LOCALIZATION OF STABILITY AND p-FRAMES IN THE FOURIER DOMAIN

QIYU SUN

Abstract. In this paper, we introduce and study the localization of stability and p-frame properties of a finitely generated shift-invariant system in the Fourier domain, and then provide more information to that shift invariant system. Especially for a shift-invariant system generated by finitely many compactly supported functions, we show that it has p-frame property at almost all frequencies, and that it either has stability property at almost all frequencies or does not have stability property at all frequencies.

1. Introduction

In this paper, we are interested in the localization of stability and p-frame properties of the shift invariant system generated by finitely many functions in the Fourier domain. The localization of stable shifts and p-frames in the Fourier domain provides us more information to that shift invariant system, especially when the system is generated by compactly supported functions. For instance, we show that a vector-valued compactly supported distribution either has stable shifts at almost all frequencies or does not have stable shifts at all frequencies, and that a vector-valued compactly supported bounded function generates a p-frame at almost all frequencies.

Let $\ell^p, 1 \le p \le \infty$, be the space of all p-summable sequences on $\mathbb{Z}^d$, and let $\|\cdot\|_{\ell^p}$ denote the usual ℓ^p norm. For a linear space X, we denote its direct sum of r copies by $X^{(r)}$. For compactly supported bounded functions $f_1, \ldots, f_r$ on $\mathbb{R}^d$, we define the *semi-convolution* $F*'$ on $(\ell^p)^{(r)}, 1 \le p \le \infty$, by

$$F*' : \ (\ell^p)^{(r)} \ni D := \{D(k)\} \longmapsto \sum_{k\in\mathbb{Z}^d} D(k)^T F(\cdot - k) := F *' D,$$

and denote the range of the semi-convolution $F*'$ on $(\ell^p)^{(r)}$ by

$$V_p(F) := \{F *' D : \ D \in (\ell^p)^{(r)}\},$$

Date: July 21, 2003.

where $F=(f_1,\ldots,f_r)^T$. The space $V_p(F)$ is the set of all linear combinations of $f_i(\cdot-k), 1\le i\le r, k\in \mathbb{Z}^d$, using ℓ^p coefficients. Thus it is a shift invariant space generated by the shift invariant system

$$\mathcal{F}:=\{f_i(\cdot-k):\ 1\le i\le r, k\in\mathbb{Z}^d\}.$$

Here we say that a linear space V of functions on $\mathbb{R}^d$ is *shift invariant* if $f\in V$ implies that $f(\cdot-k)\in V$ for all $k\in\mathbb{Z}^d$. The shift invariant space $V_p(F)$ is used in wavelet analysis for the case that the generator is a scaling function or a mother wavelet ([7, 9, 19]), and is also used as the model space for sampling, where the sinc function is the generator (see for instance [1, 5]).

Before starting to discuss localization in the Fourier domain, we recall the definition of Fourier series. For any summable sequence $D=\{D(k)\}$, its Fourier series is defined by $\mathcal{F}(D)(\xi):=\sum_{k\in\mathbb{Z}^d}D(k)e^{-ik\xi}$. The above definition of Fourier series can be extended to any sequence D with polynomial growth, that is, $|D(k)|\le p(k), k\in\mathbb{Z}^d$, for some polynomial p. In this situation, the Fourier series is a periodic tempered distribution, which we still denote by $\mathcal{F}(D)$. For any measurable set E, we let $\ell^p_E, 1\le p\le\infty$, be the set of all ℓ^p sequences whose Fourier series are supported in $E+2\pi\mathbb{Z}^d$.

The semi-convolution $F*'$ from $(\ell^\infty)^{(r)}$ to $V_\infty(F)$ is the first operator we want to localize in the Fourier domain. We say that F has *stable shifts* if the semi-convolution $F*'$ is one-to-one on $(\ell^\infty)^{(r)}$. For the case that F has compact support, it was known that F has stable shifts if and only if the rank of the $r\times\mathbb{Z}^d$ matrix $(\widehat{F}(\xi+2k\pi))_{k\in\mathbb{Z}^d}$ is r for all $\xi\in\mathbb{R}^d$ ([16, 21]). Here we define the *Fourier transform* $\hat f$ of an integrable function f by $\hat f(\xi)=\int_{\mathbb{R}^d}e^{-ix\xi}f(x)dx$, and understand the one of a tempered distribution as usual. Moreover, by the Possion formula, it follows that if the $r\times\mathbb{Z}^d$ matrix $(\widehat{F}(\xi_0+2k\pi))_{k\in\mathbb{Z}^d}$ has rank strictly less than r, then $F*'(vE_{\xi_0})=0$, where $E_{\xi_0}:=\{e^{ik\xi_0}\}\in\ell^\infty$ and v is a nonzero vector in $\mathbb{C}^r$ so chosen that $v^T\widehat{F}(\xi_0+2k\pi)=0$ for all $k\in\mathbb{Z}^d$. We observe that the Fourier series of E_{ξ_0} is supported in $\xi_0+2\pi\mathbb{Z}^d$. On the other hand, if the $r\times\mathbb{Z}^d$ matrix $(\widehat{F}(\xi_0+2k\pi))_{k\in\mathbb{Z}^d}$ is of full rank, then $F*'D\not\equiv 0$ for any nonzero sequence $D\in(\ell^\infty)^{(r)}$ whose Fourier transform is supported in $B(\xi_0,\delta)+2\pi\mathbb{Z}^d$ for some small $\delta>0$, and hence the semi-convolution $F*'$ is one-to-one on $(\ell^\infty_{B(\xi_0,\delta)})^{(r)}$ for some $\delta>0$. That property is called *stable shifts at the frequency* ξ_0 in this paper. Note that F has stable shifts if and only if it has stable shifts at any frequency (Corollary 2.3). We may then consider the stable shifts at some frequency as the localization of the stable shifts in the Fourier domain. In Section 2, we discuss the stable shifts of F at a frequency when F is a tempered distribution having ℓ^1 decay, while the class of tempered distributions with ℓ^1 decay contains all integrable functions, compactly supported distributions, and also refinable distributions with smooth symbols (see Section 2 for more details about that function space). In Theorem 2.2, we characterize the stable shifts at a frequency and hence generalize the corresponding result for the stable shifts in [24]. In Theorem 2.4, we discuss the stable shifts of F at a frequency when F is a compactly supported distribution, and we show that a compactly supported distribution F either does not have stable shifts at all frequencies, or has stable shifts at almost all frequencies (Corollary 2.5).

Let $L^p, 1 \le p \le \infty$, be the space of all p-integrable functions on $\mathbb{R}^d$, and let $\|\cdot\|_p$ be the usual L^p norm. The semi-convolution operator $F*'$ from $(\ell^p)^{(r)}$ to $V_p(F), 1 \le p \le \infty$, is the second operator we want to localize in the Fourier domain. We say that F has ℓ^p *stable shifts* if $F*'$ is an isomorphism between $(\ell^p)^{(r)}$ and $V_p(F)$, that is, there exists a positive constant C so that

$$C^{-1}\|D\|_{\ell^p} \le \|F *' D\|_p \le C\|D\|_{\ell^p} \quad \forall\ D \in (\ell^p)^{(r)}. \tag{1.1}$$

The concept of ℓ^p stable shifts plays an important role in the approximation by shift invariant spaces, the regularity of scaling functions, and the convergence of cascade algorithms (see for instance [13, 14, 25, 27] and references therein). Comparing the definitions of the stable shifts and the ℓ^p stable shifts of F, we see that (a) the semi-convolution $F*'$ is required to be a bounded operator from $(\ell^p)^{(r)}$ to L^p in the definition of the ℓ^p stable shifts of F, but it is only required to be well defined in the distributional sense in the definition of the stable shifts of F, and (b) the semi-convolution $F*'$ is assumed to be one-to-one and to have bounded inverse in the definition of the ℓ^p stable shifts of F, but it is assumed to be one-to-one only in the definition of the stable shifts of F. For a compactly supported L^p function F, the semi-convolution is a bounded operator from $(\ell^p)^{(r)}$ to L^p because

$$\begin{aligned}\|F *' D\|_p &\le \|F\|_p\|D\|_p \times \sup_{x\in\mathbb{R}^d}\Big(\sum_{k\in\mathbb{Z}^d}\chi_{[-K,K]^d}(x-k)\Big)^{(p-1)/p}\\ &\le (2K+2)^{d(p-1)/p}\|F\|_p\|D\|_p,\end{aligned}$$

where K is so chosen that F is supported in $[-K,K]^d$ and χ_E is the characteristic function on a measurable set E (see (3.1) for a more general result). Moreover, it is shown that the ℓ^p stable shifts of F and the stable shifts of F are equivalent to each other ([15]). We say that F has ℓ^p *stable shifts at the frequency* $\xi_0 \in \mathbb{R}^d$ if the semi-convolution $F*'$ is an isomorphism between $(\ell^p_{B(\xi_0,\delta)})^{(r)}$ and its image for some $\delta > 0$, that is, (1.1) holds for all sequences $D \in (\ell^p_{B(\xi_0,\delta)})^{(r)}$. Note that $F \in (\mathcal{L}^p)^{(r)}$ has ℓ^p stable shifts if and only if it has ℓ^p stable shifts at any frequency (Corollary 3.2). Then we may consider the ℓ^p stable shifts at a certain frequency as the localization of the ℓ^p stable shifts in the Fourier domain. In Theorem 3.1 of Section 3, we generalize the equivalence between the ℓ^p stable shifts and the stable shifts in [15] to the equivalence of their localization in the Fourier domain for those functions F in $(\mathcal{L}^p)^{(r)}$, which contains all compactly supported L^p functions.

