Z}} c_k \tilde{c}_{k+2l} = 2\delta_{l0}. \tag{48}$$

Let H and $\tilde{H}$ be the symbol of the refinement equation (45) and (47) respectively. Then we may write (48) as

$$H(\xi)\overline{\tilde{H}(\xi)} + H(\xi+\pi)\overline{\tilde{H}(\xi+\pi)} = 1. \tag{49}$$

Furthermore there exists a trigonomometric polynomial $d(\xi)$ such that

$$\begin{cases} \tilde{H}(\xi) = 1 + d(\xi)H(\xi+\pi), \\ d(\xi) = d(-\xi), \\ d(\xi) + d(\xi+\pi) = 0, \\ d(0) = -d(\pi) = 1. \end{cases} \tag{50}$$

Set $\tilde{\phi}_{n,k}(x) = 2^n\tilde{\phi}(2^n x - k)$. Define operators $P_{n,2}$ by

$$P_{n,2}f(x) = \sum_{k\in\boldsymbol{Z}} \phi_{n,k}(x) \int_{\mathbb{R}} f(y)\tilde{\phi}_{n,k}(y)dy.$$

Then by the biorthogonality of ϕ and $\tilde{\phi}$, we have

$$P_{n,2}f(x) = f(x), \quad \forall\ f \in V_n.$$

Set

$$\psi(x) = \sum_{k\in\boldsymbol{Z}} \tilde{c}_k(-1)^k\phi(2x-1-k)$$

and

$$\tilde{\psi}(x) = \sum_{k\in \boldsymbol{Z}} c_k(-1)^k \tilde{\phi}(2x-1-k)$$

Define $Q_{n,2} = P_{n+1,2} - P_{n,2}$. By computation, we have

$$Q_{n,2}f(x) = \sum_{k\in \boldsymbol{Z}} \psi_{n,k}(x) \int_{\mathbb{R}} f(y)\tilde{\psi}_{n,k}(y)dy$$

where $\psi_{n,k}(x) = \psi(2^n x - k)$ and $\tilde{\psi}_{n,k}(x) = 2^n\tilde{\psi}(2^n x - k)$. Obviously $Q_{n,2}f$ is well defined for square integrable function f. Furthermore we can use operators $Q_{n,2}$ to characterize L^2 and certain type of Besov spaces. In fact the L^2 norm of $\left(\sum_{n\geq 0} |Q_{n,2}f(\cdot)|^2 + |P_{0,2}f(\cdot)|^2\right)^{1/2}$ is equivalent to the one of f when ϕ and $\tilde{\phi} \in L^{2,\alpha}$ for some $\alpha > 0$.

Now we restrict ourselves to consider the construction of biorthogonal wavelets on the following simple non-uniform grid: the length of the grid of scale n at left side of the origin is $2^{-n}b$ and the one of the right hand side is $2^{-n}a$, where $a \neq b$. The construction of biorthogonal wavelets on non-uniform initial grid with uniform dyadic subdivision will be given in Remark 5.8.

First we modify the primal refinable function $\phi_{n,k}$ on the uniform grid with grid length 1 to the one on the non-uniform grid above. Set

$$\phi_{n,k}^{new}(x) = \begin{cases} \phi(2^n b^{-1}x - k), & x \leq 0, \\ \phi(2^n a^{-1}x - k), & x \geq 0. \end{cases}$$

Then by (46) $\phi_{n,k}^{new}$ satisfies the refinement equation

$$\phi_{n,k}^{new}(x) = \sum_{l\in \boldsymbol{Z}} c_{l-2k}\phi_{n+1,l}^{new}(x). \tag{51}$$

By the unit decomposition $1 = \sum_{k\in \boldsymbol{Z}} \phi_{n,k}(x)$ for $\phi_{n,k}$, we obtain the unit decomposition

$$1 = \sum_{k\in \boldsymbol{Z}} \phi_{n,k}^{new}(x)$$

for $\phi_{n,k}^{new}$.

A lazy method to construct dual refinable function $\tilde{\phi}^{lazy}_{n,k}$ is to set

$$\tilde{\phi}^{lazy}_{n,k}(x) = \begin{cases} 2^n b^{-1}\tilde{\phi}(2^n b^{-1}x - k), & x \le 0, \\ 2^n a^{-1}\tilde{\phi}(2^n a^{-1}x - k), & x \ge 0. \end{cases}$$

Then

$$\tilde{\phi}^{lazy}_{n,k}(x) = \frac{1}{2}\sum_{l\in \mathbb{Z}} \tilde{c}_{l-2k}\tilde{\phi}^{lazy}_{n+1,l}(x), \quad \forall\ n \ge 1,\ k \in \mathbb{Z}$$

and

$$\int_{\mathbb{R}} \phi^{new}_{n,k}(x)\tilde{\phi}^{lazy}_{j,k'}(x)dx = \delta_{kk'}.$$

In wavelet approximation, the unit decomposition is very important. But in this case the unit decomposition does not hold for $\tilde{\phi}^{lazy}_{n,k}$.

