Characterizations of Tight Over-sampled Affine Frame Systems and Over-sampling Rates

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Let M be a dilation matrix, Ψ a finite family of L^2-functions, and $\mathcal{P}$ the collection of all nonsingular matrices P such that M, P, and PMP^{-1} have integer entries. The objective of this paper is two-fold. Firstly, for each P in $\mathcal{P}$, we characterize all tight affine frames $X(\Psi, M)$ generated by Ψ such that the over-sampled affine systems $X^P(\Psi, M)$ relative to the "over-sampling rate" P remain to be tight frames. Secondly, we characterize all over-sampling rates $P \in \mathcal{P}$, such that the over-sampled affine systems $X^P(\Psi, M)$ are tight frames whenever the affine system $X(\Psi, M)$ is a tight frame. Our second result therefore provides a general and precise formulation of the Second Over-sampling Theorem for tight affine frames.

1. INTRODUCTION

In this paper, a d-dimensional square matrix M is called a *dilation matrix* if all entries of M are integers and all eigenvalues λ of M satisfy $|\lambda| > 1$. Let M be a dilation matrix, $\Psi := \{\psi_l : 1 \le l \le L\}$ a finite family of square-integrable functions, and K an *over-sampling set*,

$$K = K_N := \{y_n \in \mathbf{R}^d : 0 \le n \le N\}$$

with $y_0 = 0$. Then the collection of functions

$$X(\Psi, M, K) := \Big\{ |\det M|^{j/2} \psi(M^j \cdot -k - y) : \psi \in \Psi, y \in K, j \in \mathbf{Z}, k \in \mathbf{Z}^d \Big\} \quad (1)$$

is called an *affine system*. For convenience, we denote $X(\Psi, M, K_0)$ by $X(\Psi, M)$, which is usually called an affine system generated by Ψ. Hence, the affine system $X(\Psi, M, K)$ is indeed an *over-sampled affine system* obtained by over-sampling the affine system $X(\Psi, M)$ with "over-sampling rates" governed by the over-sampling set K. We are particularly interested in sets of the form $K = [0, 1)^d \cap P^{-1}\mathbf{Z}^d$, where P, to be called the *over-sampling rate* of the affine system $X(\Psi, M, K)$, is a nonsingular matrix with integer entries. For

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this choice of K, $X(\Psi, M, K)$ becomes the familiar over-sampled affine system

$$X^P(\Psi, M) := \Big\{ |\det M|^{j/2}\psi(M^j \cdot -P^{-1}k) : \ \psi \in \Psi, j \in \mathbf{Z}, k \in \mathbf{Z}^d \Big\} \tag{2}$$

with over-sampling rate P (see [5, 8, 19, 21, 22]).

Recall that a family of functions $\{e_\lambda : \lambda \in \Lambda\}$ in $L^2 := L^2(\mathbf{R}^d)$ is called a *frame* of L^2, if there exist positive constants A and B, with $A \le B$, such that

$$A\|f\| \le \Big(\sum_{\lambda \in \Lambda} |\langle f, e_\lambda \rangle|^2 \Big)^{1/2} \le B\|f\| \quad \forall\, f \in L^2, \tag{3}$$

where $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote, as usual, the inner product and norm for L^2, respectively. The constants A, B in (3) are referred to as lower and upper frame bounds, respectively. If A and B can be so chosen that $A = B$, then the frame is called a tight frame, with frame (bound) constant A.

For a dilation matrix M, $K := \{y_n \in \mathbf{R}^d : 0 \le n \le N\}$ with $y_0 = 0$, and $\Psi := \{\psi_l \in L^2 : \ 1 \le l \le L\}$, we consider the *affine operators* Q_j, defined by

$$Q_j f := |\det M|^j \sum_{\psi \in \Psi, y \in K, k \in \mathbf{Z}^d} \langle f, \psi(M^j \cdot -k - y) \rangle \psi(M^j \cdot -k - y), \ \ f \in L^2, \tag{4}$$

for the j-th scale levels (see [13, 14]). Observe that each Q_j is related to the affine operator Q_0 for the ground level by

$$Q_j = D_{-j} Q_0 D_j, \tag{5}$$

where $D_j, j \in \mathbf{Z}$, are the dilation operators associated with the dilation matrix M, defined by

$$D_j f(x) = |\det M|^{j/2} f(M^j x), \ \ f \in L^2.$$

Moreover, for an affine frame $X(\Psi, M, K)$, it is easy to verify that $X(\Psi, M, K)$ is a tight frame of L^2 if and only if the family of affine operators $Q_j, j \in \mathbf{Z}$, satisfies

$$\sum_{j \in \mathbf{Z}} Q_j = \sum_{j \in \mathbf{Z}} D_{-j} Q_0 D_j = CI \tag{6}$$

for some positive constant C, where I is the identity operator on L^2.

In this paper, we will address the following problems on frame over-sampling.

Problem 1: *Given a dilation matrix M and nonsingular matrix P, characterize all tight affine frames $X(\Psi, M)$ such that the over-sampled affine systems $X^P(\Psi, M)$ remain to be tight frames.*

Problem 2: *Given a dilation matrix M and nonsingular matrix P, characterize all affine frames $X(\Psi, M)$ such that over-sampling of the affine system $X(\Psi, M)$ by P preserves the frame bounds.*

Problem 3: *Characterize the over-sampling rates P for which the over-sampled affine systems $X^P(\Psi, M)$ are tight frames whenever $X(\Psi, M)$ is a tight frame.*

Problem 1 was studied recently for the one-variable setting in [4] for $d = 1$ and $P = 2^r, r \geq r_0$, and [13] for $d = 1$ and gcd(M,P) = P, where M and P are integers.

In this paper, we consider the general multivariate setting, where the over-sampling rate P is a non-singular matrix such that all the matrices M, P and PMP^{-1} have integer entries. We give complete characterizations of all tight frames $X(\Psi, M)$ in the frequency domain, in terms of Fourier transform of Ψ; as well as in the time domain, in terms of certain "over-sampled frame operators" with over-sampling rate P, so that the over-sampled affine systems $X^P(\Psi, M)$ remain to be tight frames. This result will be stated precisely in Theorem 2.1, with relevant comments to be given in Remarks 2.3 – 2.5. Since the assumption of $P^{-1}MP$ to have integer entries becomes trivial for one-dimensional matrices, the characterization results in Theorem 2.1, in both the frequency and time domains, provide a complete solution of Problem 1 for the one-variable setting.

Problem 2 is perhaps a new problem. In this paper, we provide a sufficient condition (9) on the generator Ψ of the affine frame $X(\Psi, M)$ for which over-sampling of the affine frame $X(\Psi, M)$ by P preserves the frame bounds (see Theorem 3.1 for a precise statement). The sufficient condition (9) on the generator Ψ is also necessary for frame tightness preservation (see Theorem 2.1); but we do not know, in general, if it is necessary for the preservation of frame bounds in frame over-sampling.

Problem 3 is an older problem, first considered in [9], where it is shown that odd over-sampling preserves 2-dilated tight affine frames. This result was later extended and generalized in other works (see, for instance, [4, 5, 8, 9, 10, 13, 19, 21, 22, 23], including the Second Over-sampling Theorem in [8]).

In this paper, we solve the third problem again under the assumption that the matrices M, P, and PMP^{-1} have integer entries, and we show that over-sampling with over-sampling rate P preserves tight frames if and only if the matrices M and P for dilations and over-sampling rates satisfy $M^{-1}\mathbf{Z}^d \cap P^{-1}\mathbf{Z}^d = \mathbf{Z}^d$. Hence, in some sense, this result gives a complete and precise extension of the Second Over-sampling Theorem for tight affine frames in both one and several variables. This result will be stated precisely in Theorem 4.1, with relevant comments to be given in Remark 4.2 – 4.5.