We see from the definition of the ℓ^p stable shifts that $F = (f_1, \ldots, f_r)^T$ has stable shifts if and only if $\{f_i(\cdot - k) : 1 \le i \le r, k \in \mathbb{Z}^d\}$ is a Riesz basis of $V_p(F)$. This observation inspires us to introduce the concept of a p-frame of a finitely generated shift invariant space in [3]. We say that F generates a *p-frame* if $\{f_i(\cdot - k) : 1 \le i \le r, k \in \mathbb{Z}^d\}$ is a p-frame for $V_p(F)$, that is, there exist positive constants A and B so that

$$A\|f\|_p \le \sum_{i=1}^{r} \|\{\langle f, f_i(\cdot - k)\rangle\}\|_{\ell^p} \le B\|f\|_p \quad \forall\, f \in V_p(F), \tag{1.2}$$

which, in turn, is equivalent to the operator T from $V_p(F)$ to $(\ell^p)^{(r)}$,

$$T : V_p(F) \ni f \to \left\{(\langle f, f_1(\cdot - k)\rangle, \ldots, \langle f, f_r(\cdot - k)\rangle)^T\right\} \in (\ell^p)^{(r)},$$

is an isomorphism on $V_p(F)$. The operator T is called the *analysis operator* T. The third operator we want to localize in the Fourier domain is the analysis operator T. For a shift invariant space V and a measurable set E, we define

$$V_E = \left\{f \in V : \ \hat{f} \text{ is supported in } E + 2\pi\mathbb{Z}^d\right\}.$$

If $V = V_p(F)$ for some generator F and $p \in [1, \infty]$, we use $V_{p,E}(F)$ to denote V_E. We say that F generates a *p-frame at the frequency* $\xi_0 \in \mathbb{R}^d$ if (1.2) holds for all $f \in V_{p,B(\xi_0,\delta)}(F)$, where $\delta > 0$. As we show later that for those functions F with certain decay at infinity, it generates a p-frame if and only if it generates a p-frame at any frequency (Corollary 4.4). Then we may consider the p-frame property at some frequency as the localization of the p-frame property in the Fourier domain. In [3], it is shown that a function F in $(W(L^\infty, \ell^1))^{(r)}$, (a function in that space is locally bounded and globally L^1, see Section 4 for the precise definition), generates a p-frame if and only if the rank of the $r \times \mathbb{Z}^d$ matrix $(\widehat{F}(\xi + 2k\pi))_{k\in\mathbb{Z}^d}$ is independent of $\xi \in \mathbb{R}^d$ (Proposition 4.1).

In Section 4, we establish a corresponding result for the p-frame at any frequency (Theorem 4.2). Applying Theorem 4.2, we show that any compactly supported bounded function F generates a p-frame for $V_p(F)$ at almost all frequencies (Theorem 4.5).

2. Stable Shifts at Certain Frequency

In this section, we consider the localization of the stable shifts of finitely generated shift invariant system in Fourier domain.

First let us recall a class of function space from which the generators of the shift invariant system are chosen. Let $\mathcal{S}$ to be the space of all Schwartz functions, and let $\ll \cdot, \cdot \gg$ be the action between a tempered distribution and a Schwartz function. We say that a tempered distribution f has ℓ^1 *decay* if $\ll f(\cdot - x), h \gg$ is continuous about x for any $h \in \mathcal{S}$, and if there exist positive constants C and k_0 independent of $h \in \mathcal{S}$ such that

$$\sum_{k \in \mathbb{Z}^d} | \ll f(\cdot - k), h \gg | \le C \sum_{|\alpha| \le k_0} \|D^\alpha h(\cdot)(1 + |\cdot|)^{k_0}\|_\infty$$

for any $h \in \mathcal{S}$ ([24]). The class of tempered distributions with ℓ^1 decay contains most of the functions (tempered distributions) we are interested in for the study of stable shifts. In fact, any integrable function f on $\mathbb{R}^d$ is a tempered distribution with ℓ^1 decay because for any $h \in \mathcal{S}$,

$$\begin{aligned} &\sum_{k \in \mathbb{Z}^d} | \ll f(\cdot - k), h \gg | = \sum_{k \in \mathbb{Z}^d} | \ll f, h(\cdot + k) \gg | \\ \le\ & \int_{\mathbf{R}} |f(x)| \times \Big(\sum_{k \in \mathbb{Z}^d} |h(x + k)| \Big) dx \\ \le\ & \|h(1 + |\cdot|)^{d+1}\|_\infty \times \int_{\mathbf{R}^d} |f(x)| \Big(\sum_{k \in \mathbb{Z}^d} (1 + |x + k|)^{-d-1} \Big) dx \\ \le\ & \|h(1 + |\cdot|)^{d+1}\|_\infty \Big(1 + \sum_{0 \ne k \in \mathbb{Z}^d} |k|^{-d-1} \Big) \|f\|_1. \end{aligned}$$

The class of tempered distributions with ℓ^1 decay also contains all compactly supported distributions and globally supported refinable distributions with smooth symbol ([24]), but it does not contain the sinc function $\mathrm{sinc}(x) := \frac{\sin \pi x}{\pi x}$ on the line because

$$\sum_{k\in\mathbb{Z}} | \ll f(\cdot - k), h \gg | = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}^d} \Big| \int_{-\pi}^{\pi} e^{-i(k+1/2)\xi} d\xi \Big|$$
$$= \sum_{k\in\mathbb{Z}} \Big| \frac{\sin(k+1/2)\pi}{(k+1/2)\pi} \Big| = +\infty$$

for any Schwartz function h whose Fourier transform satisfies $\hat{h}(\xi) = e^{-i\xi/2}$ for $|\xi| \le \pi$.

For $F = (f_1, \ldots, f_r)^T$ having ℓ^1 decay, $\{\ll F(\cdot - k), h \gg\} \in \ell^1$ for any $h \in \mathcal{S}$, and hence the sum $\sum_{k\in\mathbb{Z}^d} D(k)^T \ll F(\cdot - k), h \gg$ is well defined for any bounded sequence $D = \{D(k)\}$. So we may define the *semi-convolution* $F *' D$ of a vector-valued tempered distribution F and a bounded sequence $D = \{D(k)\}$ by

$$\ll F *' D, h \gg := \sum_{k\in\mathbb{Z}^d} D(k)^T \ll F(\cdot - k), h \gg \quad \text{for any } h \in \mathcal{S}. \tag{2.1}$$

Thus the semi-convolution $F *' D$ is a tempered distribution ([24]).

We define the stable shifts of a tempered distribution $F = (f_1, \ldots, f_r)^T$ with ℓ^1 decay and its localization in Fourier domain as follows.

Definition 2.1. Let $F = (f_1, \ldots, f_r)^T$ be a vector-valued tempered distribution with ℓ^1 decay.

- We say that F has *stable shifts* if $F*'$ is one-to-one on $(\ell^\infty)^{(r)}$, that is, the only bounded sequence D such that $F *' D = 0$ is the zero sequence ([24]).
- We say that F has *stable shifts at frequency* $\xi_0 \in \mathbb{R}^d$ if there exists $\delta > 0$ so that the only sequence $D \in (\ell^\infty_{B(\xi_0,\delta)})^{(r)}$ so that $F *' D = 0$ is the zero sequence.

