# WIENER'S LEMMA FOR INFINITE MATRICES II

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ABSTRACT. In this paper, we introduce a class of infinite matrices related to the Beurling algebra of periodic functions, and we show that it is an inverse-closed subalgebra of $\mathcal{B}(\ell^q_w)$, the algebra of all bounded linear operators on the weighted sequence space $\ell^q_w$, for any $1 \le q < \infty$ and any discrete Muckenhoupt $A_q$-weight $w$.

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## 1. INTRODUCTION

Let us begin this sequel to [44] by introducing a new class of infinite matrices,

$$\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d) := \Big\{ A := (a(i,j))_{i,j\in\mathbb{Z}^d} \ \Big|\ \|A\|_{\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)} < \infty \Big\}, \tag{1.1}$$

where $d \ge 1$, $|x|_\infty := \max(|x_1|, \ldots, |x_d|)$ for $x := (x_1, \ldots, x_d) \in \mathbb{R}^d$, and

$$\|A\|_{\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)} := \sum_{k\in\mathbb{Z}^d} \Big( \sup_{|i-j|_\infty \ge |k|_\infty} |a(i,j)| \Big).$$

It is observed that a Laurent matrix $A := (a(i-j))_{i,j\in\mathbb{Z}}$ associated with a sequence $a := (a(n))_{n\in\mathbb{Z}}$ belongs to $\mathcal{B}(\mathbb{Z}, \mathbb{Z})$ if and only if the Fourier series $\hat{a}(\xi) := \sum_{n\in\mathbb{Z}} a(n) \exp(-\sqrt{-1}\ n\xi)$ belongs to the Beurling algebra

$$A^*(\mathbb{T}) := \Big\{ \sum_{n=-\infty}^{\infty} a(n) e^{-\sqrt{-1}\ n\xi} \ \Big|\ \sum_{k=0}^{\infty} \sup_{|n|\ge k} |a(n)| < \infty \Big\}.$$

The algebra $A^*(\mathbb{T})$ was introduced by A. Beurling for establishing contraction properties of periodic functions [9], and was used in considering pointwise summability of Fourier series [11, 16, 17, 41, 49]. So the class $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ of infinite matrices can be interpreted as a noncommutative matrix extension of the *Beurling algebra* $A^*(\mathbb{T})$.

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Define the *Gröchenig-Schur class* $\mathcal{S}(\mathbb{Z}^d, \mathbb{Z}^d)$ by

$$\begin{aligned}\mathcal{S}(\mathbb{Z}^d, \mathbb{Z}^d) \ := \ & \Big\{ \big(a(i,j)\big)_{i,j\in\mathbb{Z}^d} \ \Big| \\ & \max\Big(\sup_{i\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}^d}|a(i,j)|, \sup_{j\in\mathbb{Z}^d}\sum_{i\in\mathbb{Z}^d}|a(i,j)|\Big) < \infty\Big\} \end{aligned} \tag{1.2}$$

[26, 38, 42, 44], and the *Gohberg-Baskakov-Sjöstrand class* $\mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d)$ by

$$\mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d) := \Big\{ \big(a(i,j)\big)_{i,j\in\mathbb{Z}^d} \ \Big| \ \sum_{k\in\mathbb{Z}^d}\big(\sup_{i-j=k}|a(i,j)|\big) < \infty\Big\} \tag{1.3}$$

[6, 21, 26, 33, 40, 44]. The above two classes of infinite matrices appeared in the study of Gabor time-frequency analysis, nonuniform sampling, and algebra of pseudo-differential operators etc (see [2, 4, 25, 29, 40, 43] for a sample of papers). From (1.1), (1.2) and (1.3) it follows that

$$\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d) \subset \mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d) \subset \mathcal{S}(\mathbb{Z}^d, \mathbb{Z}^d).$$

Hence any matrix in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ belongs to the Gröchenig-Schur class $\mathcal{S}(\mathbb{Z}^d, \mathbb{Z}^d)$ and also the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d)$.

An equivalent way of defining an element $A := (a(i,j))_{i,j\in\mathbb{Z}^d}$ in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ is the existence of a radially decreasing sequence $\{b(i)\}_{i\in\mathbb{Z}^d}$ such that

$$|a(i,j)| \le b(i-j) \quad \text{for all } i,j\in\mathbb{Z}^d,$$

$$\sum_{i\in\mathbb{Z}^d} b(i) < \infty,$$

and

$$b(i) = h(|i|_\infty) \text{ for some decreasing sequence } \{h(n)\}_{n=0}^\infty.$$

Therefore any infinite matrix in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ is dominated by a convolution operator associated with a summable, radial and (radially) decreasing sequence. We remark that any infinite matrix in the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d)$ is dominated by a convolution operator associated with a summable sequence [6, 21, 26, 33, 40, 44].

A positive sequence $w := (w(i))_{i\in\mathbb{Z}^d}$ is said to be a *discrete $A_q$-weight* for $1 < q < \infty$ if there exists a positive constant $C$ such that

$$\Big(N^{-d}\sum_{i\in a+[0,N-1]^d} w(i)\Big)\Big(N^{-d}\sum_{i\in a+[0,N-1]^d}(w(i))^{-1/(q-1)}\Big)^{q-1} \le C \tag{1.4}$$

hold for all $a \in \mathbb{Z}^d$ and $1 \le N \in \mathbb{Z}$, and to be a *discrete $A_1$-weight* if there exists a positive constant $C$ such that

$$N^{-d} \sum_{i\in a+[0,N-1]^d} w(i) \le C \inf_{i\in a+[0,N-1]^d} w(i) \tag{1.5}$$

hold for all $a \in \mathbb{Z}^d$ and $1 \le N \in \mathbb{Z}$ [19, 41]. The *discrete $A_q$-bound*, denoted by $A_q(w)$, is the smallest constant $C$ for which (1.4) holds when $1 < q < \infty$, and respectively for which (1.5) holds when $q = 1$. The positive sequences $((1 + |i|_\infty)^\alpha)_{i\in\mathbb{Z}^d}$ with $-d < \alpha < d(q - 1)$ if $1 < q < \infty$, and respectively with $-d < \alpha \le 0$ if $q = 1$, are discrete $A_q$-weights.

For $1 < q < \infty$, a positive locally-integrable function $w$ on $\mathbb{R}^d$ is said to be an *$A_q$-weight* if there exists a positive constant $C$ such that

$$\Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{-1/(q-1)}dx\Big)^{q-1} \le C \tag{1.6}$$

for all cubes $Q \subset \mathbb{R}^d$, where $|Q|$ denotes the Lebesgue measure of the cubic $Q$ [37]. Similarly for $q = 1$, a positive locally-integrable function $w$ is said to be an *$A_1$-weight* if there exists a positive constant $C$ such that

$$\frac{1}{|Q|}\int_Q w(y)dy \le Cw(x), \qquad x \in Q \tag{1.7}$$

for all cubes $Q \subset \mathbb{R}^d$ [15]. One may then verify that for $1 \le q < \infty$, a positive sequence $w := \big(w(i)\big)_{i\in\mathbb{Z}^d}$ is a discrete $A_q$-weight if and only if $\tilde{w}(x) := \sum_{i\in\mathbb{Z}^d} w(i)\chi_{[-1/2,1/2)^d}(x - i)$ is an $A_q$-weight, where $\chi_E$ is the characteristic function on a set $E$ [34].

For $1 \le q < \infty$ and a positive sequence $w := (w(i))_{i\in\mathbb{Z}^d}$ on $\mathbb{Z}^d$, let $\ell^q_w := \ell^q_w(\mathbb{Z}^d)$ be the space of all weighted $q$-summable sequences on $\mathbb{Z}^d$, i.e.,

$$\ell^q_w(\mathbb{Z}^d) := \Big\{(c(i))_{i\in\mathbb{Z}^d}\ \Big|\ \|c\|_{q,w} := \Big(\sum_{i\in\mathbb{Z}^d} |c(i)|^q w(i)\Big)^{1/q} < \infty\Big\}.$$

For the trivial weight $w_0$ (i.e. $w_0(i) = 1$ for all $i \in \mathbb{Z}^d$), we will use $\ell^q$ and $\|\cdot\|_q$ instead of $\ell^q_{w_0}$ and $\|\cdot\|_{q,w_0}$ for brevity. Define the *discrete maximal function* by

$$Mc(i) := \sup_{0\le N\in\mathbb{Z}} \frac{1}{(2N+1)^d} \sum_{k\in i+[-N,N]^d} |c(k)| \quad \text{for } c := (c(i))_{i\in\mathbb{Z}^d}.$$

Similar to the characterization of an $A_q$-weight on $\mathbb{R}^d$ via the standard maximal operator [35], the discrete $A_q$-weight can be characterized via the discrete maximal function on the weighted $\ell^q$ space. More precisely, a positive sequence $w := (w(i))_{i\in\mathbb{Z}^d}$ is a discrete $A_q$-weight if and only if the discrete maximal operator $c \longmapsto Mc$ is of weak-type $(\ell^q_w, \ell^q_w)$, i.e., there exists a positive constant $C$ such that

$$\sum_{Mc(i)\geq\alpha} w(i) \leq \frac{C}{\alpha^q}\|c\|^q_{q,w} \quad \text{for all } \alpha > 0 \text{ and } c \in \ell^p_w.$$

Moreover for $1 < q < \infty$, the discrete maximal operator $M$ is of strong type $(\ell^q_w, \ell^q_w)$ for a discrete $A_q$-weight $w$, i.e., there exists a positive constant $C'$ such that

$$\|Mc\|_{q,w} \leq C'\|c\|_{q,w} \quad \text{for all } c \in \ell^p_w.$$

The reader may refer to [19] for a complete account of the theory of weighted inequalities and its ramifications.

