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p-frames and Shift Invariant Subspaces of L^p

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Abstract

We investigate the frame properties and closedness for the shift invariant space

$$V_p(\Phi) = \Big\{ \sum_{i=1}^{r} \sum_{j \in \mathbf{Z}^d} d_i(j)\phi_i(\cdot - j) :\ (d_i(j))_{j\in\mathbf{Z}^d} \in \ell^p \Big\}, \quad 1 \le p \le \infty.$$

We derive necessary and sufficient conditions for an indexed family $\{\phi_i(\cdot - j) : 1 \le i \le r, j \in \mathbf{Z}^d\}$ to constitute a p-frame for $V_p(\Phi)$, and to generate a closed shift invariant subspace of L^p. A function in the L^p-closure of $V_p(\Phi)$ is not necessarily generated by ℓ^p coefficients. Hence we often hope that $V_p(\Phi)$ itself is closed, i.e., a Banach space. For $p \ne 2$, this issue is complicated, but we show that under the appropriate conditions on the frame vectors, there is an equivalence between the concept of p-frames, Banach frames, and the closedness of the space they generate. The relation between a function $f \in V_p(\Phi)$ and the coefficients of its representations is neither obvious, nor unique, in general. For the case of p-frames, we are in the context of normed linear spaces, but we are still able to give a characterization of p-frames in terms of the equivalence between the norm of f and an ℓ^p-norm related to its representations. A Banach frame does not have a dual Banach frame in general, however, for the shift invariant spaces $V_p(\Phi)$, dual Banach frames exist and can be constructed.

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1 Introduction

A frame for a Hilbert space $\mathcal{H}$ is an indexed family of vectors $\{g_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{H}$ that can be used to represent, in a stable way, any vector $f \in \mathcal{H}$ as a linear combination $f = \sum_{\lambda\in\Lambda} d(\lambda) g_\lambda$ with $D = (d(\lambda)) \in \ell^2(\Lambda)$, where Λ is a countable index set. In particular, any vector $f \in \mathcal{H}$ can be represented as

$$f = \sum_{\lambda\in\Lambda} \langle f, \tilde{g}_\lambda \rangle g_\lambda, \tag{1.1}$$

where $\{\tilde{g}_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{H}$ is any dual frame. Frames were introduced by Duffin and Schaeffer in the context of nonharmonic Fourier series [14]. The relatively recent interest in frame theory is due to its use in wavelet theory, time frequency analysis, and sampling theory (see for example [2, 5, 6, 8, 9, 10, 11, 12, 15, 20, 21, 26, 27, 28, 29]). In these contexts, the frames $\{g_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{H}$ have some additional structure. Specifically, in wavelet theory $\{g_\lambda\}_{\lambda\in\Lambda}$ are of the form $\{\phi\left((x-nb)/ma\right)\}_{m\in\mathbf{Z}^+, n\in\mathbf{Z}^d} \subset L^2(\mathbf{R}^d)$, in time frequency analysis the frames are of the form $\{e^{imbx}\phi(x-na)\}_{(m,n)\in\mathbf{Z}^{2d}} \subset L^2(\mathbf{R}^d)$, and in sampling theory they are of the form $\{\phi(x-na)\}_{n\in\mathbf{Z}^d} \subset L^2(\mathbf{R}^d)$, generating shift invariant spaces. While the concept of Riesz basis has been generalized to L^p spaces and studied in the context of shift invariant spaces [23, 24], the systematic study of frames in Banach spaces is relatively recent [9, 16, 19], and we are not aware of any investigation of frames in shift invariant subspaces of L^p ($p \neq 2$). Aside from their theoretical appeal, frames in L^p spaces and other Banach function spaces are effective tools for modeling a variety of natural signals and images [13, 26]. They are also used in the numerical computation of integral and differential equations. Hence, one goal of this paper is to study frames for certain shift invariant subspaces of L^p which are generated by finitely many functions. Specifically, we investigate frames of the form $\{\phi_i(\cdot - j) : \ j \in \mathbf{Z}^d, 1 \le i \le r\} \subset L^p$, that we call p-frames, for the spaces $V_p(\Phi)$ generated by linear combinations of the frame vectors using ℓ^p coefficient sequences. Under certain conditions on the decay at infinity for $\phi_i, 1 \le i \le r$, we derive necessary and sufficient conditions for the indexed family $\{\phi_i(\cdot - j) : \ 1 \le i \le r, j \in \mathbf{Z}^d\}$ to constitute a p-frame, and to generate a shift invariant, closed subspace of L^p. For the case $p = 2$, this problem has been considered by [6, 7, 25, 28].

Let $V_0(\Phi)$ be the space of finite linear combination of integer translates of $\phi_i, 1 \le i \le r$, and let $V_{0,p}(\Phi)$ be the L^p closure of $V_0(\Phi)$. The space $V_{0,p}(\Phi)$ has been studied by [6, 7, 22, 23, 28]. Obviously, we have $V_0(\Phi) \subset V_p(\Phi) \subset V_{0,p}(\Phi)$. We remark that a function f in $V_{0,p}(\Phi)$ is not necessarily generated by ℓ^p coefficients. So the closedness of the shift invariant spaces $V_p(\Phi)$ is an important issue in some applications. Hence we often hope that $V_p(\Phi)$ itself is closed, i.e., a Banach space. In that case, $V_p(\Phi) = V_{0,p}(\Phi)$. For the case $p = 2$, this problem has been considered by many investigators, see for example [4]. In the case of $p \neq 2$, this issue is more complicated since L^p is not a Hilbert space and the Parseval identity is no longer applicable. But we show that under the appropriate conditions on the frame vectors, there is an equivalence between the concept of p-frames, Banach frames (with respect to ℓ^p), and the closedness of the space they generate.

For the vectors $\{g_\lambda\}_{\lambda\in\Lambda}$ of a frame, the relation between a function $f = \sum_{\lambda\in\Lambda} d(\lambda) g_\lambda$ and the coefficients of $D = (d(\lambda))_{\lambda\in\Lambda}$ is not obvious. In Hilbert spaces, it is well known that the coefficients sequence $(d(\lambda)) = (\langle f, \tilde{g}_\lambda\rangle)$ obtained by using the Duffin-Schaeffer dual frame $\{\tilde{g}_\lambda\}_{\lambda\in\Lambda}$ has minimal size [14]. Moreover, the ℓ^2-norm of D is equivalent to the norm of f. For the case of p-frames, we are in the context of Banach spaces, but we are still able to give a characterization of p-frames in terms of the equivalence between $\inf\{\sum_{i=1}^r \|D_i\|_{\ell^p} \,:\, f = \sum_{i=1}^r \phi_i *' D_i\}$ and the L^p-norm of f. Actually, when $\{\phi_i(\cdot - j) :\ 1 \le i \le r, j \in \mathbf{Z}^d\}$ is a p-frame for $V_p(\Phi)$, we construct functions ψ_i, $1 \le i \le r$, independent of p with $\psi_i \in V_1(\Phi) \subset V_p(\Phi)$ such that

$$\begin{aligned} f &= \sum_{i=1}^r \sum_{j\in\mathbf{Z}^d} \langle f, \psi_i(\cdot - j)\rangle \phi_i(\cdot - j) \\ &= \sum_{i=1}^r \sum_{j\in\mathbf{Z}^d} \langle f, \phi_i(\cdot - j)\rangle \psi_i(\cdot - j) \quad \forall\ f \in V_p(\Phi). \end{aligned}$$

Moreover, $V_p(\Phi) = V_p(\Psi)$ and the family $\{\psi_i(\cdot - j) :\ 1 \le i \le r, j \in \mathbf{Z}^d\}$ is also a p-frame for $V_p(\Phi)$. So the family $\{\psi_i(\cdot - j) :\ 1 \le i \le r, j \in \mathbf{Z}^d\}$ can be thought of as a "dual Banach frame".

The paper is organized as follows: Section 2 will introduce some notation, definitions and preliminaries. We present the main theorem and some of its corollaries in Section 3. Several technical lemmas are given in Section 4. In particular, Lemma 1 gives some equivalent relations to the condition (iii) of Theorem 1. Lemma 2 provides a localization technique in Fourier domain which is essential to the proofs of our main theorem, and Lemma 3 is crucial for two key estimates in the proofs of (i) $\Longrightarrow$ (iii) and (ii) $\Longrightarrow$ (iii) of Theorem 1. The localization technique in Lemma 2 and the estimate in Lemma 3 are not necessary if we restrict ourselves to the case $p = 2$. All the proofs are gathered in Section 5.

2 Notation, Definitions and Preliminaries

2.1 Periodic Distributions

We say that T is a 2π-periodic distribution if it is a tempered distribution on $\mathbf{R}^d$ and $T = T(\cdot + 2j\pi)$ for all $j \in \mathbf{Z}^d$. A 2π-periodic distribution can also be thought of as a continuous linear functional on the space of all 2π-periodic C^∞ functions on $\mathbf{R}^d$. For any 2π-periodic distribution T, there exist an integer N_0 and a positive constant C such that

$$|T(f)| \le C \sum_{|\alpha| \le N_0} \|D^\alpha f\|_{L^\infty} \tag{2.1}$$

for any 2π-periodic C^∞ function f on $\mathbf{R}^d$. Note that $e^{-ij\xi}$ is a 2π-periodic C^∞ function on $\mathbf{R}^d$ for any $j \in \mathbf{Z}^d$. We define $T(e^{ij\cdot})$ as the j-th Fourier coefficient of T and formally write $T = \sum_{j \in \mathbf{Z}^d} T(e^{ij\cdot})e^{-ij\xi}$. It follows from (2.1) that there exists a polynomial P such that $|T(e^{ij\cdot})| \le P(j)$ for all $j \in \mathbf{Z}^d$. Conversely for any sequence $D = \{d(j)\}_{j \in \mathbf{Z}^d}$ dominated by some polynomial, $\mathcal{F}(D) = \sum_{j \in \mathbf{Z}^d} d(j)e^{-ij\cdot}$ is a 2π-periodic distribution. For a 2π-periodic distribution T, we say that T is supported in $A + 2\pi\mathbf{Z}^d$ if $T(f) = 0$ for any 2π-periodic C^∞ function f supported in $\mathbf{R}^d \backslash (A + 2\pi\mathbf{Z}^d)$.

