Spatially Distributed Sampling and Reconstruction

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Summary

A spatially distributed system contains a large amount of agents with limited sensing, data processing, and communication capabilities. Recent technological advances have opened up possibilities to deploy spatially distributed systems for signal sampling and reconstruction. We introduce a graph structure for such distributed sampling and reconstruction systems (DSRS). We build up a locally verifiable stability criterion for overlapping smaller subsystems. We propose an exponentially convergent distributed algorithm for signal reconstruction.

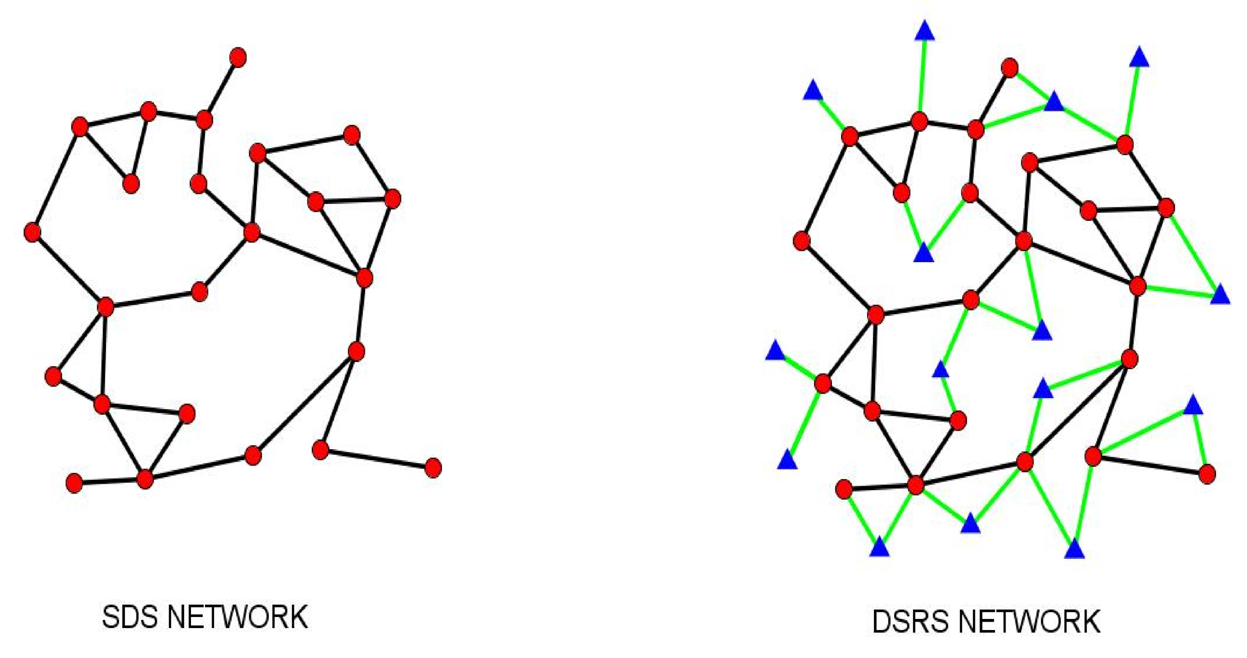
Spatially Distributed Systems

Spatially distributed systems (SDS) have been widely used in wireless sensor network, smart grid, (underwater) multivehicle and multirobot networks, etc. Comparing with a traditional centralized system that has a powerful central processor and reliable communication between agents and the central processor, an SDS could give unprecedented capabilities especially when creating a data exchange network requires significant efforts, or when establishing a centralized processor presents the daunting challenge of processing all the information (such as big-data problems). We consider SDSs for signal sampling and reconstruction.

Graph Structure of a DSRS

We consider spatial signals $f := \sum_{i\in V} c(i)\varphi_i$, where amplitudes $c(i), i \in V$, are bounded, and $\varphi_i, i \in V$ are generators at the innovative position i.

We describe the SDS by an undirected graph $\mathcal{G} := (G, S)$, where a vertex represents an agent and an edge between two vertices means that a direct communication link exists. We introduce a graph structure for the DSRS by coupling every innovative position $i \in V$ of the spatial signal f with some anchor agents $\lambda \in G$, and we generate an undirected graph $\mathcal{V} = (V, E)$ for signals.



Sensing Matrix of a DSRS

Define the sensing matrix associated with our DSRS by

$$\mathbf{S} := (\langle \psi_\lambda, \phi_i \rangle)_{\lambda\in G, i\in V},$$

where ψ_λ is the impulse response of the agent $\lambda \in G$.

