CONVOLUTION SAMPLING AND RECONSTRUCTION OF SIGNALS IN A REPRODUCING KERNEL SUBSPACE

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Abstract. We consider convolution sampling and reconstruction of signals in certain reproducing kernel subspaces of $L^p, 1 \le p \le \infty$. We show that signals in those subspaces could be stably reconstructed from their convolution samples taken on a relatively-separated set with small gap. Exponential convergence and error estimates are established for the iterative approximation-projection reconstruction algorithm.

1. Introduction

In this paper, we consider convolution sampling of signals in a reproducing kernel subspace of $L^p := L^p(R^d), 1 \le p \le \infty$. Here $L^p, 1 \le p \le \infty$, is the space of all p-integrable functions on the d-dimensional Euclidean space with its standard norm denoted by $\|\cdot\|_p$.

Convoluting a signal $f \in L^p$ with an integrable convolutor ψ, the continuous analog of filtering a digital signal, gives

$$\psi * f := \int_{\mathbb{R}^d} f(y)\psi(\cdot - y)dy.$$

For given convolutor ψ and sampling set $\Gamma \subset \mathbb{R}^d$, the associated convolution sampling of a signal f yields the data $\{f * \psi(\gamma)\}_{\gamma\in\Gamma}$. This is the ideal sampling of the convoluted signal $\psi * f$ taken on a sampling set Γ,

$$f \stackrel{\text{convoluting}}{\longmapsto} f * \psi \stackrel{\text{sampling}}{\longmapsto} \{f * \psi(\gamma)\}_{\gamma\in\Gamma}.$$

In this paper, a sampling set Γ means a relatively-separated discrete subset of $\mathbb{R}^d$; i.e.,

$$B_\Gamma(\delta) := \sup_{x\in\mathbb{R}^d} \sum_{\gamma\in\Gamma} \chi_{[-\delta,\delta]^d}(x-\gamma) < \infty \text{for some } \delta > 0, \tag{1.1}$$

where χ_E is the characteristic function on a set E [2, 4, 5].

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An *idempotent operator* T on L^p is a bounded linear operator T on L^p that satisfies

$$T^2 f = Tf \quad \text{for all} \quad f \in L^p. \tag{1.2}$$

The reproducing kernel subspace V_p of L^p for our signals to live in is the range space of an idempotent integral operator T,

$$V_p := \{Tf \mid \ f \in L^p\}, \tag{1.3}$$

whose kernel K has certain regularity and decay at infinity [13]. Particularly, we assume that the kernel K of the integral operator T,

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y) dy, \ f \in L^p,$$

satisfies

$$\max\Big(\sup_{x\in\mathbb{R}^d} \|K(x,\cdot)\|_{W^1}, \sup_{y\in\mathbb{R}^d} \|K(\cdot,y)\|_{W^1}\Big) < \infty \tag{1.4}$$

and

$$\lim_{\delta\to 0} \max\Big(\sup_{x\in\mathbb{R}^d} \|\omega_\delta(K)(x,\cdot)\|_{W^1}, \sup_{y\in\mathbb{R}^d} \|\omega_\delta(K)(\cdot,y)\|_{W^1}\Big) = 0. \tag{1.5}$$

Here

$$W^1 = \Big\{ f \mid \|f\|_{W^1} := \Big\| \sup_{z\in[-1/2,1/2)^d} |f(\cdot + z)| \Big\|_1 < \infty \Big\} \tag{1.6}$$

is the *Wiener amalgam space*, and

$$\omega_\delta(K)(x,y) := \sup_{z_1,z_2\in[-\delta,\delta]^d} |K(x+z_1, y+z_2) - K(x,y)|. \tag{1.7}$$

is the *modulus of continuity* of a kernel function K on $\mathbb{R}^d \times \mathbb{R}^d$.

We recall that the range space V_p in (1.3) is a reproducing kernel space [13]. Here a closed subspace V of L^p is a *reproducing kernel subspace of* L^p such that $\sup_{0\neq f\in V} |f(x)|/\|f\|_p < \infty$ for all $x \in \mathbb{R}^d$ [6, 14]. Examples of reproducing kernel subspace V_p of L^p include the space of non-uniform splines of order n having $n-1$ continuity at each knot [15, 26], the shift-invariant space generated by finitely many functions with certain regularity and decay at infinity [2, 25], and the space modeling signals with finite rate of innovation [7, 20, 23].

In this paper, we study the convolution sampling of signals in the reproducing kernel space V_p on a sampling set Γ with small gap (Theorems 2.1 and 2.2), and the exponential convergence and error estimates of the iterative approximation-projection algorithm for reconstructing signals from their convolution samples (Theorems 3.1, 4.1 and 4.2).

2. Stability of convolution sampling

In this section, we study stability of the convolution sampling for signals in a reproducing kernel subspace of L^p. We show that any signal in the reproducing kernel subspace V_p can be stably reconstructed from its convolution samples taken on a sampling set with small gap, provided that the convolution is stable on V_p.