Now let us present our results for infinite matrices in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$. In Section 3, we establish the following algebraic properties for the class $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ of infinite matrices.

**Theorem 1.1.** *Let $1 \leq q < \infty$, and let $w$ be a discrete $A_q$-weight. Then $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ is a unital Banach algebra under matrix multiplication, and it is also a subalgebra of $\mathcal{B}(\ell^q_w)$, the algebra of all bounded linear operators on the weighted sequence space $\ell^q_w$.*

By Theorem 1.1, every infinite matrix $A \in \mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ defines a bounded operator on $\ell^q_w$ for any $1 \leq q < \infty$ and for any discrete $A_q$-weight $w$, i.e., there exists a positive constant $C$ such that

$$\|Ac\|_{q,w} \leq C\|c\|_{q,w} \quad \text{for all } c \in \ell^q_w,$$

see (3.16) for an estimate of the above constant $C$. Besides the above boundedness of an infinite matrix on the weighted sequence space $\ell^q_w$, it is natural to consider $\ell^q_w$-stability. Here for $1 \leq q < \infty$ and a positive sequence $w$ on $\mathbb{Z}^d$, we say that an infinite matrix $A$ has *$\ell^q_w$-stability* if there exists a positive constant $C$ such that

$$C^{-1}\|c\|_{q,w} \leq \|Ac\|_{q,w} \leq C\|c\|_{q,w} \quad \text{for all } c \in \ell^q_w.$$

The $\ell^q_w$-stability is one of the basic assumptions for infinite matrices arising in the study of spline approximation, Gabor time-frequency analysis, nonuniform sampling, and algebra of pseudo-differential operators etc (see [2, 4, 25, 29, 30, 40, 43, 44] and the references therein.) In Section 4, we establish the equivalence of $\ell^q_w$-stabilities of any infinite matrix in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ for different exponents $1 \leq q < \infty$ and for different discrete $A_q$-weights $w$.

**Theorem 1.2.** *Let $A \in \mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$. If $A$ has $\ell^q_w$-stability for some $1 \le q < \infty$ and for some discrete $A_q$-weight $w$, then it has $\ell^{q'}_{w'}$-stability for all $1 \le q' < \infty$ and for all discrete $A_{q'}$-weights $w'$.*

The reader may refer to [1, 39, 50] for the equivalence of unweighted $\ell^q_w$-stability of infinite matrices in the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d)$. If $A \in \mathcal{B}(\ell^q_w)$ has a *left inverse* $B \in \mathcal{B}(\ell^q_w)$, i.e., $BA = I$, then $A$ has $\ell^q_w$-stability. The converse is not true in general, unless $q = 2$. As an application of Theorem 1.2, we show that the converse holds for any infinite matrix $A$ in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$.

**Corollary 1.3.** *Let $1 \le q < \infty$, and let $w$ be a discrete $A_q$-weight. Then an infinite matrix in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ has $\ell^q_w$-stability if and only if it has a left inverse in $\mathcal{B}(\ell^q_w)$.*

Given a Banach algebra $\mathcal{B}$, a subalgebra $\mathcal{A}$ of $\mathcal{B}$ is said to be *inverse-closed* if $A \in \mathcal{A}$ and the inverse $A^{-1}$ of the element $A$ exists in $\mathcal{B}$ implies that $A^{-1} \in \mathcal{A}$ [20, 36, 48]. The next question following the $\ell^q_w$-stability of an infinite matrix in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ is whether its inverse, if exists in $\mathcal{B}(\ell^q_w)$, belongs to $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$, or in the other words, whether $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^q_w)$.

The inverse-closedness for the subalgebra of absolutely convergent Fourier series in the algebra of bounded periodic functions was first studied in [10, 20, 36, 51]. The inverse-closed property (=Wiener's lemma) has been established for infinite matrices satisfying various off-diagonal decay conditions, see [3, 5, 6, 18, 21, 24, 26, 30, 40, 42, 44] for a sample of papers. Inverse-closedness occurs under various names (such as spectral invariance, Wiener pair, local subalgebra) in many fields of mathematics, see the survey [22].

The inverse-closed property of non-commutative matrix subalgebras has been shown to be crucial for the well-localization of dual wavelet frames and dual Gabor frames [4, 25, 30, 31, 32], the algebra of pseudo-differential operators [23, 29, 40], the fast implementation in numerical analysis [13, 14, 28], and the local reconstruction in sampling theory [2, 43, 46].

In [24], Gröchenig and Klotz considered the forms of off-diagonal decay that can be inherited under matrix inversion and said that "*The answers so far mix art and hard mathematical work. The art is to guess a suitable decay condition, the work is then to prove that this decay condition is preserved under inversion.*" Particularly in our case, the art part is to find suitable algebras $\mathcal{A}$ of infinite matrices and the mathematical part is to prove the inverse-closedness of the algebra $\mathcal{A}$ in $\mathcal{B}(\ell^q_w)$. There are several approaches to prove the inverse-closedness of a subalgebra of $\mathcal{B}(\ell^q_w)$. Here are three of them: (i) the indirect approach, such as the Gelfand's technique [18, 20, 26]; (ii) the semi-direct approach, such as the bootstrap argument [30] and the derivation trick [24]; (iii) the direct approach, such as the commutator trick [6, 40], the decomposition technique based on the Bochner-Phillips theorem [6, 7, 21], and the asymptotic estimate technique [39, 42, 44]. Each approach has its advantages and limitations. For instance, the Gelfand technique and the asymptotic estimate technique work well for inverse-closed subalgebras of $\mathcal{B}(\ell^q)$ with $q = 2$, but they are not directly applicable for inverse-closed subalgebras of $\mathcal{B}(\ell^q)$ with $q \neq 2$. The commutator trick is applicable to establish Wiener's lemma for subalgebra of $\mathcal{B}(\ell^q), 1 \le q \le \infty$ [6, 40]. In Section 5, we combine the commutator trick, the asymptotic estimate technique and the equivalence of $\ell^q_w$-stability for different exponents $q$ and for different discrete $A_q$-weights $w$, and then establish Wiener's lemma for subalgebras of $\mathcal{B}(\ell^q_w)$, where $1 \le q < \infty$ and $w$ is a discrete $A_q$-weight.

**Theorem 1.4.** *Let $1 \le q < \infty$ and let $w$ be a discrete $A_q$-weight. Then $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^q_w)$.*

**Remark 1.5.** Let $1 \le q < \infty$, $w$ be a discrete $A_q$-weight, and $A = (a(i,j))_{i,j\in\mathbb{Z}^d} \in \mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ have a bounded inverse in $\mathcal{B}(\ell^q_w)$. Inspection of the proof of Theorem 1.4 shows that the norm $\|A^{-1}\|_{\mathcal{B}}$ of the inverse matrix $A^{-1}$ depends on the exponent $q$, the weight $w$ and the matrix $A$ not only through the discrete $A_q$-bound $A_q(w)$, the norm $\|A\|_{\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)}$ in the algebra $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$, the operator norm $\|A^{-1}\|_{\mathcal{B}(\ell^q_w)}$ in the operator algebra $\mathcal{B}(\ell^q_w)$, and also through a rather **implicit** quantity $N$ such that $A_q(w)^{2/q} \sum_{|\mathbf{k}|_\infty \ge \sqrt{N}/2} \sup_{|i-j|_\infty \ge |k|_\infty} |a(i,j)|$ is sufficiently small.

As an application of Theorem 1.4, we obtain Wiener's lemma for the Beurling algebra $A^*(\mathbb{T})$ of periodic functions [8].

**Corollary 1.6.** *If $f \in A^*(\mathbb{T})$ and $f(\xi) \neq 0$ for all $\xi \in \mathbb{R}$ then $1/f \in A^*(\mathbb{T})$.*

As applications of Theorems 1.2 and 1.4, we establish the equivalence between the $\ell^q_w$-stability of an infinite matrix in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ and the existence of its left inverse in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$.

**Corollary 1.7.** *Let $1 \le q < \infty$, and let $w$ be a discrete $A_q$-weight. Then an infinite matrix in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ has $\ell^q_w$-stability if and only if it has a left inverse in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$.*

## 2. A class of infinite matrices

In this section, we introduce a class of infinite matrices with off-diagonal decay, which includes the class $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ in the Introduction as a special case.

A *weight matrix on* $\mathbb{Z}^d \times \mathbb{Z}^d$, or a *weight matrix* for brevity, is a positive matrix $u := (u(i,j))_{i,j\in\mathbb{Z}^d}$ with each entry not less than one, i.e., $u(i,j) \ge 1$ for all $i, j \in \mathbb{Z}^d$. For $1 \le p \le \infty$ and a weight matrix $u := (u(i,j))_{i,j\in\mathbb{Z}^d}$, define

$$\mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) := \Big\{ A := (a(i,j))_{i,j\in\mathbb{Z}^d} \ \Big| \ \|A\|_{\mathcal{B}_{p,u}(\mathbb{Z}^d,\mathbb{Z}^d)} < \infty \Big\}, \tag{2.1}$$

where

$$\|A\|_{\mathcal{B}_{p,u}(\mathbb{Z}^d,\mathbb{Z}^d)} := \Big\| \Big( \sup_{|i-j|_\infty \ge |k|_\infty} |a(i,j)|u(i,j) \Big)_{k\in\mathbb{Z}^d} \Big\|_p. \tag{2.2}$$

If $p = 1$ and $u \equiv 1$ (i.e., all entries of the weight matrix $u$ are equal to 1), then

$$\mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) = \mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d) \quad \text{and} \quad \|\cdot\|_{\mathcal{B}_{p,u}(\mathbb{Z}^d,\mathbb{Z}^d)} = \|\cdot\|_{\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)}.$$

In this paper, we use $\mathcal{B}_{p,u}, \mathcal{B}, \|\cdot\|_{\mathcal{B}_{p,u}}, \|\cdot\|_{\mathcal{B}}$ instead of $\mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d), \mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$, $\|\cdot\|_{\mathcal{B}_{p,u}(\mathbb{Z}^d,\mathbb{Z}^d)}, \|\cdot\|_{\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)}$ for brevity.