In this paper, we will always assume that dilation matrices M and over-sampling rates P satisfy

$$M, P, PMP^{-1} \in GL_d(\mathbf{Z}), \tag{7}$$

where, as usual, $GL_d(\mathbf{Z})$ denotes the collection of all nonsingular d-dimensional square matrices with integer entries.

2. TIGHTNESS OF OVER-SAMPLED AFFINE SYSTEMS

This section is devoted to the solution of Problem 1 stated in the Introduction section. The usual normalization

$$\hat{f}(\xi) = \int_{\mathbf{R}^d} f(x) e^{-ix\xi} dx$$

for the Fourier transform of an integrable function f will be used. The key concept in our study of this problem relies on the notation

$$I_{M,P} := \sup_{s\in \mathbf{Z}^d\backslash M^T\mathbf{Z}^d} I_{M,P}(s),$$

where $I_{M,P}(s)$ denotes the smallest integer j for which $(M^T)^i s\in P^T\mathbf{Z}^d$ for all $i\ge j$, provided that such an integer j exists, and is assigned the value 0, otherwise.

To formulate a time-domain characterization, we need the notion of *over-sampled frame operators* $R_{j,0}$ *and* $R_{j,1}$ *with over-sampling rate* P for $0\le j\in \mathbf{Z}$, defined by

$$\begin{aligned} R_{j,0}f \ &= \sum_{p\in P[0,1)^d\cap \mathbf{Z}^d,\psi\in\Psi,k\in\mathbf{Z}^d} \langle f, \psi(\cdot-M^{-j}k-P^{-1}p)\rangle \\ &\qquad\qquad \times\psi(\cdot-M^{-j}k-P^{-1}p), \end{aligned}$$

where in terms of over-sampling sets, we have $K=([0,1)^d\cap M^{-j}\mathbf{Z}^d)+([0,1)^d\cap P^{-1}\mathbf{Z}^d)$; and

$$\begin{aligned} R_{j,1}f \ &= \sum_{p\in P[0,1)^d\cap \mathbf{Z}^d,\psi\in\Psi,k\in\mathbf{Z}^d} \langle f, \psi(\cdot-M^{-j}k-M^{I_{M,P}}P^{-1}p)\rangle \\ &\qquad\qquad \times\psi(\cdot-M^{-j}k-M^{I_{M,P}}P^{-1}p), \end{aligned}$$

where in terms of over-sampling sets, we have $K=([0,1)^d\cap M^{-j}\mathbf{Z}^d)+M^{I_{M,P}}([0,1)^d\cap P^{-1}\mathbf{Z}^d)$. We remark that under the assumption (7), we have

$$R_{j,0}=R_{j,1} \quad \text{for all} \quad j\ge I_{M,P}. \tag{8}$$

This will follow from Proposition 2.2, to be established later.

THEOREM 2.1. *Let* M *be a dilation matrix, and* M, P *satisfy (7). Also, let* $\Psi=\{\psi_1,\ldots,\psi_L\}$ *generate a tight affine frame of* L^2 *with dilation matrix* M. *Then the following statements are equivalent.*

(i) The over-sampled affine system $X^P(\Psi,M)$ *is a tight frame of* L^2.

(ii) For any $s\in\mathbf{Z}^d\backslash M^T\mathbf{Z}^d$,

$$\sum_{j=0}^{I_{M,P}(s)-1}\sum_{\psi\in\Psi}\widehat{\psi}((M^T)^j\xi)\overline{\widehat{\psi}((M^T)^j(\xi+2\pi s))}=0 \quad \text{a.e. } \xi\in\mathbf{R}^d. \tag{9}$$

(iii) The difference of the two over-sampled frame operators $R_{j,0}$ *and* $R_{j,1}$ *satisfies the identity:*

$$\begin{aligned} R_{0,1}-R_{0,0}+\sum_{j=1}^{I_{M,P}-1}|\det M|^{-j}\Big[D_{-j}(R_{j,1}-R_{j,0})D_j \\ -D_{-j+1}(R_{j,1}-R_{j,0})D_{j-1}\Big]=0. \end{aligned} \tag{10}$$

From the formulations of the above tight over-sampling frame characterizations, it is clear that we need to have a good understanding of the integer-valued function $I_{M,P}$. In Proposition 2.2 to be stated next, we list five equivalent formulations, with derivations to be given in the Appendix. But first we need the notion of set equivalence, as follows.

Two sets $A = \{a_\lambda : \lambda \in \Lambda_1\}$ and $B = \{b_\mu, \mu \in \Lambda_2\}$ are said to be equal, modulus some set C, with the notation

$$A = B \mod C,$$

if there exists a one-to-one map σ from Λ_1 onto Λ_2 such that $b_{\sigma(\lambda)} - a_\lambda \in C$ for all $\lambda \in \Lambda_1$. They are said to be equivalent, with the notation $A \equiv B$, if $A = B \bmod \{0\}$.

PROPOSITION 2.2. *Let M, P and PMP^{-1} have integer entries. Then the following statements hold.*

(i) $I_{M,P} = \sup_{s\in \mathbf{Z}^d} I_{M,P}(s) = \sup_{s\in P^T[0,1)^d\cap \mathbf{Z}^d} I_{M,P}(s)$.

(ii) $I_{M,P}$ *is the smallest nonnegative integer* n_0 *for which* $\#(\{r : (M^T)^n r \in P^T\mathbf{Z}^d, r \in P^T[0,1)^d \cap \mathbf{Z}^d\})$ *remain unchanged for* $n \geq n_0$.

(iii) $I_{M,P}$ *is the smallest nonnegative integer* n_0 *for which* $\#(\{q : M^nP^{-1}q \in \mathbf{Z}^d, q \in P[0,1)^d \cap \mathbf{Z}^d\})$ *remain unchanged for* $n \geq n_0$.

(iv) $I_{M,P}$ *is the smallest nonnegative integer* n_0 *for which*

$$M^nP^{-1}(P[0,1)^d \cap \mathbf{Z}^d) \equiv M^{n_0}P^{-1}(P[0,1)^d \cap \mathbf{Z}^d) \mod \mathbf{Z}^d \quad \forall\, n \geq n_0. \tag{11}$$

REMARK 2.3.

(a) For the univariate setting (i.e. d = 1 and M, P are integers), under the assumption that gcd(M,P) = P, it follows that equation (9) becomes

$$\sum_{\psi\in\Psi} \widehat{\psi}(\xi)\overline{\widehat{\psi}(\xi + 2\pi s)} = 0 \quad \text{a.e.} \quad \xi \in \mathbf{R}^d \quad \forall\, s \in \mathbf{Z}\backslash P^T\mathbf{Z}, \tag{12}$$

which is the third equivalence statement in Theorem 1 of [13], since $I_{M,P}(s) = 1$ for $s \in \mathbf{Z}\backslash P^T\mathbf{Z}$ and $I_{M,P}(s) = 0$ for $s \in P^T\mathbf{Z}$.

(b) For the multivariate setting, under the assumption that $P = M^r$ for some $1 \leq r \in \mathbf{Z}$, it follows that equation (9) becomes

$$\sum_{j=0}^{r-1}\sum_{\psi\in\Psi} \widehat{\psi}((M^T)^j\xi)\overline{\widehat{\psi}((M^T)^j(\xi + 2\pi s))} = 0 \quad \text{a.e.} \quad \xi \in \mathbf{R}^d \tag{13}$$

for all $s \in \mathbf{Z}^d\backslash M^T\mathbf{Z}^d$, since $I_{M,P}(s) = r$ for $s \in \mathbf{Z}^d\backslash M^T\mathbf{Z}^d$. Hence, over-sampling by any over-sampling rate $M^r, r \geq r_0$, preserves tight frames if and only if

$$\sum_{\psi\in\Psi} \widehat{\psi}(\xi)\overline{\widehat{\psi}(\xi + 2\pi s)} = 0 \quad \text{a.e.} \quad \xi \in \mathbf{R}^d \quad \forall\, s \in (M^T)^{r_0}\mathbf{Z}^d, \tag{14}$$

which is a generalization of the result in [4] from one-dimension ($d = 1$) to higher dimensions, and from a single frame generator ($\#(\Psi) = 1$) to multiple frame generators ($\#(\Psi) \geq 1$).