To state our main result of this section, we recall the gap of a sampling set and the space of all p-summable sequences. A sampling set Γ is said to have *gap* $\delta > 0$ if

$$A_\Gamma(\delta) := \inf_{x\in\mathbb{R}^d} \sum_{\gamma\in\Gamma} \chi_{[-\delta,\delta]^d}(x-\gamma) \geq 1 \tag{2.1}$$

([2, 4, 5]). The space of all p-summable sequences on a sampling set Γ is denoted by $\ell^p := \ell^p(\Gamma)$ with its standard norm by $\|\cdot\|_{\ell^p(\Gamma)}$ (or $\|\cdot\|_p$ for short).

Theorem 2.1. *Let $1 \leq p \leq \infty, \psi_1, \cdots, \psi_L$ be integrable functions on $\mathbb{R}^d$, V_p be the reproducing kernel subspace of L^p in* (1.3) *with the kernel K of the associated idempotent operator satisfying* (1.4) *and* (1.5)*, and set $\Psi = (\psi_1, \cdots, \psi_L)^T$. Then the following statements are equivalent:*

(i) *Ψ is a stable convolution sampler on V_p for all sampling sets having sufficiently small gap; i.e., there exists $\delta_0 > 0$ such that*

$$0 < \inf_{0\neq f\in V_p} \frac{\sum_{l=1}^L \left\|\left(\psi_l * f(\gamma)\right)_{\gamma\in\Gamma}\right\|_p}{\|f\|_p} \leq \sup_{0\neq f\in V_p} \frac{\sum_{l=1}^L \left\|\left(\psi_l * f(\gamma)\right)_{\gamma\in\Gamma}\right\|_p}{\|f\|_p} < \infty$$

holds for any sampling set Γ satisfying $1 \leq A_\Gamma(\delta) \leq B_\Gamma(\delta) < \infty$ for some $\delta \in (0, \delta_0)$.

(ii) *Ψ is a stable convolutor on V_p; i.e.,*

$$0 < \inf_{g\in V_p, \|g\|_p=1} \sum_{l=1}^L \|\psi_l * g\|_p \leq \sup_{g\in V_p, \|g\|_p=1} \sum_{l=1}^L \|\psi_l * g\|_p < \infty. \tag{2.2}$$

We remark that the above equivalence between a stable convolutor on V_p and a stable convolution sampler on V_p for sampling sets with sufficiently small gap is established in [4] provided that V_p is assumed to be a finitely-generated shift-invariant space. The readers may refer to [2, 3, 4, 5, 8, 12, 19, 20, 23, 24, 25, 27, 28, 29, 30] and references therein for sampling and reconstruction of signals in a shift-invariant space and in a reproducing kernel space.

Let V be a closed subspace of L^p, and $R_1, \ldots, R_L$ be bounded operators on $L^p, 1 \leq p \leq \infty$. We say that $R_1, \ldots, R_L$ are *collectively stable* on V if

$$0<\inf_{g\in V,\|g\|_p=1}\sum_{l=1}^L \|R_l g\|_p\le \sup_{g\in V,\|g\|_p=1}\sum_{l=1}^L \|R_l g\|_p<\infty, \tag{2.3}$$

and that $R_1,\ldots,R_L$ are *stable average samplers* on V for sampling sets having small gap if there exists a sufficiently small positive number δ_0 such that

$$0<\inf_{g\in V,\|g\|_p=1}\sum_{l=1}^L \big\|\big(R_l g(\gamma)\big)_{\gamma\in\Gamma}\big\|_p\le \sup_{g\in V,\|g\|_p=1}\sum_{l=1}^L \big\|\big(R_l g(\gamma)\big)_{\gamma\in\Gamma}\big\|_p<\infty \tag{2.4}$$

for any sampling set Γ with $1\le A_\Gamma(\delta)\le B_\Gamma(\delta)<\infty$ where $\delta\in(0,\delta_0)$. In this section, we will prove the following slight generalization of Theorem 2.1.

Theorem 2.2. *Let $1\le p\le\infty$, V be a closed subspace of L^p, and let $R_1,\ldots,R_L$ be integral operators with their kernels $K_1,\ldots,K_L$ satisfying*

$$\max\Big(\sup_{x\in\mathbb{R}^d}\|K_l(x,\cdot)\|_{W^1},\ \sup_{y\in\mathbb{R}^d}\|K_l(\cdot,y)\|_{W^1}\Big)<\infty \tag{2.5}$$

and

$$\lim_{\delta\to0}\max\Big(\sup_{x\in\mathbb{R}^d}\|\omega_\delta(K_l)(x,\cdot)\|_{W^1},\ \sup_{y\in\mathbb{R}^d}\|\omega_\delta(K_l)(\cdot,y)\|_{W^1}\Big)=0,\quad 1\le l\le L. \tag{2.6}$$

Then integral operators $R_1,\ldots,R_L$ are collectively stable on V if and only if they are stable average samplers on V for all sampling sets having sufficiently small gap.

For a moment, we assume that Theorem 2.2 holds and proceed the proof of Theorem 2.1.