**Remark 2.1.** Let $1 \le p \le \infty$ and $u$ be a weight matrix. Define the *Gröchenig-Schur class* $\mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ of infinite matrices by

$$\mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) := \Big\{ A := (a(i,j))_{i,j\in\mathbb{Z}^d} \ \Big| \ \|A\|_{\mathcal{S}_{p,u}(\mathbb{Z}^d,\mathbb{Z}^d)} < \infty \Big\},$$

where

$$\|A\|_{\mathcal{S}_{p,u}(\mathbb{Z}^d,\mathbb{Z}^d)} := \max\Big( \sup_{i\in\mathbb{Z}^d} \big\| \big(a(i,j)u(i,j)\big)_{j\in\mathbb{Z}^d} \big\|_p, \sup_{j\in\mathbb{Z}^d} \big\| \big(a(i,j)u(i,j)\big)_{i\in\mathbb{Z}^d} \big\|_p \Big)$$

[26, 38, 42, 44]. For $p = 1$, the class $\mathcal{S}_{1,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ was introduced by Schur [38] for weight matrices $u := (w(i)/w(j))_{i,j\in\mathbb{Z}^d}$ generated by positive sequences $w := (w(i))_{i\in\mathbb{Z}^d}$, and by Gröchenig and Leinert [26] for weight matrices $u := (v(i-j))_{i,j\in\mathbb{Z}^d}$ associated with positive functions $v$ on $\mathbb{R}^d$. For $1 \le p \le \infty$, the class $\mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ was introduced by Sun for polynomial weights $u := ((1 + |i-j|_\infty)^\alpha)_{i,j\in\mathbb{Z}^d}$ with $\alpha > d(1-1/p)$ in [42] and for any weight matrix $u$ in [44]. From the above definition of the Gröchenig-Schur class $\mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$, the following inclusion follows:

$$\mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) \subset \mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) \tag{2.3}$$

for any $1 \le p \le \infty$ and for any weight matrix $u$.

**Remark 2.2.** Let $1 \le p \le \infty$ and $u$ be a weight matrix. Define the *Gohberg-Baskakov-Sjöstrand class* $\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ of infinite matrices by

$$\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) := \Big\{ A := (a(i,j))_{i,j\in\mathbb{Z}^d} \ \Big| \ \|A\|_{\mathcal{C}_{p,u}(\mathbb{Z}^d,\mathbb{Z}^d)} < \infty \Big\},$$

where

$$\|A\|_{\mathcal{C}_{p,u}(\mathbb{Z}^d,\mathbb{Z}^d)} := \Big\| \Big( \sup_{i-j=k} \big(|a(i,j)|u(i,j)\big) \Big)_{k\in\mathbb{Z}^d} \Big\|_p$$

[6, 21, 26, 27, 33, 40, 44]. For $p = 1$ and the trivial weight matrix $u_0$ (i.e., $u_0(i,j) = 1$ for all $i, j \in \mathbb{Z}^d$), the class $\mathcal{C}_{1,u_0}(\mathbb{Z}^d, \mathbb{Z}^d)$ was introduced by Gohberg, Kaashoek, and Woerdeman [21] as a generalization of the class of Laurent matrices associated with summable sequences. It was reintroduced by Sjöstrand [40] in considering an algebra of pseudo-differential operators. For $p = 1$ and nontrivial weight matrices $u := (v(i-j))_{i,j\in\mathbb{Z}^d}$ associated with positive functions $v$ on $\mathbb{R}^d$, the class $\mathcal{C}_{1,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ was introduced and studied by Baskakov [6] and Kurbatov [33] independently, see also [26]. The above definition of the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ is given by Sun [44] for any $1 \le p \le \infty$ and any weight matrix $u$. From the definition of the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$, we have the following inclusion:

$$\mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) \subset \mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) \tag{2.4}$$

for any $1 \le p \le \infty$ and for any weight matrix $u$.

**Remark 2.3.** The inclusions (2.3) and (2.4) become equalities when $p = \infty$, i.e.,

$$\mathcal{B}_{\infty,u}(\mathbb{Z}^d, \mathbb{Z}^d) = \mathcal{C}_{\infty,u}(\mathbb{Z}^d, \mathbb{Z}^d) = \mathcal{S}_{\infty,u}(\mathbb{Z}^d, \mathbb{Z}^d) =: \mathcal{J}_u(\mathbb{Z}^d, \mathbb{Z}^d) \tag{2.5}$$

The class $\mathcal{J}_u(\mathbb{Z}^d, \mathbb{Z}^d)$ of infinite matrices is usually known as the *Jaffard class*, [6, 14, 24, 26, 30, 42, 44]. The Jaffard class $\mathcal{J}_u(\mathbb{Z}^d, \mathbb{Z}^d)$ with polynomial weight matrix $u := ((1 + |i-j|)^\alpha)_{i,j\in\mathbb{Z}^d}$ was introduced by Jaffard [30] in considering wavelets on an open domain. The Jaffard class $\mathcal{J}_u(\mathbb{Z}^d, \mathbb{Z}^d)$ with weight matrices $u := (v(i-j))_{i,j\in\mathbb{Z}^d}$ associated with positive functions $v$ on $\mathbb{R}^d$ was introduced by Baskakov [6] independently, and later applied nontrivially in the study of localization of frames [26], adaptive computation [14], and nonuniform sampling [43].

For the class $\mathcal{B}_{p,u}$ of infinite matrices, we have

**Proposition 2.4.** *Let $\alpha \in \mathbb{C}$, $1 \le p \le \infty$, $u := (u(i,j))_{i,j\in\mathbb{Z}^d}$ be a weight matrix, and let $A := ((a(i,j))_{i,j\in\mathbb{Z}^d}$ and $B := (b(i,j))_{i,j\in\mathbb{Z}^d}$ belong to $\mathcal{B}_{p,u}$. Then*

(i) $\|A + B\|_{\mathcal{B}_{p,u}} \le \|A\|_{\mathcal{B}_{p,u}} + \|B\|_{\mathcal{B}_{p,u}}$.
(ii) $\|\alpha A\|_{\mathcal{B}_{p,u}} = |\alpha|\|A\|_{\mathcal{B}_{p,u}}$.

(iii) $\|A^*\|_{\mathcal{B}_{p,u}} = \|A\|_{\mathcal{B}_{p,u}}$ *if* $u(i,j) = u(j,i)$ *for all* $i, j \in \mathbb{Z}^d$, *where* $A^* := \big(\overline{a(j,i)}\big)_{i,j\in\mathbb{Z}^d}$ *is the conjugate transpose of the matrix* $A$.

(iv) $\|A\|_{\mathcal{B}_{p,u}} \le \|B\|_{\mathcal{B}_{p,u}}$ *if* $|A| \le |B|$, *i.e.,* $|a(i,j)| \le |b(i,j)|$ *for all* $i, j \in \mathbb{Z}^d$.

All conclusions in the above proposition follow directly from (2.1) and (2.2). From the conclusions (i) and (ii) of the above proposition, we see that $\|\cdot\|_{\mathcal{B}_{p,u}}$ is a norm on the class $\mathcal{B}_{p,u}$ of infinite matrices. The properties in the conclusion (iv) is usually known as the *solidness* of the matrix norm $\|\cdot\|_{\mathcal{B}_{p,u}}$.

## 3. Algebraic properties

In this section, we establish some algebraic properties for the class $\mathcal{B}_{p,u}$ of infinite matrices and give a proof of Theorem 1.1.

Let us first recall the concept of a $p$-submultiplicative weight matrix $u$ [26, 44, 45]. For $1 \le p \le \infty$, a weight matrix $u := (u(i,j))_{i,j\in\mathbb{Z}^d}$ is said to be *$p$-submultiplicative* if there exists another weight matrix $v := (v(i,j))_{i,j\in\mathbb{Z}^d}$ such that

$$v(i,j) \ge 1 \quad \text{for all } i, j \in \mathbb{Z}^d, \tag{3.1}$$

$$u(i,j) \le u(i,k)v(k,j) + v(i,k)u(k,j) \quad \text{for all } i, j, k \in \mathbb{Z}^d, \tag{3.2}$$

and

$$C_p(v,u) := \Big\| \Big( \sup_{|i-j|_\infty \ge |k|_\infty} \big( v(i,j)(u(i,j))^{-1} \big) \Big) \Big\|_{p/(p-1)} < \infty. \tag{3.3}$$

For $p = 1$, we simply say that a weight matrix is *submultiplicative* instead of 1-submultiplicative. We call the weight matrix $v$ satisfying (3.1), (3.2) and (3.3) a *companion weight matrix* of the $p$-submultiplicative weight matrix $u$. Denote by $C(u)$ the set of all companion weights of a $p$-submultiplicative weight matrix $u$, and define the *$p$-submultiplicative bound* $M_p(u)$ by $M_p(u) := \inf_{v\in C(u)} C_p(v,u)$. One may verify that $C(u)$ is a convex set and the infinimum of $C_p(v,u)$ in the set $C(u)$ can be attained for some companion weight matrix $v$. So from now on, except stated explicitly, we **always** assume that the companion weight $v$ of a $p$-submultiplicative weight matrix $u$ is the one satisfying

$$M_p(u) = C_p(v,u). \tag{3.4}$$

**Remark 3.1.** From the definitions of $p$-submultiplicative weight matrices on $\mathbb{Z}^d \times \mathbb{Z}^d$, we have the following:

(i) A $p$-submultiplicative weight matrix is $q$-submultiplicative for all $1 \le q \le p$.