(c) Assume that both $M^{I_{M,P}}P^{-1}$ and $PM^{I_{M,P}-1}$ have integer entries. Then

$$R_{j,0} = |\det M|^j Q_{0,P} \quad \text{and} \quad R_{j,1} = |\det P| Q_{0,M^j}$$

for all $0 \le j \le I_{M,P} - 1$, where the notation of the operator $Q_{0,T}$, defined for any matrix $T \in GL_d(\mathbf{Z})$ by

$$Q_{0,T} f := \sum_{\psi \in \Psi, k \in \mathbf{Z}^d} \langle f, \psi(\cdot - T^{-1}k)\rangle \psi(\cdot - T^{-1}k) \quad \text{for } f \in L^2,$$

is used. Therefore, equation (10) becomes

$$\begin{aligned} Q_{0,P} &= |\det P| D_{I_{M,P}-1} Q_{0,I} D_{-I_{M,P}+1} + \sum_{j=1}^{I_{M,P}-1} \frac{|\det P|}{|\det M|^j} \\ &\quad \times \Big\{ D_{I_{M,P}-j-1} Q_{0,M^j} D_{j+1-I_{M,P}} - D_{I_{M,P}-j} Q_{0,M^j} D_{j-I_{M,P}} \Big\}, \quad (15) \end{aligned}$$

where I is the unit matrix. The result in (15) then generalizes the one-dimensional result in [13] to higher dimensions.

REMARK 2.4. *Let $K = M^{I_{M,P}}([0,1)^d \cap P^{-1}\mathbf{Z}^d)$ and assume that (7) is satisfied. Then from the proof of Theorem 2.1, it follows that the over-sampled affine system $X(\Psi, M, K)$ is a tight frame, whenever $X(\Psi, M)$ is a tight frame. For instance, for the univariate setting, when $M = 2^2 \times 3 \times 5$ and $P = 2 \times 3^2 \times 7$, the over-sampled affine system $X(\Psi, M, K)$ can be re-formulated as the over-sampled affine system $X^7(\Psi, M)$ with over-sampling rate 7 and multiplicity 2×3^2. Hence, the affine system $X(\Psi, M, K)$ with over-sampling set $K = M^{I_{M,P}}([0,1)^d \cap P^{-1}\mathbf{Z}^d)$ can be treated as odd-over-sampled affine system in [9], even though the dilation factor is not 2. Also we note that for such dilation M and over-sampling rate P,*

$$X^{2\times 3^2 \times 7}(\Psi, M) = \{|\det M|^{j/2} \psi(M^j \cdot - k - l/7 - m/(2\times 3^2)) : 0 \le l \le 6, 0 \le m \le 17\}.$$

As an extension of this observation, let $\tilde{K}_0$ denote the set of all different elements in $M^{I_{M,P}}([0,1)^d \cap P^{-1}\mathbf{Z}^d)$, and $\tilde{K}_1$ denote the set of all elements $x \in [0,1)^d \cap P^{-1}\mathbf{Z}^d$ with $M^{I_{M,P}} x \in \mathbf{Z}^d$. Then we have

$$X^P(\Psi, M) = X(\Psi, M, \tilde{K}),$$

where $\tilde{K} = \tilde{K}_0 + \tilde{K}_1$. Thus, the over-sampled affine system $X^P(\Psi, M)$ can be thought of as a combination of odd over-sampling (with the over-sampling set $\tilde{K}_0$) and even over-sampling (with the over-sampling set $\tilde{K}_1$), of the given affine system $X(\Psi, M)$ generated by Ψ, even though the dilation factor is different from 2.

REMARK 2.5.

(a) For the univariate setting, since (7) is valid when M and P are integers, Theorem 2.1 gives a complete characterization of all tight affine frames with integer dilations, for which over-sampling with an arbitrarily given integer over-sampling rate preserves tight frames.

(b) For higher dimensions, however, the assumption (7) is no longer trivial. As an example, consider the dilation matrix and over-sampling rate

$$M = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, \; P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

respectively. Then

$$P^{-1}M^nP = \frac{1}{2}\begin{bmatrix} 4^n+3^n & 4^n-3^n \\ 4^n-3^n & 4^n+3^n \end{bmatrix} \notin GL_2(\mathbf{Z}) \quad \forall\, n \geq 1.$$

In this example, however, we have that

$$\begin{cases} (M^T)^j s \notin P^T\mathbf{Z}^2 & \forall\, j \geq 0 \text{ and } s \in (2\mathbf{Z}+1)\times \mathbf{Z}, \\ (M^T)^j s \in P^T\mathbf{Z}^2 & \forall j \geq 0 \text{ and } s \in (2\mathbf{Z})\times (2\mathbf{Z}), \\ (M^T)^j s \in P^T\mathbf{Z}^2 & \forall\, j \geq 1 \text{ and } s \in (2\mathbf{Z})\times (2\mathbf{Z}+1), \\ (M^T)^0 s \notin P^T\mathbf{Z}^2 & \forall\, s \in (2\mathbf{Z})\times (2\mathbf{Z}+1). \end{cases}$$

It follows that

$$I_{M,P}(s) = \begin{cases} 0 & \text{if } s \notin (2\mathbf{Z})\times(2\mathbf{Z}+1), \\ 1 & \text{if } s \in (2\mathbf{Z})\times(2\mathbf{Z}+1). \end{cases}$$

Therefore, by following the argument to be given in the proof of Theorem 2.1, we may conclude that over-sampling with this particular over-sampling rate P preserves tight frames, if and only if

$$\sum_{\psi\in\Psi} \widehat{\psi}(\xi)\overline{\widehat{\psi}(\xi+2\pi s)} = 0 \quad \forall\, s \in (2\mathbf{Z})\times(2\mathbf{Z}+1). \tag{16}$$

To prove Theorem 2.1, let us first recall the following well-known result on the characterization of tight affine frames [1, 2, 3, 5, 16].

LEMMA 2.6. *Let $\Psi = \{\psi_1,\ldots,\psi_L\}$ be a finite family of L^2 functions. Then Ψ generates a tight affine frame for L^2 if and only if*

$$\sum_{j\in\mathbf{Z}}\sum_{\psi\in\Psi} |\widehat{\psi}((M^T)^j\xi)|^2 = C > 0 \;\; a.e.\; \xi \in \mathbf{R}^d$$

for some constant C, and

$$\sum_{j\geq 0}\sum_{\psi\in\Psi} \widehat{\psi}((M^T)^j\xi)\overline{\widehat{\psi}((M^T)^j(\xi+2\pi s))} = 0 \;\; a.e.\; \xi\in\mathbf{R}^d \quad \forall\, s\in \mathbf{Z}^d\backslash M^T\mathbf{Z}^d.$$

We also need the following general formula on summation over the “roots of unity” that can be found in [20, Theorem 23.19].