Proof of Theorem 2.1. Given integrable functions $\psi_1,\ldots,\psi_L$ and a reproducing kernel subspace of L^p with the kernel K of the associated idempotent operator satisfying (1.4) and (1.5), we let

$$K_l(x,y)=\int_{\mathbb{R}^d}\psi_l(z)K(x-z,y)dz$$

and define

$$R_l f(x)=\int_{\mathbb{R}^d}K_l(x,y)f(y)dy,\quad 1\le l\le L. \tag{2.7}$$

Then for any $1\le l\le L$, it is easy to show that

$$\|R_l f\|_p\le\max\Big(\sup_{x\in\mathbb{R}^d}\|K(x,\cdot)\|_{W^1},\ \sup_{y\in\mathbb{R}^d}\|K(\cdot,y)\|_{W^1}\Big)\|\psi_l\|_1\|f\|_p \tag{2.8}$$

for all $f \in L^p$, and hence the integral operator R_l is a bounded operator on L^p. Also for any $1 \le l \le L$, the integral operator R_l coincides with the convolution operator ψ_l on V_p,

$$R_l f = \psi_l * f \quad \text{for all } f \in V_p; \tag{2.9}$$

and its kernel K_l satisfies (2.5) and (2.6). These facts together with Theorem 2.2 lead to the equivalence in Theorem 2.1. □

Now we proceed to prove Theorem 2.2. For a kernel function K on $\mathbb{R}^d \times \mathbb{R}^d$, define

$$\|K\|_{\mathcal{W}} := \max\big(\sup_{x\in\mathbb{R}^d} \|K(x,\cdot)\|_1, \sup_{y\in\mathbb{R}^d} \|K(\cdot,y)\|_1\big) < \infty. \tag{2.10}$$

The proof of Theorem 2.2 is based on the following two lemmas. The first lemma shows the necessity, while the second one establishes the sufficiency.

Lemma 2.3. *Let $1 \le p \le \infty$, V be a closed subspace of L^p, and let $R_1, \cdots, R_L$ be integral operators with their kernels $K_1, \cdots, K_L$ satisfying*

$$\|K_l\|_{\mathcal{W}} < \infty \quad \text{and} \quad \lim_{\delta\to 0} \|\omega_\delta(K_l)\|_{\mathcal{W}} = 0, \quad 1 \le l \le L. \tag{2.11}$$

If $R_1, \cdots, R_L$ are collectively stable on V, then for any sampling set Γ with $1 \le A_\Gamma(\delta_0) \le B_\Gamma(\delta_0) < \infty$ for some $\delta_0 > 0$, we have that

$$\begin{aligned}
&(A_\gamma(\delta_0))^{1/p}\Big(\inf_{g\in V, \|g\|_p=1} \sum_{l=1}^L \big\|R_l g\|_p - \sum_{l=1}^L \|\omega_{\delta_0}(K_l)\|_{\mathcal{W}}\Big)\|f\|_p \\
\le\ & \delta_0^{1/p} \sum_{l=1}^L \big\|\big(R_l f(\gamma)\big)_{\gamma\in\Gamma}\big\|_p \\
\le\ & (B_\tau(\delta_0))^{1/p}\Big(\sup_{g\in V, \|g\|_p=1} \sum_{l=1}^L \big\|R_l g\|_p + \sum_{l=1}^L \|\omega_{\delta_0}(K_l)\|_{\mathcal{W}}\Big)\|f\|_p
\end{aligned} \tag{2.12}$$

for all $f \in V$.

Proof. Let $\delta_0 > 0$ and Γ be a sampling set with

$$1 \le A_\Gamma(\delta_0) \le B_\Gamma(\delta_0) < \infty. \tag{2.13}$$

Hence $\{\gamma + [-\delta_0, \delta_0]^d \mid \gamma \in \Gamma\}$ is a finite covering of $\mathbb{R}^d$, and the collection of functions

$$u_\gamma(x) = \frac{\chi_{[-\delta_0,\delta_0]^d}(x-\gamma)}{\sum_{\gamma'\in\Gamma} \chi_{[-\delta_0,\delta_0]^d}(x-\gamma')} \tag{2.14}$$

defines a *bounded uniform partition of the unity (BUPU)* $U = \{u_\gamma\}_{\gamma\in\Gamma}$; i.e.,

$$\begin{cases} 0 \le u_\gamma(x) \le 1 \quad \text{for all } x \in \mathbb{R}^d \text{ and } \gamma \in \Gamma, \\ u_\gamma(x) = 0 \text{ if } x \notin \gamma + [-\delta_0, \delta_0]^d, \\ \sum_{\gamma\in\Gamma} u_\gamma(x) = 1 \quad \text{for all } x \in \mathbb{R}^d. \end{cases} \tag{2.15}$$

Moreover,

$$\frac{\delta_0^d}{B_\Gamma(\delta_0)} \le \|u_\gamma\|_1 \le \frac{\delta_0^d}{A_\Gamma(\delta_0)} \quad \text{for all } \gamma \in \Gamma. \tag{2.16}$$

For all $1 \le l \le L$ and $f \in L^p, 1 \le p \le \infty$, we have that

$$\|\omega_{\delta_0}(R_l f)\|_p \le \|\omega_{\delta_0}(K_l)\|_{\mathcal{W}} \|f\|_p \tag{2.17}$$

and

$$|R_l f(x)| - \omega_{\delta_0}(R_l f)(x) \le R_l f(\gamma) \le |R_l f(x)| + \omega_{\delta_0}(R_l f)(x), \tag{2.18}$$

where $x \in \gamma + [-\delta_0, \delta_0]^d, \gamma \in \Gamma$. Then the conclusion (2.12) for $p = \infty$ follows directly from (2.13), (2.17) and (2.18).