2.2 Sequence Spaces

For two sequences $D_1 = (d_1(j))_{j \in \mathbf{Z}^d} \in \ell^{p_1}$ and $D_2 = (d_2(j))_{j \in \mathbf{Z}^d} \in \ell^{p_2}$ with $1/p_1 + 1/p_2 \ge 1$, define their convolution $D_1 * D_2$ as

$$D_1 * D_2(j) = \sum_{j' \in \mathbf{Z}^d} d_1(j - j')d_2(j') \quad \forall\, j \in \mathbf{Z}^d.$$

It is easy to check that $D_1 * D_2 \in \ell^r$ with $1/r = 1/p_1 + 1/p_2 - 1$ for all $D_1 \in \ell^{p_1}$ and $D_2 \in \ell^{p_2}$. Denote by $\mathcal{WC}^p$, $1 \le p \le \infty$, the space of all 2π-periodic distributions with their sequences of Fourier coefficients in ℓ^p. For $p = 1$, $\mathcal{WC}^1$ is simply the Wiener class $\mathcal{WC}$. For a 2π-periodic distribution $T \in \mathcal{WC}^p$, define $\|T\|_{\ell^p_*}$ to be the ℓ^p norm of its sequence of Fourier coefficients. For a vector $T = (T_1, ..., T_r)^T$ of 2π-periodic distributions, we say that $T \in \mathcal{WC}^p$ if $T_i \in \mathcal{WC}^p, 1 \le i \le r$, and define $\|T\|_{\ell^p_*} = \sum_{i=1}^r \|T_i\|_{\ell^p_*}$. For 2π-periodic distributions T_1 and T_2 with their sequences of Fourier coefficients D_1 and D_2 respectively, define their product T_1T_2 as the 2π-periodic distribution with its sequence of Fourier coefficients given by $D_1 * D_2$ if it is well defined. Thus the product of two 2π-periodic distributions $T_1 \in \mathcal{WC}^{p_1}$ and $T_2 \in \mathcal{WC}^{p_2}$ is well defined and belongs to $\mathcal{WC}^r$ when $1/r = 1/p_1 + 1/p_2 - 1 \ge 0$. In particular, the product between a 2π-periodic distribution in $\mathcal{WC}^p$ and a 2π-periodic function in the Wiener class $\mathcal{WC}$ still belongs to $\mathcal{WC}^p$.

2.3 Function Spaces

Let

$$\begin{aligned} \mathcal{L}^p &= \Big\{f:\ \|f\|_{\mathcal{L}^p} = \Big(\int_{[0,1]^d}\Big(\sum_{j\in\mathbf{Z}^d}|f(x+j)|\Big)^p dx\Big)^{1/p} < \infty\Big\} \quad \text{if} \quad 1\le p<\infty, \\ \mathcal{L}^\infty &= \Big\{f:\ \|f\|_{\mathcal{L}^\infty} = \sup_{x\in[0,1]^d}\sum_{j\in\mathbf{Z}^d}|f(x+j)| < \infty\Big\} \quad \text{if} \quad p=\infty, \end{aligned}$$

and

$$\mathcal{W} = \Big\{f:\ \|f\|_{\mathcal{W}} = \sum_{j\in\mathbf{Z}^d}\sup_{x\in[0,1]^d}|f(x+j)| < \infty\Big\}.$$

For $F = (f_1, ..., f_r)^T$, we set $\|F\|_X = \sum_{i=1}^r \|f_i\|_X$ and say that $F \in X$ if $\|F\|_X < \infty$, where $X = L^p, \mathcal{L}^p$ or $\mathcal{W}$. Here A^T denotes the transpose of A. Obviously we have $\mathcal{W} \subset \mathcal{L}^\infty \subset \mathcal{L}^q \subset \mathcal{L}^p \subset L^p$, where $1 \le p \le q \le \infty$ (for instance see [2, 24] for more properties). For $p = 1$, we also have that $\mathcal{L}^1 = L^1$. For any $1 \le p \le \infty$, $f \in L^p$ and $g \in \mathcal{L}^\infty$, by straightforward computation we have

$$\Big\|\Big\{\int_{\mathbf{R}^d} f(x)g(x-j)dx\Big\}_{j\in\mathbf{Z}^d}\Big\|_{\ell^p} \le \|f\|_{L^p}\|g\|_{L^1}^{1-1/p}\|g\|_{\mathcal{L}^\infty}^{1/p} \tag{2.2}$$

Define the Fourier transform $\hat{f}$ of an integrable function f by $\hat{f}(\xi) = \int_{\mathbf{R}^d} f(x)e^{-ix\xi}dx$ and that of a vector-valued tempered distribution by the usual interpretation. Denote the inverse Fourier transform of f by $\mathcal{F}^{-1}(f)$. By the Riemann-Lebesgue Lemma, the Fourier transform of an integrable function is continuous. Hence the Fourier transform of an $\mathcal{L}^p$ function, $1 \le p \le \infty$, is continuous.

2.4 Semi-convolution

For any sequence $D = (d(j))_{j\in\mathbf{Z}^d} \in \ell^p$ and $f \in \mathcal{L}^p$, define their *semi-convolution* $f *' D$ by $f *' D = \sum_{j\in\mathbf{Z}^d} d(j)f(\cdot - j)$ (For $p = \infty$, the convergence of the series is pointwise convergence, but not L^∞ convergence). It is not difficult to check that $f*'$ is a continuous map from ℓ^p to L^p, and also from ℓ^1 to $\mathcal{L}^p$ if $f \in \mathcal{L}^p$, $1 \le p \le \infty$, and it is a continuous map from ℓ^1 to $\mathcal{W}$ if $f \in \mathcal{W}$. Indeed, we have

$$\|f *' D\|_{L^p} \le \|D\|_{\ell^p}\|f\|_{\mathcal{L}^p}, \tag{2.3}$$

$$\|f *' D\|_{\mathcal{L}^p} \leq \|D\|_{\ell^1} \|f\|_{\mathcal{L}^p}, \tag{2.4}$$

and

$$\|f *' D\|_{\mathcal{W}} \leq \|D\|_{\ell^1} \|f\|_{\mathcal{W}}. \tag{2.5}$$

2.5 Shift Invariant Space

For $\Phi = (\phi_1, ..., \phi_r)^T \in \mathcal{L}^p$, let

$$V_p(\Phi) = \Big\{ \sum_{i=1}^{r} \phi_i *' D_i : \ D_i \in \ell^p, 1 \leq i \leq r \Big\}.$$

It follows from (2.3) that $V_p(\Phi)$ is a shift invariant linear subspace of L^p for $1 \leq p \leq \infty$. Here the shift invariance of a linear space V means that $g \in V$ implies $g(\cdot - j) \in V$ for all $j \in \mathbf{Z}^d$. The space $V_p(\Phi)$ is said to be the shift invariant space generated by Φ.

2.6 p-frames and Banach frames

Let $1 \leq p \leq \infty$, Λ be a countable set, and let B be a normed linear space and B^* be its dual. We say that $\{g_\lambda : \ \lambda \in \Lambda\} \subset B^*$ is a *p-frame* for B if the map T defined by

$$T : B \ni f \longmapsto \{\langle f, g_\lambda \rangle\}_{\lambda \in \Lambda} \in \ell^p(\Lambda),$$

is both bounded and bounded below, i.e., there exists a positive constant C such that

$$C^{-1} \|f\|_B \leq \Big(\sum_{\lambda \in \Lambda} |\langle f, g_\lambda \rangle|^p \Big)^{1/p} \leq C \|f\|_B \quad \forall \ f \in B \tag{2.6}$$

for $1 \leq p < \infty$, and

$$C^{-1} \|f\|_B \leq \sup_{\lambda \in \Lambda} |\langle f, g_\lambda \rangle| \leq C \|f\|_B \quad \forall \ f \in B \tag{2.7}$$

for $p = \infty$.

A *Banach frame* (with respect to ℓ^p) is a p-frame with a bounded left inverse R of the operator T [16, p. 148].

For Hilbert spaces, (2.6) or (2.7) guarantees the existence of a reconstruction operator R that allows the reconstruction of a function $f \in B$ from the sequence $\{\langle f, g_\lambda \rangle\}_{\lambda \in \Lambda}$. However, for Banach spaces the operator R does not exist in general. Thus, for Banach spaces, the existence of a reconstruction operator R is included as part of the definition of Banach frames (see [9, 16, 19]). However, in the definition of p-frame above, the existence of a reconstruction operator is not required. Instead for finitely generated shift invariant spaces $V_p(\Phi)$ considered in this paper, the p-frame condition (2.6), or (2.7), is sufficient to prove the existence of a reconstruction operator R. Therefore a p-frame for $V_p(\Phi)$ is a Banach frame (the converse is obvious).

2.7 p-Riesz Basis

Let $1 \le p < \infty$, B a normed linear space and Λ a countable index set. We say that a collection $\{g_\lambda : \lambda \in \Lambda\} \subset B$ is a *p-Riesz basis in B* if the map defined by

$$\ell^p(\Lambda) \ni (c_\lambda)_\lambda \longmapsto \sum_{\lambda\in\Lambda} c_\lambda g_\lambda \in B$$

is both bounded and bounded below, i.e., there exists a positive constant C such that

$$C^{-1}\|c\|_{\ell^p} \le \Big\|\sum_{\lambda\in\Lambda} c_\lambda g_\lambda\Big\|_B \le C\|c\|_{\ell^p} \qquad \forall\ c = (c_\lambda)_{\lambda\in\Lambda} \in \ell^p(\Lambda). \tag{2.8}$$

For $p = 2$, this definition is consistent with the standard definition of a Riesz basis for the closed span of its elements. Definition (2.8) implies that the space $V_p = \Big\{\sum_{\lambda\in\Lambda} c_\lambda g_\lambda\ : c \in \ell^p(\Lambda)\Big\}$ is a complete subspace of B. Thus, V_p is a Banach space even though B may not be complete. In fact, Definition (2.8) implies that $\ell^p(\Lambda)$ and V_p are isomorphic Banach spaces. Obviously, a p-Riesz basis is unconditional, i.e., the sum in the middle term of (2.8) is independent of the order in which the sum is performed.

2.8 Bracket Product

For any functions $\Phi = (\phi_1, \ldots, \phi_r)^T$ and $\Psi = (\psi_1, \ldots, \psi_s)^T$ such that $\widehat{\phi}_i(\xi)\overline{\widehat{\psi}_{i'}(\xi)}$ is integrable for any $1 \le i \le r$ and $1 \le i' \le s$, define an $r \times s$ matrix

$$[\widehat{\Phi}, \widehat{\Psi}](\xi) = \Big(\sum_{j\in\mathbf{Z}^d} \widehat{\phi}_i(\xi + 2j\pi)\overline{\widehat{\psi}_{i'}(\xi + 2j\pi)}\Big)_{1\le i\le r, 1\le i'\le s}.$$

Observe that for any $\phi, \psi \in \mathcal{L}^2$, we have

$$\sum_{j\in\mathbf{Z}^d} \int_{\mathbf{R}^d} |\phi(x)\psi(x-j)|dx \le \int_{[0,1]^d} \sum_{k\in\mathbf{Z}^d} |\phi(x-k)| \sum_{j\in\mathbf{Z}^d} |\psi(x-j)|dx \le \|\phi\|_{\mathcal{L}^2}\|\psi\|_{\mathcal{L}^2}$$

Thus for any $\Phi, \Psi \in \mathcal{L}^2$, Poisson's summation formula implies that all entries of $[\widehat{\Phi}, \widehat{\Psi}](\xi)$ belong to the Wiener class, and are continuous.