The matrix $\mathbf{S}$ is stored by agents in a distributed manner. Due to the storage limitation, each agent in our SDS stores its corresponding row (and perhaps also its neighboring rows), but it does not have the whole matrix available. The matrix $\mathbf{S}$ has certain off-diagonal decay, since each agent has limited acquisition ability and it could essentially catch signals not far from its physical locations.

Sampling problem for a DSRS

A fundamental problem in sampling theory is whether and how to find an approximation $\tilde{\mathbf{d}}$ to the amplitude $\mathbf{c}$ of the signal f, when sampling data $\mathbf{z} = \mathbf{S}\mathbf{c} + \boldsymbol{\eta}$ is corrupted by the bounded noise $\boldsymbol{\eta}$, where $\mathbf{c}, \boldsymbol{\eta} \in \ell^\infty$.

Theorem 1. *If the sensing matrix* $\mathbf{S}$ *has* ℓ^2*-stability,* $\|\mathbf{S}\mathbf{c}\|_2 \approx \|\mathbf{c}\|_2$*, then the "least squares" solution* $\tilde{\mathbf{d}} := (\mathbf{S}^T\mathbf{S})^{-1}\mathbf{S}^T\mathbf{z}$ *is a sub-optimal approximation to* $\mathbf{c}$*,*

$$\|\tilde{\mathbf{d}} - \mathbf{c}\|_\infty \le C\|\boldsymbol{\eta}\|_\infty.$$

Stability of a DSRS

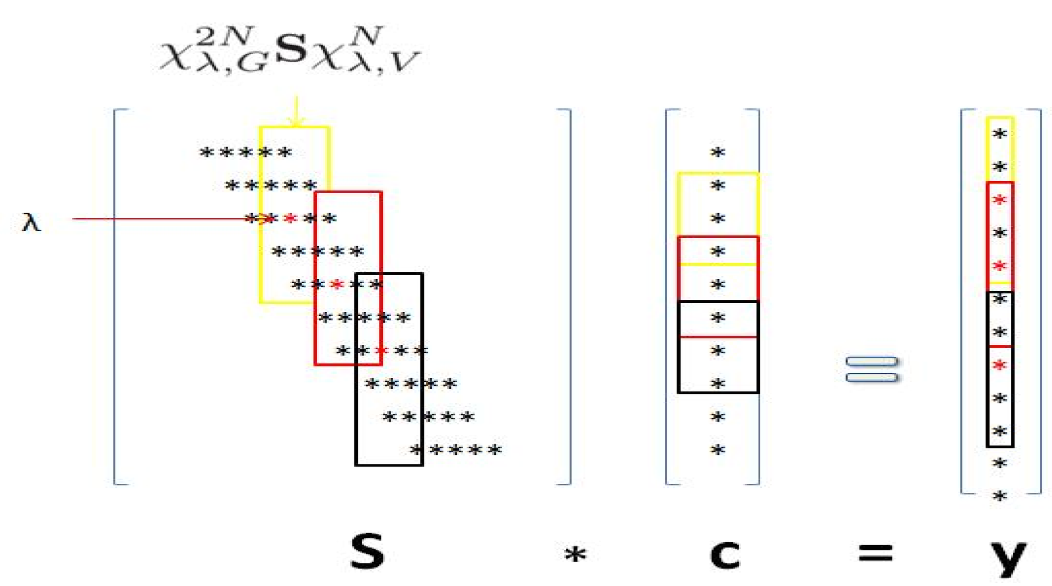
Define truncation operators $\chi_{\lambda'}^N$ by

$$\chi_{\lambda'}^N : (c(\lambda)) \longmapsto (c(\lambda)\chi_{B(\lambda',N)}(\lambda)),$$

where $B(\lambda', N)$ is the ball with center λ' and radius N.

Theorem 2. *The sensing matrix* $\mathbf{S}$ *has* ℓ^2*-stability if and only if its quasi-main submatrices* $\chi_\lambda^{2N}\mathbf{S}\chi_\lambda^N$*,* $\lambda \in G$*, have uniform* ℓ^2*-stability for large* N*, i.e., there exists* $A > 0$ *such that*

$$\|\chi_\lambda^{2N}\mathbf{S}\chi_\lambda^N\mathbf{c}\|_2 \ge A\|\chi_\lambda^N\mathbf{c}\|_2, \ \lambda \in G.$$



The above stability criterion could be pivotal for the design of a robust DSRS against supplement, replacement and impairment of agents, as we only need to verify the uniform stability of affected subsystems.

Distributed Algorithm

The DSRS does not have a central processor and it has huge amounts of agents and large number of innovative positions. So it is infeasible to find the pseudo-inverse of the sensing matrix $\mathbf{S}$, and then use it to obtain the suboptimal approximation $\tilde{\mathbf{d}} := (\mathbf{S}^T\mathbf{S})^{-1}\mathbf{S}^T\mathbf{z}$.