Define

$$L(x) = \begin{cases} a^{-1}x, & x \ge 0, \\ b^{-1}x, & x \ge 0. \end{cases}$$

Then $\phi^{new}_{n,k}(x) = \phi_{n,k}(L(x))$ and $\phi^{lazy}_{n,k}(x) = |L'(x)|^{-1}\tilde{\phi}_{n,k}(L(x))$. This shows that $\tilde{\phi}^{lazy}_{j,k}$ is constructed by using the Lipschitz transform L from the non-uniform grid above to the uniform grid with length 1. The Lipschitz transform above doesn't always exist for the domain decomposition in high dimensions, such as triangulation of bounded polygon. This inspires us to construct new type of biorthogonal wavelet basis. Here we use the lifting scheme in [SS1] to construct new class of biorthogonal wavelets and find that the dual refinable function at the intersection of different type of grid satisfies a nonhomogeneous refinment equation of the form

$$\phi(x) = \phi(2x) + G(x).$$

For the construction of dual refinable functions, we define $d_l(\xi), l \in \mathbb{Z}$ at first.

- For $l \ge N_3$, we use $d(\xi)$ in (50) as $d_l(\xi)$.
- For $2N_2 + 1 \le l \le N_3$, $d_l(\xi) = \sum_{k\in \mathbb{Z}} d_{l,k}e^{ik\xi}$ are chosen that $d_l(\xi) + d_l(\xi + \pi) = 0$, $deg^+ d_l \le N_3 - 2N_2 + 1$, $1 \ge deg^- d_l \ge -l + 2N_2 + 2$ and $\sum_{l\ge 2N_2+1} d_{l,j-2l} = 1$ when $j \ge 4N_2 + 3$ is odd, where deg^+P and deg^-P denote the upper and lower degree of a trigonometric polynomial P respectively.

- For $0 < l \leq 2N_2$, $d_l(\xi)$ is chosen as 0.
- For $l = 0$, $d_0(\xi)$ is defined by

$$d_0(\xi) = \sum_{k=0}^{2N_2} (d_{0,2k+1} e^{i(2k+1)\xi} + d_{0,-2k-1} e^{-i(2k+1)\xi}),$$

where

$$d_{0,2k+1} = 2(b+a)^{-1}(a \int_{-2k-1}^{\infty} \phi(x)dx + b \int_{-\infty}^{-2k-1} \phi(x)dx)$$

and

$$d_{0,-2k-1} = 2(b+a)^{-1}(a \int_{2k+1}^{\infty} \phi(x)dx + b \int_{-\infty}^{2k+1} \phi(x)dx), \quad 0 \leq k \leq 2N_2.$$

- For $l < 0$, we set $d_l(\xi) = d_{-l}(-\xi)$.

Write

$$d_l(\xi) = \sum_{k \in \boldsymbol{Z}} d_{l,k} e^{ik\xi}$$

and

$$\tilde{H}_l(\xi) = 1 + d_l(\xi) H(\xi + \pi) = \sum_{k \in \boldsymbol{Z}} \tilde{h}_{l,k} e^{ik\xi}, \quad l \in \mathbb{Z}.$$

Then

$$\tilde{h}_{l,k} = \delta_{k0} + \frac{1}{2} \sum_{s \in \boldsymbol{Z}} d_{l,s} c_{k-s} (-1)^{k-s} \tag{52}$$

and

$$\tilde{h}_{l,k} = 0 \tag{53}$$

when $k \leq -l$ and $l > 0$ or when $k \geq l$ and $l < 0$.

Let $\tilde{\phi}_{n,k}^{new}$ be the solution of the refinement equation

$$\tilde{\phi}_{n,k}^{new}(x) = \frac{1}{2} \sum_{l \in \boldsymbol{Z}} \tilde{h}_{k,l-2k} \tilde{\phi}_{n+1,l}^{new}(x) \tag{54}$$

with

$$\tilde{\phi}_{n,k}^{new}(x) = \begin{cases} 2^n b^{-1} \tilde{\phi}(2^n b^{-1} x - k), & when \quad k \leq -N_3, \\ 2^n a^{-1} \tilde{\phi}(2^n a^{-1} x - k), & when \quad k \geq N_3. \end{cases} \tag{55}$$