2.9 Rank of a $r \times \infty$ Matrix

For $w(j) = (w_1(j), \ldots, w_r(j))^T \in \mathbf{C}^r$ satisfying $\sum_{i=1}^r \sum_{j\in\mathbf{Z}^d} |w_i(j)|^2 < \infty$, define an $r \times \infty$ matrix G by

$$G = (w(j))_{j\in\mathbf{Z}^d}. \tag{2.9}$$

For the matrix G in (2.9), define its column rank as the maximal number of linearly independent columns (as vectors in $\mathbf{C}^r$) of G, and its row rank as the maximal number of linearly independent rows (as vectors in ℓ^2) of G. It is obvious that the column rank and the row rank of G are equal, which is denoted by rank G. For the $r \times \infty$ matrix G in (2.9), define the $r \times r$ matrix $G\overline{G^T} = \Big(\sum_{j\in\mathbf{Z}^d} w_i(j)\overline{w_{i'}(j)}\Big)_{1\le i,i'\le r}$. It can be shown that

$$\text{rank } G = \text{rank } G\overline{G^T}. \tag{2.10}$$

Recall that $[\widehat{\Phi}, \widehat{\Phi}](\xi)$ is continuous for any $\Phi \in \mathcal{L}^2$. Then by (2.10),

$$\Omega_N = \Big\{\xi \in \mathbf{R}^d : \text{ rank } (\widehat{\Phi}(\xi + 2k\pi))_{k\in\mathbf{Z}^d} > N\Big\} \tag{2.11}$$

is an open set for any $N \ge 0$ and $\Phi \in \mathcal{L}^2$.

3 Main Results

In this paper, we shall prove

Theorem 1 *Let $\Phi = (\phi_1, ..., \phi_r)^T \in \mathcal{L}^\infty$ if $1 < p < \infty$, and $\Phi \in \mathcal{W}$ if $p = 1, \infty$. Then the following statements are equivalent to each other.*

(i) $V_p(\Phi)$ is closed in L^p.

(ii) $\{\phi_i(\cdot - j) : j \in \mathbf{Z}^d, 1 \le i \le r\}$ is a p-frame for $V_p(\Phi)$, i.e., there exists a positive constant A (depending on Φ and p) such that

$$A^{-1}\|f\|_{L^p} \le \sum_{i=1}^r \Big\|\Big(\int_{\mathbf{R}^d} f(x)\overline{\phi_i(x-j)}dx\Big)_{j\in\mathbf{Z}^d}\Big\|_{\ell^p} \le A\|f\|_{L^p} \quad \forall\ f \in V_p(\Phi). \tag{3.1}$$

(iii) There exists a positive constant C such that

$$C^{-1}[\widehat{\Phi}, \widehat{\Phi}](\xi) \le [\widehat{\Phi}, \widehat{\Phi}](\xi)\overline{[\widehat{\Phi}, \widehat{\Phi}](\xi)^T} \le C[\widehat{\Phi}, \widehat{\Phi}](\xi) \quad \forall\ \xi \in [-\pi, \pi]^d.$$

(iv) There exists a positive constant B (depending on Φ and p) such that

$$B^{-1}\|f\|_{L^p} \le \inf_{f=\sum_{i=1}^r \phi_i *' D_i} \sum_{i=1}^r \|D_i\|_{\ell^p} \le B\|f\|_{L^p} \qquad \forall\ f \in V_p(\Phi). \tag{3.2}$$

(v) There exists $\Psi = (\psi_1, \ldots, \psi_r)^T \in \mathcal{L}^\infty$ if $1 < p < \infty$, and $\Psi \in \mathcal{W}$ if $p = 1, \infty$, such that

$$f = \sum_{i=1}^r \sum_{j\in\mathbf{Z}^d} \langle f, \psi_i(\cdot - j)\rangle \phi_i(\cdot - j) = \sum_{i=1}^r \sum_{j\in\mathbf{Z}^d} \langle f, \phi_i(\cdot - j)\rangle \psi_i(\cdot - j) \qquad \forall\ f \in V_p(\Phi). \tag{3.3}$$

Remark 1 From (v) of Theorem 1 it follows that $V_p(\Psi) = V_p(\Phi)$. This together with the implication (v) $\Longrightarrow$ (ii) in Theorem 1 leads to the conclusion that $\{\psi_i(\cdot - j) :\ 1 \le i \le r, j \in \mathbf{Z}^d\}$ is also a p-frame for $V_p(\Psi) = V_p(\Phi)$. Thus, $\{\psi_i(\cdot - j) :\ 1 \le i \le r, j \in \mathbf{Z}^d\}$ can be thought of as a "dual p-frame" of the family $\{\phi_i(\cdot - j) :\ 1 \le i \le r, j \in \mathbf{Z}^d\}$. Hence a p-frame for $V_p(\Phi)$ is a Banach frame (for definitions of Banach frames see [9, 16, 19]).

Remark 2 The functions ψ_i, $1 \le i \le r$, in (v) of Theorem 1 belong to $V_1(\Phi) \subset V_p(\Phi)$ and are independent of p, $1 \le p \le \infty$. In fact, from the proof of (v) in Theorem 1, we have

$$\psi_i = \sum_{i'=1}^r \sum_{j\in\mathbf{Z}^d} c_{ii'}(j)\phi_{i'}(\cdot - j)$$

for some ℓ^1 sequences $\{c_{ii'}(j) : j \in \mathbf{Z}^d\}$, $1 \le i, i' \le r$, independent of p.

Remark 3 Let $\psi_i, 1 \le i \le r$, be as in (v) of Theorem 1. Define an operator P on L^p by

$$Pf = \sum_{i=1}^r \sum_{j\in\mathbf{Z}^d} \langle f, \psi_i(\cdot - j)\rangle \phi_i(\cdot - j).$$

Then P is a bounded operator from L^p to $V_p(\Phi)$, and $Pf = f$ for all $f \in V_p(\Phi)$. Therefore P is a bounded projection operator. Let

$$W = \{f \in L^p :\ Pf = 0\}.$$

Then W is a closed linear subspace of L^p, and $L^p = V_p(\Phi) \bigoplus W$. In other words, $V_p(\Phi)$ is complemented in L^p, with W as a complement.

Remark 4 It is easy to construct examples of shift invariant spaces $V_p(\Phi)$ that are not closed. For example, if $\Phi = \chi_{[0,1]} - \chi_{[1,2]}$ where χ_E is the characteristic function of a set E, then $V_p(\Phi)$ is not closed.

Note that the condition (iii) in Theorem 1 is independent of p. Therefore, the p-frame property ((ii) and (iv) in Theorem 1) and closedness property ((i) in Theorem 1) of $V_p(\Phi)$ is independent of p.

Corollary 1 *Let $\Phi = (\phi_1, ..., \phi_r)^T \in \mathcal{W}$, and $1 \le p_0 \le \infty$.*

(i) If $\{\phi_i(\cdot - j) : \ j \in \mathbf{Z}^d, 1 \le i \le r\}$ is a p_0-frame for $V_{p_0}(\Phi)$, then $\{\phi_i(\cdot - j) : \ j \in \mathbf{Z}^d, 1 \le i \le r\}$ is a p-frame for $V_p(\Phi)$ for any $1 \le p \le \infty$.

(ii) If $V_{p_0}(\Phi)$ is closed in L^{p_0}, then $V_p(\Phi)$ is closed in L^p for any $1 \le p \le \infty$.

(iii) If (3.2) holds for p_0, then (3.2) holds for any $1 \le p \le \infty$.

Note that the p-Riesz basis for the shift invariant spaces $V_p(\Phi)$ can be characterized in the following way (see for instance [24]).

Proposition 1 *Let $1 \le p < \infty$ and Φ be as in Theorem 1. Then $\{\phi_i(\cdot - j) : \ j \in \mathbf{Z}^d, 1 \le i \le r\}$ is a p-Riesz basis for $V_p(\Phi)$ if and only if there exists a positive constant C such that*

$$C^{-1} I_r \le [\widehat{\Phi}, \widehat{\Phi}](\xi) \le C I_r \quad \forall\, \xi \in [-\pi, \pi]^d, \tag{3.4}$$

where I_r denotes the $r \times r$ identity matrix.

Then as a consequence of Theorem 1 and Proposition 1, we obtain the expected result below.

Corollary 2 *Let $\Phi = (\phi_1, ..., \phi_r)^T \in \mathcal{L}^\infty$ if $1 < p < \infty$, and $\Phi \in \mathcal{W}$ if $p = 1$. If $\{\phi_i(\cdot - j) : \ j \in \mathbf{Z}^d, 1 \le i \le r\}$ is a p-Riesz basis for $V_p(\Phi)$, then $\{\phi_i(\cdot - j) : \ j \in \mathbf{Z}^d, 1 \le i \le r\}$ is a p-frame for $V_p(\Phi)$.*

As further consequences of Theorem 1 and Proposition 1, a p-frame for $V_p(\Phi)$ is a p-Riesz basis for $V_p(\Phi)$ if we impose more restrictions on Φ. Actually the additional condition on Φ is that the matrix $[\widehat{\Phi}, \widehat{\Phi}](\xi)$ is invertible for some ξ. Obviously, this condition always holds if $r = 1$ (except for the trivial case $\Phi = 0$).

Corollary 3 *Let $\Phi = (\phi_1, ..., \phi_r)^T \in \mathcal{L}^\infty$ if $1 < p < \infty$, and $\Phi \in \mathcal{W}$ if $p = 1$. Assume that $\{\phi_i(\cdot - j) : j \in \mathbf{Z}^d, 1 \le i \le r\}$ is a p-frame for $V_p(\Phi)$.*

(i) If there exists a $\xi_0 \in [-\pi, \pi]^d$ such that the $r \times r$ matrix $[\widehat{\Phi}, \widehat{\Phi}](\xi_0)$ is invertible, then $\{\phi_i(\cdot - j) : j \in \mathbf{Z}^d, 1 \le i \le r\}$ is a p-Riesz basis for $V_p(\Phi)$.

(ii) If $r = 1$ and $\phi_1 \ne 0$, then $\{\phi_1(\cdot - j) : j \in \mathbf{Z}^d\}$ is a p-Riesz basis for $V_p(\phi_1)$.

Part (ii) of Corollary 3 states that if ϕ_1 is regular and $r = 1$, then it is not possible to construct frames that are not Riesz bases. For example if $|\phi_1| \le M < \infty$ and has compact support, or if ϕ_1 is continuous and decays faster than $|x|^{-2}$ at infinity, then any frame of $V_p(\phi_1)$ is also a Riesz basis of $V_p(\phi_1)$. Thus, for $r = 1$, frames of $V_p(\phi_1)$ that are not Riesz bases must be generated by ϕ_1 that are not regular, e.g., if ϕ_1 has compact support, then ϕ_1 must have an infinite singularity, or if ϕ_1 is continuous with global support, then ϕ must have slow decay. For example for $p = 2$ and $r = 1$, Benedetto and Li construct frames that are not Riesz bases [6]. However, in their construction the generator ϕ_1 has slow decay at infinity.

In view of Corollary 2, we may consider the converse problem: Given a p-frame $\{\phi_i(\cdot - j) : 1 \le i \le r, j \in \mathbf{Z}^d\}$ for $V_p(\Phi)$, can we find $\tilde{\phi}_i, 1 \le i \le s$, such that the closed span of $\{\tilde{\phi}_i(\cdot - j) : 1 \le i \le s, j \in \mathbf{Z}^d\}$ using ℓ^p coefficient sequences is $V_p(\Phi)$ and such that $\{\tilde{\phi}_i(\cdot - j) : 1 \le i \le s, j \in \mathbf{Z}^d\}$ is a p-Riesz basis for $V_p(\Phi)$.