LEMMA 2.7. *Let $A \in GL_d(\mathbf{Z})$. Then*

$$\sum_{k\in A^T[0,1)^d\cap\mathbf{Z}^d} e^{-2\pi i k^T A^{-1}s} = \begin{cases} |\det A| & \text{if } s\in A\mathbf{Z}^d, \\ 0 & \text{if } s \in \mathbf{Z}^d\backslash A\mathbf{Z}^d. \end{cases}$$

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. For convenience, we set $B = M^T$. To prove (i)$\Longrightarrow$(ii), we first apply Lemma 2.6 and the tight frame property of Ψ to conclude

$$\sum_{j=0}^{\infty}\sum_{\psi\in\Psi}\widehat{\psi}(B^j\xi)\overline{\widehat{\psi}(B^j(\xi+2\pi s))}=0 \quad \text{a.e. } \xi\in\mathbf{R}^d \tag{17}$$

for all $s\in\mathbf{Z}^d\backslash B\mathbf{Z}^d$.

Observe that by setting

$$\Psi^P=\Big\{\psi(\cdot-P^{-1}q):\ \psi\in\Psi, q\in P[0,1)^d\cap\mathbf{Z}^d\Big\},$$

we have

$$X^P(\Psi,M)=X(\Psi^P,M). \tag{18}$$

Then by applying Lemma 2.6 and the tight frame property to the over-sampled affine system $X^P(\Psi,M)$, we have

$$\sum_{j=0}^{\infty}\Big(\sum_{\psi\in\Psi}\widehat{\psi}(B^j\xi)\overline{\widehat{\psi}(B^j(\xi+2\pi s))}\Big)\times\Big(\sum_{q\in P[0,1)^d\cap\mathbf{Z}^d}e^{-2\pi i s^T M^j P^{-1}q}\Big)=0 \tag{19}$$

for almost all $\xi\in\mathbf{R}^d$, where $s\in\mathbf{Z}^d\backslash B\mathbf{Z}^d$. Note that if $B^js\in P^T\mathbf{Z}$ for some $j\ge 0$, then $B^{j+1}s\in P^T\mathbf{Z}$ by (7). Therefore, it follows, by an application of Lemma 2.7, that

$$\sum_{q\in P[0,1)^d\cap\mathbf{Z}^d}e^{-2\pi i s^T M^j P^{-1}q}=\begin{cases}|\det P| & \text{if } j\ge I_{M,P}(s),\\ 0 & \text{if } 0\le j<I_{M,P}(s),\end{cases} \tag{20}$$

provided $I_{M,P}(s)>0$; that

$$\sum_{q\in P[0,1)^d\cap\mathbf{Z}^d}e^{-2\pi i s^T M^j P^{-1}q}=|\det P| \quad \forall\, j\ge 0 \tag{21}$$

provided $I_{M,P}(s)=0$ and $s\in P^T\mathbf{Z}$; and that

$$\sum_{q\in P[0,1)^d\cap\mathbf{Z}^d}e^{-2\pi i s^T M^j P^{-1}q}=0 \quad \forall\, j\ge 0 \tag{22}$$

provided $I_{M,P}(s)=0$ and $s\notin P^T\mathbf{Z}$. Hence, equation (9) follows from (17) and (19) – (22).

To prove (ii)$\Longrightarrow$(i), we observe that in view of (18) and Lemma 2.6, it suffices to prove that

$$\sum_{j\in\mathbf{Z}}\sum_{\psi\in\Psi}\sum_{q\in P[0,1)^d\cap\mathbf{Z}^d}|\widehat{\psi}(B^j\xi)e^{-i\xi^T M^j P^{-1}q}|^2=C \quad \text{a.e. } \xi\in\mathbf{R}^d \tag{23}$$

for some positive constant C, and that

$$\sum_{j=0}^{\infty}\sum_{\psi\in\Psi}\sum_{q\in P[0,1)^d\cap\mathbf{Z}^d}\widehat{\psi}(B^j\xi)e^{-i\xi^T M^j P^{-1}q} \times\overline{\widehat{\psi}(B^j(\xi+2\pi s))e^{-i(\xi+2\pi s)^T M^j P^{-1}q}} = 0 \quad \text{a.e. } \xi\in\mathbf{R}^d \tag{24}$$

for all $s\in\mathbf{Z}^d\backslash B\mathbf{Z}^d$. In this regard, (23) follows from Lemma 2.6 and the tight frame property of the affine system $X(\Psi,M)$, while the equality (24) holds because of (9), (17), and (20) – (22).

To prove (ii)$\Longleftrightarrow$(iii), we first conclude, in view of (11) and (19) – (22), that equation (9) has an equivalent formulation:

$$\begin{aligned} 0 \;=\; & \sum_{j=0}^{I_{M,P}-1}\Big(\sum_{\psi\in\Psi}\widehat{\psi}(B^j\xi)\overline{\widehat{\psi}(B^j(\xi+2\pi s))}\Big)(1-\chi_{B\mathbf{Z}^d}(s)) \\ & \times\Big(\sum_{q\in P[0,1)^d\cap\mathbf{Z}^d} e^{-2\pi i s^T M^j P^{-1}q} - \sum_{q\in P[0,1)^d\cap\mathbf{Z}^d} e^{-2\pi i s^T M^{j+I_{M,P}} P^{-1}q}\Big)\end{aligned}$$

for almost all $\xi\in\mathbf{R}^d$, where $s\in\mathbf{Z}^d$. Hence, multiplying both sides by $\hat{f}(\xi+2s\pi)$ and then summing over $s\in\mathbf{Z}^d$, we have

$$\begin{aligned} & \sum_{s\in\mathbf{Z}^d}\hat{f}(\xi+2s\pi)\sum_{j=0}^{I_{M,P}-1}\Big(\sum_{\psi\in\Psi}\widehat{\psi}(B^j\xi)\overline{\widehat{\psi}(B^j(\xi+2\pi s))}\Big) \\ & \quad\times\Big(\sum_{q\in P[0,1)^d\cap\mathbf{Z}^d} e^{-2\pi i s^T M^j P^{-1}q} - \sum_{q\in P[0,1)^d\cap\mathbf{Z}^d} e^{-2\pi i s^T M^{j+I_{M,P}} P^{-1}q}\Big) \\ & -\sum_{s\in\mathbf{Z}^d}\hat{f}(\xi+2Bs\pi)\sum_{j=0}^{I_{M,P}-1}\Big(\sum_{\psi\in\Psi}\widehat{\psi}(B^j\xi)\overline{\widehat{\psi}(B^j(\xi+2\pi Bs))}\Big) \\ & \quad\times\Big(\sum_{q\in P[0,1)^d\cap\mathbf{Z}^d} e^{-2\pi i s^T M^{j+1} P^{-1}q} - \sum_{q\in P[0,1)^d\cap\mathbf{Z}^d} e^{-2\pi i s^T M^{j+I_{M,P}+1} P^{-1}q}\Big) \\ & \quad = 0 \quad \text{a.e. } \xi\in\mathbf{R}^d. \end{aligned} \tag{25}$$

Since this holds for all L^2-functions f with compactly supported Fourier transform, we may conclude, by taking the inverse Fourier transform of both sides of the equation (25) and by applying (11), the formulation (10) is indeed the time-domain formulation of the equation (25). This proves the equivalence of statements (ii) and (iii). ■

3. FRAME BOUND PRESERVATION

In this section, we consider Problem 2 as stated in the Introduction section.

THEOREM 3.1. *Let M be a dilation matrix and that M,P satisfy (7). If $\Psi=\{\psi_1,\dots,\psi_L\}$ satisfies (9) and $X(\Psi,M)$ is an affine frame of L^2 with upper frame bound B and lower frame bound A, then the over-sampled affine system $X^P(\Psi,M)$ is an affine frame of L^2 with upper frame bound $|\det P|^{1/2}B$ and lower frame bound $|\det P|^{1/2}A$.*

To prove Theorem 3.1, we need the following Second Over-sampling Theorem, with over-sampling rates governed by an over-sampling set K.