Then $\tilde{\phi}_{n,k}^{new}$ is linear combination of $\tilde{\phi}_{n+1,l}^{new}$ with $|l| \geq |k|+1$ when $k \neq 0$ by (53) and (54). Thus there exists an integer N_4 such that $\tilde{\phi}_{n,k}^{new}$ is linear combination of $\tilde{\phi}_{n+N_4,l}^{new}$ with $|l| \geq N_3$ when $k \neq 0$. Hence $\tilde{\phi}_{n,k}^{new}, k \neq 0$ is well-defined for $k \neq 0$ and satisfies the refinement equation (54)

By (52), the symmetry of ϕ and $\sum_{k=0}^{N_2} c_{2k+1} = \frac{1}{2}$, we obtain

$$\begin{aligned}\tilde{h}_{0,0} &= 1 + \frac{1}{2}\sum_{k=0}^{2N_2}(d_{0,2k+1}c_{-2k-1}(-1)^{-2k-1} + d_{0,-2k-1}c_{2k+1}(-1)^{2k+1}) \\ &= 1 - \frac{1}{2}\sum_{k=0}^{2N_2} c_{2k+1}(d_{0,2k+1} + d_{0,-2k-1}) = \frac{1}{2}.\end{aligned}$$

Thus

$$\tilde{\phi}_{n,0}^{new}(x) = \frac{1}{2}\tilde{\phi}_{n+1,0}^{new}(x) + G_n(x), \tag{56}$$

where $G_n(x) = \sum_{l \neq 0} \tilde{h}_{0,l}\tilde{\phi}_{n+1,l}^{new}(x)$. Observe that $\tilde{\phi}_{n+1,0}^{new}$ is the corresponding function of $\tilde{\phi}_{n,0}^{new}$ of level n with corresponding initial grid lengths $b/2$ and $a/2$. Thus $\tilde{\phi}_{n+1,0}^{new}(x) = 2\tilde{\phi}_{n,0}^{new}(2x)$ and $\tilde{\phi}_{n,0}^{new}$ satisfies the nonhomogeneous refinement equation

$$\tilde{\phi}_{n,0}^{new}(x) = \tilde{\phi}_{n,0}^{new}(2x) + G_n(x).$$

By Theorem 2.1 and Theorem 3.1, the nonhomogeneous refinement equation above is solvable in L^2 when $G_n \in L^{2,\alpha}$ for some $\alpha > 0$. Furthermore $\tilde{\phi}_{n,0}^{new}(x)$ belongs to $L^{2,min(\alpha,1/2-\epsilon)}$ for any $\epsilon > 0$ and for any $0 \leq \gamma < \min(\alpha, 1/2)$ there exists a constant C such that

$$\|\tilde{\phi}_{n,k}^{new}\|_{L^{2,\gamma}} \leq C2^{n(1/2+\gamma)}$$

when $\tilde{\phi}(x) \in L^{2,\alpha}$. This completes the construction of dual refinable function $\tilde{\phi}_{n,k}^{new}$. For $\tilde{\phi}_{n,k}^{new}$, we have the biorthogonality with $\phi_{n,k}^{new}$ (Proposition 5.1), the unit decomposition (Proposition 5.6), regularity (Proposition 5.7) and the characterization of Sobolev spaces (Theorem 5.8). To our surprise, the regularity of $\tilde{\phi}_{n,k}^{new}$ is the same as $\tilde{\phi}$. This shows that our construction is better than the one by lazy method, where $\tilde{\phi}_{n,k}^{lazy}$ is not continuous even if $\tilde{\phi}$ is for really non-uniform grid.

Proposition 5.1 *Let $\tilde{\phi}^{new}_{n,k}$ be defined by (54) and (55). Then we have*

$$\int_{\mathbb{R}} \phi^{new}_{n,k}(x)\tilde{\phi}^{new}_{n,k'}(x)dx = \delta_{kk'}, \quad \forall\ k, k' \in \mathbb{Z}. \tag{57}$$

Proof By (52), we have

$$\sum_{l\in\boldsymbol{Z}} c_{l-2k}\tilde{h}_{k',l-2k'} = \sum_{l\in\boldsymbol{Z}} c_{l-2k}\delta_{l(2k')} + \frac{1}{2}\sum_{s \quad odd} d_{k',s}(\sum_{l\in\boldsymbol{Z}} c_{l-2k}c_{l-2k'-s}(-1)^{l-s}) = \delta_{k,k'}.$$

For $m \geq 1$, by (51) and (54) we may write

$$\phi^{new}_{n,k} = \sum_{l\in\boldsymbol{Z}} c^k_{m,l}\phi^{new}_{n+m,l},$$

and

$$\tilde{\phi}^{new}_{n,k} = \sum_{l\in\boldsymbol{Z}} \tilde{c}^k_{m,l}\tilde{\phi}^{new}_{n+m,l}.$$