Conjecture *Let $\Phi = (\phi_1, ..., \phi_r)^T \in \mathcal{L}^\infty$ if $1 < p < \infty$, and $\Phi \in \mathcal{W}$ if $p = 1$. Assume that $\{\phi_i(\cdot - j) : 1 \le i \le r, j \in \mathbf{Z}^d\}$ is a p-frame for $V_p(\Phi)$. Then there exist $s \ge 1$, and $\tilde{\Phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_s)^T \in \mathcal{L}^\infty$ if $1 < p < \infty$, and $\tilde{\Phi} \in \mathcal{W}$ if $p = 1$, such that $\{\tilde{\phi}_i(\cdot - j) : 1 \le i \le s, j \in \mathbf{Z}^d\}$ is a p-Riesz basis for $V_p(\tilde{\Phi})$ and $V_p(\tilde{\Phi}) = V_p(\Phi)$.*

The assertion in the above conjecture is true under the additional assumption that $\phi_i, 1 \le i \le r$, are compactly supported bounded functions on $\mathbf{R}$ (see [22] for $1 < p < \infty$, and [3] for $1 \le p < \infty$ with a completely different proof). We feel strongly that the assertion of the above conjecture for higher dimensions would not be true.

Remark 5 We remark that the condition (iii) in Theorem 1 is different from the following condition about quasi-stability in [7]:

$(*)$ *There exists a positive constant C such that*

$$C^{-1} I_r \le [\widehat{\Phi}, \widehat{\Phi}](\xi) \le C I_r \quad \forall \ \xi \in \Omega,$$

where

$$\Omega = \{\xi \in [-\pi,\pi]^d : \sum_{j\in\mathbf{Z}^d} \hat{\phi}_i(\xi+2\pi j)\overline{\hat{\phi}_i(\xi+2\pi j)} \neq 0, \quad \text{for some } 1 \le i \le r\}.$$

Hence the p-frame property in this paper is different from the quasi-stability in [7]. For instance, let ϕ_1, ϕ_2 be two compactly supported L^2 functions with their shifts being orthonormal, and let $\phi_3 = \phi_1(\cdot - 1) - \phi_2$. Define $\Phi = (\phi_1, \phi_2, \phi_3)^T$. Then

$$[\widehat{\Phi}, \widehat{\Phi}](\xi) = \begin{pmatrix} 1 & 0 & e^{-i\xi} \\ 0 & 1 & -1 \\ e^{i\xi} & -1 & 2 \end{pmatrix}, \quad \xi \in \mathbf{R}^d.$$

By direct computation, for each $\xi \in \mathbf{R}^d$, the eigenvalues of $[\widehat{\Phi}, \widehat{\Phi}](\xi)$ are $0, 1$ and 3. Therefore Condition (iii) of Theorem 1 holds for Φ, but Condition $(*)$ is not true for Φ since $[\widehat{\Phi}, \widehat{\Phi}](\xi)$ is not invertible for any $\xi \in [-\pi,\pi]^d$.

Remark 6 We note that from the proof of Theorem 1 together with some modifications, we only need to assume that $\Phi \in \mathcal{L}^p, 1 \le p \le \infty$, for the equivalence between (i) and (iv) in Theorem 1; $\Phi \in \mathcal{L}^2 \cap \mathcal{L}^p$ if $1 \le p < \infty$ and $\Phi \in \mathcal{W}$ if $p = \infty$ for the equivalence between (i) and (iii); and $\Phi \in \mathcal{L}^\infty$ if $1 < p \le \infty$ and $\Phi \in \mathcal{W}$ if $p = 1$ for the equivalence between (ii) and (iii).

4 Technical Lemmas

The first lemma will give some equivalent relations of the condition (iii) of Theorem 1. Corresponding results involving Riesz bases can be found in [1, 7, 17, 18, 24, 30].

Lemma 1 *Let $\Phi = (\phi_1, \ldots, \phi_r)^T \in \mathcal{L}^2$. Then the following statements are equivalent to each other.*

(i) $\operatorname{rank} \left(\widehat{\Phi}(\xi + 2j\pi)\right)_{j\in\mathbf{Z}^d}$ *is a constant function on* $\mathbf{R}^d$*.*

(ii) $\operatorname{rank} [\widehat{\Phi}, \widehat{\Phi}](\xi)$ *is a constant function on* $\mathbf{R}^d$*.*

(iii) There exists a positive constant C independent of ξ such that

$$C^{-1}[\widehat{\Phi}, \widehat{\Phi}](\xi) \le [\widehat{\Phi}, \widehat{\Phi}](\xi)\overline{[\widehat{\Phi}, \widehat{\Phi}](\xi)^T} \le C[\widehat{\Phi}, \widehat{\Phi}](\xi) \quad \forall\, \xi \in [-\pi,\pi]^d.$$

Proof of Lemma 1 Obviously the equivalence of (i) and (ii) follows from (2.10), where $G = (\widehat{\Phi}(\xi + 2j\pi))_{j\in\mathbf{Z}^d}$. Now we start to prove the equivalence of (ii) and (iii). Let $\lambda_i(\xi), 1 \le i \le r$, be all eigenvalues of $[\widehat{\Phi}, \widehat{\Phi}](\xi)$ ordered such that $\lambda_1(\xi) \ge \lambda_2(\xi) \ge \ldots \ge \lambda_r(\xi)$. Note that $[\widehat{\Phi}, \widehat{\Phi}](\xi)$ is a positive semi-definite matrix. Thus $\lambda_i(\xi) \ge 0$ for all $i = 1, \ldots, r$ and there exists an $r \times r$ matrix $A(\xi)$ such that $\overline{A(\xi)^T} A(\xi) = I_r$, the $r \times r$ identity matrix, and

$$\overline{A(\xi)^T}[\widehat{\Phi}, \widehat{\Phi}](\xi)A(\xi) = \mathrm{diag}(\lambda_1(\xi), ..., \lambda_r(\xi)). \tag{4.1}$$

Recall that $[\widehat{\Phi}, \widehat{\Phi}](\xi)$ is in the Wiener class, since $\Phi \in \mathcal{L}^2$. Then $\lambda_i(\xi)$ are continuous and 2π-periodic for all $1 \le i \le r$. Denote the rank of $[\widehat{\Phi}, \widehat{\Phi}](\xi)$ by $k_1(\xi)$.

Assume that (ii) holds. Then $k_1(\xi)$ is a constant, which is denoted by k_1. Thus $\lambda_i(\xi) > 0$ for all $\xi \in \mathbf{R}^d$ and $1 \le i \le k_1$, and

$$\lambda_i(\xi) \equiv 0 \quad \forall\, \xi \in \mathbf{R}^d \quad \text{and} \quad k_1 + 1 \le i \le r. \tag{4.2}$$

Hence it follows from the continuity and periodicity of $\lambda_i(\xi)$ that there exists a positive constant C such that

$$C^{-1} \le \lambda_i(\xi) \le C \quad \forall\, \xi \in \mathbf{R}^d \quad \text{and } 1 \le i \le k_1. \tag{4.3}$$

Combining (4.1) – (4.3), we get (iii).

Now assume that (iii) holds. Then by (4.1),

$$C^{-1}\lambda_i(\xi) \le \lambda_i(\xi)^2 \le C\lambda_i(\xi) \quad \forall\ 1 \le i \le r \quad \text{and} \quad \xi \in \mathbf{R}^d.$$

Thus either $\lambda_i(\xi) = 0$ or $C^{-1} \le \lambda_i(\xi) \le C$. Hence $\mathrm{rank}[\widehat{\Phi}, \widehat{\Phi}](\xi)$ is a constant by the continuity and periodicity of $\lambda_i(\xi)$. This completes the proof of (ii) $\Longleftrightarrow$ (iii). $\square$

The next lemma and the technique introduced in its proof are crucial for our subsequent discussion. In Fourier domain, it allows us to replace locally the generator $\widehat{\Phi}$ of size r by a local generator $\widehat{\Psi}_{1,\lambda}$ of size k_0.

Lemma 2 *Let $\Phi \in \mathcal{L}^2$ satisfy $\mathrm{rank}\Big(\widehat{\Phi}(\xi + 2k\pi)\Big)_{k\in\mathbf{Z}^d} = k_0 \ge 1$ for all $\xi \in \mathbf{R}^d$. Then there exist a finite index set Λ, $\eta_\lambda \in [-\pi, \pi]^d$, $0 < \delta_\lambda < 1/4$, nonsingular 2π-periodic $r\times r$ matrix $P_\lambda(\xi)$ with all entries in the Wiener class and $K_\lambda \subset \mathbf{Z}^d$ with cardinality$(K_\lambda) = k_0$ for all $\lambda \in \Lambda$, having the following properties:*

(i)

$$\cup_{\lambda\in\Lambda} B(\eta_\lambda, \delta_\lambda/2) \supset [-\pi, \pi]^d, \tag{4.4}$$

where $B(x_0, \delta)$ denotes the open ball in $\mathbf{R}^d$ with center x_0 and radius δ;

(ii)

$$P_\lambda(\xi)\widehat{\Phi}(\xi) = \begin{pmatrix} \widehat{\Psi}_{1,\lambda}(\xi) \\ \widehat{\Psi}_{2,\lambda}(\xi) \end{pmatrix}, \quad \xi \in \mathbf{R}^d \quad \text{and} \quad \lambda \in \Lambda, \tag{4.5}$$

where $\Psi_{1,\lambda}$ and $\Psi_{2,\lambda}$ are functions from $\mathbf{R}^d$ to $\mathbf{C}^{k_0}$ and $\mathbf{C}^{r-k_0}$ respectively satisfying

$$\text{rank}\Big(\widehat{\Psi}_{1,\lambda}(\xi + 2k\pi)\Big)_{k\in K_\lambda} = k_0 \quad \forall\, \xi \in B(\eta_\lambda, 2\delta_\lambda) \tag{4.6}$$

and

$$\widehat{\Psi}_{2,\lambda}(\xi) = 0 \quad \forall\, \xi \in B(\eta_\lambda, 8\delta_\lambda/5) + 2\pi\mathbf{Z}^d. \tag{4.7}$$

Further there exist 2π-periodic C^∞ functions $h_\lambda(\xi), \lambda \in \Lambda$, on $\mathbf{R}^d$ such that

$$\sum_{\lambda\in\Lambda} h_\lambda(\xi) = 1 \quad \forall\, \xi \in \mathbf{R}^d \tag{4.8}$$

and

$$\text{supp } h_\lambda(\xi) \subset B(\eta_\lambda, \delta_\lambda/2) + 2\pi\mathbf{Z}^d. \tag{4.9}$$

Proof For any $\eta_0 \in [-\pi, \pi]^d$, there exist a nonsingular $r \times r$ matrix P_{η_0}, a $k_0 \times k_0$ nonsingular matrix A_{η_0} and $K_{\eta_0} \subset \mathbf{Z}^d$ with cardinality$(K_{\eta_0}) = k_0$ such that

$$P_{\eta_0}\Big(\widehat{\Phi}(\eta_0 + 2k\pi)\Big)_{k\in K_{\eta_0}} = \begin{pmatrix} A_{\eta_0} \\ 0 \end{pmatrix}.$$

Write

$$P_{\eta_0}\Big(\widehat{\Phi}(\xi + 2k\pi)\Big)_{k\in K_{\eta_0}} = \begin{pmatrix} A_{\eta_0} + R_1(\xi) \\ R_2(\xi) \end{pmatrix}.$$

By the continuity of $\widehat{\Phi}$, $R_1(\xi)$ and $R_2(\xi)$ are continuous, and $R_1(\eta_0) = 0$ and $R_2(\eta_0) = 0$. Thus $\sup_{\xi\in B(\eta_0,4\delta)} \|R_1(\xi)\| + \|R_2(\xi)\|$ is sufficiently small for any sufficiently small positive δ. Let $H(x)$ be a nonnegative C^∞ function on $\mathbf{R}^d$ such that

$$H(x) = \begin{cases} 1, & \text{if } |x| \le 4/5, \\ 0, & \text{if } |x| \ge 1. \end{cases} \tag{4.10}$$

Then $A_{\eta_0} + H((\xi - \eta_0)/4\delta)R_1(\xi)$ is a $k_0 \times k_0$ nonsingular matrix for all $\xi \in \mathbf{R}^d$ when δ is chosen sufficiently small.