For $1 \le p < \infty$, we obtain from (2.16), (2.17), (2.18) and the stability of $R_1, \cdots, R_L$ on V that

$$\begin{aligned} & \sum_{l=1}^L \big\| \big(R_l f(\gamma)\big)_{\gamma\in\Gamma} \big\|_p \\ \le\ & (\delta_0^{-d} B_\Gamma(\delta_0))^{1/p} \sum_{l=1}^L \Big(\sum_{\gamma\in\Gamma} |R_l f(\gamma)|^p \|u_\gamma\|_1 \Big)^{1/p} \\ \le\ & (\delta_0^{-d} B_\Gamma(\delta_0))^{1/p} \sum_{l=1}^L \big\| |R_l f| + \omega_{\delta_0}(R_l f) \big\|_p \\ \le\ & (\delta_0^{-d} B_\Gamma(\delta_0))^{1/p} \big(\sum_{l=1}^L \big\|R_l f\|_p + \sum_{l=1}^L \|\omega_{\delta_0}(K_l)\|_{\mathcal{W}} \|f\|_p \big) \\ \le\ & (\delta_0^{-d} B_\Gamma(\delta_0))^{1/p} \Big(\sup_{g\in V, \|g\|_p = 1} \sum_{l=1}^L \big\|R_l g\|_p + \sum_{l=1}^L \|\omega_{\delta_0}(K_l)\|_{\mathcal{W}} \Big) \|f\|_p \end{aligned}$$

and

$$\begin{aligned}
&\sum_{l=1}^{L} \big\| \big(R_l f(\gamma)\big)_{\gamma\in\Gamma} \big\|_p \\
\geq\ & (\delta_0^{-d} A_\Gamma(\delta_0))^{1/p} \sum_{l=1}^{L} \Big(\sum_{\gamma\in\Gamma} |R_l f(\gamma)|^p \|u_\gamma\|_1 \Big)^{1/p} \\
\geq\ & (\delta_0^{-d} A_\Gamma(\delta_0))^{1/p} \sum_{l=1}^{L} \big(\|R_l f\|_p - \|\omega_{\delta_0}(R_l f)\|_p \big) \\
\geq\ & (\delta_0^{-d} A_\gamma(\delta_0))^{1/p} \Big(\inf_{g\in V, \|g\|_p=1} \sum_{l=1}^{L} \big\|R_l g\|_p - \sum_{l=1}^{L} \|\omega_{\delta_0}(K_l)\|_{\mathcal{W}} \Big) \|f\|_p
\end{aligned}$$

for all $1 \leq p < \infty$ and $f \in V$. This proves (2.12) for $1 \leq p < \infty$ and completes the proof. □

Lemma 2.4. *Let $1 \leq p \leq \infty$, V be a closed subspace of L^p, $R_1, \cdots, R_L$ be integral operators with their kernels $K_1, \cdots, K_L$ satisfying* (2.5) *and* (2.6)*, and let Γ be a sampling set with $1 \leq A_\Gamma(\delta_0) \leq B_\Gamma(\delta_0) < \infty$ for some $\delta_0 > 0$. If*

$$0 < \inf_{f\in V, \|f\|_p=1} \sum_{l=1}^{L} \big\| \big(R_l f(\gamma)\big)_{\gamma\in\Gamma} \big\|_p \leq \sup_{f\in V, \|f\|_p=1} \sum_{l=1}^{L} \big\| \big(R_l f(\gamma)\big)_{\gamma\in\Gamma} \big\|_p < \infty, \tag{2.19}$$

then $R_1, \cdots, R_L$ are collectively stable on V.

Proof. The stability of $R_1, \ldots, R_L$ for $p = \infty$ follows directly from (2.19) and the assumption on their kernels. Now we assume that $1 \leq$