For the stable shifts, there is a long list of publications on the characterizations and applications, especially for compactly supported distributions and refinable distributions, (see, for instance, [12, 16, 21, 23, 26] for compactly supported distributions, [8, 10, 11, 18, 22, 28] for compactly supported refinable distributions, [15] for globally supported functions in $\mathcal{L}^p$, and [24] for tempered distributions with ℓ^1 decay).

We say that a function f is a C^∞ *function with* ℓ^1 *decay* if

$$\sum_{k\in\mathbb{Z}^d} \|D^n f\|_{L^\infty(k+[0,1)^d)} < \infty \text{ for any } n \in (\mathbb{Z}_+)^d.$$

We may understand that C^∞ functions with ℓ^1 decay are functions locally C^∞ and globally ℓ^1. Thus a Schwartz function is a C^∞ function with ℓ^1 decay, and so is a linear combination of the integer shifts of a

Schwartz function using ℓ^1 coefficients. In [24], the author established the following result for the stable shifts of a tempered distribution with ℓ^1 decay (The original result is slightly different from the one stated below).

Proposition 2.1. *Let $F = (f_1, \ldots, f_r)^T$ be a tempered distribution with ℓ^1 decay. Then the following statements are equivalent:*

(i) *F has stable shifts.*

(ii) *The matrix $(\widehat{F}(\xi + 2k\pi))_{k\in\mathbb{Z}^d}$ is of full rank for any $\xi \in \mathbb{R}^d$.*

(iii) *There exist C^∞ functions $h_1, \ldots, h_r$ with ℓ^1 decay so that*

$$d_i(k) = \ll F *' D, h_i(\cdot - k) \gg,\ 1 \le i \le r,\ k \in \mathbb{Z}^d$$

for all bounded sequences $D = \{(d_1(k), \ldots, d_r(k))^T\}$.

We remark that the equivalence between the first and second statement is established in [21] for a compactly supported function F with $r = 1$ and in [16] with $r \ge 1$, and in [15] for an integrable function F.

In this section, we establish a corresponding version of Proposition 2.1 for the stable shifts at a frequency.

Theorem 2.2. *Let $\xi_0 \in \mathbb{R}^d$, and let $F = (f_1, \ldots, f_r)^T$ be a tempered distribution with ℓ^1 decay. Then the following three statements are equivalent to each other.*

(i) *F has stable shifts at the frequency $\xi_0 \in \mathbb{R}^d$.*

(ii) *The $r \times \mathbb{Z}^d$ matrix $(\widehat{F}(\xi_0 + 2k\pi))_{k\in\mathbb{Z}^d}$ is of full rank.*

(iii) *There exist C^∞ functions $h_1, \ldots, h_r$ with ℓ^1 decay and a positive constant δ so that*

$$d_i(k) = \ll F *' D, h_i(\cdot - k) \gg,\ 1 \le i \le r,\ k \in \mathbb{Z}^d \tag{2.2}$$

for any bounded sequences $D := \{(d_1(k), \ldots, d_r(k))^T\}$ whose Fourier series $\mathcal{F}(D)$ are supported in $B(\xi_0, \delta) + 2\pi\mathbb{Z}^d$.

We may understand the above theorem as the equivalence among stable shifts at a certain frequency, full dimension of the Fourier fiber bundle in the neighborhood of that frequency, and the inverse process of the semi-convolution at that frequency. Combining Theorem 2.2 and Proposition 2.1, we have:

Corollary 2.3. *A tempered distribution with ℓ^1 decay has stable shifts if and only if it has stable shifts at any frequency.*

Then Theorem 2.2 can be thought as the localization version of Proposition 2.1 in the Fourier domain.

For a compactly supported distribution $F = (f_1, \ldots, f_r)^T$, we say that F has *finitely linearly independent shifts* if there does not exist a nonzero sequence D with finite support so that $F *' D \equiv 0$ ([28]). In other words, there does not exist a linear dependent finite subsystem of $\mathcal{F} := \{f_i(\cdot - k) : 1 \le i \le r, k \in \mathbb{Z}^d\}$. In the Fourier domain, we may interpret the finitely linearly independent shifts as the nonexistence of a nonzero vector $Q(\xi) = (q_1(\xi), \ldots, q_r(\xi))^T$ so that all entries of $Q(\xi)$ are trigonometric polynomials and that $Q(\xi)^T \widehat{F}(\xi) \equiv 0$. Applying Theorem 2.2, we have the following result about the stable shifts of a compactly supported distribution at a frequency.

Theorem 2.4. *Let $F = (f_1, \ldots, f_r)^T$ be a compactly supported nonzero distribution. Then:*

(i) *F has finitely linearly dependent shifts if and only if F does not have stable shifts at all frequencies.*

(ii) *F has finitely linearly independent shifts if and only if F has stable shifts at almost all frequencies.*

As an easy consequence of Theorem 2.4, we have:

Corollary 2.5. *Let $F = (f_1, \ldots, f_r)^T$ be a compactly supported nonzero distribution. Then either F does not have stable shifts at all frequencies, or F has stable shifts at almost all frequencies. Especially if $r = 1$ then F has stable shifts at almost all frequencies.*

2.1. **Proof of Theorem 2.2.** We recall some properties of a tempered distribution with ℓ^1 decay in [24].

Lemma 2.6. *Let F be a vector-valued tempered distribution on $\mathbb{R}^d$ with ℓ^1 decay. Then:*

(i) *$\widehat{F}$ is continuous.*

(ii) *$\widehat{F}$ has polynomial increase at infinity, i.e., there exists a polynomial $Q(\xi)$ such that $|\widehat{F}(\xi)| \le Q(\xi)$ for all $\xi \in \mathbb{R}^d$.*

(iii) *For any $g \in \mathcal{S}$, the following Poisson summation formula holds*

$$\sum_{k \in \mathbb{Z}^d} \widehat{F}(\xi + 2k\pi)\overline{\hat{g}(\xi + 2k\pi)} = \sum_{j \in \mathbb{Z}^d} \ll F(\cdot + j), g \gg e^{-ij\xi}.$$

Proof of Theorem 2.2. The implication (iii)$\Longrightarrow$(i) is obvious. Then it suffices to prove (i) $\Longrightarrow$ (ii)$\Longrightarrow$ (iii). First we prove (i) $\Longrightarrow$ (ii). Suppose, on the contrary, that the rank of $r \times \mathbb{Z}^d$ matrix $(\widehat{F}(\xi_0 + 2k\pi))_{k \in \mathbb{Z}^d}$ is strictly less than r. Then there exists a (complex-valued) nonzero vector $v = (v_1, \ldots, v_r)^T$ so that the function $g := v^T F$ satisfies $\hat{g}(\xi_0 + 2k\pi) = 0$ for all $k \in \mathbb{Z}^d$. The sequence $E_{\xi_0} = (e^{ij\xi_0})_{j \in \mathbb{Z}^d}$ belongs to ℓ^∞, has its Fourier series supported in $\xi_0 + 2\pi\mathbb{Z}^d$, and satisfies $g *' E_{\xi_0} = 0$. This contradicts the assumption (i).

Next we prove (ii)$\Longrightarrow$(iii). By (ii), there exist $k_1, \ldots, k_r \in \mathbb{Z}^d$ such that

$$A := \big(\widehat{F}(\xi_0 + 2k_1\pi), \ldots, \widehat{F}(\xi_0 + 2k_r\pi)\big) \quad \text{is nonsingular.} \tag{2.3}$$

Let $e_i, 1 \le i \le r$, be vectors with i-th component one and other component zero. By (2.3), $Av_i = e_i$ for some $v_i = (v_i(k_1), \ldots, v_i(k_r))^T \in \mathbb{C}^r, 1 \le i \le r$. Let w_i be a C^∞ function so that w_i is supported in $\cup_{i'=1}^{l} B(\xi_0, \delta) + 2k_{i'}\pi$ and $w_i(\xi) = v_i(k_{i'})$ for all $\xi \in B(\xi_0, \delta/2) + 2k_{i'}\pi$, where $\delta > 0$ is a sufficiently small positive number chosen later. Recall that $\widehat{f}_i, 1 \le i \le r$, are continuous on $\mathbb{R}^d$ by Lemma 2.6. Then for sufficiently small $\delta > 0$, the matrix

$$A(\xi) := \Big(\sum_{k \in \mathbb{Z}^d} \widehat{f}_i(\xi + 2k\pi)\overline{w_{i'}(\xi + 2k\pi)}\Big)_{1 \le i, i' \le r}$$

has its entries in the Wiener class, and is nonsingular in a neighborhood of $B(\xi_0, \delta) + 2\pi\mathbb{Z}^d$. Hence there exists another matrix $B(\xi)$ whose entries belong to the Wiener class so that

$$A(\xi)B(\xi) = I_r \tag{2.4}$$

on a smaller neighborhood of $\xi + 2\pi\mathbb{Z}^d$, say $B(\xi_0, \delta_1) + 2\pi\mathbb{Z}^d$ for some $0 < \delta_1 < \delta/2$. Define $h_1, \ldots, h_r$ by

$$\big(\widehat{h}_1(\xi), \ldots, \widehat{h}_r(\xi)\big)^T = \overline{B(\xi)}^T (w_1(\xi), \ldots, w_r(\xi))^T. \tag{2.5}$$