For $\xi \in \mathbf{R}^d$, set

$$\alpha_{\eta_0}(\xi) = A_{\eta_0} + \sum_{j\in\mathbf{Z}^d} H\Big(\frac{\xi + 2j\pi - \eta_0}{4\delta}\Big)R_1(\xi + 2j\pi)$$

and

$$\beta_{\eta_0}(\xi) = \sum_{j\in\mathbf{Z}^d} H\Big(\frac{\xi + 2j\pi - \eta_0}{2\delta}\Big) R_2(\xi + 2j\pi).$$

Then $\alpha_{\eta_0}(\xi)$ is 2π-periodic and nonsingular when δ is chosen sufficiently small. Note that the ℓ^1-norms of the sequences of the Fourier coefficients of $\alpha_{\eta_0}(\xi)$ and $\beta_{\eta_0}(\xi)$ are dominated by

$$\begin{aligned} & C + C\sum_{j\in\mathbf{Z}^d} \Big|\mathcal{F}^{-1}\Big(H(\cdot/(4\delta))\widehat{\Phi}\Big)(j)\Big| + C\sum_{j\in\mathbf{Z}^d} \Big|\mathcal{F}^{-1}\Big(H(\cdot/(2\delta))\widehat{\Phi}\Big)(j)\Big| \\ \leq\ & C\|\Phi\|_{L^1} < \infty, \end{aligned}$$

where C is a positive constant. This shows that all entries of $\alpha_{\eta_0}(\xi)$ and $\beta_{\eta_0}(\xi)$ belong to the Wiener class.

Let

$$P_{\eta_0}(\xi) = P_{\eta_0} + \begin{pmatrix} 0 & 0 \\ -\beta_{\eta_0}(\xi)(\alpha_{\eta_0}(\xi))^{-1} & 0 \end{pmatrix} P_{\eta_0}, \quad \xi \in \mathbf{R}^d.$$

Then $P_{\eta_0}(\xi)$ is a 2π-periodic nonsingular $r\times r$ matrix for any $\xi\in\mathbf{R}^d$, and all entries of $P_{\eta_0}(\xi)$ belong to the Wiener class. Note that $\alpha_{\eta_0}(\xi) = A_{\eta_0} + R_1(\xi)$ and $\beta_{\eta_0}(\xi) = R_2(\xi)$ for all $\xi \in B(\eta_0, 8\delta/5)$. Thus

$$P_{\eta_0}(\xi)\Big(\widehat{\Phi}(\xi + 2k\pi)\Big)_{k\in K_{\eta_0}} = \begin{pmatrix} A_{\eta_0} + R_1(\xi) \\ 0 \end{pmatrix} \quad \forall\ \xi \in B(\eta_0, 8\delta/5), \tag{4.11}$$

when $1/4 > \delta > 0$ is chosen sufficiently small.

Define $\Psi_{1,\eta_0} = (\psi_{1,\eta_0,1}, \ldots, \psi_{1,\eta_0,k_0})^T$ and $\Psi_{2,\eta_0} = (\psi_{2,\eta_0,1}, \ldots, \psi_{2,\eta_0,r-k_0})^T$ by

$$\begin{pmatrix} \widehat{\Psi}_{1,\eta_0}(\xi) \\ \widehat{\Psi}_{2,\eta_0}(\xi) \end{pmatrix} = P_{\eta_0}(\xi)\widehat{\Phi}(\xi), \quad \xi \in \mathbf{R}^d. \tag{4.12}$$

Recall that all entries of $P_{\eta_0}(\xi)$ belong to the Wiener class. Thus by (2.4), $\Psi_{1,\eta_0} \in \mathcal{L}^1$ and $\Psi_{2,\eta_0} \in \mathcal{L}^p$ if $\Phi \in \mathcal{L}^p$. By (4.11) and (4.12), we have

$$\widehat{\Psi}_{2,\eta_0}(\xi + 2k\pi) = 0 \quad \forall\ \xi \in B(\eta_0, 8\delta/5) \quad \text{and} \quad k \in K_{\eta_0}. \tag{4.13}$$

Also for $\xi \in B(\eta_0, 8\delta/5)$, rank $(A_{\eta_0} + R_1(\xi)) = k_0$ and

$$\begin{aligned} & \text{rank} \begin{pmatrix} A_{\eta_0} + R_1(\xi) & \widehat{\Psi}_{1,\eta_0}(\xi + 2k'\pi) \\ 0 & \widehat{\Psi}_{2,\eta_0}(\xi + 2k'\pi) \end{pmatrix}_{k' \in \mathbf{Z}^d \setminus K_{\eta_0}} \\ = \ & \text{rank} \begin{pmatrix} \widehat{\Psi}_{1,\eta_0}(\xi + 2k\pi) \\ \widehat{\Psi}_{2,\eta_0}(\xi + 2k\pi) \end{pmatrix}_{k \in \mathbf{Z}^d} = \text{rank} \left(\widehat{\Phi}(\xi + 2k\pi) \right)_{k \in \mathbf{Z}^d} = k_0. \end{aligned}$$

Thus $\widehat{\Psi}_{2,\eta_0}(\xi + 2k'\pi) = 0$ for all $\xi \in B(\eta_0, 8\delta/5)$ and $k' \in \mathbf{Z}^d \setminus K_{\eta_0}$. This together with (4.13) lead to

$$\widehat{\Psi}_{2,\eta_0}(\xi) = 0 \quad \forall\, \xi \in B(\eta_0, 8\delta/5) + 2\pi \mathbf{Z}^d.$$

Let δ_{η_0} be chosen such that $1 < \delta_{\eta_0} < 1/4$, $P_{\eta_0}(\xi)$ and $A_{\eta_0} + H((\xi - \eta_0)/(4\delta_{\eta_0}))R_1(\xi)$ as defined above are nonsingular for all $\xi \in \mathbf{R}^d$, and $A_{\eta_0} + R_1(\xi)$ is nonsingular for $\xi \in B(\eta_0, 4\delta_{\eta_0})$. For the family $\{B(\eta_0, \delta_{\eta_0}/2) : \eta_0 \in [-\pi, \pi]^d\}$ of open balls covering $[-\pi, \pi]^d$, there exists a finite index set Λ such that $\cup_{\lambda \in \Lambda} B(\eta_\lambda, \delta_\lambda/2) \supset [-\pi, \pi]^d$ by the compactness of $[-\pi, \pi]^d$. For such a finite covering, the corresponding $P_\lambda(\xi)$, $\Psi_{1,\lambda}$ and $\Psi_{2,\lambda}$ constructed above satisfy the desired properties in Lemma 2. The existence of the desired 2π-periodic functions $h_\lambda(\xi), \lambda \in \Lambda$, follows easily from (4.4). $\square$

The following result is about $\mathcal{L}^p$ and $\mathcal{W}$ estimates of ϕ near a point ξ_0 such that $\hat{\phi}(\xi_0 + 2k\pi) = 0$ for all $k \in \mathbf{Z}^d$. This lemma is crucial to obtaining (5.6) and (5.19), the key estimates in the proofs of (i) $\Longrightarrow$ (iii) and (ii) $\Longrightarrow$ (iii). We remark that for $p = 2$, these estimates follow easily from L^2 Fourier theory.

Lemma 3 *Let $\phi \in \mathcal{L}^p$ if $1 \le p < \infty$, and $\phi \in \mathcal{W}$ if $p = \infty$. Assume that $\sum_{j \in \mathbf{Z}^d} \phi(\cdot - j) = 0$. Then for any function h on $\mathbf{R}^d$ satisfying*

$$|h(x)| \le C(1+|x|)^{-d-1} \quad \text{and} \quad |h(x) - h(y)| \le C|x-y| \left(1 + \min(|x|, |y|)\right)^{-d-1}, \quad (4.14)$$

we have

$$\lim_{n \to \infty} 2^{-nd} \Big\| \sum_{j \in \mathbf{Z}^d} h(2^{-n}j)\phi(\cdot - j) \Big\|_{\mathcal{L}^p} = 0.$$

Note that any Lipschitz function with compact support and any Schwartz function satisfy (4.14).

Proof Here we only give the proof of the assertion for $\phi \in \mathcal{L}^p, 1 \le p < \infty$, in detail. The case $\phi \in \mathcal{W}$ when $p = \infty$ can be proved by similar arguments.