$p < \infty$. In this case, we have that for $0 < \delta \le \delta_0$,

$$\begin{aligned}
& \sum_{\gamma\in\Gamma} \int_{\gamma+[-\delta/2,\delta/2]^d} |\omega_\delta(R_l f)(x)|^p dx \\
\le\ & \sum_{\gamma\in\Gamma} \int_{\gamma+[-\delta/2,\delta/2]^d} \Big| \int_{\mathbb{R}^d} \omega_\delta(K_l)(x,y)|f(y)|dy\Big|^p dx \\
\le\ & \big(\sup_{x\in\mathbb{R}^d} \|\omega_\delta(K_l)(x,\cdot)\|_1\big)^{p-1} \sum_{\gamma\in\Gamma} \int_{\gamma+[-\delta/2,\delta/2]^d} \int_{\mathbb{R}^d} \omega_\delta(K_l)(x,y)|f(y)|^p dy dx \\
\le\ & \Big(\sup_{y\in\mathbb{R}^d} \sum_{k\in\mathbb{Z}^d} \sum_{\gamma\in\Gamma} \int_{(\gamma+[-\delta/2,\delta/2]^d)\cap(k+[-1/2,1/2]^d)} \omega_\delta(K_l)(x,y)dx\Big) \\
& \times\Big(\sup_{x\in\mathbb{R}^d} \|\omega_\delta(K_l)(x,\cdot)\|_{W^1}\Big)^{p-1} \|f\|_p^p \\
\le\ & \Big(\sup_{z\in\mathbb{R}^d} \sum_{\gamma\in\Gamma} |(\gamma+[-\delta/2,\delta/2]^d)\cap(z+[-1/2,1/2]^d)|\Big) \\
& \times\Big(\sup_{x\in\mathbb{R}^d} \|\omega_\delta(K_l)(x,\cdot)\|_{W^1}\Big)^{p-1} \Big(\sup_{y\in\mathbb{R}^d} \|\omega_\delta(K_l)(\cdot,y)\|_{W^1}\Big) \|f\|_p^p \\
\le\ & B_\Gamma(1+\delta_0)\Big(\sup_{x\in\mathbb{R}^d} \|\omega_\delta(K_l)(x,\cdot)\|_{W^1}\Big)^{p-1} \Big(\sup_{y\in\mathbb{R}^d} \|\omega_\delta(K_l)(\cdot,y)\|_{W^1}\Big)\delta^d \|f\|_p^p.
\end{aligned}$$

This implies that

$$\begin{aligned}
& (B_\Gamma(\delta_0))^{1/p} \sum_{l=1}^L \|R_l f\|_p \\
\ge\ & \sum_{l=1}^L \Big(\sum_{\gamma\in\Gamma} \int_{\gamma+[-\delta/2,\delta/2]^d} |R_l f(x)|^p dx\Big)^{1/p} \\
\ge\ & \delta^{d/p} \sum_{l=1}^L \Big(\sum_{\gamma\in\Gamma} |R_l f(\gamma)|^p\Big)^{1/p} - \sum_{l=1}^L \Big(\sum_{\gamma\in\Gamma} \int_{\gamma+[-\delta/2,\delta/2]^d} |\omega_\delta(R_l f)(x)|^p dx\Big)^{1/p} \\
\ge\ & \delta^{d/p} \|f\|_p \Big(\inf_{g\in V, \|g\|_p=1} \sum_{l=1}^L \|(R_l g(\gamma))_{\gamma\in\Gamma}\|_p - (B_\Gamma(1+\delta_0))^{1/p} \\
& \times\Big(\sum_{l=1}^L \Big(\sup_{x\in\mathbb{R}^d} \|\omega_\delta(K_l)(x,\cdot)\|_{W^1}\Big)^{(p-1)/p} \Big(\sup_{y\in\mathbb{R}^d} \|\omega_\delta(K_l)(\cdot,y)\|_{W^1}\Big)^{1/p}\Big) \\
\ge\ & \frac{1}{2}\delta^{d/p} \Big(\inf_{g\in V, \|g\|_p=1} \sum_{l=1}^L \|(R_l g(\gamma))_{\gamma\in\Gamma}\|_p\Big) \|f\|_p
\end{aligned}$$

by (2.6), when δ is chosen sufficiently small. Hence the stability of the integral operators $R_1, \cdots, R_L$ on V for $1 \le p < \infty$ follows. $\square$

Finally we prove Theorem 2.2.

Proof of Theorem 2.2. The necessity follows from Lemma 2.3, while the sufficiency holds by Lemma 2.4. $\square$

3. Exponential convergence of the iterative approximation-projection reconstruction algorithm

Let $\mathcal{B}(L^p), 1 \le p \le \infty$, be the Banach algebra of bounded linear operators on L^p and denote by $\|\cdot\|_{\mathcal{B}(L^p)}$ its norm. In this section, we establish exponential convergence of the iterative approximation-projection algorithm to reconstruct signals in a reproducing kernel subspace V_p of $L^p, 1 \le p \le \infty$, from their convolution samples on a sampling set with small gap.