Then $h_1, \ldots, h_r$ are linear combinations of the integer shifts of some Schwartz functions using ℓ^1 sequences, and hence are C^∞ functions with ℓ^1 decay. By (2.4) and (2.5),

$$\sum_{k \in \mathbb{Z}^d} \widehat{f}_i(\xi + 2k\pi)\overline{\widehat{h}_{i'}(\xi + 2k\pi)} = \delta_{ii'}, \ 1 \le i, i' \le r \tag{2.6}$$

for all $\xi \in B(\xi_0, \delta_1) + 2\pi\mathbb{Z}^d$, where $\delta_{ii'}$ is the usual Kronecker symbol. Let $D = \{(d_1(k), \ldots, d_r(k))^T\}$ be any bounded sequence with its Fourier series $\mathcal{F}(D)$ supported in $B(\xi_0, \delta_1) + 2\pi\mathbb{Z}^d$. Multiplying $\mathcal{F}(D_i)$ at both sides of the equation (2.6) and then summing up for i from 1 to r, we obtain

$$\mathcal{F}(D_{i'}) = \sum_{k \in \mathbb{Z}^d} \widehat{F *' D}(\cdot + 2k\pi)\overline{\widehat{h}_{i'}(\cdot + 2k\pi)}, \ 1 \le i' \le r,$$

where $D_i = \{d_i(k)\}, 1 \le i \le r$. Then the assertion (iii) follows. $\square$

2.2. **Proof of Theorem 2.4.** To prove Theorem 2.4, we need a characterization of finitely linearly dependent shifts of a compactly supported distribution, which is essentially given in [28] for the one dimensional case.

Lemma 2.7. *Let $F = (f_1, \ldots, f_r)^T$ be a compactly supported distribution. Then F has finitely linearly dependent shifts if and only if the matrix $(\widehat{F}(\xi + 2k\pi))_{k\in\mathbb{Z}^d}$ has rank less than or equal to $r-1$ for all $\xi \in \mathbb{R}^d$.*

To prove Lemma 2.7, we recall a result in [2].

Lemma 2.8. *Let $F = (f_1, \ldots, f_r)^T$ be a compactly supported distribution. Then there exists a compactly supported distribution $G = (g_1, \ldots, g_s)^T$ so that G has stable shifts, and such that $f_1, \ldots, f_r$ are finite linear combinations of $\{g_i(\cdot - k) : \ 1 \le i \le s, k \in \mathbb{Z}^d\}$.*

Proof of Lemma 2.7. First the sufficiency. Let $Q(\xi) = (q_1(\xi), \ldots, q_r(\xi))^T$ be a nonzero vector with polynomial entries so that $Q(\xi)^T \widehat{F}(\xi) = 0$ for all $\xi \in \mathbb{R}^d$. Therefore

$$Q(\xi)^T A(\xi) = 0 \quad \forall\, \xi \in \mathbb{R}^d, \tag{2.7}$$

where $A(\xi) = (\widehat{F}(\xi + 2k\pi))_{k\in\mathbb{Z}^d}$. By our assumption, $Q(\xi) \ne 0$ for almost all $\xi \in \mathbb{R}^d$, which together with (2.7) implies that the matrix $A(\xi)$ has rank less than or equals to $r-1$ for almost all $\xi \in \mathbb{R}^d$. On the other hand, $\widehat{F}$ is an analytic function by the assumption on F, and hence either the rank of $(\widehat{F}(\xi + 2k\pi))_{k\in\mathbb{Z}^d}$ is strictly less than r for all $\xi \in \mathbb{R}^d$, or is equal to r for almost all $\xi \in \mathbb{R}^d$. Then the assertion follows.

Then the necessity. Let G be the function as in Lemma 2.8, and $P(\xi)$ be the trigonometric polynomial matrix so chosen that $\widehat{F}(\xi) = P(\xi)\widehat{G}(\xi)$. By the stable shifts of G, $(\widehat{G}(\xi + 2k\pi))_{k\in\mathbb{Z}^d}$ has rank s for all $\xi \in \mathbb{R}^d$ ([16, 21]). This together with our assumption on F implies that the matrix $P(\xi)$ has rank less than or equal to $r-1$ for all $\xi \in \mathbb{R}^d$. Thus there exists a nonzero trigonometric polynomial vector $Q(\xi) = (q_1(\xi), \ldots, q_r(\xi))^T$ so that $Q(\xi)^T P(\xi) = 0$ for all $\xi \in \mathbb{R}^d$. Hence $Q(\xi)^T \widehat{F}(\xi) = 0$ for all $\xi \in \mathbb{R}^d$ and the assertion follows. □

Proof of Theorem 2.4. By the assumption on F, the Fourier transform $\widehat{F}(\xi)$ of F is an analytic function. Hence either the rank of $(\widehat{F}(\xi + 2k\pi))_{k\in\mathbb{Z}^d}$ is strictly less than r for all $\xi \in \mathbb{R}^d$, or is equal to r for almost all $\xi \in \mathbb{R}^d$. The assertion then follows from Theorem 2.2 and Lemma 2.7. □

3. ℓ^p stable Shifts at Certain Frequencies

In this section, we consider ℓ^p stable shifts at some frequency for the finitely generated shift invariant system, and show the equivalence between the stable shifts at some frequency and ℓ^p stable shifts at that frequency when the generators have a certain decay property at infinity.

The generators of the finitely generated shift invariant system are chosen from the function space $\mathcal{L}^p, 1 \le p \le \infty$, which contains all functions f with finite $\|f\|_{\mathcal{L}^p}$, where $\|f\|_{\mathcal{L}^p} = \big\| \sum_{j\in\mathbb{Z}^d} |f(\cdot+j)|\big\|_{L^p([0,1)^d)}$ ([15]). Here $L^p(K), 1 \le p \le \infty$, is the space of all p-integrable functions on a measurable set K, and $\|\cdot\|_{L^p(K)}$ is the usual $L^p(K)$ norm. Clearly for $1 \le p \le \infty$, a compactly supported L^p belongs to $\mathcal{L}^p$, and an $\mathcal{L}^p$ function is integrable. Thus a function in $\mathcal{L}^p$ is a tempered distribution with ℓ^1 decay.

For any $F = (f_1, \ldots, f_r)^T \in (\mathcal{L}^p)^{(r)}$, we have

$$\begin{aligned}
\|F *' D\|_\infty &\le \sup_{x\in\mathbb{R}^d} \sum_{j\in\mathbb{Z}^d} |D(j)||F(x-j)| \\
&\le \|D\|_{\ell^\infty} \sup_{x\in\mathbb{R}^d} \sum_{j\in\mathbb{Z}^d} |F(x-j)| = \|D\|_{\ell^\infty}\|F\|_{\mathcal{L}^\infty}
\end{aligned}$$

when $p = \infty$, and

$$\begin{aligned}
\|F *' D\|_p^p &\le \int_{\mathbb{R}^d} \Big(\sum_{j\in\mathbb{Z}^d} |D(j)||F(x-j)|\Big)^p dx \\
&\le \int_{\mathbb{R}^d} \Big(\sum_{j\in\mathbb{Z}^d} |D(j)|^p|F(x-j)|\Big) \times \Big(\sum_{j\in\mathbb{Z}^d} |F(x-j)|\Big)^{p-1} dx \\
&= \|D\|_{\ell^p}^p \int_{[0,1]^d} \Big(\sum_{j\in\mathbb{Z}^d} |F(x-j)|\Big)^p dx = \|D\|_{\ell^p}^p \|F\|_{\mathcal{L}^p}^p
\end{aligned}$$

when $1 \le p < \infty$. Therefore

$$\|F *' D\|_p \le \|D\|_{\ell^p}\|F\|_{\mathcal{L}^p} \quad \text{for all } D \in (\ell^p)^{(r)} \tag{3.1}$$

([15]). Thus we may consider the semi-convolution $F*'$ as a bounded operator from $(\ell^p)^{(r)}$ to $L^p, 1 \le p \le \infty$.