By the Lebesgue dominated convergence theorem, for any $\epsilon > 0$ there exists $N_0 \geq 2$ such that

$$\Big(\int_{x\in[0,1]^d}\Big(\sum_{|j|\geq N_0}|\phi(x+j)|\Big)^p dx\Big)^{1/p} \leq \epsilon.$$

Set

$$\phi_1(x) = \phi(x)\chi_{O_{N_0}}(x) + \sum_{|j|\geq N_0}\phi(x+j)\chi_{[0,1]^d}(x),$$

where χ_E denotes the characteristic function of a set E, and $O_{N_0} = \cup_{|j|<N_0}(j+[0,1]^d)$. Thus $\sum_{j\in\mathbf{Z}^d}\phi_1(\cdot+j) = \sum_{j\in\mathbf{Z}^d}\phi(\cdot+j) = 0$ and

$$\|\phi_1-\phi\|_{\mathcal{L}^p} \leq 2\Big(\int_{[0,1]^d}\Big(\sum_{|j|\geq N_0}|\phi(x+j)|\Big)^p dx\Big)^{1/p} \leq 2\epsilon.$$

Therefore, using (2.4), (4.14), and the fact that $\operatorname{supp}\phi_1 \subset \{x : |x| \leq N_0 + d\}$, we get

$$\begin{aligned} & \|2^{-nd}\sum_{j\in\mathbf{Z}^d}h(2^{-n}j)(\phi(x-j)-\phi_1(x-j))\|_{\mathcal{L}^p} \\ \leq\ & 2^{-nd}\sum_{j\in\mathbf{Z}^d}|h(2^{-n}j)|\ \|\phi-\phi_1\|_{\mathcal{L}^p} \leq C\epsilon, \end{aligned}$$

and

$$\begin{aligned} & 2^{-ndp}\int_{[0,1]^d}\Big(\sum_{k\in\mathbf{Z}^d}\Big|\sum_{j\in\mathbf{Z}^d}h(2^{-n}j)\phi_1(x-j+k)\Big|\Big)^p dx \\ \leq\ & 2^{-ndp}\int_{[0,1]^d}\Big(\sum_{k\in\mathbf{Z}^d}\Big|\sum_{j\in\mathbf{Z}^d}(h(2^{-n}j)-h(2^{-n}k))\phi_1(x-j+k)\Big|\Big)^p dx \\ \leq\ & 2^{-n(d+1)p}C_1(N_0)\int_{[0,1]^d}\Big(\sum_{k\in\mathbf{Z}^d}(1+2^{-n}|k|)^{-d-1}\sum_{|j-k|\leq N_0+d}|\phi_1(x-j+k)|\Big)^p dx \\ \leq\ & 2^{-n(d+1)p}C_2(N_0)(\sum_{k\in\mathbf{Z}^d}(1+2^{-n}|k|)^{-d-1})^p\|\phi_1\|^p_{\mathcal{L}^p} \\ \leq\ & 2^{-np}C_3(N_0)(\|\phi\|_{\mathcal{L}^p}+2\epsilon)^p \leq (\|\phi\|_{\mathcal{L}^p}+2\epsilon)^p\epsilon^p, \end{aligned}$$

when $n > \frac{1}{p}\ln\Big(C_3(N_0)/\epsilon\Big)$, where $C_i(N_0),\ i = 1,2,3$, are positive constants depending only on N_0, d and the constant C in (4.14). Hence the assertion follows for $1 \leq p < \infty$ and $\phi \in \mathcal{L}^p$. $\Box$

For $p = \infty$ the hypothesis $\phi \in \mathcal{W}$ in Lemma 3 cannot be weakened to $\phi \in \mathcal{L}^\infty$, as is shown by the following example.

Example Let $d=1$, $\phi(x)=\chi_{[0,1]}(x)-\sum_{j=1}^{\infty}\chi_{2^j+[2^{-j},2^{-j+1}]}(x)$ and $h(x)=(1+x^2)^{-1}$. Obviously ϕ is in $\mathcal{L}^\infty$ but not in $\mathcal{W}$, and $\sum_{j\in\mathbf{Z}}\phi(x-j)=0$. Set

$$g_n(x)=2^{-n}\sum_{k\in\mathbf{Z}}h(2^{-n}k)\phi(x-k).$$

Thus we get

$$g_n(x+k)=2^{-n}(h(2^{-n}k)-h(2^{-n}k+2^{l-n})),\quad \forall\ k\in\mathbf{Z},\ x\in[2^{-l},2^{-l+1}]$$

and

$$\sup_{x\in[0,1]}\sum_{k\in\mathbf{Z}}|g_n(x+k)|\geq 2^{-n}\sum_{0\leq k\leq 2^{n-2}}|h(2^{-n}k)-h(2^{-n}k+1)|\geq\frac{1}{10}.$$

This shows that $\|g_n\|_{\mathcal{L}^\infty}, n\geq 1$, does not tend to zero as n tends to infinity.

5 Proofs

In this section, we shall give the proof of Theorem 1. We divide the proof into the following steps: (v) $\Longrightarrow$ (iv) $\Longrightarrow$ (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (v) and (iii) $\Longrightarrow$ (v) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). The proofs of (i) $\Longrightarrow$ (iii) and (ii) $\Longrightarrow$ (iii) are the most difficult and technical parts in our proof.

5.1 Proof of (v) $\Longrightarrow$(iv)

Let $f=\sum_{i=1}^r\sum_{j\in\mathbf{Z}^d}\langle f,\psi_i(\cdot-j)\rangle\phi_i(\cdot-j)$. Then using (2.2) we have

$$\inf_{f=\sum_{i=1}^r\phi_i*'D_i}\sum_{i=1}^r\|D_i\|_{\ell^p}\leq\|\{\sum_{i=1}^r\langle f,\psi_i(\cdot-j)\rangle\}_{j\in\mathbf{Z}^d}\|_{\ell^p}\leq B\|f\|_{L^p}$$

which is the right hand side of inequality (3.2) .

To prove the left hand side of inequality (3.2), we simply note that if $f=\sum_{i=1}^r D_i*'\phi_i\in V_p(\Phi)$, we have, using (2.3),

$$\|f\|_{L^p}=\|\sum_{i=1}^r D_i*'\phi_i\|_{L^p}\leq C\sum_{i=1}^r\|D_i\|_{\ell^p}.$$

Taking the infimum on both sides of the inequality above, we obtain the left hand side of inequality (3.2). □

5.2 Proof of (iv) $\Longrightarrow$(i)

The implication (iv) $\Longrightarrow$ (i) follows from standard functional analytic arguments. We include it here for the sake of completeness, which is a consequence of the following general result.

Theorem 2 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces, and let T be a bounded linear operator from X to Y. If there is a positive constant C such that*

$$C^{-1}\|y\|_Y \le \inf_{y=Tx} \|x\|_X \le C\|y\|_Y \quad \forall\ y \in \mathrm{Ran}(T),$$

then the range $\mathrm{Ran}(T)$ *of T is closed.*

Clearly Theorem 2 in turn follows from the following lemma.

Lemma 4 *Let $(X, \|\cdot\|_X)$ be a Banach space, $(Y, \|\cdot\|_Y)$ a normed linear space, and T a bounded linear operator from X to Y. Define*

$$|||y||| = \inf_{y=Tx} \|x\|_X \quad \forall\ y \in \mathrm{Ran}(T).$$

Then $(\mathrm{Ran}(T), |||\cdot|||)$ *is a Banach space.*

Proof It is routine to check that $|||\cdot|||$ is a norm on $\mathrm{Ran}(T)$. Let $y_n, n \ge 1$, be a Cauchy sequence in $\mathrm{Ran}(T)$. Without loss of generality, we assume that $|||y_{n+1} - y_n||| < 2^{-n}$. By the definition of the norm $|||\cdot|||$, there exists $x_n \in X, n \ge 1$, such that $Tx_n = y_{n+1} - y_n$ and $\|x_n\|_X \le 2^{-n}$ for all $n \ge 1$. Since X is complete and $\sum_{n=1}^{\infty} \|x_n\|_X < \infty$, we have $z = \sum_{n=1}^{\infty} x_n \in X$ and $y_1 + Tz \in \mathrm{Ran}(T)$. Note that $|||Tx||| \le \|x\|_X$ for any $x \in X$. Hence

$$|||y_n - y_1 - Tz||| = \Big|\Big|\Big|T\Big(\sum_{k=n}^{\infty} x_k\Big)\Big|\Big|\Big| \le \sum_{k=n}^{\infty} \|x_k\|_X \to 0 \quad \text{as} \quad n \to \infty.$$

This leads to the assertion. $\square$

5.3 Proof of (i)$\Longrightarrow$(iii)

Let $k_0 = \min_{\xi\in\mathbf{R}^d} \mathrm{rank}\Big(\widehat{\Phi}(\xi+2k\pi)\Big)_{k\in\mathbf{Z}^d}$ and let $\Omega_{k_0} = \{\xi : \mathrm{rank}\Big(\widehat{\Phi}(\xi+2k\pi)\Big)_{k\in\mathbf{Z}^d} > k_0\}$. Then $\Omega_{k_0} \ne \mathbf{R}^d$. By Lemma 1, it suffices to prove that $\Omega_{k_0} = \emptyset$. Suppose on the contrary that $\Omega_{k_0} \neq \emptyset$. Then it suffices to construct a function G in the L^p closure of $V_p(\Phi)$ such that G cannot be written as $\sum_{i=1}^r \phi_i *' D_i$ for some $D_i \in \ell^p, 1 \le i \le r$. Recall that Ω_{k_0} is open set by (2.11). Then the boundary $\partial\Omega_{k_0}$ of Ω_{k_0} is nonempty. Also for any $\xi_0 \in \partial\Omega_{k_0}$, $\mathrm{rank}\Big(\widehat{\Phi}(\xi_0+2k\pi)\Big)_{k\in\mathbf{Z}^d} = k_0$ and

$$\max_{\xi\in B(\xi_0,\delta)} \mathrm{rank}\Big(\widehat{\Phi}(\xi+2k\pi)\Big)_{k\in\mathbf{Z}^d} > k_0 \quad \forall\ \delta > 0.$$

As in the proof of Lemma 2, there exist a $r\times r$ matrix $P_{\xi_0}(\xi)$, $\delta_0 > 0$ and $K_{\xi_0} \subset \mathbf{Z}^d$ with cardinality$(K_{\xi_0}) = k_0$ such that $P_{\xi_0}(\xi)$ is a 2π-periodic nonsingular matrix with its entries in the Wiener class, and Ψ_{ξ_0} defined by

$$\widehat{\Psi}_{\xi_0}(\xi) = \begin{pmatrix} \widehat{\Psi}_{1,\xi_0}(\xi) \\ \widehat{\Psi}_{2,\xi_0}(\xi) \end{pmatrix} = P_{\xi_0}(\xi)\widehat{\Phi}(\xi)$$

satisfies

$$\mathrm{rank}(\widehat{\Psi}_{1,\xi_0}(\xi+2k\pi))_{k\in K_{\xi_0}} = k_0 \quad \forall\ \xi \in B(\xi_0, 2\delta_0), \tag{5.1}$$

$$\widehat{\Psi}_{2,\xi_0}(\xi_0+2k\pi) = 0 \quad \forall\ k \in \mathbf{Z}^d, \tag{5.2}$$

$$\widehat{\Psi}_{2,\xi_0}(\xi+2k\pi) = 0 \quad \forall\ k \in K_{\xi_0},\ \xi \in B(\xi_0, 2\delta_0), \tag{5.3}$$

and

$$\widehat{\Psi}_{2,\xi_0}(\xi) \not\equiv 0 \quad \text{on } B(\xi_0,\delta) + 2\pi\mathbf{Z}^d \tag{5.4}$$

for all $0 < \delta < 2\delta_0$. Since $P_{\xi_0}(\xi) \in \mathcal{WC}$ and $\Phi \in \mathcal{L}^\infty$ (respectively $\Phi \in \mathcal{W}$), we have $\Psi_{\xi_0} \in \mathcal{L}^\infty$ (respectively $\Psi_{\xi_0} \in \mathcal{W}$). This together with (5.2) and the Poisson summation formula lead to

$$\sum_{j\in\mathbf{Z}^d} e^{-i\xi_0(\cdot+j)}\Psi_{2,\xi_0}(\cdot+j) = 0. \tag{5.5}$$

Let H be a nonnegative C^∞ function satisfying (4.10), and let

$$H_{n,\xi_0}(\xi) = \sum_{k\in\mathbf{Z}^d} H(2^n(\xi+2k\pi-\xi_0)), \quad n \ge 2.$$