Theorem 3.1. *Let $1 \le p \le \infty$ and V be a closed subspace of L^p. Assume that integral operators $R_1, \ldots, R_L$ are stable convolutor on V whose kernels $K_1, \ldots, K_L$ satisfy* (2.5) *and* (2.6)*, and that $\tilde{R}_1, \cdots, \tilde{R}_L$ are bounded operators from L^p to V satisfying*

$$\sum_{l=1}^{L} \tilde{R}_l(R_l f) = f \quad \text{for all } f \in V. \tag{3.1}$$

Given a sample set Γ with $1 \le A_\Gamma(\delta_0) \le B_\Gamma(\delta_0) < \infty$ for some $\delta_0 > 0$, let $\{u_\gamma\}_{\gamma\in\Gamma}$ be a bounded uniform partition of the unity in (2.15) *and define an operator P from L^p to V by*

$$Pf(x) = \sum_{l=1}^{L} \sum_{\gamma\in\Gamma} R_l f(\gamma) \tilde{R}_l u_\gamma(x), \quad f \in L^p. \tag{3.2}$$

Then given samples $R_l g(\gamma), \gamma \in \Gamma$, of a signal $g \in V$, the following iterative algorithm,

$$g_0 = Pg \quad \text{and} \quad g_n = g_0 + g_{n-1} - Pg_{n-1} \text{ when } n \ge 1, \tag{3.3}$$

converges exponentially to $g \in V$, provided that

$$r := \sum_{l=1}^{L} \|\tilde{R}_l\|_{\mathcal{B}(L^p)} \|\omega_{\delta_0}(K_l)\|_{\mathcal{W}} < 1. \tag{3.4}$$

Moreover

$$\|g_n - g\|_p \le \frac{r^{n+1}}{1-r} \|Pg\|_p \quad \text{for all } \; g \in V. \tag{3.5}$$

Let T be an integral idempotent operator whose kernel satisfies (1.4) and (1.5), and V_p be the corresponding reproducing kernel subspace of L^p in (1.3). Using an argument similar to the one in the proof of Theorem 2.1 and applying Theorem 3.1 with $R_l f$ replaced by $\psi_l * Tf, 1 \le l \le L$, we obtain the exponential convergence of the iterative approximation-projection algorithm for reconstructing signals in a reproducing kernel subspace V_p of L^p from their convolution samples.

Corollary 3.2. *Let $1 \le p \le \infty$, T be an integral idempotent operator whose kernel satisfies* (1.4) *and* (1.5)*, and V_p be the corresponding reproducing kernel subspace of L^p in* (1.3)*. Assume that $\Psi := (\psi_1, \dots, \psi_L)$ is a stable convolutor on V_p and has its components $\psi_1, \dots, \psi_L$ being integrable, and that $\tilde{R}_1, \cdots, \tilde{R}_L$ are bounded operators from L^p to V_p satisfying $\sum_{l=1}^L \tilde{R}_l(\psi_l * f) = f$ for all $f \in V_p$. Given a sample set Γ with $1 \le A_\Gamma(\delta_0) \le B_\Gamma(\delta_0) < \infty$ for some $\delta_0 > 0$, let $\{u_\gamma\}_{\gamma\in\Gamma}$ be a bounded uniform partition of the unity in* (2.15) *and define an operator Q from L^p to V_p by*

$$Qf(x) = \sum_{l=1}^{L} \sum_{\gamma\in\Gamma} \psi_l * f(\gamma) \tilde{R}_l u_\gamma(x), \quad f \in L^p. \tag{3.6}$$

Then given convolution samples $\psi_l * g(\gamma), \gamma \in \Gamma, 1 \le l \le L$, of a signal $g \in V_p$, the iterative approximation-projection algorithm,

$$g_0 = Qg \quad \text{and} \quad g_n = g_0 + g_{n-1} - Qg_{n-1} \text{ when } n \ge 1, \tag{3.7}$$

converges exponentially to $g \in V$ provided that

$$r := \Big(\sum_{l=1}^{L} \|\tilde{R}_l\|_{\mathcal{B}(L^p)} \|\psi_l\|_1\Big) \|\omega_{\delta_0}(K)\|_{\mathcal{W}} < 1.$$

Moreover $\|g_n - g\|_p \le \frac{r^{n+1}}{1-r} \|Qg\|_p$ for all $g \in V_p$.

Remark 3.1. The iterative algorithm (3.3) can be thought as a generalization of the approximation-projection reconstruction algorithm in [9] for convolution sampling. We remark that the approximation-projection algorithm was originally introduced in [9] for reconstructing band-limited signals, and was later generalized in [1] to the recovery of signals in a shift-invariant space, see [2, 4, 5, 13] and the references therein for various generalizations and applications of that reconstruction algorithm.

Remark 3.2. The bounded operators $\tilde{R}_1, \dots, \tilde{R}_l$ in (3.1) can be interpreted as the inverse of integral operators $R_1, \dots, R_L$ on the closed subspace V. The existence problem of such bounded operators is open, except that V is a finitely generated shift-invariant space and $R_1, \dots, R_L$

are convolution operators. As integral operators $R_1, \ldots, R_l$ has their kernels with certain regularity and decay at infinity, we expect that $\tilde{R}_1, \ldots, \tilde{R}_L$ can be chosen to have the same property. This is a very interesting topic closely related to Wiener's lemma for localized integral operators. The readers may refer to [10, 16, 18, 21, 22] and references therein for Wiener's lemma for infinite matrices and localized integral operators.