Then $c^k_{m,l} = \sum_{s\in\boldsymbol{Z}} c^k_{m-1,s}c_{l-2s}$ and $\tilde{c}^k_{m,l'} = \sum_{s'\in\boldsymbol{Z}} \tilde{c}^k_{m-1,s'}\tilde{h}_{s',l'-2s'}$. By induction, we have

$$\sum_{l\in\boldsymbol{Z}} c^k_{m,l}\tilde{c}^{k'}_{m,l} = \delta_{kk'}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}} \phi^{new}_{n,k}(x)\tilde{\phi}^{new}_{n,k'}(x)dx - \delta_{kk'} \\
= &\sum_{l,l'\in\boldsymbol{Z}} c^k_{m,l}\tilde{c}^{k'}_{m,l'}(\int_{\mathbb{R}} \phi^{new}_{n+m,l}(x)\tilde{\phi}^{new}_{n+m,l'}(x)dx - \delta_{ll'}).
\end{aligned} \tag{58}$$

By the biorthogonality of ϕ and $\tilde{\phi}$ and the definition $\tilde{\phi}^{new}_{n,k'}$ for $|k'| \geq N_3$, we have

$$\int_{\mathbb{R}} \phi^{new}_{n,k}(x)\tilde{\phi}^{new}_{n,k'}(x)dx = \delta_{kk'}, \quad \forall\ |k'| \geq N_3.$$

Recall that for $k' \neq 0$, $\tilde{\phi}^{new}_{n,k'}$ is linear combination of $\tilde{\phi}^{new}_{n+N_4,l}$ with $|l| \geq N_3$. Then by letting $m = N_4$ in (58), we obtain

$$\int_{\mathbb{R}} \phi^{new}_{n,k}(x)\tilde{\phi}^{new}_{n,k'}(x)dx = \delta_{kk'}, \quad \forall\ 1 \leq |k'| \leq N_3.$$

For $k' = 0$, we have

$$
\begin{aligned}
& |\int_{\mathbb{R}} \phi_{n,k}^{new}(x)\tilde{\phi}_{n,0}^{new}(x)dx - \delta_{k0}| \\
\leq\ & |\sum_{l\in\boldsymbol{Z}} c_{m,l}^{k}\tilde{c}_{m,0}^{0}\Big(\int_{\mathbb{R}} \phi_{n+m,l}^{new}(x)\tilde{\phi}_{n+m,0}^{new}(x)dx - \delta_{l0}\Big)| \\
=\ & 2^{-m}|\int_{\mathbb{R}} \phi_{n,k}^{new}(x)\tilde{\phi}_{n+m,0}^{new}(x)dx - \int_{\mathbb{R}} \phi_{n,k}(x)\tilde{\phi}_{n+m,0}(x)dx| \\
\leq\ & 2^{-m/2}\Big(\|\phi_{n,k}^{new}\|_2\|\tilde{\phi}_{n,0}^{new}\|_2 + \|\phi_{n,k}\|_2\|\tilde{\phi}_{n,0}\|_2\Big) \to 0, \quad as \quad m\to\infty,
\end{aligned}
$$

where the equality holds because of $\tilde{c}_{m,0} = 2^{-m}$ and

$$\int_{\mathbb{R}} \phi_{n,k}(x)\tilde{\phi}_{n+m,0}(x)dx = \sum_{l\in\boldsymbol{Z}} c_{m,l}^{k}\int_{\mathbb{R}} \phi_{n+m,l}(x)\tilde{\phi}_{n+m,0}(x)dx = c_{m,0}^{k}.$$

Hence

$$\int_{\mathbb{R}} \phi_{n,k}^{new}(x)\tilde{\phi}_{n,k'}^{new}(x)dx = \delta_{kk'}$$

and Proposition 5.1 is proved.

Proposition 5.2 *Let $\tilde{\phi}_{n,k}^{new}$ be defined by (54) and (55). Then $\tilde{\phi}_{n,k}^{new}$ satisfies the normalized condition*

$$\int_{\mathbb{R}} \tilde{\phi}_{n,k}^{new}(x)dx = 1. \tag{59}$$

Proof For $k \neq 0$, we prove the assertion (59) by induction. By the definition of $\tilde{\phi}_{n,k}^{new}$ for $k \geq N_3$, the equality holds for $k \geq N_3$. Inductively we assume that (59) holds for all $k \geq k_0 \geq 2$. Recall that

$$\tilde{\phi}_{n,k_0-1}^{new} = \sum_{l\geq k_0} \tilde{h}_{k_0-1,l-2k_0+2}\tilde{\phi}_{n+1,l}^{new}$$

and $\sum_{l\geq k_0} \tilde{h}_{k_0-1,l-2k_0+2} = \tilde{H}_{k_0-1}(0) = 1$. Then (59) holds for $k = k_0 - 1$. This proves the assertion for $k \geq 1$. By the same procedure as above, we may prove the assertion for $k \leq -1$.