Define an operator T_{n,ξ_0} from $(\ell^p)^{r-k_0}$ to L^p by

$$T_{n,\xi_0} : (D_1,\ldots,D_{r-k_0})^T \longrightarrow \sum_{i=1}^{r-k_0} (\psi_{2,\xi_0,i} *' H_{n,\xi_0}) *' D_i,$$

where H_{n,ξ_0} denotes the sequence of the Fourier coefficients of the 2π-periodic function $H_{n,\xi_0}(\xi)$ and $\Psi_{2,\xi_0} = (\psi_{2,\xi_0,1}, \ldots, \psi_{2,\xi_0,r-k_0})^T$. By (2.3), (5.5) and Lemma 3 with taking $h = \mathcal{F}^{-1}H$, we obtain

$$\lim_{n\to\infty} \|T_{n,\xi_0}\| = 0. \tag{5.6}$$

Let $\tilde{H}(x) = H(2x) - H(8x)$ and define $\tilde{H}_{n,\xi_0}(\xi) = \sum_{k\in\mathbf{Z}^d} \tilde{H}(2^n(\xi+2k\pi-\xi_0)),\ n \geq 2$. Then for $n \geq 4$, we have

$$\tilde{H}_{n,\xi_0}(\xi) H_{n,\xi_0}(\xi) = \tilde{H}_{n,\xi_0}(\xi). \tag{5.7}$$

Define an operator $\tilde{T}_{n,\xi_0}$ from $(\ell^p)^{r-k_0}$ to L^p by

$$\tilde{T}_{n,\xi_0} : (D_1, \ldots, D_{r-k_0})^T \longrightarrow \sum_{i=1}^{r-k_0} (\psi_{2,\xi_0,i} *' \tilde{H}_{n,\xi_0}) *' D_i,$$

where $\tilde{H}_{n,\xi_0}$ denotes the sequence of Fourier coefficients of the 2π-periodic function $\tilde{H}_{n,\xi_0}(\xi)$. ¿From (5.6), (5.7), and the fact that $\|\tilde{H}_{n,\xi_0}(\xi)\|_{\ell^1_*} \leq C$ for some positive constant C independent of n, it follows that $\|\tilde{T}_{n,\xi_0}\| \leq \|\tilde{H}_{n,\xi_0}(\xi)\|_{\ell^1_*} \|T_{n,\xi_0}\|$, and

$$\lim_{n\to\infty} \|\tilde{T}_{n,\xi_0}\| = 0. \tag{5.8}$$

By (5.4) and (5.6), there exists a subsequence $n_l, l \geq 1$ such that $n_{l+1} \geq n_l + 8$,

$$\|\tilde{T}_{n_l,\xi_0}\| \neq 0 \quad \text{and} \quad \sum_{k=0}^{5} \|T_{n_l+k,\xi_0}\| \leq 2^{-l}. \tag{5.9}$$

Let $D_{n_l} \in (\ell^p)^{r-k_0}$ be chosen such that

$$\|D_{n_l}\|_{\ell^p} = 1 \quad \text{and} \quad \|\tilde{T}_{n_l,\xi_0} D_{n_l}\|_{L^p} \geq \|\tilde{T}_{n_l,\xi_0}\|/2, \tag{5.10}$$

and let $D_{n_l}(\xi)$ be the Fourier series having D_{n_l} as its sequence of Fourier coefficients. For sufficiently large l_1, define $G_s, s \geq l_1$, by

$$\begin{aligned} \widehat{G}_s(\xi) &= \sum_{l=l_1}^{s} l(H_{n_l,\xi_0}(\xi) - H_{n_l+4,\xi_0}(\xi)) D_{n_l}(\xi)^T \widehat{\Psi}_{2,\xi_0}(\xi) \\ &= \sum_{l=l_1}^{s} l(H_{n_l,\xi_0}(\xi) - H_{n_l+4,\xi_0}(\xi))(0, D_{n_l}(\xi)^T) P_{\xi_0}(\xi) \widehat{\Phi}(\xi) \end{aligned}$$

and G by

$$\widehat{G}(\xi) = \sum_{l=l_1}^{\infty} l(H_{n_l,\xi_0}(\xi) - H_{n_l+4,\xi_0}(\xi))D_{n_l}(\xi)^T \widehat{\Psi}_{2,\xi_0}(\xi).$$

In fact, we only need $2^{-l_1} \le \delta_0$.

Now it remains to prove that G is in the L^p closure of $V_p(\Phi)$ and that G cannot be written as $\sum_{i=1}^r \phi_i *' D_i$ for some $D_i \in \ell^p, 1 \le i \le r$. ¿From the construction of G_s and G, $G_s \in V_p(\Phi)$ for all $s \ge l_1$, and

$$\begin{aligned}
\|G_s - G\|_{L^p} &\le \sum_{l=s+1}^{\infty} l \left\| \mathcal{F}^{-1}\Big(\Big(H_{n_l,\xi_0}(\xi) - H_{n_l+4,\xi_0}(\xi)\Big)D_{n_l}(\xi)^T \widehat{\Psi}_{2,\xi_0}(\xi)\Big)\right\|_{L^p} \\
&\le \sum_{l=s+1}^{\infty} l(\|T_{n_l+4,\xi_0}\| + \|T_{n_l,\xi_0}\|) \\
&\to 0 \quad \text{as} \quad s \to \infty,
\end{aligned}$$

where we have used (5.9). This shows that G is in the L^p closure of $V_p(\Phi)$.

Finally we prove that $G \notin V_p(\Phi)$. On the contrary, suppose that

$$\widehat{G}(\xi) = A(\xi)^T \widehat{\Phi}(\xi) \tag{5.11}$$

for some vector-valued 2π-periodic distribution $A(\xi) \in \mathcal{WC}^p$. Note that supp $\widehat{G}_s(\xi) \subset B(\xi_0, 2^{-l_1}) + 2\pi\mathbf{Z}^d$ for all $s \ge l_1$ and so is supp $\widehat{G}(\xi)$. Hence we may assume that $A(\xi)$ in (5.11) is supported in $B(\xi_0, \delta_0) + 2\pi\mathbf{Z}^d$ when l_1 is chosen sufficiently large. Write $A(\xi)^T (P_{\xi_0}(\xi))^{-1} = (A_1(\xi)^T, A_2(\xi)^T)$. Then we may write (5.11) as

$$A_1(\xi)^T \widehat{\Psi}_{1,\xi_0}(\xi) = \Big(- A_2(\xi)^T + \sum_{l=l_1}^{\infty} l\Big(H_{n_l,\xi_0}(\xi) - H_{n_l+4,\xi_0}(\xi)\Big)D_{n_l}(\xi)^T\Big)\widehat{\Psi}_{2,\xi_0}(\xi). \tag{5.12}$$

Since $A_1(\xi)$ is a 2π-periodic distribution in $\mathcal{WC}^p$ and supp $A_1(\xi) \subset B(\xi_0, \delta_0) + 2\pi\mathbf{Z}^d$, it follows from (5.1), (5.3) and (5.12) that $A_1(\xi) \equiv 0$. Substituting this in (5.12),

$$A_2(\xi)^T \widehat{\Psi}_{2,\xi_0}(\xi) = \sum_{l=l_1}^{\infty} l\Big(H_{n_l,\xi_0}(\xi) - H_{n_l+4,\xi_0}(\xi)\Big)D_{n_l}(\xi)^T \widehat{\Psi}_{2,\xi_0}(\xi). \tag{5.13}$$

By direct computation, and using the fact that $n_{l+1} \ge n_l + 8$, we have

$$\tilde{H}_{n_l,\xi_0}(\xi)\Big(H_{n_{l'},\xi_0}(\xi) - H_{n_{l'}+4,\xi_0}(\xi)\Big) = \begin{cases} \tilde{H}_{n_l,\xi_0}(\xi) & l' = l \\ 0 & l' \ne l. \end{cases}$$

Multiplying $\tilde{H}_{n_l,\xi_0}(\xi)$ on both sides of (5.13),

$$\tilde{H}_{n_l,\xi_0}(\xi)A_2(\xi)^T\widehat{\Psi}_{2,\xi_0}(\xi) = l\tilde{H}_{n_l,\xi_0}(\xi)D_{n_l}(\xi)^T\widehat{\Psi}_{2,\xi_0}(\xi). \tag{5.14}$$

By (5.10) and $\|\tilde{T}_{n_l,\xi_0}\| \neq 0$, we get

$$\left\|\mathcal{F}^{-1}\Big(l\tilde{H}_{n_l,\xi_0}(\xi)D_{n_l}(\xi)^T\widehat{\Psi}_{2,\xi_0}(\xi)\Big)\right\|_{L^p} \geq l\|\tilde{T}_{n_l,\xi_0}\|/2$$

and

$$\left\|\mathcal{F}^{-1}\Big(\tilde{H}_{n_l,\xi_0}(\xi)A_2(\xi)^T\widehat{\Psi}_{2,\xi_0}(\xi)\Big)\right\|_{L^p} \leq \|\tilde{T}_{n_l,\xi_0}\|\ \|A_2(\xi)\|_{\ell^p_*},$$

which contradicts (5.14). This completes the proof of (i) $\Longrightarrow$ (iii). $\Box$

5.4 Proof of (iii) $\Longrightarrow$ (v)

Let $h_\lambda(\xi), P_\lambda(\xi)$ and $\widehat{\Psi}_{1,\lambda}$ be as in Lemma 2. Define

$$B_\lambda(\xi) = H_\lambda(\xi)\overline{P_\lambda(\xi)^T}\begin{pmatrix} [\widehat{\Psi}_{1,\lambda}, \widehat{\Psi}_{1,\lambda}](\xi)^{-1} & 0 \\ 0 & I \end{pmatrix} P_\lambda(\xi), \tag{5.15}$$

where $H_\lambda(\xi)$ is a 2π-periodic C^∞ function such that $H_\lambda(\xi) = 1$ on $\operatorname{supp} h_\lambda$, and H_λ is supported in $B(\eta_\lambda, \delta_\lambda) + 2\pi\mathbf{Z}^d$. Then $B_\lambda(\xi) \in \mathcal{WC}$. Define $\Psi = (\psi_1, \ldots, \psi_r)^T$ by

$$\widehat{\Psi}(\xi) = \sum_{\lambda\in\Lambda} h_\lambda(\xi)B_\lambda(\xi)\widehat{\Phi}(\xi). \tag{5.16}$$

Then $\Psi \in \mathcal{L}^\infty$ if $1 < p < \infty$ and $\Psi \in \mathcal{W}$ if $p = 1, \infty$. Now we start to prove (3.3) for such a Ψ. For any $f \in V_p(\Phi)$, define

$$g = \sum_{i=1}^{r}\sum_{j\in\mathbf{Z}^d}\langle f, \psi_i(\cdot - j)\rangle\phi_i(\cdot - j).$$