(ii) A necessary condition for a weight matrix $u := (u(i,j))_{i,j\in\mathbb{Z}^d}$ to be $p$-submultiplicative is $u(i,j) \le Cu(i,k)u(k,j)$ for all $i,j,k \in \mathbb{Z}^d$ and for some positive constant $C$. When $p=1$, the above necessary condition is also a sufficient condition [26].

(iii) Let $1 \le p \le \infty, \delta \in (0,1)$, and let $\alpha$ be a number with the property that $\alpha > d - d/p$ if $1 < p \le \infty$, and $\alpha \ge 0$ if $p=1$. Then the Laurent matrices $p_\alpha := \big((1+|i-j|_\infty)^\alpha\big)_{i,j\in\mathbb{Z}^d}$ generated by the polynomial weight $(1+|x|_\infty)^\alpha$, and $e_\delta := \big(\exp(|i-j|_\infty^\delta)\big)_{i,j\in\mathbb{Z}^d}$ generated by the sub-exponential weight $\exp(|x|_\infty^\delta)$, are $p$-submultiplicative [44].

Now we state the main result of this section, an extension of Theorem 1.1.

**Theorem 3.2.** *Let $1 \le p \le \infty, 1 \le q < \infty$, $u$ be a $p$-submultiplicative weight matrix with the $p$-submultiplicative bound $M_p(u)$, and let $w$ be a discrete $A_q$-weight with the $A_q$-bound $A_q(w)$. Then the following statements hold.*

(i) *If $v$ is a companion weight matrix of the $p$-submultiplicative weight matrix $u$, then*

$$\|AB\|_{\mathcal{B}_{p,u}} \le 2^{2/p}5^{(d-1)/p}\big(\|A\|_{\mathcal{B}_{p,u}}\|B\|_{\mathcal{B}_{1,v}} + \|A\|_{\mathcal{B}_{1,v}}\|B\|_{\mathcal{B}_{p,u}}\big) \tag{3.5}$$

*for all $A,B \in \mathcal{B}_{p,u}$.*

(ii) *$\mathcal{B}_{p,u}$ is (and hence $\mathcal{B}$ is also) an algebra. Moreover*

$$\|AB\|_{\mathcal{B}_{p,u}} \le 2^{1+2/p}5^{(d-1)/p}M_p(u)\|A\|_{\mathcal{B}_{p,u}}\|B\|_{\mathcal{B}_{p,u}} \quad \text{for all } A,B \in \mathcal{B}_{p,u}. \tag{3.6}$$

(iii) *$\mathcal{B}_{p,u}$ is a subalgebra of $\mathcal{B}$. Moreover*

$$\|A\|_{\mathcal{B}} \le M_p(u)\|A\|_{\mathcal{B}_{p,u}} \quad \text{for all } A \in \mathcal{B}_{p,u}. \tag{3.7}$$

(iv) *$\mathcal{B}_{p,u}$ is (and hence $\mathcal{B}$ is also) a subalgebra of $\mathcal{B}(\ell^q_w)$. Moreover*

$$\|Ac\|_{q,w} \le 2^{2d}3^{d/q}(A_q(w))^{1/q}M_p(u)\|A\|_{\mathcal{B}_{p,u}}\|c\|_{q,w} \tag{3.8}$$

*for all $A \in \mathcal{B}_{p,u}$ and $c \in \ell^q_w$.*

Before we give the proof of the above theorem, let us next make some remarks on the unital Banach algebra property of the algebra $\mathcal{B}_{p,u}$, on the equality of spectral radii in the algebras $\mathcal{B}_{p,u}$ and $\mathcal{B}_{1,v}$, and on the inclusion $\mathcal{B}_{p,u} \subset \mathcal{B}(\ell^q_w)$.

**Remark 3.3.** For $1 \le p \le \infty$ and a $p$-submultiplicative weight matrix $u := (u(i,j))_{i,j\in\mathbb{Z}^d}$, following the standard procedure [20, 36] we define $\|A\|'_{\mathcal{B}_{p,u}} := \sup_{\|B\|_{\mathcal{B}_{p,u}}=1}\|AB\|_{\mathcal{B}_{p,u}}$ for $A \in \mathcal{B}_{p,u}$. Then

$$\|AB\|'_{\mathcal{B}_{p,u}} \le \|A\|'_{\mathcal{B}_{p,u}}\|B\|'_{\mathcal{B}_{p,u}} \quad \text{for all } A,B \in \mathcal{B}_{p,u}.$$

If the weight matrix $u$ further satisfies

$$M := \sup_{i\in\mathbb{Z}^d} u(i,i) < \infty, \tag{3.9}$$

then the identity matrix $I$ belongs to $\mathcal{B}_{p,u}$, and the norms $\|\cdot\|_{\mathcal{B}_{p,u}}$ and $\|\cdot\|'_{\mathcal{B}_{p,u}}$ on $\mathcal{B}_{p,u}$ are equivalent to each other, because

$$M^{-1}\|A\|_{\mathcal{B}_{p,u}} \le \|A\|'_{\mathcal{B}_{p,u}} \le 2^{1+2/p}5^{(d-1)/p}M_p(u)\|A\|_{\mathcal{B}_{p,u}} \quad \text{for all } A \in \mathcal{B}_{p,u}$$

by the conclusion (ii) of Theorem 3.2 and the fact that $\|I\|_{\mathcal{B}_{p,u}} = M$. Therefore if $1 \le p \le \infty$ and $u$ is a $p$-submultiplicative weight matrix satisfying (3.9), then the class $\mathcal{B}_{p,u}$ of infinite matrices endowed with the norm $\|\cdot\|'_{\mathcal{B}_{p,u}}$ becomes a unital Banach algebra.

**Remark 3.4.** Let $1 \le p \le \infty$, $u$ be a $p$-submultiplicative weight matrix satisfying (3.9), and $v$ be its companion weight matrix. If the companion weight matrix $v$ is submultiplicative, then both $\mathcal{B}_{p,u}$ and $\mathcal{B}_{1,v}$ are algebras by the conclusion (ii) of Theorem 3.2, and $\mathcal{B}_{p,u}$ is a subalgebra of $\mathcal{B}_{1,v}$ since

$$\|A\|_{\mathcal{B}_{1,v}} \le C_p(v,u)\|A\|_{\mathcal{B}_{p,u}} \quad \text{for all } A \in \mathcal{B}_{p,u}. \tag{3.10}$$

Applying (3.10) with $A$ replaced by $A^n$ and then taking $n$-th roots and the limit as $n \to \infty$ yields

$$\rho_{\mathcal{B}_{1,v}}(A) := \limsup_{n\to\infty}(\|A^n\|_{\mathcal{B}_{1,v}})^{1/n} \le \limsup_{n\to\infty}(\|A^n\|_{\mathcal{B}_{p,u}})^{1/n} =: \rho_{\mathcal{B}_{p,u}}(A).$$

From the conclusion (i) of Theorem 3.2 it follows that

$$\|A^{2n}\|_{\mathcal{B}_{p,u}} \le 2^{1+2/p}5^{(d-1)/p}\|A^n\|_{\mathcal{B}_{p,u}}\|A^n\|_{\mathcal{B}_{1,v}}.$$

Taking $n$-th roots on both sides of the above inequality and then letting $n \to \infty$ lead to the inequality $\rho_{\mathcal{B}_{p,u}}(A) \le \rho_{\mathcal{B}_{1,v}}(A)$. This implies that if $u$ is a $p$-submultiplicative weight matrix and its companion weight matrix $v$ is submultiplicative, then the spectral radii $\rho_{\mathcal{B}_{p,u}}(A)$ and $\rho_{\mathcal{B}_{1,v}}(A)$ are the same for any $A \in \mathcal{B}_{p,u}$, i.e., $\rho_{\mathcal{B}_{1,v}}(A) = \rho_{\mathcal{B}_{p,u}}(A)$ for all $A \in \mathcal{B}_{p,u}$. The above procedure to establish the equality of spectral radii in the algebras $\mathcal{B}_{p,u}$ and $\mathcal{B}_{1,v}$ from the inequality in the conclusion (i) of Theorem 3.2 is known as *Brandenburg's trick* [12, 24]. Another technique to prove the equality of spectral radii in two algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ with the same unit element is by showing that

$$\|A\|_{\mathcal{A}_2} \le C\|A\|_{\mathcal{A}_1} \tag{3.11}$$

and

$$\|A^2\|_{\mathcal{A}_1} \le C\|A\|^{1+\theta}_{\mathcal{A}_1}\|A\|^{1-\theta}_{\mathcal{A}_2} \quad \text{for all } A \in \mathcal{A}_1, \tag{3.12}$$

where $\|\cdot\|_{\mathcal{A}_1}$ and $\|\cdot\|_{\mathcal{A}_2}$ are norms in the algebra $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, and where $C \in (0,\infty)$ and $\theta \in [0,1)$ are constants independent of

$A \in \mathcal{A}$. The estimates in (3.11) and (3.12) for $\mathcal{A}_2 = \mathcal{B}(\ell^2)$ and $\mathcal{A}_1 = \mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ or $\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ are established in [42, 44], while those for $\mathcal{A}_2 = \mathcal{B}(\ell^2)$ and $\mathcal{A}_1 = \mathcal{B}_{p,u}, 1 \le p \le \infty$, are given in Lemma 5.3.