By (56), we have

$$
\begin{aligned}
\int_{\mathbb{R}} \tilde{\phi}_{n,0}^{new}(x)dx &= \int_{\mathbb{R}} \tilde{\phi}_{n,0}^{new}(2x)dx + \sum_{l\neq 0}\tilde{h}_{0,l} \\
&= \frac{1}{2}\int_{\mathbb{R}} \tilde{\phi}_{n,0}^{new}(x)dx + \frac{1}{2}.
\end{aligned}
$$

This proves the assertion for $k=0$.

Define wavelets $\psi_{n,k}^{new}$ and $\tilde{\psi}_{n,k}^{new}$ by

$$\left\{ \begin{array}{l} \psi_{n,k}^{new} = 2\phi_{n+1,2k+1}^{new} - \sum_{s\in\boldsymbol{Z}} d_{s,2k+1-s}\phi_{n,s}^{new}, \\ \tilde{\psi}_{n,k}^{new} = \frac{1}{2}\sum_{l\in\boldsymbol{Z}}(-1)^{l+1}c_{2k+1-l}\tilde{\phi}_{n+1,l}^{new}. \end{array} \right. \tag{60}$$

Then we have

Proposition 5.3 *Let $\tilde{\phi}_{n,k}^{new}$, $\tilde{\psi}_{n,k}^{new}$ and $\psi_{n,k}^{new}$ be defined by (54), (55) and (60). Then*

$$\int_{\mathbb{R}} \phi_{n,k}^{new}(x)\tilde{\psi}_{n,k'}^{new}(x)dx = \int_{\mathbb{R}} \psi_{n,k}^{new}(x)\tilde{\phi}_{n,k'}^{new}(x)dx = 0, \ \forall\ k,k' \in \mathbb{Z}$$

and

$$\int_{\mathbb{R}} \psi_{n,k}^{new}(x)\tilde{\psi}_{n,k'}^{new}(x)dx = \delta_{kk'}, \forall\ k,k' \in \mathbb{Z}.$$

Proof By Proposition 5.1 and (60), we have

$$\begin{aligned} & \int_{\mathbb{R}} \phi_{n,k}^{new}(x)\tilde{\psi}_{n,k'}^{new}(x)dx \\ = & \frac{1}{2}\sum_{l,l'\in\boldsymbol{Z}} c_{l-2k}(-1)^{l'+1}c_{2k'+1-l'}\int_{\mathbb{R}} \phi_{n+1,l}^{new}(x)\tilde{\phi}_{n+1,l'}^{new}(x)dx \\ = & \frac{1}{2}\sum_{l\in\boldsymbol{Z}} c_{l-2k}(-1)^{l+1}c_{2k'+1-l} = 0, \end{aligned}$$

and by (54)

$$\begin{aligned} & \int_{\mathbb{R}} \psi_{n,k}^{new}(x)\tilde{\phi}_{n,k'}^{new}(x)dx \\ = & \sum_{l\in\boldsymbol{Z}} \Big(2\delta_{l(2k+1)} - \sum_{s\in\boldsymbol{Z}} d_{s,2k+1-2s}c_{l-2s}\Big) \times \Big(\delta_{l(2k')} + \frac{1}{2}\sum_{t\in\boldsymbol{Z}} d_{k',t}c_{l-2k'-t}(-1)^{l-t}\Big) \\ = & \sum_{t\ \ odd} d_{k',t}c_{2k+1-2k'-t} - \sum_{s\in\boldsymbol{Z}} d_{s,2k+1-2s}c_{2k'-2s} \\ & -\frac{1}{2}\sum_{s\in\boldsymbol{Z},\ t\ odd} d_{s,2k+1-2s}d_{k',t}\big(\sum_{l\in\boldsymbol{Z}} c_{l-2s}c_{l-2k'-t}(-1)^{l-t}\big) \\ = & d_{k',2k+1-2k'} - d_{k',2k+1-2k'} + 0 = 0. \end{aligned}$$