We introduce a distributed algorithm to find the suboptimal approximation $\tilde{\mathbf{d}}$. Our algorithm is reconstructing signal locally via subsystems and then patching solutions of subsystems to form an approximation.

$$\begin{cases} \mathbf{y}_{n;\lambda,N} = \mathbf{R}_{\lambda,N}\mathbf{z}_n, \\ \mathbf{y}_n = \dfrac{\sum_{\lambda\in G}\chi_\lambda^N \mathbf{y}_{n;\lambda,N}}{\sum_{\lambda\in G}\chi_{B(\lambda,N)}}, \\ \mathbf{d}_{n+1} = \mathbf{d}_n + \mathbf{y}_n, \\ \mathbf{z}_{n+1} = \mathbf{z}_n - \mathbf{S}\mathbf{y}_n. \end{cases}$$

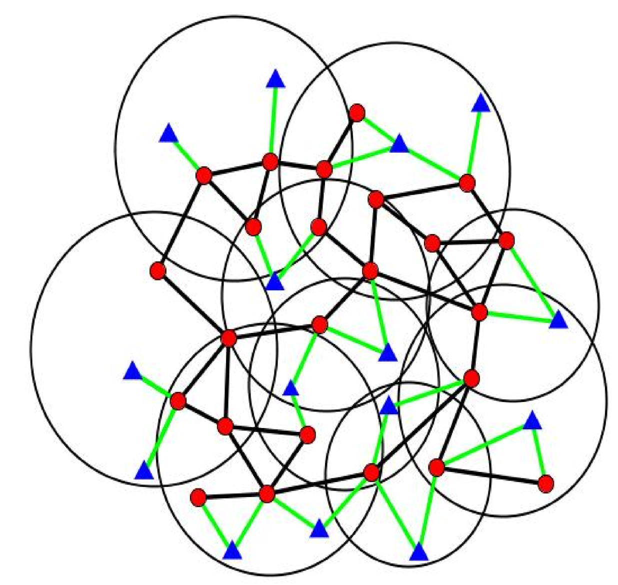


Figure 1: On the left is the distributed algorithm, where $\mathbf{R}_{\lambda,N} := (\chi_\lambda^{2N}\mathbf{S}^T\mathbf{S}\chi_\lambda^{2N})^{-1}\chi_\lambda^{2N}\mathbf{S}^T\chi_\lambda^{4N}$, $\mathbf{d}_0 = \mathbf{0}$ and $\mathbf{z}_0 = \mathbf{z}$ is the noisy sampling data. Illustrated on the right is overlapping smaller subsystems of our DSRS.

Simulations

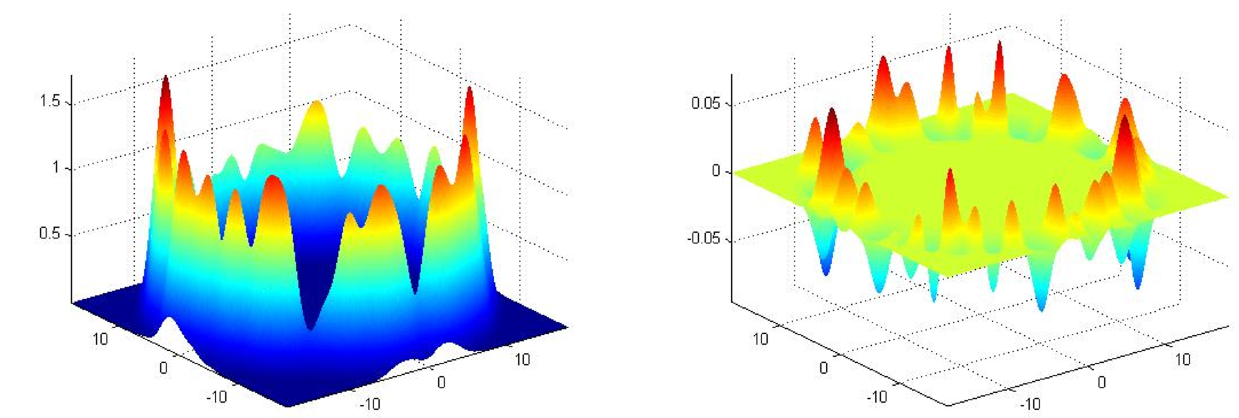


Figure 2: Agents are uniformly deployed on a circle, and innovative positions of the 2D Gaussian signal (plotted on the right) are almost uniformly near the circle. On the right is the difference between the original signal and the reconstructed signal from its noisy sampling data, where $N = 6$.

Main References

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