LEMMA 3.2. *Let M be a dilation matrix, M and P satisfy (7), and that $K := \{y_n \in \mathbf{R}^d : 0 \le n \le N\}$ be a finite set with $y_0 = 0$ and*

$$MK = K \bmod \mathbf{Z}^d. \tag{26}$$

If $\Psi = \{\psi_1, \ldots, \psi_L\}$ generates an affine frame of L^2 with upper frame bound B and lower frame bound A, then the over-sampled affine system $X(\Psi, M, K)$ is an affine frame of L^2 with upper frame bound $(N+1)^{1/2}B$ and lower frame bound $(N+1)^{1/2}A$.

Proof. Lemma 3.2 is a generalization of the Second Over-sampling Theorem in [21, Lemma 3.1 and Theorem 3.2] and can be proved by some similar argument. We omit the details of the proof here. ■

REMARK 3.3. *The Second Over-sampling Theorem was first studied by Chui and Shi in [9], where it was shown that odd over-sampling of 2-dilated affine frames preserves frame bounds (i.e., $d = 1$, $M = 2$ and $K = \{P^{-1}p :\ 0 \le p \le P-1\}$ for some odd integer P in Lemma 3.2). This result was later extended and generalized to higher dimensions with over-sampling rate P (that is, $K = \{P^{-1}p :\ p \in P[0,1)^d \cap \mathbf{Z}^d\}$ in Lemma 3.2). See, for instance, [8, 21, 22, 23] for dilation matrices M and [19] for non-dilation matrices M.*

In the following, we prove Theorem 3.1.

Proof of Theorem 3.1. Let τ_y denote the translation operator defined by $\tau_y f(x) = f(x-y)$. Then for any compactly supported function $f \in L^2$ and $y \in \mathbf{R}^d$, we have

$$\begin{aligned}
&\Big(\sum_{\psi\in\Psi}\sum_{k\in\mathbf{Z}^d} |\det M|^j |\langle f, \psi(M^j\cdot -k-y)\rangle|^2\Big)^{1/2}\\
\le\ &\Big(\sum_{\psi\in\Psi}\sum_{k\in\mathbf{Z}^d} |\det M|^j |\langle f, \psi(M^j\cdot -k)\rangle|^2\Big)^{1/2}\\
&\quad+\Big(\sum_{\psi\in\Psi}\sum_{k\in\mathbf{Z}^d} |\det M|^j |\langle (\tau_{-M^{-j}y}f - f), \psi(M^j\cdot -k)\rangle|^2\Big)^{1/2}\\
\le\ &\Big(\sum_{\psi\in\Psi}\sum_{k\in\mathbf{Z}^d} |\det M|^j |\langle f, \psi(M^j\cdot -k)\rangle|^2\Big)^{1/2} + B\|\tau_{-M^{-j}y}f - f\|\\
\to\ &0 \quad \text{as } j \to +\infty,
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
&\sum_{\psi\in\Psi}\sum_{k\in\mathbf{Z}^d} |\det M|^j |\langle f, \psi(M^j\cdot -k-y)\rangle|^2\\
\le\ &\|f\|^2 \sum_{\psi\in\Psi}\sum_{k\in\mathbf{Z}^d} \int_{M^jK_0(f)} |\psi(x-y-k)|^2 dx\\
\le\ &\|f\|^2 \sum_{\psi\in\Psi} \int_{y+\cup_{k\in\mathbf{Z}^d}(M^jK_0(f)+k)} |\psi(x)|^2 dx
\end{aligned}$$

$$\rightarrow\ 0 \text{ as } j \rightarrow -\infty, \tag{28}$$

where B is the upper frame bound of the affine frame $X(\Psi, M)$, and $K_0(f)$ denotes the support of the function f. By (26), (27), and the equivalence of the equations (9) and (10), we obtain

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty}\sum_{\psi\in\Psi}\sum_{k\in\mathbf{Z}^d} |\langle f, |\det M|^{j/2}\psi(M^j\cdot - P^{-1}k)\rangle|^2 \\
=\ & \lim_{N_1,N_2\to+\infty}\sum_{j=-N_1}^{N_2}\langle D_{-j}R_{0,1}D_j f, f\rangle \\
=\ & \lim_{N_1,N_2\to+\infty}\sum_{j=-N_1}^{N_2}\langle D_{-j}R_{0,0}D_j f, f\rangle + \sum_{i=1}^{I_{M,P}-1}|\det M|^{-i} \\
&\times\Big(\langle D_{-i-N_2}(R_{i,1}-R_{i,0})D_{i+N_2}f, f\rangle - \langle D_{-i+1+N_1}(R_{i,1}-R_{i,0})D_{i-N_1-1}f, f\rangle\Big) \\
=\ & \lim_{N_1,N_2\to+\infty}\sum_{j=-N_1}^{N_2}\langle D_{-j}R_{0,0}D_j f, f\rangle \\
=\ & \sum_{j=-\infty}^{\infty}\sum_{\psi\in\Psi}\sum_{p\in P[0,1)^d\cap\mathbf{Z}^d}\sum_{k\in\mathbf{Z}^d}|\langle f, |\det M|^{j/2}\psi(M^j\cdot -k - M^{I_{M,P}}P^{-1}p)\rangle|^2.
\end{aligned} \tag{29}$$

Since this holds for all compactly supported L^2 functions f, the proof of the theorem is complete by applying Proposition 2.2 and Lemma 3.2. ■

4. OVER-SAMPLING RATES FOR TIGHT FRAME PRESERVATION

In this section we will give a solution of Problem 3 as stated in the Introduction section. A lot of effort has been devoted to find over-sampling rates P for the purpose of tight-frame preservation over-sampling in the literature (see, for example, [5, 8, 9, 10, 13, 19, 21, 22, 23, 25]).

We say that $X(\Psi, M)$ is a *compactly supported MRA tight frame* of L^2 if it is a tight frame of L^2, and $\Psi := \{\psi_1, \dots, \psi_L\}$ is a finite family of compactly supported L^2-functions such that

$$\widehat{\psi}_l(M^T\xi) = H_l(\xi)\widehat{\phi}(\xi),\ 1 \le l \le L \tag{30}$$

for some trigonometrical polynomials $H_l(\xi), 1 \le l \le L$, and some compactly supported refinable function ϕ, that satisfies

$$\widehat{\phi}(M^T\xi) = H_0(\xi)\widehat{\phi}(\xi) \quad \text{and} \quad \widehat{\phi}(0) = 1, \tag{31}$$

where $H_0(\xi)$ is a trigonometric polynomial with $H_0(0) = 1$ (see [1, 6, 7, 15, 23] and references therein).

THEOREM 4.1. *Let M be a dilation matrix, and M, P satisfy (7). Then the following statements are equivalent:*

(i) The over-sampled affine system $X^P(\Psi, M)$ is a tight frame of L^2 whenever Ψ generates a tight affine frame of L^2 with dilation matrix M.

(ii) The over-sampled affine system $X^P(\Psi, M)$ is a tight frame of L^2 for some compactly supported MRA tight frame $X(\Psi, M)$.

(iii) $I_{M,P} = 0$.