Similarly by (60) and Proposition 5.1, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \psi_{n,k}^{new}(x)\tilde{\psi}_{n,k'}^{new}(x)dx \\
=& \sum_{l\in\boldsymbol{Z}}\Big(2\delta_{l(2k+1)} - \sum_{s\in\boldsymbol{Z}} d_{s,2k+1-2s}c_{l-2s}\Big)(\frac{1}{2})(-1)^{l+1}c_{2k'+1-l} \\
=& \delta_{kk'} - \frac{1}{2}\sum_{s\in\boldsymbol{Z}} d_{s,2k+1-2s}(\sum_{l\in\boldsymbol{Z}}(-1)^{l+1}c_{l-2s}c_{2k'+1-l}) = \delta_{kk'}.
\end{aligned}$$

Proposition 5.4 *Let* $\tilde{\psi}_{n,k}^{new}$ *and* $\psi_{n,k}^{new}$ *be defined by (60). Then*

$$\int_{\mathbb{R}} \tilde{\psi}_{n,k}^{new}(x)dx = 0$$

and

$$\int_{\mathbb{R}} \psi_{n,k}^{new}(x)dx = 0.$$

Proof Obviously the first assertion follows from (60), Proposition 5.2 and $\sum_{k\in\boldsymbol{Z}} c_{2k+1} = \sum_{k\in\boldsymbol{Z}} c_{2k} = 1$.

By (50) and

$$\int_{\mathbb{R}} \phi_{n,k}^{new}(x)dx = 2^{-n}b\int_{-\infty}^{-2k-1}\phi(x)dx + 2^{-n}a\int_{-2k-1}^{\infty}\phi(x)dx,$$

we have

$$\begin{aligned}
&\int_{\mathbb{R}} \psi_{n,k}^{new}(x)dx \\
=& 2^{-n}ba_k + 2^{-n}a(1-a_k) - d_{0,2k+1}2^{-n-1}(b+a) \\
&-2^{-n}\sum_{s=2N_2+1}^{\infty}(bd_{s,2k+1-2s} + ad_{s,-2k-1-2s}), \\
=& \begin{cases} 2^{-n}a - 2^{-n}a\sum_{s=2N_2+1}^{\infty} d_{s,2k+1-2s}, & k \geq 2N_2+1 \\ 2^{-n}ba_k + 2^{-n}a(1-a_k) - d_{0,2k+1}2^{-n-1}(b+a), & -2N_2-1 \leq k \leq 2N_2, \\ 2^{-n}b - 2^{-n}b\sum_{s=2N_2+1}^{\infty} d_{s,-2k-1-2s}, & k \leq -2N_2-2, \end{cases} \\
=& 0,
\end{aligned}$$

where $a_k = \int_{-\infty}^{-2k-1}\phi(x)dx$. This completes the proof.

By computation, we get

$$\sum_{k\in\boldsymbol{Z}} c_{l-2k}(\delta_{l'(2k)} + \frac{1}{2}\sum_{s\in\boldsymbol{Z}} d_{k,s}c_{l'-2k-s}(-1)^{l'-s}) + \sum_{k\in\boldsymbol{Z}}(-1)^{l'+1}c_{2k+1-l} \times$$
$$(2\delta_{l(2k+1)} - \sum_{s\in\boldsymbol{Z}} d_{s,2k+1-2s}c_{l-2s}) = \delta_{ll'}, \quad \forall\ l, l' \in \mathbb{Z}.$$

Thus we have the reconstruction formula.

Proposition 5.5 *Let $\tilde{\phi}^{new}_{n,k}$ be defined by (54) and (55), and let $\tilde{\psi}^{new}_{n,k}$ and $\psi^{new}_{n,k}$ be defined by (60). Then we have the reconstruction formula*

$$\phi^{new}_{n+1,l} = \sum_{k\in\boldsymbol{Z}}(\delta_{l(2k)} + \frac{1}{2}\sum_{s\in\boldsymbol{Z}} d_{k,s}c_{l-2k-s}(-1)^{l-s})\phi^{new}_{n,k} + \sum_{k\in\boldsymbol{Z}}(-1)^{l+1}c_{2k+1-l}\psi^{new}_{n,k}$$

and

$$\tilde{\phi}^{new}_{n+1,l} = \sum_{k\in\boldsymbol{Z}} c_{l-2k}\tilde{\phi}^{new}_{n,k} + \sum_{k\in\boldsymbol{Z}}(2\delta_{l(2k+1)} - \sum_{s\in\boldsymbol{Z}} d_{s,2k+1-2s}c_{l-2s})\tilde{\psi}^{new}_{n,k}.$$