**Remark 3.5.** The conclusion (iv) of Theorem 3.2 about the boundedness of an infinite matrix in $\mathcal{B}$ on the weighted sequence space $\ell^q_w$ is a simplified discrete version of the second conclusion in [41, Proposition 2 of Chapter 10]. The reader may refer to [27, Lemma 3.1] for a general result on the boundedness of an infinite matrix on sequence spaces.

We conclude this section by giving the proof of Theorem 3.2.

*Proof of Theorem 3.2.* **(i)**: Let $1 \le p \le \infty$, $u$ be a $p$-submultiplicative weight matrix, and let $v$ be a companion weight matrix of the weight matrix $u$. Take $A := (a(i,j))_{i,j\in\mathbb{Z}^d}$ and $B := (b(i,j))_{i,j\in\mathbb{Z}^d}$ in $\mathcal{B}_{p,u}$, and write $AB := (c(i,j))_{i,j\in\mathbb{Z}^d}$. Then it follows from (3.2) that

$$\begin{aligned} |c(i,j)|u(i,j) &= \Big|\sum_{k\in\mathbb{Z}^d} a(i,k)b(k,j)\Big|u(i,j) \\ &\le \sum_{k\in\mathbb{Z}^d} |a(i,k)|u(i,k)|b(k,j)|v(k,j) \\ &\quad + \sum_{k\in\mathbb{Z}^d} |a(i,k)|v(i,k)|b(k,j)|u(k,j) \quad \text{for all } i,j \in \mathbb{Z}^d. \end{aligned} \tag{3.13}$$

For $1 \le p < \infty$, we obtain from (3.13) that

$$
\begin{aligned}
& \sum_{k\in\mathbb{Z}^d} |a(i,k)|u(i,k)|b(k,j)|v(k,j) \\
\le \ & \Big(\sum_{k'\in\mathbb{Z}^d} (|a(i,k')|u(i,k'))^p |b(k',j)|v(k',j)\Big)^{1/p} \\
& \quad \times \Big(\sum_{k''\in\mathbb{Z}^d} |b(k'',j)|v(k'',j)\Big)^{(p-1)/p} \\
\le \ & \big(\|B\|_{\mathcal{B}_{1,v}}\big)^{(p-1)/p} \Big\{ \Big(\sum_{k'\in\mathbb{Z}^d} |b(k',j)|v(k',j)\Big) \\
& \qquad \times \Big(\sup_{|i'-j'|_\infty \ge |i-j|_\infty/2} \big(|a(i',j')|u(i',j')\big)^p\Big) \\
& \quad + \Big(\sup_{|i'-j'|_\infty \ge |i-j|_\infty/2} \big(|b(i',j')|v(i',j')\big)\Big) \\
& \qquad \times \Big(\sum_{k'\in\mathbb{Z}^d} \big(|a(i,k')|u(i,k')\big)^p\Big)\Big\}^{1/p} \\
\le \ & \big(\|B\|_{\mathcal{B}_{1,v}}\big)^{(p-1)/p} \Big\{ \|B\|_{\mathcal{B}_{1,v}} \Big(\sup_{|i'-j'|_\infty \ge |i-j|_\infty/2} \big(|a(i',j')|u(i',j')\big)^p\Big) \\
& \quad + \big(\|A\|_{\mathcal{B}_{p,u}}\big)^p \Big(\sup_{|i'-j'|_\infty \ge |i-j|_\infty/2} \big(|b(i',j')|v(i',j')\big)\Big)\Big\}^{1/p},
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{k\in\mathbb{Z}^d} |a(i,k)|v(i,k)|b(k,j)|u(k,j) \\
\le \ & \big(\|A\|_{\mathcal{B}_{1,v}}\big)^{(p-1)/p} \Big\{ \|A\|_{\mathcal{B}_{1,v}} \Big(\sup_{|i'-j'|_\infty \ge |i-j|_\infty/2} \big(|b(i',j')|u(i',j')\big)^p\Big) \\
& \quad + \big(\|B\|_{\mathcal{B}_{p,u}}\big)^p \Big(\sup_{|i'-j'|_\infty \ge |i-j|_\infty/2} \big(|a(i',j')|v(i',j')\big)\Big)\Big\}^{1/p}.
\end{aligned}
$$

Combining the above two estimates with (3.13) leads to

$$\begin{aligned}
\|AB\|_{\mathcal{B}_{p,u}} &= \Big\|\Big(\sup_{|i-j|_\infty\ge |k|_\infty}\big(|c(i,j)|u(i,j)\big)\Big)_{k\in\mathbb{Z}^d}\Big\|_p\\
&\le \big(\|B\|_{\mathcal{B}_{1,v}}\big)^{(p-1)/p}\Big\{\|B\|_{\mathcal{B}_{1,v}}\\
&\qquad\times\Big\|\Big(\sup_{|i-j|_\infty\ge |k|_\infty/2}\big(|a(i',j')|u(i',j')\big)^p\Big)_{k\in\mathbb{Z}^d}\Big\|_1\\
&\qquad+\big(\|A\|_{\mathcal{B}_{p,u}}\big)^p\Big\|\Big(\sup_{|i-j|_\infty\ge |k|_\infty/2}\big(|b(i',j')|v(i',j')\big)\Big)_{k\in\mathbb{Z}^d}\Big\|_1\Big\}^{1/p}\\
&\quad+\big(\|A\|_{\mathcal{B}_{1,v}}\big)^{(p-1)/p}\Big\{\|A\|_{\mathcal{B}_{1,v}}\\
&\qquad\times\Big\|\Big(\sup_{|i-j|_\infty\ge |k|_\infty/2}\big(|b(i',j')|u(i',j')\big)^p\Big)_{k\in\mathbb{Z}^d}\Big\|_1\\
&\qquad+\big(\|B\|_{\mathcal{B}_{p,u}}\big)^p\Big\|\Big(\sup_{|i-j|_\infty\ge |k|_\infty/2}\big(|a(i',j')|v(i',j')\big)\Big)_{k\in\mathbb{Z}^d}\Big\|_1\Big\}^{1/p}\\
&\le 2^{2/p}5^{(d-1)/p}\Big(\|A\|_{\mathcal{B}_{p,u}}\|B\|_{\mathcal{B}_{1,v}}+\|A\|_{\mathcal{B}_{1,v}}\|B\|_{\mathcal{B}_{p,u}}\Big),
\end{aligned}$$

where we have used the fact that

(3.14)
$$\Big\|\Big(\sup_{|i-j|_\infty\ge |k|_\infty/N}|a(i,j)|\Big)\Big\|_1\le N(2N+1)^{d-1}\Big\|\Big(\sup_{|i-j|_\infty\ge |k|_\infty}|a(i,j)|\Big)\Big\|_1$$

for any integer $N\ge 1$ and $A:=(a(i,j))\in\mathcal{B}$. This proves (3.5) for $1\le p<\infty$.

For $p=\infty$, it follows from (3.13) that

$$\|AB\|_{\mathcal{B}_{\infty,u}}\le \|A\|_{\mathcal{B}_{\infty,u}}\|B\|_{\mathcal{B}_{1,v}}+\|A\|_{\mathcal{B}_{1,v}}\|B\|_{\mathcal{B}_{\infty,u}}.$$

Hence (3.5) for $p=\infty$ is proved.

**(ii)** Let $v$ be the companion weight matrix of the $p$-submultiplicative weight $u$ that satisfies (3.1)–(3.4). Then

$$\|A\|_{\mathcal{B}_{1,v}}\le M_p(u)\|A\|_{\mathcal{B}_{p,u}}\quad\text{for all } A\in\mathcal{B}_{p,u}, \tag{3.15}$$

because

$$\begin{aligned}
\sup_{|i-j|_\infty\ge|k|_\infty}|a(i,j)|v(i,j) &\le \Big(\sup_{|i'-j'|_\infty\ge|k|_\infty}|a(i',j')|u(i',j')\Big)\\
&\quad\times\Big(\sup_{|i'-j'|_\infty\ge|k|_\infty}v(i',j')(u(i',j'))^{-1}\Big)
\end{aligned}$$

hold for all $k\in\mathbb{Z}^d$. Combining (3.5) and (3.15) proves (3.6).

**(iii)** Let $v$ be the companion weight matrix of the $p$-submultiplicative weight $u$ that satisfies (3.1)–(3.4). Then

$$\|A\|_{\mathcal{B}} \le \|A\|_{\mathcal{B}_{1,v}} \quad \text{for all } A \in \mathcal{B}_{1,v}$$

by (3.1) for the weight matrix $v$. This together with (3.15) gives (3.7) and hence proves the conclusion (iii).