REMARK 4.2. *For the univariate setting (i.e. d = 1 and M, P are integers), the above theorem says that P-times over-sampling preserves tight frames if and only if gcd(M,P) = 1. That is, Theorem 4.1 implies that the Second Over-sampling Theorem for tight frames in [8] is sharp.*

For the multivariate setting, various sufficient conditions on the over-sampling rate P, for which the first statement in Theorem 4.1 holds, have been derived in the literature. For example,

- *(a) $P^T\mathbf{Z}^d \cap (M^T)^j\mathbf{Z}^d \subset (M^T)^j P^T\mathbf{Z}^d$ for all $j \geq 0$ in [23];*
- *(b) $PMP^{-1} \in GL_d(\mathbf{Z})$ and $M^T\mathbf{Z}^d \cap P^T\mathbf{Z}^d \subset M^T P^T\mathbf{Z}^d$ in [5, 22];*
- *(c) $PMP^{-1} \in GL_d(\mathbf{Z})$ and $M^{-1}\mathbf{Z}^d \cap P^{-1}\mathbf{Z}^d = \mathbf{Z}^d$ in [19, 21]; and*
- *(d) $P = pI$ with the great common divisor between p and $|\det M|$ being 1 ([10]).*

In the following result, the proof of which will be given in the Appendix, we see that under the assumption (7), the three statements (a), (b), (c), and the third statement in Theorem 4.1 are equivalent.

PROPOSITION 4.3. *Let M be a dilation matrix and P be nonsingular, such that both M and P have integer entries. Then the following statements are equivalent.*

- *(i) $PMP^{-1} \in GL_d(\mathbf{Z})$ and $M^{-1}\mathbf{Z}^d \cap P^{-1}\mathbf{Z}^d = \mathbf{Z}^d$.*
- *(ii) $PMP^{-1} \in GL_d(\mathbf{Z})$ and $(M^T)^{-1}\mathbf{Z}^d \cap (P^T)^{-1}\mathbf{Z}^d = \mathbf{Z}^d$.*
- *(iii) $PMP^{-1} \in GL_d(\mathbf{Z})$ and $M^T\mathbf{Z}^d \cap P^T\mathbf{Z}^d \subset M^T P^T\mathbf{Z}^d$.*
- *(iv) $P^T\mathbf{Z}^d \cap (M^T)^j\mathbf{Z}^d \subset (M^T)^j P^T\mathbf{Z}^d$ for all $j \geq 0$.*
- *(v) $MP^{-1}\mathbf{Z}^d + \mathbf{Z}^d = P^{-1}\mathbf{Z}^d$.*
- *(vi) $PMP^{-1} \in GL_d(\mathbf{Z})$ and $I_{M,P} = 0$.*

REMARK 4.4. *There exist compactly supported MRA tight frames of L^2 for any dilation matrix M. Let us give a constructive proof of this claim. For any dilation matrix M, let Γ be the set of representors of the group $\mathbf{Z}^d/M^T\mathbf{Z}^d$. Also, let $(u_{l,k})_{0\leq l\leq |\det M|-1, k\in\Gamma}$ be a unitary matrix, with $u_{0,k} = 1, k \in \Gamma$. Define trigonometric polynomials $H_l(\xi), 0 \leq l \leq |\det M| - 1$, by*

$$H_l(\xi) = |\det M|^{-1} \sum_{k\in\Gamma} u_{lk} e^{-ik\xi},\ 0 \leq l \leq |\det M| - 1.$$

Let ϕ and $\psi_l, 1 \leq l \leq L$, be compactly supported functions that satisfy

$$\widehat{\phi}(M^T\xi) = H_0(\xi)\widehat{\phi}(\xi) \quad \text{and} \quad \widehat{\phi}(0) = 1,$$

and

$$\widehat{\psi}_l(M^T\xi) = H_l(\xi)\widehat{\phi}(\xi),\ 1 \le l \le |\det M| - 1.$$

Since the trigonometric polynomials $H_l, 0 \le l \le |\det M| - 1$, *satisfy*

$$\sum_{l=0}^{|\det M|-1} H_l(\xi)\overline{H_l(\xi + 2\pi(M^T)^{-1}s)} = \begin{cases} 1 & \text{if } s \in M^T\mathbf{Z}^d, \\ 0 & \text{if } s \in \mathbf{Z}^d\backslash M^T\mathbf{Z}^d, \end{cases}$$

the functions $\psi_1, \ldots, \psi_{|\det M|-1}$ *generate a tight affine frame of* L^2 *([17, 18]). We remark that those functions* $\psi_1, \ldots, \psi_{|\det M|-1}$ *are not continuous. A general construction of compactly supported frame generators* $\psi_1, \ldots, \psi_L$ *with arbitrary regularity can be found in [18]. In the univariate setting, the frame generators can also be constructed to possess an arbitrary order of vanishing moments up to the the order of polynomial reproduction by* ϕ *(see [6, 7, 15]).*

REMARK 4.5. *For the univariate setting, let* $M = 2$ *and* $P \ge 1$ *be an arbitrary integer. There exists an MRA tight affine frame such that its over-sampled affine system with over-sampling rate* P *remains to be a tight frame. The frame generator of this tight frame, however, does not have compact support. In other words, the compact support assumption on the frame generator* Ψ *in the second statement of Theorem 4.1 cannot be dropped. To prove this claim, we define* ψ *by*

$$\widehat{\psi}(\xi) = \chi_{[-\pi,-\pi/2]}(\xi) + \chi_{[\pi/2,\pi]}(\xi).$$

Then for any $f \in L^2$,

$$\begin{aligned} & \sum_{j,k\in\mathbf{Z}} |\langle f, 2^{j/2}\psi(2^j \cdot -k/P)\rangle|^2 \\ = \ & \frac{1}{2\pi} \sum_{j,k\in\mathbf{Z}} \Big| \int_{\mathbf{R}} \widehat{f}(\xi)\overline{\widehat{\psi}(2^{-j}\xi)} 2^{-j/2} e^{i2^{-j}k\xi/P} d\xi \Big|^2 \\ = \ & P \sum_{j\in\mathbf{Z}} \int_{\mathbf{R}} |\widehat{f}(\xi)\widehat{\psi}(2^{-j}\xi)|^2 d\xi \\ = \ & 2\pi P \|f\|_2^2, \end{aligned}$$

where the second equality follows from the fact that $\widehat{\psi}(2^{-j}\xi)$ *is supported in* $[-2^jP\pi, 2^jP\pi]$ *and* $\{(2^{j+1}P\pi)^{-1/2}e^{-i2^{-j}k\xi/P} : k \in \mathbf{Z}\}$ *is an orthonormal basis of* $L^2([-2^jP\pi, 2^jP\pi])$ *for all* $j \ge 1$. *This implies that both the affine system* $X(\psi, 2)$ *generated by* ψ *and its over-sampled affine system* $X^P(\psi, 2)$ *with over-sampling rate* P *are tight frames for* L^2. *Moreover, let* ϕ *be the Meyer scaling function whose Fourier transform* $\widehat{\phi}$ *takes the value* 1 *on* $[-2\pi/3, 2\pi/3]$ *and the value* 0 *on* $\mathbf{R}\backslash[-4\pi/3, 4\pi/3]$. *Then we have*

$$\widehat{\psi}(2\xi) = H_1(\xi)\widehat{\phi}(\xi)$$

for some 2π*-periodic function* H_1. *Hence, the affine system* $X(\psi, 2)$ *generated by* ψ *is an MRA tight frame.*

In the following, we will prove Theorem 4.1.

Proof of Theorem 4.1. The implication (i)⟹(ii) is obvious, while the implication (iii)⟹(i) is valid in view of Theorem 2.1 and the fact that (9) holds under the assumption (iii).