Let V^{new}_n be the closed subspace of L^2 spanned by $\phi^{new}_{n,k}, k \in \mathbb{Z}$ and $\tilde{V}^{new}_n$ be the closed subspace of L^2 spanned by $\tilde{\phi}^{new}_{n,k}, k \in \mathbb{Z}$. Then

$$V^{new}_n \subset V^{new}_{n+1}$$

and

$$\tilde{V}^{new}_n \subset \tilde{V}^{new}_{n+1}$$

by (51) and (54). Define

$$P^{new}_n f(x) = \sum_{k\in\boldsymbol{Z}} \phi^{new}_{n,k}(x)\int_{\mathbb{R}} f(y)\tilde{\phi}^{new}_{n,k}(y)dy$$

and

$$\tilde{P}^{new}_n f(x) = \sum_{k\in\boldsymbol{Z}} \tilde{\phi}^{new}_{n,k}(x)\int_{\mathbb{R}} f(y)\phi^{new}_{n,k}(y)dy.$$

Then by Proposition 5.1, we have

$$P^{new}_n f(x) = f(x), \quad \forall\ f \in V^{new}_n$$

and

$$\tilde{P}_n^{new} f(x) = f(x), \quad \forall\ f \in \tilde{V}_n^{new}.$$

By Propositions 5.1, 5.3 and 5.5, we have

$$P_n^{new} P_{n+1}^{new} = P_n^{new}$$

and

$$\tilde{P}_n^{new} \tilde{P}_{n+1}^{new} = \tilde{P}_n^{new}.$$

Let W_n^{new} and $\tilde{W}_n^{new}$ be the orthogonal complement of V_n^{new} in V_{n+1}^{new} and $\tilde{V}_n^{new}$ in $\tilde{V}_{n+1}^{new}$ respectively. Set

$$Q_n^{new} f(x) = \sum_{k \in \boldsymbol{Z}} \psi_{n,k}^{new}(x) \int_{\mathbb{R}} f(y) \tilde{\psi}_{n,k}^{new}(y) dy$$

and

$$\tilde{Q}_n^{new} f(x) = \sum_{k \in \boldsymbol{Z}} \tilde{\psi}_{n,k}^{new}(x) \int_{\mathbb{R}} f(y) \psi_{n,k}^{new}(y) dy.$$

Then

$$Q_n^{new} f(x) = f(x), \quad \forall\ f \in W_n^{new}$$

and

$$\tilde{Q}_n^{new} f(x) = f(x), \quad \forall\ f \in \tilde{W}_n^{new}.$$

Furthermore

$$P_{n+1}^{new} - P_n^{new} = Q_n^{new} \tag{61}$$

and

$$\tilde{P}_{n+1}^{new} - \tilde{P}_n^{new} = \tilde{Q}_n^{new} \tag{62}$$

by Proposition 5.5.

Observe that $(P_n^{new} 1)(x) = (\tilde{P}_n^{new} 1)(x) = 1$ when $2^{-n} N_3 \leq |x| \neq 0$. Then by (61), (62) and Proposition 5.4 we have

Proposition 5.6 *Let P_n^{new} and $\tilde{P}_n^{new}$ be defined above. Then we have the unit decomposition*

$$P_n^{new} 1 = 1$$

and

$$\tilde{P}_n^{new} 1 = 1.$$

Proposition 5.7 *Let $\alpha > 0$ and $\tilde{\phi} \in L^{2,\alpha}$. Then for any $\gamma \leq \alpha$ there exists a constant C such that*

$$\|\tilde{\phi}_{n,k}^{new}\|_{2,\gamma} \leq C2^{n(1/2+\gamma)}\|\tilde{\phi}\|_{2,\gamma}.$$

Proof The assertion for $k \neq 0$ follows from the fact that $\tilde{\phi}_{n,k}^{new}$ is finitely combination of $2^{n+N_3}\tilde{\phi}(2^{n+N_3}\cdot -k)$ with $|k| \geq N_3$ by the construction of $\tilde{\phi}_{n,k}^{new}$.

For $k = 0$, we have

$$\tilde{\phi}_{n,0}^{new} = 1 - \frac{2b}{a+b}\sum_{k>0}\tilde{\phi}_{n,k}^{new} - \frac{2a}{a+b}\sum_{k<0}\tilde{\phi}_{n,k}^{new}$$

by Proposition 5.6. Thus the assertion holds for $k = 0$.

Obviously $P_n^{new}f$ and $\tilde{P}_n^{new}f$ is well-defined for locally square integrable function f. Then by the procedure in [Da], we have the following characterization of $L^{2,\alpha}$.