Proof of Theorem 3.1. The operator P in (3.2) is well defined because $\tilde{R}_l, 1 \le l \le L$, are bounded operators on L^p by the assumption, and $\sum_{\gamma\in\Gamma} R_l f(\gamma) u_\gamma, 1 \le l \le L$, belongs to L^p as $(R_l f(\gamma))_{\gamma\in\Gamma} \in \ell^p(\Gamma)$ by Lemma 2.3. Let $f \in V \subset L^p, 1 \le p \le \infty$. Then for $1 \le p < \infty$,

$$\begin{aligned}
\|f - Pf\|_p &= \Big(\int_{\mathbb{R}^d} \big|f(x) - \sum_{l=1}^{L}\sum_{\gamma\in\Gamma} R_l f(\gamma)\tilde{R}_l u_\gamma(x)\big|^p dx\Big)^{1/p} \\
&= \Big(\int_{\mathbb{R}^d} \Big|\sum_{l=1}^{L} \tilde{R}_l\big(R_l f - \sum_{\gamma\in\Gamma} R_l f(\gamma) u_\gamma\big)(x)\Big|^p dx\Big)^{1/p} \\
&\le \sum_{l=1}^{L} \|\tilde{R}_l\|_{\mathcal{B}(L^p)} \\
&\quad \times \Big(\int_{\mathbb{R}^d} \Big|\int_{\mathbb{R}^d} \big(K_l(x,y) - \sum_{\gamma\in\Gamma} K_l(\gamma,y) u_\gamma(x)\big) f(y) dy\Big|^p dx\Big)^{1/p} \\
&\le \Big(\sum_{l=1}^{L} \|\tilde{R}_l\|_{\mathcal{B}(L^p)} \|\omega_{\delta_0}(K_l)\|_{\mathcal{W}}\Big) \|f\|_p, \qquad (3.8)
\end{aligned}$$

and for $p = \infty$,

$$\begin{aligned}
\|f - Pf\|_p &= \sup_{x\in\mathbb{R}^d} \big|f(x) - \sum_{l=1}^{L}\sum_{\gamma\in\Gamma} R_l f(\gamma)\tilde{R}_l u_\gamma(x)\big| \\
&\le \sum_{l=1}^{L} \|\tilde{R}_l\|_{\mathcal{B}(L^\infty)} \sup_{x\in\mathbb{R}^d} \Big|\int_{\mathbb{R}^d} \big(K_l(x,y) - \sum_{\gamma\in\Gamma} K_l(\gamma,y) u_\gamma(x)\big) f(y) dy\Big| \\
&\le \Big(\sum_{l=1}^{L} \|\tilde{R}_l\|_{\mathcal{B}(L^\infty)} \|\omega_{\delta_0}(K_l)\|_{\mathcal{W}}\Big) \|f\|_\infty. \qquad (3.9)
\end{aligned}$$

From (3.3) it follows that

$$f_{n+1} - f_n = (I-P)(f_n - f_{n-1}) = \cdots = (I-P)^{n+1} Pf \qquad (3.10)$$

for all $n \ge 0$. This together with (3.4), (3.8) and (3.9) proves (3.5), and hence the exponential convergence of $f_n, n \ge 0$, defined in (3.3). □

4. Error estimates of the iterative approximation-projection reconstruction algorithm

In this section, we first consider the iterative approximation-projection algorithm (3.3) when samples are corrupted, that is $\{R_l g(\gamma) + \epsilon_l(\gamma)\}$, where $\{\epsilon_l(\gamma)\}$ is noise .

Theorem 4.1. *Let* $1 \le p \le \infty$*,* V *be a closed subspace of* L^p*, integral operators* $R_1, \ldots, R_L$ *be stable convolutor on* V *whose kernels* $K_1, \cdots, K_L$ *satisfy* (2.5) *and* (2.6)*, and* $\tilde{R}_1, \cdots, \tilde{R}_L$ *be bounded operators from* L^p *to* V *satisfying* (3.1)*. Given a sample set* Γ *with* $1 \le A_\Gamma(\delta_0) \le B_\Gamma(\delta_0) < \infty$ *for some* $\delta_0 > 0$*, let* $\{u_\gamma\}_{\gamma\in\Gamma}$ *be a bounded uniform partition of the unity in* (2.15) *and define an operator* P *from* L^p *to* V *as in* (3.2)*. Then given noisy samples* $R_l g(\gamma)+\epsilon_l(\gamma), \gamma \in \Gamma$ *of a signal* $g \in V$ *with* $\epsilon_l := (\epsilon_l(\gamma))_{\gamma\in\Gamma} \in \ell^p(\Gamma)$*, the iterative approximation-projection algorithm,*

$$\begin{cases} \tilde{g}_0 = \sum_{l=1}^L \sum_{\gamma\in\Gamma} (R_l g(\gamma) + \epsilon_l(\gamma)) \tilde{R}_l u_\gamma \\ \tilde{g}_n = \tilde{g}_0 + \tilde{g}_{n-1} - P\tilde{g}_{n-1} \text{ when } n \ge 1, \end{cases}$$

converges exponentially to $\tilde{g} \in V$ *provided that* (3.4) *holds. Moreover*

$$\|g - \tilde{g}\|_p \le \frac{\delta_0^{d/p} \sum_{l=1}^L \|\tilde{R}_l\|_{\mathcal{B}(L^p)} \|(\epsilon_l(\gamma))_{\gamma\in\Gamma}\|_p}{\big(A_\Gamma(\delta_0)\big)^{1/p} \big(1 - \sum_{l=1}^L \|\tilde{R}_l\|_{\mathcal{B}(L^p)} \|\omega_{\delta_0}(K_l)\|_{\mathcal{W}}\big)}. \tag{4.1}$$