Next, we prove the implication (ii)⟹(iii) by an indirect argument. Suppose, on the contrary, that $I_{M,P}(s_0) \geq 1$ for some $s_0 \in \mathbf{Z}^d$. Set $B = M^T$. By (A.3) and (A.4) to be derived in the Appendix, we may assume that $s_0 \in \mathbf{Z}^d \backslash B\mathbf{Z}^d$. Let $X(\Psi, M)$ be a compactly supported MRA tight frame with $\Psi := \{\psi_1, \ldots, \psi_L\}$ that satisfies (30) and (31) and that the over-sampled affine system $X^P(\Psi, M)$ is a tight frame of L^2. Then we have, by applying Theorem 2.1,

$$\sum_{j=0}^{I_{M,P}(s_0)-1} \sum_{l=1}^{L} \widehat{\psi}_l(B^j\xi)\overline{\widehat{\psi}_l(B^j(\xi + 2s_0\pi))} = 0 \quad \text{a.e. } \xi \in \mathbf{R}^d. \tag{32}$$

Recalling the characterization of compactly supported MRA tight affine frame (see [1, 6, 7, 15, 24]), we can find a rational trigonometric polynomial $S(\xi)$ such that $S(0) = 1$, $S(\xi) \geq 0$ for all $\xi \in \mathbf{R}^d$, and

$$\begin{aligned} & S(M^T\xi)H_0(\xi)\overline{H_0(\xi + 2\pi B^{-1}s)} + \sum_{l=1}^{L} H_l(\xi)\overline{H_l(\xi + 2\pi B^{-1}s)} \\ = & \begin{cases} S(\xi) & \text{if } s \in B\mathbf{Z}^d, \\ 0 & \text{if } s \in \mathbf{Z}^d \backslash B\mathbf{Z}^d. \end{cases} \end{aligned} \tag{33}$$

Hence, it follows from (30)–(33) that

$$\begin{aligned} 0 = & \sum_{j=0}^{I_{M,P}(s_0)-1} \sum_{l=1}^{L} \widehat{\psi}_l(B^j\xi)\overline{\widehat{\psi}_l(B^j(\xi + 2s_0\pi))} \\ = & \Big(\sum_{l=1}^{L} H_l(B^{-1}\xi)\overline{H_l(B^{-1}\xi + 2\pi B^{-1}s_0)}\Big)\widehat{\phi}(B^{-1}\xi)\overline{\widehat{\phi}(B^{-1}(\xi + 2s_0\pi))} \\ & + \sum_{j=1}^{I_{M,P}(s_0)-1} \Big(\sum_{l=1}^{L} H_l(B^{j-1}\xi)\overline{H_l(B^{j-1}\xi)}\Big) \\ & \times \prod_{t=-1}^{j-2} H_0(B^t\xi)\overline{H_0(B^t(\xi + 2\pi s_0))}\widehat{\phi}(B^{-1}\xi)\overline{\widehat{\phi}(B^{-1}(\xi + 2s_0\pi))} \\ = & -S(\xi)H_0(B^{-1}\xi)\overline{H_0(B^{-1}(\xi + 2\pi s_0))}\widehat{\phi}(B^{-1}\xi)\overline{\widehat{\phi}(B^{-1}(\xi + 2s_0\pi))} \\ & + \sum_{j=1}^{I_{M,P}(s_0)-1} \Big(S(B^{j-1}\xi) - S(B^j\xi)|H_0(B^{j-1}\xi)|^2\Big) \\ & \times \prod_{t=-1}^{j-2} H_0(B^t\xi)\overline{H_0(B^t(\xi + 2\pi s_0))}\widehat{\phi}(B^{-1}\xi)\overline{\widehat{\phi}(B^{-1}(\xi + 2s_0\pi))} \\ = & -S(B^{I_{M,P}(s_0)-1}\xi)\widehat{\phi}(B^{I_{M,P}(s_0)-1}\xi)\overline{\widehat{\phi}(B^{I_{M,P}(s_0)-1}(\xi + 2\pi s_0))} \end{aligned}$$

for almost all $\xi \in \mathbf{R}^d$. This is a contradiction, since $S(\xi)$ is a nonzero rational function and $\widehat{\phi}(\xi)$ is analytic function in $\mathbf{R}^d$. Hence, we have completed the proof of the theorem. ■

APPENDIX: PROPERTIES OF OVER-SAMPLING RATES

In the appendix, we discuss some interesting properties of the over-sampling rates P that satisfy (7). In particular, we also include the proofs of Propositions 2.2 and 4.3.

Proof of Proposition 2.2. In the following we again use the notation $B = M^T$. To prove assertion (i), we first observe that

$$B^j P^T = P^T((PMP^{-1})^T)^j \quad \forall\, j \geq 0, \tag{A.1}$$

so that

$$I_{M,P}(s + P^T t) = I_{M,P}(s) \ \ \forall\, s, t \in \mathbf{Z}^d; \tag{A.2}$$

that $I_{M,P}(s)$ is the minimal nonnegative integer j for which $B^j s \in P^T\mathbf{Z}^d$ if such integers j exist; and that $I_{M,P}(s)$ = 0 if $B^j s \notin P^T\mathbf{Z}^d$ for all $j \geq 0$. Hence, by direct computations, we see that

$$I_{M,P}(s) = 0 \quad \text{if } s = 0, \tag{A.3}$$

and

$$I_{M,P}(s) = \begin{cases} 0 & \text{if } j \geq I_{M,P}(s'), \\ I_{M,P}(s') - j & \text{if } j \leq I_{M,P}(s'), \end{cases} \tag{A.4}$$

for $s = B^j s' \in B^j\mathbf{Z}^d \backslash B^{j+1}\mathbf{Z}^d$, where $s' \in \mathbf{Z}^d \backslash B\mathbf{Z}^d$ and $j \geq 0$. Therefore assertion (i) follows from (A.2) – (A.4).

To prove assertion (ii), let

$$A_n = \{r : \ B^n r \in P^T\mathbf{Z}^d, r \in P^T[0,1)^d \cap \mathbf{Z}^d\},$$

and n_0 be the minimal nonnegative integer such that $\#(A_n)$ remains unchanged for all $n \geq n_0$. Since we have either $B^j r \in P^T\mathbf{Z}^d$ or $B^j r \notin P^T\mathbf{Z}^d$ for all $j \geq I_{M,P}$, where $r \in P^T[0,1) \cap \mathbf{Z}^d$, it follows that $I_{M,P} \geq n_0$. On the other hand, we also have

$$A_{n+1} \subset A_n, \quad \forall\, n \geq 0 \tag{A.5}$$

by (A.1). Hence, $A_n = A_{n_0}$ for all $n \geq n_0$; so that $I_{M,P}(r) \leq n_0$ for $r \in A_{n_0}$, and $I_{M,P}(r) = 0$ for $r \notin A_{n_0}$. Therefore, $I_{M,P} \leq n_0$ by assertion (i). This implies the validity of assertion (ii).

To prove assertion (iii), let

$$B_n = \{q : \ M^n P^{-1} q \in \mathbf{Z}^d, q \in P[0,1)^d \cap \mathbf{Z}^d\}.$$

Then by assertion (ii), it suffices to show

$$\#(A_n) = \#(B_n) \quad \forall\, n \geq 0. \tag{A.6}$$

For this purpose, we set

$$C_n = \sum_{r\in P^T[0,1)^d\cap \mathbf{Z}^d} \sum_{q\in P[0,1)^d\cap \mathbf{Z}^d} e^{-2\pi i r^T M^n P^{-1} q},$$

and observe, by applying Lemma 2.6, that

$$C_n = \sum_{q\in B_n} \sum_{r\in P^T[0,1)^d\cap \mathbf{Z}^d} 1 + \sum_{q\notin B_n} \sum_{r\in P^T[0,1)^d} e^{-2\pi i((P^T)^{-1}r)^T P M^n P^{-1} q} = \#(B_n)|\det P|,$$

where we have used the fact that $PM^nP^{-1}q \in \mathbf{Z}^d\backslash P\mathbf{Z}^d$ for $q \notin B_n$. By applying Lemma 2.6 again, we also have

$$C_n = \sum_{r\in A_n} \sum_{q\in P[0,1)^d\cap \mathbf{Z}^d} 1 + \sum_{r\notin A_n} \sum_{q\in P[0,1)^d} e^{-2\pi i (B^n r)^T P^{-1} q} = \#(A_n)|\det P|,$$

due to the fact that $B^n r \in \mathbf{Z}^d\backslash P^T\mathbf{Z}^d$ for $r \notin A_n$. Combining the above two formulations of C_n, we have proved that (A.6) holds, and hence confirm that assertion (iii) holds.