For any function $F = (f_1, \ldots, f_r)^T \in (\mathcal{L}^p)^{(r)}$ and $1 \le p \le \infty$, it is known that F has ℓ^p stable shifts if and only if F has stable shifts ([15]). In this section, we show the equivalence between the ℓ^p stable shifts at a certain frequency and the stable shifts at that frequency.

Theorem 3.1. *Let $F = (f_1, \ldots, f_r)^T \in (\mathcal{L}^p)^{(r)}, 1 \le p \le \infty$, and $\xi_0 \in \mathbb{R}^d$. Then F has ℓ^p stable shifts at the frequency ξ_0 if and only if F has stable shifts at the frequency ξ_0.*

Combining Corollary 2.3 and Theorem 3.1, we have

Corollary 3.2. *A function in* $(\mathcal{L}^p)^{(r)}$ *has* ℓ^p *stable shifts if and only if it has* ℓ^p *stable shifts at any frequency.*

To prove Theorem 3.1, we recall a lemma in [3].

Lemma 3.3. *Let* $f \in \mathcal{L}^p$ *for some* $1 \le p < \infty$. *Assume that* $\sum_{j\in\mathbb{Z}^d} f(\cdot - j) = 0$. *Then for any Schwartz function* h *on* $\mathbb{R}^d$,

$$\lim_{\delta\to 0} \delta^d \Big\| \sum_{j\in\mathbb{Z}^d} h(\delta j) f(\cdot - j)\Big\|_{\mathcal{L}^p} = 0.$$

Note that the Fourier transform of $\delta^d \sum_{j\in\mathbb{Z}^d} h(\delta j) f(\cdot - j)$ is $\hat f(\xi) \times \sum_{k\in\mathbb{Z}^d} \hat h(\delta^{-1}(\xi + 2k\pi))$, which can be considered as a periodic smooth cutoff of $\hat f$ at the origin. Then the result in Lemma 3.3 can be understood as: if $\hat f(2k\pi) = 0$ for all $k \in \mathbb{Z}^d$, then the $\mathcal{L}^p$ norm of the periodic smooth cutoff of a function $f \in \mathcal{L}^p$ at the δ neighborhood of the origin tends to zero as δ tends to zero. A periodic smooth cutoff is crucial when we consider the shift invariant problem in $L^p, 1 \le p \le \infty$, instead of in L^2, where a bounded periodic cutoff is usually used. The limit result in Lemma 3.3 is not true for $p = \infty$ and a counterexample is given in [3].

Proof of Theorem 3.1. First we prove the necessity, which is essentially given in [15]. Suppose, on the contrary, that F does not have stable shifts at the frequency ξ_0. Then the matrix $(\widehat F(\xi_0 + 2k\pi))_{k\in\mathbb{Z}^d}$ is not of full rank by Theorem 2.2, and hence there exists a (complex-valued) nonzero vector $v = (v_1, \ldots, v_r)^T$ so that the function $g := v^T F$ satisfies

$$\hat g(\xi_0 + 2k\pi) = 0 \quad \forall\ k \in \mathbb{Z}^d. \tag{3.2}$$

If $g \equiv 0$, then the proof is done since $\sum_{i=1}^r f_i *' D_i = 0$ for sequences $D_i, 1 \le i \le r$, chosen so that $D_i = v_i D$ for some $D \in \ell^p$, which contradicts the stable assumption on F. Now we suppose $g \not\equiv 0$. Set $E_{\xi_0} = (e^{-ij\xi_0})_{j\in\mathbb{Z}^d} \in \ell^\infty$. For $p = \infty$, it follows from (3.2) that

$$g *' E_{\xi_0} = 0, \tag{3.3}$$

which contradicts the stable shifts at the frequency ξ_0 since the Fourier series of E_{ξ_0} is supported in $\xi_0 + 2\pi\mathbb{Z}^d$. Let h_0 be a Schwartz function so chosen that $\widehat{h}_0$ is supported in $B(0,1)$ and $\widehat{h}_0(x) = 1$ for all $x \in B(0,1/2)$. For $1 \le p < \infty$, it follows from (3.3) and Lemma 3.3 that

$$\lim_{\delta\to 0} \delta^d \Big\| \sum_{j\in\mathbb{Z}^d} h_0(\delta j) e^{-i(\cdot - j)\xi_0} g(\cdot - j)\Big\|_{\mathcal{L}^p} = 0. \tag{3.4}$$

Note that the Fourier transform of $\delta^d \sum_{j\in\mathbb{Z}^d} h_0(\delta j)e^{-i(\cdot-j)\xi_0}g(\cdot-j)$ is $\hat{g}(\xi+\xi_0)\sum_{k\in\mathbb{Z}^d}\widehat{h}_0(\delta^{-1}(\xi+2k\pi))$,which equals $\hat{g}(\xi+\xi_0)$ on $B(0,\delta/2)$. This together with (3.4) shows that for any $\epsilon>0$ there exists $\delta_0>0$ so that

$$\|g *' D\|_p \le \delta^d \Big\| \sum_{j\in\mathbb{Z}^d} h_0(\delta j)e^{-i(\cdot-j)\xi_0}g(\cdot-j)\Big\|_{\mathcal{L}^p} \times \|D\|_{\ell^p} \le \epsilon\|D\|_{\ell^p}$$

for any ℓ^p sequence D whose Fourier series is supported in $B(\xi_0,\delta/2)+2\pi\mathbb{Z}^d$ and $\delta\in(0,\delta_0)$. This is a contradiction since $\epsilon>0$ can be chosen arbitrarily.

Now we prove the sufficiency. By Theorem 2.2, there exist C^∞ functions $h_1,\ldots,h_r$ with ℓ^1 decay and $\delta>0$ so that

$$d_i(k) = \ll F *' D, h_i(\cdot-k)\gg,\ 1\le i\le r,\ k\in\mathbb{Z}^d, \tag{3.5}$$

where $D=\{(d_1(k),\ldots,d_r(k))^T\}\in(\ell^p)^{(r)}$ and $\mathcal{F}(D)$ is supported in $B(\xi_0,\delta)+2\pi\mathbb{Z}^d$. Therefore

$$\|D\|_{\ell^\infty} \le \sum_{i=1}^r \|F *' D\|_\infty \|h_i\|_1 \le C_0\|F *' D\|_\infty$$

for $p=\infty$, and

$$\|D\|_{\ell^p}^p \le C_1\sum_{i=1}^r \|h_i\|_1^{p-1}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d}|F *' D(x)|^p|h_i(x-k)|dx \le C_2\|F * D\|_p^p$$

for $1\le p<\infty$, where C_0, C_1, C_2 are positive constants independent of $D\in\ell^p$. Hence the sufficiency follows. □

4. p-frame at Certain Frequency

In this section, we consider localization of a p-frame in the finitely generated shift invariant system in the Fourier domain. The generators of the shift invariant system are chosen from the function space $W(L^p,\ell^q), 1\le p,q\le\infty$, which contains all functions f whose norm $\|f\|_{W(L^p,\ell^q)}$ is finite, where

$$\|f\|_{W(L^p,\ell^q)} := \|\{\|f\|_{L^p(k+[0,1)^d)}\}\|_{\ell^q}.$$

For any $1\le p,q\le\infty$, we have

$$W(L^\infty,\ell^q)\subset W(L^p,\ell^q)$$

since $\|f\|_{L^p(k+[0,1)^d)}\le\|f\|_{L^\infty(k+[0,1)^d)}$ for all $k\in\mathbb{Z}^d$, and

$$W(L^p,\ell^1)\subset\mathcal{L}^p\subset W(L^p,\ell^p)=L^p$$

because

$$\|f\|_{\mathcal{L}^p} = \Big\| \sum_{k\in\mathbb{Z}^d} |f(k+\cdot)| \Big\|_{L^p([0,1)^d)} \le \sum_{k\in\mathbb{Z}^d} \|f(k+\cdot)\|_{L^p([0,1)^d)} = \|f\|_{W(L^p,\ell^1)}.$$