**(iv)** By (iii), it suffices to prove

$$\|Ac\|_{q,w} \le 2^{2d} 3^{d/q} (A_q(w))^{1/q} \|A\|_{\mathcal{B}} \|c\|_{q,w} \tag{3.16}$$

for all $A := (a(i,j))_{i,j\in\mathbb{Z}^d} \in \mathcal{B}$ and $c \in \ell^q_w$. Set $h(n) := \sup_{|i-j|_\infty \ge n} |a(i,j)|$. Then $\{h(n)\}_{n=0}^\infty$ is a decreasing sequence, i.e., $h(n+1) \le h(n)$ for all $n \ge 0$, and

$$\begin{aligned}
\sum_{l=1}^\infty h(2^{l-1}) 2^{(l+1)d} & \le 2^{2d} h(1) + 2^{d+2} \sum_{l=2}^\infty \Big( \sum_{2^{l-2} < s \le 2^{l-1}} h(s) \Big) 2^{l(d-1)} \\
& \le 2^{2d} h(1) + 2^{3d} \sum_{s=2}^\infty h(s) s^{d-1} \\
& \le 2^{2d} h(1) + 2^{2d} d^{-1} \sum_{s=2}^\infty \sum_{k\in\mathbb{Z}^d \text{ with } |k|_\infty = s} h(|k|_\infty) \\
& \le 2^{2d} (\|A\|_{\mathcal{B}} - h(0)).
\end{aligned}$$

For $1 < q < \infty$ and a discrete $A_q$-weight $w$,

$$\begin{aligned}
\|Ac\|_{q,w} & \le \Big\{ \sum_{i\in\mathbb{Z}^d} \Big( \sum_{j\in\mathbb{Z}^d} h(|i-j|_\infty) |c(j)| \Big)^q w(i) \Big\}^{1/q} \\
& \le h(0) \Big\{ \sum_{i\in\mathbb{Z}^d} |c(i)|^q w(i) \Big\}^{1/q} + \Big\{ \sum_{i\in\mathbb{Z}^d} w(i) \Big( \sum_{l=1}^\infty h(2^{l-1}) 2^{(l+1)d} \Big)^{q-1} \\
& \quad \times \Big( \sum_{l=1}^\infty h(2^{l-1}) 2^{-(l+1)d(q-1)} \Big( \sum_{2^{l-1} \le |i-j|_\infty < 2^l} |c(j)| \Big)^q \Big) \Big\}^{1/q}.
\end{aligned}$$

Thus

$$\begin{aligned}\|Ac\|_{q,w} &\le h(0)\|c\|_{q,w} + 2^{2d(q-1)/q}(\|A\|_{\mathcal B}-h(0))^{(q-1)/q}\\ &\quad \times\Big\{\sum_{i\in\mathbb Z^d}\sum_{l=1}^\infty w(i)h(2^{l-1})2^{-(l+1)d(q-1)}\\ &\quad \times\Big(\sum_{2^{l-1}\le|i-j|_\infty<2^l}|c(j)|^q w(j)\Big)\\ &\quad \times\Big(\sum_{2^{l-1}\le|i-j'|_\infty<2^l}(w(j'))^{-1/(q-1)}\Big)^{q-1}\Big\}^{1/q}.\end{aligned}$$

This together with the discrete $A_q$-weight assumption leads to

$$\begin{aligned}\|Ac\|_{q,w} &\le h(0)\|c\|_{q,w} + 2^{2d(q-1)/q}(\|A\|_{\mathcal B}-h(0))^{(q-1)/q}(A_q(w))^{1/q}\\ &\quad \times\Big\{\sum_{l=1}^\infty h(2^{l-1})2^{(l+1)d}\sum_{i\in\mathbb Z^d}\frac{w(i)}{\sum_{|i-j'|_\infty<2^l} w(j')}\\ &\quad \times\Big(\sum_{2^{l-1}\le|i-j|_\infty<2^l}|c(j)|^q w(j)\Big)\Big\}^{1/q}\\ &\le h(0)\|c\|_{q,w} + 2^{2d(q-1)/q}(\|A\|_{\mathcal B}-h(0))^{(q-1)/q}(A_q(w))^{1/q}\\ &\quad \times\Big\{\sum_{l=1}^\infty h(2^{l-1})2^{(l+1)d}\Big(\sum_{j\in\mathbb Z^d}|c(j)|^q w(j)\\ &\quad \Big(\sum_{\epsilon\in\{-1,0,1\}^d}\sum_{|i-j-\epsilon 2^{l-1}|_\infty<2^{l-1}}\frac{w(i)}{\sum_{|i-j'|_\infty<2^l} w(j')}\Big)\Big)\Big\}^{1/q}\\ &\le 2^{2d}3^{d/p}(A_q(w))^{1/q}\|A\|_{\mathcal B}\|c\|_{q,w},\end{aligned}$$

and hence (3.8) for $1<q<\infty$ is established.

The conclusion (3.8) for $q=1$ can be proved by similar arguments. We omit the details here. □

## 4. $\ell^q_w$-stability

In this section, we prove the following theorem (a slight generalization of Theorem 1.2) and Corollary 1.3. We also provide a characterization of the $\ell^q_w$-stability of a Laurent matrix in $\mathcal B$.

**Theorem 4.1.** *Let $1\le p\le\infty$, $1\le q,q'<\infty$, let the weight matrix $u$ be $p$-submultiplicative, and $w,w'$ be discrete $A_q$-weight and $A_{q'}$-weight respectively. If $A\in\mathcal B_{p,u}$ has $\ell^q_w$-stability, then $A$ has $\ell^{q'}_{w'}$-stability.*

As the trivial weight $w_0$ (i.e. $w_0(i) = 1$ for all $i \in \mathbb{Z}^d$) is a discrete $A_q$-weight for any $1 \le q < \infty$, we have the following corollary of Theorem 4.1.

**Corollary 4.2.** *If $A \in \mathcal{B}$ has $\ell^q$-stability for some $1 \le q < \infty$, then $A$ has $\ell^{q'}$-stability for all $1 \le q' < \infty$.*

We remark that similar results about $\ell^q$-stability for different exponents $q \in [1, \infty]$ are established by Aldroubi, Baskakov and Krishtal [1] for infinite matrices in the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_{1,p_\alpha}(\mathbb{Z}^d, \mathbb{Z}^d)$ with $\alpha > (d+1)^2$, by Tessera [50] for $\alpha > 0$, and by Shin and Sun [39] for $\alpha \ge 0$, where $p_\alpha = ((1 + |i - j|_\infty)^\alpha)_{i,j \in \mathbb{Z}^d}$.

The $\ell^q_w$-stability is one of the basic assumptions for infinite matrices arising in many fields of mathematics (see [2, 4, 25, 29, 30, 40, 43, 44] for a sample of papers), but little is known about practical criteria for the $\ell^q_w$-stability of an infinite matrix, see [47] for the diagonal-blocks-dominated criterion for the $\ell^2$-stability of infinite matrices in the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d)$. As an application of Theorem 1.2, we have the following characterization of the $\ell^q_w$-stability of a Laurent matrix in $\mathcal{B}$.

**Corollary 4.3.** *Let $1 \le q < \infty$, $A := (a(i - j))_{i,j \in \mathbb{Z}^d}$ be a Laurent matrix in $\mathcal{B}$, and let $w$ be a discrete $A_q$-weight. Then $A$ has $\ell^q_w$-stability if and only if $\hat{a}(\xi) := \sum_{n \in \mathbb{Z}^d} a(n) e^{-\sqrt{-1}\, n\xi} \ne 0$ for all $\xi \in \mathbb{R}^d$.*

To prove Theorem 4.1, we recall a characterization of discrete $A_q$-weights.

**Lemma 4.4.** [19, 41] *Let $1 \le q < \infty$. Then $w := (w(i))_{i \in \mathbb{Z}^d}$ is a discrete $A_q$-weight with the $A_q$-bound $A_q(w)$ if and only if*

$$\begin{aligned} & \Big(N^{-d} \sum_{i \in a + [0, N-1]^d} |c(i)|\Big)^q \Big(N^{-d} \sum_{i \in a + [0, N-1]^d} w(i)\Big) \\ \le\ & A_q(w) N^{-d} \sum_{i \in a + [0, N-1]^d} |c(i)|^q w(i) \end{aligned} \tag{4.1}$$

*hold for all $a \in \mathbb{Z}^d, 1 \le N \in \mathbb{Z}$ and sequences $c := (c(i))_{i \in \mathbb{Z}^d}$.*

To prove Theorem 4.1, we need a technical lemma about estimating a bounded sequence $c$ via the sequence $Ac$, which will also be used later in the proof of Theorem 1.4. Similar estimate is given in [40] when the infinite matrix $A$ belongs to the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d)$ and has $\ell^p_w$-stability for the trivial weight $w \equiv 1$.