Proof. Set $h_0 = \sum_{l=1}^L \sum_{\gamma\in\Gamma} \epsilon_l(\gamma) \tilde{R}_l u_\gamma$, and define $g_n, n \ge 0$, by

$$g_0 = Pg \quad \text{and} \quad g_n = g_0 + g_{n-1} - Pg_{n-1} \text{ when } n \ge 1. \tag{4.2}$$

Then $h_0 \in V$ and

$$\tilde{g}_n - g_n = h_0 + (I - P)h_0 + \cdots + (I - P)^n h_0, \ n \ge 0.$$

This together with (3.8), (3.9) and Theorem 3.1 leads to the exponential convergence of the sequence $\tilde{g}_n, n \ge 0$, to a function $\tilde{g} \in V$. Moreover,

$$\begin{aligned} \|\tilde{g} - g\|_p &\le \sum_{n=0}^\infty \|(I - P)^n h_0\|_p \le \frac{\|h_0\|_p}{1 - r} \\ &\le (1 - r)^{-1} \sum_{l=1}^L \|\tilde{R}_l\|_{\mathcal{B}(L^p)} \Big\| \sum_{\gamma\in\Gamma} \epsilon_l(\gamma) u_\gamma \Big\|_p \\ &\le (1 - r)^{-1} \sum_{l=1}^L \|\tilde{R}_l\|_{\mathcal{B}(L^p)} \|(\epsilon_l(\gamma))_{\gamma\in\Gamma}\|_p \big(\sup_{\gamma\in\Gamma} \|u_\gamma\|_1\big)^{1/p}. \end{aligned}$$

This together with (2.16) proves (4.1). □

We conclude this section by considering the iterative approximation-projection reconstruction algorithm (3.3) when the iterative algorithm is corrupted, that is, each iteration step is corrupted by noise $\{h_n\}$.

Theorem 4.2. *Let $1 \le p \le \infty$, V be a closed subspace of L^p, integral operators $R_1, \ldots, R_L$ be stable convolutor on V whose kernels $K_1, \ldots, K_L$ satisfy* (2.5) *and* (2.6)*, and $\tilde{R}_1, \ldots, \tilde{R}_L$ be bounded operators from L^p to V satisfying* (3.1)*. Given a sample set Γ with $1 \le A_\Gamma(\delta_0) \le B_\Gamma(\delta_0) < \infty$ for some $\delta_0 > 0$, let $\{u_\gamma\}_{\gamma\in\Gamma}$ be a bounded uniform partition of the unity in* (2.15) *and define an operator P from L^p to V as in* (3.2)*. Given samples $R_l g(\gamma), \gamma \in \Gamma$, of a signal $g \in V$, assume that the iterative approximation-projection algorithm,*

$$\tilde{g}_0 = Pg + h_0 \quad \text{and} \quad \tilde{g}_n = \tilde{g}_0 + \tilde{g}_{n-1} - P\tilde{g}_{n-1} + h_n \text{ when } n \ge 1, \tag{4.3}$$

are corrupted by $h_n \in V, n \ge 0$, during the implementation. Then

$$\|\tilde{g}_n - g\|_p \le \frac{r^n}{1-r}\|Pg\|_p + \frac{1}{1-r}\|h_0\|_p + \sum_{l=1}^{n} r^{n-l}\|h_l\|_p, \ n \ge 1, \tag{4.4}$$

provided that (3.4) *holds, where $r = \sum_{l=1}^{L} \|\tilde{R}_l\|_{\mathcal{B}(L^p)}\|\omega_{\delta_0}(K_l)\|_{\mathcal{W}} \in (0,1)$.*

Proof. Let $g_n, n \ge 0$, be as in (4.2). Then

$$\tilde{g}_n = g_n + \sum_{l=0}^{n}(I-P)^l h_0 + \sum_{l=1}^{n}(I-P)^{n-l}h_l, \ n \ge 1. \tag{4.5}$$

Then (4.4) follows from (3.8), (3.9), (3.10) and (4.5). □

Remark 4.1. We remark that $h_n, n \ge 0$, in the implementation (4.3) is assumed to belong to a subspace V of L^p instead of living in the whole space L^p with more freedom, as we notice that in that general case the error in the implementation could accumulate quickly at each step, c.f. [13]. On the other hand, the assumption that $h_n \in V, n \ge 0$, can be reached easily by replacing $h_n \in L^p$ by $\tilde{h}_n := \sum_{l=1}^{L} \tilde{R}_l(R_l h_n) \in V$.

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