Let $F=(f_1,\ldots,f_r)^T \in (W(L^{p/(p-1)},\ell^1))^{(r)}$. Then for any $f\in L^p$,

$$\sum_{i=1}^r \|\{\langle f, f_i(\cdot-k)\rangle\}\|_{\ell^p} \le \sum_{l\in\mathbb{Z}^d}\sum_{i=1}^r \|\{\langle f, f_{i,l}(\cdot-k)\rangle\}\|_{\ell^p}$$

$$\le \sum_{i=1}^r \sum_{l\in\mathbb{Z}^d} \|f\|_p \|f_{i,l}\|_{p/(p-1)} \le \sum_{i=1}^r \|f_i\|_{W(L^{p/(p-1)},\ell^1)} \|f\|_p, \tag{4.1}$$

where $f_{i,l} = f_i\chi_{l+[0,1)^d}, l\in\mathbb{Z}^d$. Thus $\{f_i(\cdot-k): 1\le i\le r, k\in\mathbb{Z}^d\}$ is the Bessel sequence for the shift invariant space $V_p(F)$ when $F=(f_1,\ldots,f_r)^T\in(\mathcal{L}^p)^{(r)}\cap(W(L^{p/(p-1)},\ell^1))^{(r)}$, and the analysis operator T is a bounded operator from L^p to $(\ell^p)^{(r)}$. So it is good to assume that the generators of the shift invariant system belong to $\mathcal{L}^p\cap W(L^{p/(p-1)},\ell^1)$ when we consider the corresponding p-frame property.

When the generators are assumed to be in the class $W(L^\infty,\ell^1)$, which is a subspace of $\mathcal{L}^p$ and also of $W(L^{p/(p-1)},\ell^1)$ for $1\le p\le\infty$, the following characterization of the p-frame property was established in [3].

Proposition 4.1. *Let $F=(f_1,\ldots,f_r)^T\in(W(L^\infty,\ell^1))^{(r)}$ and $1\le p\le\infty$. Then the following statements are equivalent to each other.*

(i) *$V_p(F)$ is a closed subspace of L^p.*

(ii) *F generates a p-frame for $V_p(F)$.*

(iii) *The rank of the $r\times\mathbb{Z}^d$ matrix $(\widehat{F}(\xi+2k\pi))_{k\in\mathbb{Z}^d}$ is independent of $\xi\in\mathbb{R}^d$.*

(iv) *There exist sequences $\{a_{ii'}(k)\}\in\ell^1, 1\le i,i'\le r$, so that*

$$f=\sum_{i,i'=1}^r \sum_{k,k'\in\mathbb{Z}^d} a_{ii'}(k-k')\langle f, f_{i'}(\cdot-k')\rangle f_i(\cdot-k)$$

for all $f\in V_p(F)$.

For any $F=(f_1,\ldots,f_r)^T\in(W(L^\infty,\ell^1))^{(r)}$, it was pointed out in [3] that if F has ℓ^p stable shifts, then it generates a p-frame for $V_p(F), 1\le p\le\infty$. In fact, when F has ℓ^p stable shifts, the shift invariant space $V_p(F)$ is closed in L^p since it is isomorphic to the sequence space ℓ^p, and hence F generates a p-frame for $V_p(F), 1\le p\le\infty$, by Proposition 4.1. The above assertion is known for $p=2$ under the weak assumption

$\sum_{k\in\mathbb{Z}^d}|\widehat{F}(\cdot+2k\pi)|^2\in L^\infty$ in [6]. So the frame property of the shift invariant system is a generalization of the stability (Riesz) property. The p-frame was introduced in [3] as a class of Banach frames related to the finitely generated shift invariant system, which preserves most of the frame properties in Hilbert space (see [4, 20] for more properties of p-frames).

In this section, we establish the corresponding version of Proposition 4.1 for a p-frame at a certain frequency.

Theorem 4.2. *Let $\xi_0\in\mathbb{R}^d$, and let $F=(f_1,\ldots,f_r)^T\in(\mathcal{L}^p)^{(r)}\cap(W(L^{p/(p-1)},\ell^1))^{(r)}$ for $1\le p<\infty$ and $F\in(W(L^\infty,\ell^1))^{(r)}$ for $p=\infty$. Then the following statements are equivalent:*

(i) *F generates a p-frame for $V_p(F)$ at the frequency ξ_0.*

(ii) *The rank of the $r\times\mathbb{Z}^d$ matrix $(\widehat{F}(\xi+2k\pi))_{k\in\mathbb{Z}^d}$ is independent of ξ at a small neighborhood of ξ_0.*

(iii) *There exist a positive constant δ and sequences $\{a_{ii'}(k)\}\in\ell^1, 1\le i,i'\le r$, so that*

$$f=\sum_{i,i'=1}^{r}\sum_{k,k'\in\mathbb{Z}^d}a_{ii'}(k-k')\langle f,f_{i'}(\cdot-k')\rangle f_i(\cdot-k)\tag{4.2}$$

for all $f\in V_{p,B(\xi_0,\delta)}(F)$.

and

Theorem 4.3. *Let $\xi_0\in\mathbb{R}^d$, and let $F=(f_1,\ldots,f_r)^T\in(\mathcal{L}^p)^{(r)}$ for $1\le p<\infty$ and $F\in(W(L^\infty,\ell^1))^{(r)}$ for $p=\infty$. Then the space $V_{p,B(\xi_0,\delta)}(F)$ is a closed subspace of L^p for sufficiently small $\delta>0$ if and only if the rank of the $r\times\mathbb{Z}^d$ matrix $(\widehat{F}(\xi+2k\pi))_{k\in\mathbb{Z}^d}$ is independent of ξ at a small neighborhood of ξ_0.*

We may understand the above two theorems as the equivalence among p-frame at certain frequency, the constant dimension of Fourier fiber bundle at a neighborhood of that frequency, inverse process of the semi-convolution at that frequency, and closedness of the shift invariant space at that frequency.

Combining Theorem 4.2 and Proposition 4.1, we have

Corollary 4.4. *A function in $(\mathcal{L}^p)^{(r)}\cap(W(L^{p/(p-1)},\ell^1))^{(r)}$ generates a p-frame if and only if it generates a p-frame at all frequencies.*

Then Theorems 4.2 and 4.3 can be considered as the localization version of Proposition 4.1 in the Fourier domain. Moreover, we see that the conditions on the generators in Theorem 4.2 are weaker than the ones in Proposition 4.1. For instance, a compactly supported function $F\in$

$L^{\max(p,p/(p-1))}$ satisfies the conditions in Theorem 4.2, but does not satisfy the conditions in Proposition 4.1.

Note that the Fourier transform of the function $h = \chi_{[0,1]} - \chi_{[1,2]}$ is $(e^{-2i\xi} - 2e^{-i\xi} + 1)/(i\xi)$. Then h generates a p-frame at any frequency $\xi_0 \in I\!R\backslash(2\pi Z\!\!Z)$ by Theorem 4.2. Moreover, applying Theorem 4.2 yields the following interesting result for compactly supported bounded function.

Theorem 4.5. *Let $1 \le p \le \infty$, $f_1, \ldots, f_r$ be compactly supported and bounded, and set $F = (f_1, \ldots, f_r)^T$. Then F generates a p-frame at almost all frequencies.*

4.1. **Proof of Theorem 4.2.** To prove Theorem 4.2, we need two lemmas.

Lemma 4.6. *Let $\xi_0 \in I\!R^d$, $F = (f_1, \ldots, f_r)^T$ be a tempered distribution with ℓ^1 decay. Then there exists a nonsingular matrix $P_{\xi_0}(\xi)$ so that all entries are periodic functions in the Wiener class, and the functions F_{1,ξ_0} and F_{2,ξ_0} defined by*

$$P_{\xi_0}(\xi)\widehat{F}(\xi) = \begin{pmatrix} \widehat{F}_{1,\xi_0}(\xi) \\ \widehat{F}_{2,\xi_0}(\xi) \end{pmatrix} \tag{4.3}$$

satisfy (i) F_{1,ξ_0} has stable shifts at the frequency ξ_0, and (ii) $\widehat{F}_{2,\xi_0}(\xi_0 + 2k\pi) = 0$ for all $k \in Z\!\!Z^d$. Furthermore, F_{2,ξ_0} can be chosen so that $\widehat{F}_{2,\xi_0}(\xi + 2k\pi) = 0$ for all $\xi \in B(\xi_0, \delta_0)$ and $k \in Z\!\!Z^d$ if the rank of $(\widehat{F}(\xi + 2k\pi))_{k\in\mathbf{Z}^d}$ is a constant at a neighborhood of ξ_0, where $\delta_0 > 0$.