**Lemma 4.5.** *Let $1 \le q < \infty$, and $w$ be a discrete $A_q$-weight. If $A \in \mathcal{B}$ has $\ell^q_w$-stability, then there exists a nonnegative sequence $\{g(i)\}_{i\in\mathbb{Z}^d}$ on $\mathbb{Z}^d$ such that*

$$\sum_{k\in\mathbb{Z}^d}\Big(\sup_{|i|_\infty\ge|k|_\infty} g(i)\Big) < \infty \tag{4.2}$$

*and*

$$|c(i)| \le \sum_{j\in\mathbb{Z}^d} g(i-j)|(Ac)(j)|, \ \ i\in\mathbb{Z}^d, \tag{4.3}$$

*where $c \in \ell^\infty$.*

*Proof.* Without loss of generality, we assume that

$$\|c\|_{q,w} \le \|Ac\|_{q,w} \quad \text{for all } c \in \ell^q_w. \tag{4.4}$$

Let $h(x) = \min(\max(2-|x|_\infty, 0), 1)$ and $N$ be a sufficiently large integer chosen later. Define linear operators $\Psi^N_n, n \in N\mathbb{Z}^d$, on $\ell^q_w$ by

$$\Psi^N_n c := \Big(h\big(\frac{j-n}{N}\big)c(j)\Big)_{j\in\mathbb{Z}^d} \quad \text{for } c := (c(j))_{j\in\mathbb{Z}^d} \in \ell^q_w.$$

Then for $c := (c(j))_{j\in\mathbb{Z}^d} \in \ell^q_w$ and $|n-n'|_\infty \le 8N$,

$$\begin{aligned}
&\|(\Psi^N_n A - A\Psi^N_n)\Psi^N_{n'}c\|_{q,w}\\
= \ & \Big\{\sum_{i\in\mathbb{Z}^d}\Big|\sum_{j\in\mathbb{Z}^d}\Big(h\big(\frac{i-n}{N}\big)-h\big(\frac{j-n}{N}\big)\Big)\\
&\quad \times a(i,j)h\big(\frac{j-n'}{N}\big)c(j)\Big|^q w(i)\Big\}^{1/q}\\
\le \ & N^{-1/2}\Big\{\sum_{i\in\mathbb{Z}^d}\Big(\sum_{|i-j|_\infty\le\sqrt{N}}|a(i,j)||c(j)|\Big)^q w(i)\Big\}^{1/q}\\
&+\Big\{\sum_{i\in\mathbb{Z}^d}\Big(\sum_{|i-j|_\infty>\sqrt{N}}|a(i,j)||c(j)|\Big)^q w(i)\Big\}^{1/q}\\
\le \ & \Big\{2^{2d+2d/q}N^{-1/2}(A_q(w))^{1/q}\|A\|_{\mathcal{B}} + 2^{3d+2d/q+1}(A_q(w))^{1/q}\\
&\quad \times\Big(\sum_{|k|_\infty\ge\sqrt{N}/2}\sup_{|i-j|_\infty\ge|k|_\infty}|a(i,j)|\Big)\Big\}\|c\|_{q,w},
\end{aligned}\tag{4.5}$$

where the last inequality follows from (3.16) and the following estimate:

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d} \sup_{|i-j|_\infty\ge \max(|k|_\infty,\sqrt{N})} |a(i,j)| \\
\le\ & (2\sqrt{N}+1)^d \sup_{|i-j|_\infty\ge\sqrt{N}} |a(i,j)| + \sum_{|k|_\infty>\sqrt{N}} \sup_{|i-j|_\infty\ge |k|_\infty} |a(i,j)| \\
\le\ & 2^{d+1} \sum_{|k|_\infty\ge\sqrt{N}/2} \sup_{|i-j|_\infty\ge|k|_\infty} |a(i,j)|.
\end{aligned}$$

Similarly for $c := (c(j))_{j\in\mathbb{Z}^d} \in \ell^q_w$ and $|n-n'|_\infty > 8N$,

$$\begin{aligned}
& \|(\Psi_n^N A - A\Psi_n^N)\Psi_{n'}^N c\|_{q,w} \\
=\ & \Big(\sum_{i\in\mathbb{Z}^d}\Big|\sum_{j\in\mathbb{Z}^d} h\big(\frac{i-n}{N}\big)a(i,j)h\big(\frac{j-n'}{N}\big)c(j)\Big|^q w(i)\Big)^{1/q} \\
\le\ & \Big(\sup_{|i'-j'|_\infty\ge|n-n'|_\infty/2} |a(i',j')|\Big)\Big(\sum_{|i-n|_\infty<2N}\Big(\sum_{|j-n'|_\infty<2N}|c(j)|\Big)^q w(i)\Big)^{1/q} \\
\le\ & 2^{2d}N^d(A_q(w))^{1/q}\Big(\sup_{|i'-j'|_\infty\ge|n-n'|_\infty/2}|a(i',j')|\Big) \\
& \times\Big(\frac{\sum_{|i'-n|_\infty<2N} w(i')}{\sum_{|i'-n'|_\infty<2N} w(i')}\Big)^{1/q}\|c\|_{q,w}. \qquad (4.6)
\end{aligned}$$

Define

$$\alpha_n := \sum_{|i'-n|_\infty<2N} w(i'), \quad n\in N\mathbb{Z}^d$$

and the linear operator $\Phi_N$ on $\ell^p_w$ by

$$\Phi_N c := \Big(\Big(\sum_{n\in N\mathbb{Z}^d}\Big(h\big(\frac{j-n}{N}\big)\Big)^2\Big)^{-1} c(j)\Big)_{j\in\mathbb{Z}^d} \text{ for } c := (c(j))_{j\in\mathbb{Z}^d}\in\ell^p_w.$$

Then for all $n'\in N\mathbb{Z}^d$ with $|n-n'|\le 8N$,

$$\alpha_n \le \sum_{|i'-n'|_\infty<10N} w(i') \le 6^{dq}A_q(w)\alpha_{n'} \qquad (4.7)$$

by (4.1), and

$$\|\Phi_N c\|_{q,w} \le \|c\|_{q,w} \quad \text{for all } c\in\ell^q_w. \qquad (4.8)$$

Note that $\Psi_n^N c\in\ell^p_w$ for any $c\in\ell^\infty$ and $n\in N\mathbb{Z}^d$, and

$$\|\Psi_n^N c\|_{q,w} \le \alpha_n^{1/q}\|c\|_\infty,\ n\in N\mathbb{Z}^d. \qquad (4.9)$$

Then for $c\in\ell^\infty$, combining (4.4), (4.5), (4.6), (4.7), and (4.8) leads to

$$
\begin{aligned}
(4.10)\quad & \alpha_n^{-1/q}\|\Psi_n^N c\|_{q,w}\le \alpha_n^{-1/q}\|A\Psi_n^N c\|_{q,w}\\
\le\ & \alpha_n^{-1/q}\|\Psi_n^N Ac\|_{q,w}+\alpha_n^{-1/q}\|(\Psi_n^N A-A\Psi_n^N)c\|_{q,w}\\
\le\ & \alpha_n^{-1/q}\|\Psi_n^N Ac\|_{q,w}+\alpha_n^{-1/q}\sum_{n'\in N\mathbb{Z}^d}\big\|(\Psi_n^N A-A\Psi_n^N)\Psi_{n'}^N\Phi_N\Psi_{n'}^N c\big\|_{q,w}\\
\le\ & \alpha_n^{-1/q}\|\Psi_n^N Ac\|_{q,w}+2^{3d+2d/q}3^d(A_q(w))^{2/q}\sum_{|n'-n|_\infty\le 8N}\alpha_{n'}^{-1/q}\|\Psi_{n'}^N c\|_{q,w}\\
& \times\Big\{\Big(N^{-1/2}\|A\|_{\mathcal B}+2^{d+1}\sum_{|k|_\infty\ge\sqrt{N}/2}\ \sup_{|i'-j'|_\infty\ge|k|_\infty}|a(i',j')|\Big)\Big\}\\
& +\sum_{|n'-n|_\infty>8N}\alpha_{n'}^{-1/q}\|\Psi_{n'}^N c\|_{q,w}\\
& \times\Big\{2^{2d}N^d(A_q(w))^{1/q}\Big(\sup_{|i'-j'|_\infty\ge|n-n'|_\infty/2}|a(i',j')|\Big)\Big\}\\
=:\ & \alpha_n^{-1/q}\|\Psi_n^N Ac\|_{q,w}+\sum_{n'\in N\mathbb{Z}^d}V_N(n-n')\alpha_{n'}^{-1/q}\|\Psi_{n'}^N c\|_{q,w}.
\end{aligned}
$$

Define sequences $V_N^l:=(V_N^l(n))_{n\in N\mathbb{Z}^d}, l\ge 1$, as follows:

$$
\begin{cases}
V_N^l(n):=V_N(n) \quad \text{if } l=1 \text{ and } n\in N\mathbb{Z}^d,\\
V_N^l(n):=\sum_{n'\in N\mathbb{Z}^d}V_N(n-n')V_N^{l-1}(n') \quad \text{if } l\ge 2 \text{ and } n\in N\mathbb{Z}^d.
\end{cases}
$$

Then for $c\in\ell^\infty$, applying (4.10) repeatedly yields

$$
\begin{aligned}
& \alpha_n^{-1/q}\|\Psi_n^N c\|_{q,w}\\
\le\ & \alpha_n^{-1/q}\|\Psi_n^N Ac\|_{q,w}+\sum_{l=1}^{l_0}\sum_{n'\in N\mathbb{Z}^d}V_N^l(n-n')\alpha_{n'}^{-1/q}\|\Psi_{n'}^N Ac\|_{q,w}\\
& +\sum_{n'\in N\mathbb{Z}^d}V_N^{l_0+1}(n-n')\alpha_{n'}^{-1/q}\|\Psi_{n'}^N c\|_{q,w},\qquad l_0\ge 1.
\end{aligned}\tag{4.11}
$$

Set

$$
\epsilon_N^l:=\sum_{k\in N\mathbb{Z}^d}\ \sup_{|n|_\infty\ge|k|_\infty}|V_N^l(n)|.
$$

Inductively for $l\ge 2$,

$$
\begin{aligned}
\epsilon_N^l\ \le\ & \epsilon_N^{l-1}\sum_{k\in N\mathbb{Z}^d}\ \sup_{|n|_\infty\ge|k|_\infty/2}|V_N(n)|\\
& +\epsilon_N^1\sum_{k\in N\mathbb{Z}^d}\ \sup_{|n|_\infty\ge|k|_\infty/2}|V_N^{l-1}(n)|\le 5^d\epsilon_N^1\epsilon_N^{l-1},
\end{aligned}
$$

where we have used (3.14) to obtain the last inequality. This shows that

$$\epsilon_N^l \le (5^d \epsilon_N^1)^l \quad \text{for all } l \ge 1. \tag{4.12}$$