Theorem 5.8 *Let $\phi \in L^{2,\alpha_0}$ and $\tilde{\phi} \in L^{2,\alpha_1}$ with $\alpha_0, \alpha_1 > 0$. Let P_n^{new}, Q_n^{new} and $\tilde{P}_n^{new}, \tilde{Q}_n^{new}$ be defined as above. Then*

$$\|f\|_{2,\alpha} \approx \|\Big(\sum_{n\geq 0} 2^{2n\alpha}|Q_n^{new}f|^2\Big)^{1/2}\|_2 + \|P_0^{new}f\|_2, \quad -\alpha_1 < \alpha < \min(\alpha_0, 1/2)$$

and

$$\|f\|_{2,\alpha} \approx \|\Big(\sum_{n\geq 0} 2^{2n\alpha}|\tilde{Q}_n^{new}f|^2\Big)^{1/2}\|_2 + \|\tilde{P}_0^{new}f\|_2, \quad -\min(\alpha_0, 1/2) < \alpha < \alpha_1,$$

where $A \approx B$ means there exists an absolute constant C such that $C^{-1}A \leq B \leq CA$.

Remark 5.9 *Let $\{x_k\}$ be initial nodes on the line such that $x_k < x_{k+1}$, $\lim_{k\to+\infty} x_k = +\infty$, $\lim_{k\to-\infty} x_k = -\infty$ and*

$$\infty > \max_{k\in\boldsymbol{Z}} |x_{k+1} - x_k| \geq \min_{k\in\boldsymbol{Z}} |x_{k+1} - x_k| > 0.$$

Define the nodes $x_k^{(n)}$ *of scale* $n \geq 1$ *by* $x_{2k}^{(n)} = x_k^{(n-1)}$ *and* $x_{2k+1}^{(n)} = (x_k^{(n-1)} + x_{k+1}^{(n-1)})/2$*, where* $x_k^{(0)} = x_k$*. Set*

$$L_n(x) = \frac{x - x_k^{(n)}}{x_{k+1}^{(n)} - x_k^{(n)}} + k, \quad x \in [x_k^{(n)}, x_{k+1}^{(n)}].$$

The grid above is appeared in [W] and is classified as non-uniform grid of semi-regular case in [DGS].

Then $2L_n(x) = L_{n+1}(x)$*. Define the primal refinable function* $\phi_{n,k}^{New}$ *on the knot* $x_k^{(n)}$ *by*

$$\phi_{n,k}^{New}(x) = \phi(L_n(x) - k).$$

Then

$$\phi_{n,k}^{New} = \sum_{l \in \mathbb{Z}} c_{l-2k} \phi_{n+1,l}^{New}.$$

By the same procedure as the one in the construction of dual refinable functions $\tilde{\phi}_{n,k}^{new}$ *and wavelets* $\tilde{\psi}_{n,k}^{new}$*, we may construct dual refinable functions* $\tilde{\phi}_{n,k}^{New}$ *and wavelets* $\tilde{\psi}_{n,k}^{New}$ *such that* $\tilde{\phi}_{n,k}^{New}$ *and* $\tilde{\psi}_{n,k}^{New}$ *uniformly in* $L^{2,\min(\alpha,1/2-\epsilon)}$ *when* $\tilde{\phi} \in L^{2,\alpha}$ *and the support* $\tilde{\phi}_{n,k}^{New}$ *and* $\tilde{\psi}_{n,k}^{New}$ *is contained in a interval with center* $x_k^{(n)}$ *and length less than* $C2^{-n}$ *for some constant* C *independent of* n *and* k*. Furthermore Propositions 5.1-5.6 and Theorem 5.7 hold when* $\phi_{n,k}^{new}$*,* $\psi_{n,k}^{new}$*,* $\tilde{\phi}_{n,k}^{new}$ *and* $\tilde{\psi}_{n,k}^{new}$ *are replaced by* $\phi_{n,k}^{New}$*,* $\psi_{n,k}^{New}$*,* $\tilde{\phi}_{n,k}^{New}$ *and* $\tilde{\psi}_{n,k}^{New}$ *respectively.*

Remark 5.10 *Let* h *be the hat function defined by*

$$h(x) = \begin{cases} 1 - |x|, & x \in [-1, 1], \\ 0, & \textit{otherwise}. \end{cases}$$

For any $\alpha > 0$*, by the construction in [D], there exists a compactly supported refinable function* $\tilde{\phi} \in L^{2,\alpha}$ *such that*

$$\int_{\mathbb{R}} h(x)\tilde{\phi}(x - k)dx = \delta_k, \forall\ k \in \mathbb{Z}.$$

Then by our construction above, we may construct refinable functions $\tilde{\phi}_{n,k}^{New} \in L^{2,\alpha}$ *on any non-uniform grids of semi-regular case such that*

$$\int_{\mathbb{R}} h_{n,k}^{New}(x)\tilde{\phi}_{n,k}^{New}(x)dx = \delta_{kk'}, \ \forall \ \ k, k' \in \mathbb{Z}.$$

This completes the construction of compactly supported refinable functions with arbitrary regularity on any non-uniform grids of semi-regular case.

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