Then it suffices to prove that $f = g$. By the definition of the space $V_p(\Phi)$, there exists a 2π-periodic distribution $A(\xi) \in \mathcal{WC}^p$ such that $\hat{f}(\xi) = A(\xi)^T\widehat{\Phi}(\xi)$. Therefore, by (5.15), (5.16) and Lemma 2, we get

$$\begin{aligned} \hat{g}(\xi) &= A(\xi)^T[\widehat{\Phi}, \widehat{\Psi}](\xi)\widehat{\Phi}(\xi) \\ &= \sum_{\lambda\in\Lambda} h_\lambda(\xi)A(\xi)^T[\widehat{\Phi}, \widehat{\Phi}](\xi)\overline{B_\lambda(\xi)^T}\widehat{\Phi}(\xi) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\lambda\in\Lambda} h_\lambda(\xi)\Big\{A(\xi)^T P_\lambda(\xi)^{-1}\times \\
&\quad \begin{pmatrix} [\widehat{\Psi}_{1,\lambda},\widehat{\Psi}_{1,\lambda}](\xi) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} [\widehat{\Psi}_{1,\lambda},\widehat{\Psi}_{1,\lambda}](\xi)^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \widehat{\Psi}_{1,\lambda}(\xi) \\ 0 \end{pmatrix}\Big\} \\
&= \sum_{\lambda\in\Lambda} h_\lambda(\xi) A(\xi)^T P_\lambda(\xi)^{-1} \begin{pmatrix} \widehat{\Psi}_{1,\lambda}(\xi) \\ 0 \end{pmatrix} \\
&= A(\xi)^T\widehat{\Phi}(\xi) = \hat{f}(\xi).
\end{aligned}
$$

Similarly we can prove that $f = \sum_{i=1}^r \sum_{j\in\mathbf{Z}^d}\langle f,\phi_i(\cdot - j)\rangle\psi_i(\cdot - j)$. This completes the proof of (iii)$\Longrightarrow$(v). $\Box$

5.5 Proof of (v) $\Longrightarrow$(ii)

Let $f = \sum_{i=1}^r \sum_{j\in\mathbf{Z}^d}\langle f,\phi_i(\cdot - j)\rangle\psi_i(\cdot - j)$. Then using (2.3) we get

$$
\|f\|_{L^p} \le C\|\{\sum_{i=1}^r \langle f,\phi_i(\cdot - j)\rangle\}_{j\in\mathbf{Z}^d}\|_{\ell^p}
$$

which is the left hand side of inequality in (ii).

The right hand side of (ii) is a direct consequence of (2.2). $\Box$

5.6 Proof of (ii) $\Longrightarrow$(iii)

Let $k_0 = \min_{\xi\in\mathbf{R}^d}\operatorname{rank}\Big(\widehat{\Phi}(\xi+2k\pi)\Big)_{k\in\mathbf{Z}^d}$ and let $\Omega_{k_0} = \{\xi :\ \operatorname{rank}\Big(\widehat{\Phi}(\xi+2k\pi)\Big)_{k\in\mathbf{Z}^d} > k_0\}$. By Lemma 1, it suffices to prove that $\Omega_{k_0} = \emptyset$. Suppose on the contrary that $\Omega_{k_0} \neq \emptyset$. Let $\xi_0 \in \partial\Omega_{k_0}, \Psi_{1,\xi_0}, \Psi_{2,\xi_0}, P_{\xi_0}(\xi), H_{n,\xi_0}(\xi), \tilde{H}_{n,\xi_0}(\xi)$ and δ_0 be as in the proof of (i) $\Longrightarrow$ (iii). Let n_0 be chosen such that $2^{-n_0} < \delta_0$ and

$$
\alpha_n(\xi) = [\widehat{\Psi}_{1,\xi_0},\widehat{\Psi}_{1,\xi_0}](\xi_0) + H_{n,\xi_0}(\xi)\Big([\widehat{\Psi}_{1,\xi_0},\widehat{\Psi}_{1,\xi_0}](\xi) - [\widehat{\Psi}_{1,\xi_0},\widehat{\Psi}_{1,\xi_0}](\xi_0)\Big)
$$

is nonsingular for all $n \ge n_0$. The existence of n_0 follows from (5.1) and the continuity of $[\widehat{\Psi}_{1,\xi_0},\widehat{\Psi}_{1,\xi_0}](\xi)$. Given any 2π-periodic distribution $F(\xi)$ in $\mathcal{WC}^p$, define $g_n, n \ge n_0+1$, by

$$
\widehat{g}_n(\xi) = \tilde{H}_{n,\xi_0}(\xi)\Big(-F(\xi)^T[\widehat{\Psi}_{2,\xi_0},\widehat{\Psi}_{1,\xi_0}](\xi)(\alpha_n(\xi))^{-1}, F(\xi)^T\Big)\begin{pmatrix} \widehat{\Psi}_{1,\xi_0}(\xi) \\ \widehat{\Psi}_{2,\xi_0}(\xi) \end{pmatrix}.
$$

Then $g_n \in V_p(\Phi)$, and

$$\begin{aligned}
& [\widehat{g}_n, \widehat{\Psi}_{1,\xi_0}](\xi) \\
= \ & \tilde{H}_{n,\xi_0}(\xi)\Big(- F(\xi)^T[\widehat{\Psi}_{2,\xi_0}, \widehat{\Psi}_{1,\xi_0}](\xi)(\alpha_n(\xi))^{-1}, F(\xi)^T\Big)\begin{pmatrix}[\widehat{\Psi}_{1,\xi_0}, \widehat{\Psi}_{1,\xi_0}](\xi)\\ [\widehat{\Psi}_{2,\xi_0}, \widehat{\Psi}_{1,\xi_0}](\xi)\end{pmatrix} \\
= \ & 0,
\end{aligned} \tag{5.17}$$

by direct computation, where we have used (5.7) and the fact that $\alpha_n(\xi) = [\widehat{\Psi}_{1,\xi_0}, \widehat{\Psi}_{1,\xi_0}](\xi)$ on the support of $\widetilde{H}_{n,\xi_0}$. Thus by (2.2), (5.7), (5.17) and

$$P_{\xi_0}(\xi)\widehat{\Phi}(\xi) = \begin{pmatrix} \widehat{\Psi}_{1,\xi_0}(\xi) \\ \widehat{\Psi}_{2,\xi_0}(\xi) \end{pmatrix},$$

we get

$$\begin{aligned}
& \|[\widehat{g}_n, \widehat{\Phi}](\xi)\|_{\ell^p_*} \\
= \ & \left\| \left[\widehat{g}_n, \begin{pmatrix} \widehat{\Psi}_{1,\xi_0} \\ \widehat{\Psi}_{2,\xi_0} \end{pmatrix}\right](\xi)\overline{P_{\xi_0}(\xi)^{-T}} \right\|_{\ell^p_*} \\
\le \ & C \left\| \left[\widehat{g}_n, \begin{pmatrix} \widehat{\Psi}_{1,\xi_0} \\ \widehat{\Psi}_{2,\xi_0} \end{pmatrix}\right](\xi) \right\|_{\ell^p_*} \le C\|[\widehat{g}_n, H_{n,\xi_0}\widehat{\Psi}_{2,\xi_0}](\xi)\|_{\ell^p_*} \\
\le \ & C\|g_n\|_{L^p}\|\mathcal{F}^{-1}(H_{n,\xi_0}(\xi)\widehat{\Psi}_{2,\xi_0}(\xi))\|_{\mathcal{L}^\infty}^{1/p}\|\mathcal{F}^{-1}(H_{n,\xi_0}(\xi)\widehat{\Psi}_{2,\xi_0}(\xi))\|_{L^1}^{1-1/p}.
\end{aligned} \tag{5.18}$$

It follows from (2.3), (2.4), (5.5) and Lemma 3 that

$$\lim_{n\to\infty} \|\mathcal{F}^{-1}(H_{n,\xi_0}(\xi)\widehat{\Psi}_{2,\xi_0}(\xi))\|_{\mathcal{L}^\infty}^{1/p}\|\mathcal{F}^{-1}(H_{n,\xi_0}(\xi)\widehat{\Psi}_{2,\xi_0}(\xi))\|_{L^1}^{1-1/p} = 0.$$

This together with (5.18) leads to the existence of $\epsilon_n, n \ge n_0$, such that

$$\|[\widehat{g}_n, \widehat{\Phi}](\xi)\|_{\ell^p_*} \le \epsilon_n \|g_n\|_{L^p} \tag{5.19}$$

and $\lim_{n\to\infty} \epsilon_n = 0$. On the other hand, by the assumption (ii) we have

$$\|[\widehat{g}_n, \widehat{\Phi}](\xi)\|_{\ell^p_*} = \Big\|\Big\{ \int_{\mathbf{R}^d} g_n(x)\overline{\Phi(x-j)}dx\Big\}_{j\in\mathbf{Z}^d}\Big\|_{\ell^p} \ge C\|g_n\|_{L^p}. \tag{5.20}$$

Combining (5.19) and (5.20), there exists an integer $n_1 \ge n_0 + 1$ such that

$$g_n \equiv 0 \quad \forall\ n \ge n_1.$$

Thus for any 2π-periodic distribution $F(\xi)$ in $\mathcal{WC}^p$ and $n \ge n_1$,

$$\tilde{H}_{n,\xi_0}(\xi)F(\xi)^T[\widehat{\Psi}_{2,\xi_0}, \widehat{\Psi}_{1,\xi_0}](\xi)(\alpha_n(\xi))^{-1}\widehat{\Psi}_{1,\xi_0}(\xi) = \tilde{H}_{n,\xi_0}(\xi)F(\xi)^T\widehat{\Psi}_{2,\xi_0}(\xi).$$

Hence

$$\tilde{H}_{n,\xi_0}(\xi)[\widehat{\Psi}_{2,\xi_0}, \widehat{\Psi}_{1,\xi_0}](\xi)(\alpha_n(\xi))^{-1}\widehat{\Psi}_{1,\xi_0}(\xi) = \tilde{H}_{n,\xi_0}(\xi)\widehat{\Psi}_{2,\xi_0}(\xi). \tag{5.21}$$

This together with (5.1) and (5.3) lead to

$$\tilde{H}_{n,\xi_0}(\xi)[\widehat{\Psi}_{2,\xi_0}, \widehat{\Psi}_{1,\xi_0}](\xi)(\alpha_n(\xi))^{-1} = 0 \quad \forall\ \xi \in B(\xi_0, 2^{-n_1})$$

and hence by 2π-periodicity,

$$\tilde{H}_{n,\xi_0}(\xi)[\widehat{\Psi}_{2,\xi_0}, \widehat{\Psi}_{1,\xi_0}](\xi)(\alpha_n(\xi))^{-1}\widehat{\Psi}_{1,\xi_0}(\xi) = 0 \quad \forall\ \xi \in B(\xi_0, 2^{-n_1}) + 2\pi\mathbf{Z}^d.$$

Substituting this into (5.21), and using the fact that (5.21) is valid for all $n \ge n_1$ we obtain

$$\widehat{\Psi}_{2,\xi_0}(\xi) \equiv 0 \quad \text{on } B(\xi_0, 2^{-n_1-2}) + 2\pi\mathbf{Z}^d,$$

which contradicts (5.4). This completes the proof of (ii)$\Longrightarrow$(iii). $\square$

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