The proof of the above lemma under the assumption that $F \in (W(L^\infty, \ell^1))^{(r)}$ was given in [3, Lemma 2]. We omit the details of the proof here since it can be done similarly.

Lemma 4.7. *Let $f \in W(L^p, \ell^1)$ for some $1 \le p \le \infty$. Assume that $\sum_{j\in\mathbf{Z}^d} f(\cdot - j) = 0$. Then for any function h on $I\!R^d$ satisfying*

$$|h(x)| \le C(1 + |x|)^{-d-1} \quad \text{for all } x \in I\!R^d, \tag{4.4}$$

and

$$|h(x) - h(y)| \le C|x - y|\,(1 + \min(|x|, |y|))^{-d-1} \quad \text{for all } x, y \in I\!R^d, \tag{4.5}$$

we have

$$\lim_{\delta\to 0} \delta^d \Big\| \sum_{j\in\mathbf{Z}^d} h(\delta j) f(\cdot - j) \Big\|_{W(L^p,\ell^1)} = 0. \tag{4.6}$$

Proof. Given a positive number ϵ, there exists $N_0 \ge 2$ such that

$$\sum_{|k|\ge N_0} \|f\|_{L^p(k+[0,1)^d)} \le \epsilon. \tag{4.7}$$

Set

$$f_1(x) = f(x)\chi_{O_{N_0}}(x) + \sum_{|k|\ge N_0} f(x+k)\chi_{[0,1]^d}(x),$$

where $O_{N_0} = \cup_{|k|<N_0}(k+[0,1]^d)$. Then

$$\|f_1 - f\|_{W(L^p,\ell^1)} \le 2 \sum_{|k|\ge N_0} \|f\|_{L^p(k+[0,1)^d)} \le 2\epsilon \tag{4.8}$$

by (4.7), and

$$\sum_{k\in\mathbb{Z}^d} f_1(x-k) = \sum_{k\in\mathbb{Z}^d} f(x-k) = 0 \quad \forall\ x \in \mathbb{R}^d \tag{4.9}$$

by the definition of f_1. By (4.4) and (4.8), we have

$$\begin{aligned}
&\Big\|\delta^d \sum_{k\in\mathbb{Z}^d} h(\delta k)(f(\cdot-k) - f_1(\cdot-k))\Big\|_{W(L^p,\ell^1)} \\
&\le \delta^d \sum_{k\in\mathbb{Z}^d} |h(\delta k)|\ \|f - f_1\|_{W(L^p,\ell^1)} \le C\epsilon, \qquad (4.10)
\end{aligned}$$

where C is a positive constant independent of $\delta \in (0,1)$. From (4.5) and (4.9) it follows that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d} \delta^d \Big\| \sum_{j\in\mathbb{Z}^d} h(\delta j) f_1(\cdot - j)\Big\|_{L^p(k+[0,1)^d)} \\
\le\ & \delta^d \sum_{k\in\mathbb{Z}^d} \Big\| \sum_{j\in\mathbb{Z}^d} (h(\delta j) - h(\delta k)) f_1(\cdot - j)\Big\|_{L^p(k+[0,1)^d)} \\
\le\ & C_1(N_0)\delta^{d+1} \\
& \times \sum_{k\in\mathbb{Z}^d} \Big\| \sum_{|j-k|\le N_0+1} \frac{|f_1(\cdot - j)|}{(1+\delta\min(|j|,|k|))^{d+1}} \Big\|_{L^p(k+[0,1)^d)} \\
\le\ & C_2(N_0)\delta \|f_1\|_{W(L^p,\ell^1)} \le C_3(N_0)\delta(\|f\|_{W(L^p,\ell^1)} + 2\epsilon), \qquad (4.11)
\end{aligned}$$

where $C_1(N_0), C_2(N_0), C_3(N_0)$ are positive constants depending only on N_0, d and the constant C in (4.5). Therefore the estimate (4.6) follows from (4.10) and (4.11) when δ is chosen sufficiently small. $\square$

Proof of Theorem 4.2. We divide the proof into the following steps: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i). The implication of (i) $\Longrightarrow$ (ii) can be done by using the same technique in [3, Section 5.6] except the estimate (5.18) and the limit following the estimate (5.18) there being replaced by the estimate (4.1) and the limit in Lemma 4.7. We omit the details of the proof here.

Now we prove (ii) $\Longrightarrow$ (iii). Let functions F_{1,ξ_0} and F_{2,ξ_0}, matrix $P_{\xi_0}(\xi)$, and the positive number δ_0 be as in Lemma 4.6. Then the rank of $(\widehat{F}_{1,\xi_0}(\xi+2k\pi))_{k\in\mathbb{Z}^d}$ is k_0 for all $\xi\in B(\xi_0,\delta_1)$ for some $\delta_1\le\delta_0$. Note that for $f\in\mathcal{L}^p$ and $g\in W(L^{p/(p-1)},\ell^1)$,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d}\Big|\int_{\mathbb{R}^d} f(x-k)g(x)dx\Big| \\
\le\ & \sum_{l\in\mathbb{Z}^d}\int_{[0,1)^d}|g(x-l)|\times\Big(\sum_{k\in\mathbb{Z}^d}|f(x-k)|\Big)dx \\
\le\ & \sum_{l\in\mathbb{Z}^d}\|g\|_{L^{p/(p-1)}(l+[0,1)^d)}\|f\|_{\mathcal{L}^p}\le\|f\|_{\mathcal{L}^p}\|g\|_{W(L^{p/(p-1)},\ell^1)}. \qquad (4.12)
\end{aligned}$$

Then the matrix $A(\xi):=\sum_{k\in\mathbb{Z}^d}\widehat{F}_{1,\xi_0}(\xi+2k\pi)\overline{\widehat{F}_{1,\xi_0}(\xi+2k\pi)}^T$ has all entries in the Wiener class. From the rank properties of the matrix $(\widehat{F}(\xi+2k\pi))_{k\in\mathbb{Z}^d}$ it follows that $A(\xi)$ is nonsingular in $B(\xi_0,\delta_1)+2\pi\mathbb{Z}^d$, which implies that there exists an inverse $B(\xi)$ of $A(\xi)$ in $B(\xi_0,\delta_1/2)+2\pi\mathbb{Z}^d$ so that its entries are still in the Wiener class. For any $f\in V_{p,B(\xi_0,\delta_1/2)}(F_{1,\xi_0})$, $\hat f(\xi)=\mathcal{F}(D)(\xi)^T\widehat{F}_{1,\xi_0}(\xi)$ for some sequence $D\in(\ell^p)^{(r)}$ with $\mathcal{F}(D)$ supported in $B(\xi_0,\delta_1/2)+2\pi\mathbb{Z}^d$. Thus the Fourier transforms of the sequences $W_i:=\{\langle f,f_{i,\xi_0}(\cdot-k)\rangle\}$, $1\le i\le k_0$, satisfies

$$(\mathcal{F}(W_1)(\xi),\ldots,\mathcal{F}(W_{k_0})(\xi))=\mathcal{F}(D)(\xi)^TA(\xi).$$

Therefore

$$\mathcal{F}(D)(\xi)^T=(\mathcal{F}(W_1)(\xi),\ldots,\mathcal{F}(W_{k_0})(\xi))B(\xi).$$

Substituting the above equation into $\hat f(\xi)=\mathcal{F}(D)(\xi)^T\widehat{F}_{1,\xi_0}(\xi)$ and using $\widehat{F}_{1,\xi_0}(\xi)=C_1(\xi)\widehat{F}(\xi)$ for some $k_0\times r$ matrix $C_1(\xi)$ with entries in the Wiener class, we obtain

$$f=\sum_{i,i'=1}^r\sum_{k,k'\in\mathbb{Z}^d}a_{ii'}(k-k')\langle f,f_{i'}(\cdot-k')\rangle f_i(\cdot-k)$$

for any $f\in V_p(F)$ with its Fourier transform supported in $B(\xi_0,\delta_1/2)+2\pi\mathbb{Z}^d$, where $\{a_{ii'}(k)\}\in\ell^1, 1\le i,i'\le r$. This proves the assertion (iii).