Finally to prove assertion (iv), let

$$K_n = \{x_{n,1}, \ldots, x_{n,|\det P|}\}$$

be the subset of $[0,1)^d$, such that

$$K_n \equiv M^n P^{-1}(P[0,1)^d \cap \mathbf{Z}^d) \mod \mathbf{Z}^d, \tag{A.7}$$

and let $\tilde{K}_n$ denote the set of all distinct elements in K_n. Then for any $y \in K_n$, we have

$$\{x \in P[0,1)^d \cap \mathbf{Z}^d : M^n P^{-1} x - y \in \mathbf{Z}^d\} \equiv x_0 + B_n \bmod P\mathbf{Z}^d, \tag{A.8}$$

where $x_0 \in P[0,1)^d \cap \mathbf{Z}^d$ is so chosen that $y - M^n P^{-1} x_0 \in \mathbf{Z}^d$. Hence, it follows from (A.7) and (A.8) that

$$\#(\tilde{K}_n) = |\det P|/\#(B_n), \tag{A.9}$$

and

$$K_n \equiv K_m \quad \text{if and only if} \quad \tilde{K}_n = \tilde{K}_m, \tag{A.10}$$

where $n, m \geq 0$. On the other hand, we have already shown, by applying (A.1), that $\tilde{K}_n \subset \tilde{K}_m$ when $n \geq m$, which implies that

$$\tilde{K}_n = \tilde{K}_m \quad \text{if and only if} \quad \#(\tilde{K}_n) = \#(\tilde{K}_m). \tag{A.11}$$

Therefore, combining (A.9) – (A.11), we may conclude that

$$\tilde{K}_n = \tilde{K}_m \quad \text{if and only if} \quad \#(B_n) = \#(B_m), \tag{A.12}$$

and hence the validity of the assertion (iv), by a direct application of (A.12) and assertion (iii). ■

REMARK A.1. *For dimension $d \geq 2$, there exist some dilation matrix M and over-sampling rate P, such that $PM^nP^{-1} \notin GL_d(\mathbf{Z})$ for all $n \geq 1$, but equation (11) still holds for some $n_0 \geq 0$. As an example, let us consider*

$$M = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix};\ P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Since

$$M^2P^{-1}(P[0,1)^2 \cap \mathbf{Z}^2) \equiv MP^{-1}(P[0,1)^2 \cap \mathbf{Z}^2) \equiv \left\{(0,0)^T, (1/2,0)^T\right\} \bmod \mathbf{Z}^2,$$

it is clear that (11) holds for $n_0 = 1$ but $P^{-1}M^nP \notin GL_2(\mathbf{Z})$ for all $n \geq 1$ (see Remark 2.5). On the other hand, we also remark that (11) may not hold for some dilation matrix M and over-sampling rate P. As an example, consider

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix};\ P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

so that

$$M^nP^{-1}(P[0,1)^2 \cap \mathbf{Z}^2) \equiv \begin{cases} \{(0,0)^T, (1/2,1/2)^T\} \bmod \mathbf{Z}^2 & \text{if } n \in 3\mathbf{Z}, \\ \{(0,0)^T, (1/2,0)^T\} \bmod \mathbf{Z}^2 & \text{if } n-1 \in 3\mathbf{Z}, \\ \{(0,0)^T, (0,1/2)^T\} \bmod \mathbf{Z}^2 & \text{if } n-2 \in 3\mathbf{Z}. \end{cases}$$

Hence, for this example, (11) does not hold for any $n_0 \geq 0$, although we have

$$M^nP^{-1}(P[0,1)^2 \cap \mathbf{Z}^2) = M^{n'}P^{-1}(P[0,1)^2 \cap \mathbf{Z}^2) \bmod \mathbf{Z}^2,$$

for $n - n' \in 3\mathbf{Z}$. Indeed, for any dimension d, any dilation matrix M, and any over-sampling rate P, there always exist integers $n_1 \geq 1$ and $n_0 \geq 0$, such that

$$M^mP^{-1}(P[0,1)^d \cap \mathbf{Z}^d) \equiv M^nP^{-1}(P[0,1)^d \cap \mathbf{Z}^d) \bmod \mathbf{Z}^d \tag{A.13}$$

for all $m, n \geq n_0$ with $m - n \in n_1\mathbf{Z}$. Clearly this is equivalent to (11) when $n_1 = 1$. The proof of the formula (A.13) follows from the same argument as that of (11). In particular, we still have (A.10), although (A.11) does not hold. From the construction of K_n, we see that

$$\tilde{K}_n \subset |\det P|^{-1}[0, |\det P|)^d \cap \mathbf{Z}^d$$

since $|\det P|M^nP^{-1} = M^nP^a$, where P^a is the adjoint matrix of P. Hence, since there are finite many subsets of $|\det P|^{-1}[0, |\det P|)^d \cap \mathbf{Z}^d$, we have

$$\tilde{K}_{l_1} = \tilde{K}_{l_2} \tag{A.14}$$

for all nonnegative integers l_1 and l_2. Hence, (A.13) follows from (A.10) and (A.14), by considering $n_0 = \min(l_1, l_2)$ and $n_1 = |l_2 - l_1|$.

Proof of Proposition 4.3. The proof of the equivalence among (i)–(v) is essentially given in [21]. The only remark we want to make is that the precise statement in [21] on the equivalence between (i) and (v) can be formulated as follows:

"Assume that $PMP^{-1} \in GL_d(\mathbf{Z})$. Then $MP^{-1}\mathbf{Z}^d + \mathbf{Z}^d = P^{-1}\mathbf{Z}^d$ if and only if $M^{-1}\mathbf{Z}^d \cap P^{-1}\mathbf{Z}^d = \mathbf{Z}^d$."

We remark, however, that $PMP^{-1} \in GL_d(\mathbf{Z})$ already follows from (v), which also implies that $PMP^{-1}k \in \mathbf{Z}^d$ for any $k \in \mathbf{Z}^d$. Hence, the equivalence between (i) and (v) immediately follows.

To prove (iv) $\Longrightarrow$ (vi), let $s \in \mathbf{Z}^d$. Then if $(M^T)^j s \in P^T\mathbf{Z}^d$ for some $j \geq 0$, we have $t = (M^T)^j s \in (M^T)^j P^T \mathbf{Z}^d$ by (iv), which implies that $s \in P^T\mathbf{Z}^d$ and hence, $I_{M,P}(s) = 0$. On the other hand, if $(M^T)^n s \notin P^T\mathbf{Z}^d$ for all $n \geq 0$, then $I_{M,P}(s) = 0$ by definition, so that $I_{M,P}(s) = 0$ for all $s \in \mathbf{Z}^d$ and hence, (vi) follows.

To prove (vi) $\Longrightarrow$ (iv), observe that, by (vi), we have, for any $s \in \mathbf{Z}^d$, either $s \in P^T\mathbf{Z}^d$ or $(M^T)^j s \notin P^T\mathbf{Z}^d$ for all $j \geq 0$. Hence, for any $t \in P^T\mathbf{Z}^d \cap (M^T)^j\mathbf{Z}^d$, we have $t' \in P^T\mathbf{Z}^d$, so that $t \in (M^T)^j P^T\mathbf{Z}^d$, where $t = (M^T)^j t'$. Therefore, we may conclude that $P^T\mathbf{Z}^d \cap (M^T)^j\mathbf{Z}^d \subset (M^T)^j P^T\mathbf{Z}^d$ for all $j \geq 0$. ■

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