Finally we prove (iii) $\Longrightarrow$ (i). By (4.2), we have

$$\begin{aligned}\|f\|_p &\le \sum_{i=1}^r \Big\|\Big\{\sum_{i'=1}^r \sum_{k'\in\mathbb{Z}^d} a_{ii'}(k-k')\langle f, f_{i'}(\cdot-k')\rangle\Big\}\Big\|_{\ell^p}\\ &\le \sum_{i,i'=1}^r \|\{a_{ii'}(k)\}\|_{\ell^1}\|\{\langle f, f_i(\cdot-k)\rangle\}\|_{\ell^p} \le C\|\{\langle f, f_i(\cdot-k)\rangle\}\|_{\ell^p}\end{aligned}$$

for some positive constant C independent of f. Hence the implication (iii) $\Longrightarrow$ (i) follows. □

4.2. **Proof of Theorem 4.3.**

Proof. First the sufficiency. By the continuity of $\widehat{F}$, the rank of $(\widehat{F}(\xi+2k\pi))_{k\in\mathbb{Z}^d}$ is at least that of $(\widehat{F}(\xi_0+2k\pi))_{k\in\mathbb{Z}^d}$ for all ξ in a small neighborhood of ξ_0. Therefore if (ii) does not hold, then the set of all $\xi\in B(\xi_0,\delta)$ such that the rank of $(\widehat{F}(\xi+2k\pi))_{k\in\mathbb{Z}^d}$ is strictly larger than the one of $(\widehat{F}(\xi_0+2k\pi))_{k\in\mathbb{Z}^d}$ is not empty for any positive δ. Using the same technique in [3, Section 5.2], the implication of (i) $\Longrightarrow$ (ii) follows.

Then the necessity. Let functions F_{1,ξ_0} and F_{2,ξ_0}, matrix $P_{\xi_0}(\xi)$ and the positive number δ_0 be as chosen in Lemma 4.6. Then $\widehat{F}_{2,\xi_0}(\xi)=0$ for all $\xi\in B(\xi_0,\delta)+2\pi\mathbb{Z}^d$, which together with the nonsingularity of the matrix P_{ξ_0} imply that

$$V_{p,B(\xi_0,\delta)}(F)=V_{p,B(\xi_0,\delta)}(F_{1,\xi_0})$$

for all $\delta\in(0,\delta_0)$. Recall that F_{1,ξ_0} has stable shifts at the frequency ξ_0, and hence it has ℓ^p stable shifts at the frequency ξ_0 by Theorem 3.1. Therefore there exists $\delta_1\in(0,\delta_0)$ so that $V_{p,B(\xi_0,\delta)}(F_{1,\xi_0})$ is a closed subspace of L^p for all $\delta<\delta_1$, and hence the assertion (i) follows. □

4.3. **Proof of Theorem 4.5.**

Proof. For a compactly supported bounded function $F=(f_1,\ldots,f_r)^T$, let k_0 be maximum of the rank of the $r\times\mathbb{Z}^d$ matrix $(\widehat{F}(\xi+2k\pi))_{k\in\mathbb{Z}^d}, \xi\in\mathbb{R}^d$. Then there exist $1\le i_1<i_2<\ldots<i_{k_0}\le r$ and $j_1,\ldots,j_{k_0}\in\mathbb{Z}^d$ so that the matrix $\big(\widehat{f_{i_s}}(\xi_0+2j_t\pi)\big)_{1\le s,t\le k_0}$ has nonzero determinant for some $\xi_0\in\mathbb{R}^d$. Recall that $\widehat{F}$ is an analytic function. Then the determinant of the above matrix is a nonzero analytic function, which implies that the set E of all $\xi\in\mathbb{R}^d$ with nonzero determinant is an open set and the complement $\mathbb{R}^d\backslash E$ has zero Lebesgue measure. Thus F generates a frame for $V_p(F)$ at any frequency in E. The assertion follows. □

Acknowledgment The research of the author was partially supported by the Texas Higher Education Coordinating Board Award ARP-003652-0071-2001. The author would like to thank the referee for his(her) suggestions.

References

[1] A. Aldroubi and K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant spaces. *SIAM Rev.*, 43(2001), 585–620.

[2] A. Aldroubi and Q. Sun, Locally finite dimensional shift-invariant spaces in $\mathbb{R}^d$, *Proc. Amer. Math. Soc.*, 130(2002), 2641–2654.

[3] A. Aldroubi, Q. Sun, and W.-S. Tang, p-frames and shift-invariant spaces of L^p. *J. Fourier Anal. Appl.*, 7(2001), 1–21.

[4] A. Aldroubi, Q. Sun and W.-S. Tang, Connection between p-frames and p-Riesz bases in locally finite SIS of $L^p(\mathbb{R}^d)$, In *Proceeding of SPIE 4119: Wavelet Applications in Signal and Image Processing VIII* (A. Aldroubi, A. F. Laine and M. A. Unser eds.), SPIE, 2000, pp. 668–674.

[5] A. Aldroubi, Q. Sun, and W.-S. Tang, Non-uniform average sampling and reconstruction in multiply generated shift-invariant spaces, *Constr. Approx.*, to appear.

[6] J. Benedetto and S. Li, The theory of multiresolution analysis frames and applications to filter banks, *Appl. Comput. Harmon. Anal.*, 5(1998), 398–427.

[7] C. K. Chui, *An Introduction to Wavelets*, Academic Press, 1992.

[8] A. Cohen and Q. Sun, An arithmetic characterization of the conjugate quadrature filters associated to orthonormal wavelet bases, *SIAM J. Math. Anal.*, 24(1993), 1355–1360.

[9] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia 1992.

[10] T. A. Hogan, Stability and independence of the shifts of finitely many refinable functions, *J. Fourier Anal. Appl.*, 3(1997), 757–774.

[11] T. A. Hogan, Stability and independence for multivariate refinable distributions, *J. Approx. Th.*, 98(1999), 248–270.

[12] T. A. Hogan and R. Q. Jia, Dependency relations among the shifts of a multivariate refinable distribution, *Constr. Approx.*, 17(2001), 19–37.

[13] G. Jetter and G. Plonka, A survey on L^2 approximation orders from shift-invariant spaces, In *Multivariate Approximation and Applications* (N. Dyn, D. Leviatan, D. Levin, A. Pinkus, eds.), Cambridge University Press, 2001, pp. 73–111.

[14] R. Q. Jia, Subdivision schemes in L^p spaces, *Adv. Comput. Math.*, 3(1995), 309–341.

[15] R. Q. Jia and C. A. Micchelli, Using the refinement equation for the construction of pre-wavelets II: powers of two, In *Curve and Surfaces* (P. J. Laurent, A. Le Méhauté and L. L. Schumaker eds.), Academic Press, New York 1991, pp. 209–246.

[16] R. Q. Jia and C. A. Micchelli, On linear independence of integer translates of a finite number of functions, *Proc. Edinburgh Math. Soc.*, 36(1992), 69–75.

[17] R. Q. Jia and J.-Z. Wang, Stability and linear independence associated with wavelet decompositions, *Proc. Amer. Math. Soc.*, 117(1993), 1115–1124.

[18] W. Lawton, Necessary and sufficient conditions for constructing orthonormal wavelet bases, *J. Math. Phys.*, 32(1991), 57–61.

[19] S. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, 1998.

[20] O. Christensen and D. T. Stoeva, p-frames in separable Banach spaces, *Adv. Comput. Math.*, 18(2003), 117–126.

[21] A. Ron, A necessary and sufficient condition for the linear independence of the integer translates of a compactly supported distribution, *Constr. Approx.*, 5(1989), 297–308.

[22] Z. Shen, Refinable function vectors, *SIAM J. Math. Anal.*, 29(1998), 239–250.

[23] Q. Sun, Sequence spaces and stability of integer translates, *Zeitschrift Analysis und ihre Anwendungen*, 12(1993), 567–584.

[24] Q. Sun, Stability of the shifts of global supported distributions, *J. Math. Anal. and Appl.*, 261(2001), 113–125.

[25] Q. Sun, Convergence of cascade algorithm and smoothness of refinable distributions, *Chinese. Ann. Math.*, to appear.

[26] Q. Sun, Linear dependence of the shifts of vector-valued compactly supported distributions, submitted.

[27] L. F. Villemoes, Wavelet analysis of refinement equations, *SIAM J. Math. Anal.*, 25(1994), 1433–1460.

[28] J.-Z. Wang, Stability and linear independence associated with scaling vectors, *SIAM J. Math. Anal.*, 29(1998), 1140–1156.

Sun, Department of Mathematics, University of Houston, Houston, TX77204, USA