Note that

$$\begin{aligned}
\epsilon_N^1 &\le 2^{4d+2d/q} 3^{3d} (A_q(w))^{2/q} \Big\{ N^d \Big( \sup_{|i'-j'|_\infty > 4N} |a(i',j')| \Big) \\
&\quad + 2^{d+1} \Big( \sum_{k' \in \mathbb{Z}^d,\ |k'|_\infty \ge \sqrt{N}/2} \sup_{|i'-j'|_\infty \ge |k'|_\infty} |a(i',j')| \Big) \\
&\quad + N^{-1/2} \|A\|_{\mathcal{B}} \Big\} + 2^{2d} (A_q(w))^{1/q} \\
&\quad \times \Big\{ \sum_{|k|_\infty > 8N, k \in N\mathbb{Z}^d} N^d \Big( \sup_{|i'-j'|_\infty \ge |k|_\infty/2} |a(i',j')| \Big) \Big\} \\
&\le 2^{4d+2d/q} 3^{3d} (A_q(w))^{2/q} \Big\{ N^{-1/2} \|A\|_{\mathcal{B}} \\
&\quad + 2^{d+2} \Big( \sum_{k' \in \mathbb{Z}^d,\ |k'|_\infty \ge \sqrt{N}/2} \sup_{|i'-j'|_\infty \ge |k'|_\infty} |a(i',j')| \Big) \Big\} \\
&\quad + 2^{2d+1} (A_q(w))^{1/q} \sum_{k' \in \mathbb{Z}^d,\ |k'| > 7N} \Big( \sup_{|i'-j'|_\infty \ge 7|k'|_\infty/16} |a(i',j')| \Big) \\
&\le 2^{6d} 3^{3d} (A_q(w))^{2/q} N^{-1/2} \|A\|_{\mathcal{B}} + 2^{7d+3} 3^{3d} (A_q(w))^{2/q} \\
&\quad \times \Big( \sum_{k' \in \mathbb{Z}^d,\ |k'|_\infty \ge \sqrt{N}/2} \big( \sup_{|i'-j'|_\infty \ge |k'|_\infty} |a(i',j')| \big) \Big) \\
&\to 0 \quad \text{as } N \to +\infty
\end{aligned}$$

by the assumption $A \in \mathcal{B}$. Let $N$ be the integer chosen sufficiently large so that

$$\epsilon_N^1 < 5^{-d}. \tag{4.13}$$

Taking the limit as $l_0 \to \infty$ in (4.11), and using (4.9), (4.12) and (4.13) lead to

$$\begin{aligned}
\alpha_n^{-1/q} \|\Psi_n^N c\|_{q,w} &\le \alpha_n^{-1/q} \|\Psi_n^N A c\|_{q,w} \\
&\quad + \sum_{n' \in N\mathbb{Z}^d} \Big( \sum_{l=1}^{\infty} V_N^l (n-n') \Big) \alpha_{n'}^{-1/q} \|\Psi_{n'}^N A c\|_{q,w} \\
&=: \sum_{n' \in N\mathbb{Z}^d} W_N(n-n') \alpha_{n'}^{-1/q} \|\Psi_{n'}^N A c\|_{q,w},
\end{aligned} \tag{4.14}$$

and

$$\sum_{k\in N\mathbb{Z}^d}\Big(\sup_{|n|_\infty\ge |k|_\infty}|W_N(n)|\Big)<\infty. \tag{4.15}$$

Given any $i\in\mathbb{Z}^d$, let $n(i)$ be the unique integer in $N\mathbb{Z}^d$ with $i\in n(i)+\{0,\ldots,N-1\}^d$. Then

$$\alpha_{n(i)}\le \sum_{|i'-i|_\infty<3N} w(i')\le (6N)^{dq}A_q(w)w(i)$$

by (4.1). This together with (4.14) implies that for any $c\in\ell^\infty$,

$$\begin{aligned}
|c(i)| &\le (6N)^d(A_q(w))^{1/q}\alpha_{n(i)}^{-1/q}\|\Psi_{n(i)}^N c\|_{q,w}\\
&\le (6N)^d(A_q(w))^{1/q}\sum_{n'\in N\mathbb{Z}^d} W_N(n(i)-n')\\
&\quad \times\Big(\sum_{j\in\mathbb{Z}^d} h\big((j-n')/N\big)|(Ac)(j)|\Big)\\
&\le (6N)^d(A_q(w))^{1/q}\\
&\quad \Big\{\sum_{j\in\mathbb{Z}^d}\Big(\sum_{\epsilon\in\{-4,\cdots,4\}^d} W_N(n(i-j)+\epsilon N)\Big)|(Ac)(j)|\Big\}\\
&=: \sum_{j\in\mathbb{Z}^d} g(i-j)|(Ac)(j)|. 
\end{aligned}\tag{4.16}$$

Then the sequence $\{g(i)\}_{i\in\mathbb{Z}^d}$ just defined satisfies (4.2) and (4.3), the requirements in Lemma 4.5, by (4.15) and (4.16). □

Now we proceed to prove Theorem 4.1.

*Proof of Theorem 4.1.* By Theorem 3.2, it suffices to prove the conclusion for any infinite matrix $A\in\mathcal{B}$.

By (3.16),

$$\|Ac\|_{q',w'}\le 2^{2d}3^{d/q'}(A_{q'}(w'))^{1/q'}\|A\|_{\mathcal{B}}\|c\|_{q',w'}\quad \text{for all } c\in\ell^{q'}_{w'}. \tag{4.17}$$

Let $\{g(i)\}_{i\in\mathbb{Z}^d}$ be the sequence in Lemma 4.5, and set

$$A_0:=\sum_{k\in\mathbb{Z}^d}\Big(\sup_{|i|_\infty\ge|k|_\infty} g(i)\Big)<\infty.$$

Then

$$\begin{aligned}
\|c\|_{q',w'} &\le \Big\|\Big(\sum_{j\in\mathbb{Z}^d} g(i-j)|(Ac)(j)|\Big)_{i\in\mathbb{Z}^d}\Big\|_{q',w'}\\
&\le 2^{2d}3^{d/q'}A_0(A_{q'}(w'))^{1/q'}\|Ac\|_{q',w'}\quad \text{for all } c\in\ell^\infty\cap\ell^{q'}_{w'},
\end{aligned}\tag{4.18}$$

where the first inequality follows from (4.3) and the second inequality holds by (3.16). Combining (4.17) and (4.18) proves the $\ell^{q'}_{w'}$-stability for the infinite matrix $A \in \mathcal{B}$. □

Finally we prove Corollary 1.3.

*Proof of Corollary 1.3.* The necessity is well known, while the sufficiency follows from Theorem 3.2 and Corollary 1.7, whose proof will be given in the next section. □

## 5. Inverse-closedness

In this section, we prove Theorem 1.4, Corollaries 1.6 and 1.7, and the following Wiener's lemma for the subalgebra $\mathcal{B}_{p,u}$ of $\mathcal{B}(\ell^q_w)$.

**Theorem 5.1.** *Let $1 \le p, q < \infty$, $w$ be a discrete $A_q$-weight, $u := (u(i,j))_{i,j\in\mathbb{Z}^d}$ be a $p$-submultiplicative weight matrix that satisfies* (3.1), (3.2), (3.3), (3.9) *and*

$$u(i,j) = u(j,i) \quad \text{for all } i,j \in \mathbb{Z}^d,$$

*and let $v := (v(i,j))_{i,j\in\mathbb{Z}^d}$ be a companion weight matrix of the $p$-submultiplicative weight matrix $u$ that satisfies* (3.4)*. If there exist $D \in (0,\infty)$ and $\theta \in (0,1)$ such that*

$$\inf_{N\ge 1} \big(A_N + B_N(p)t\big) \le Dt^\theta \quad \text{for all } t \ge 1 \tag{5.1}$$

*where*

$$A_N := \sum_{|k|_\infty \le N} \sup_{|k|_\infty \le |i'-j'|_\infty \le N} v(i',j') \tag{5.2}$$

*and*

$$B_N(p) := \Big\| \Big( \sup_{|i'-j'|_\infty \ge |k|_\infty} v(i',j')(u(i',j'))^{-1} \Big)_{|k|_\infty \ge N/2} \Big\|_{p/(p-1)}, \tag{5.3}$$

*then $\mathcal{B}_{p,u}$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^q_w)$.*

One may verify that the weight matrices $((1+|i-j|)^\alpha)_{i,j\in\mathbb{Z}^d}$ with $\alpha > d(1-1/p)$, and $(\exp(|i-j|^\delta))_{i,j\in\mathbb{Z}^d}$ with $\delta \in (0,1)$, and their companion weight matrices satisfy the conditions on weight matrices required in Theorem 5.1 [44]. Hence we have the following corollary of Theorem 5.1.

**Corollary 5.2.** *Let $1 \le p, q < \infty$, $w$ be a discrete $A_q$-weight, and let $u$ be either $((1+|i-j|_\infty)^\alpha)_{i,j\in\mathbb{Z}^d}$ with $\alpha > d(1-1/p)$ or $(\exp(|i-j|^\delta_\infty))_{i,j\in\mathbb{Z}^d}$ with $\delta \in (0,1)$. Then $\mathcal{B}_{p,u}$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^q_w)$.*

5.1. **Proof of Theorem 1.4.** Let $A\in\mathcal{B}$ have an inverse $A^{-1}\in\mathcal{B}(\ell^q_w)$. Then $\|c\|_{q,w}\le \|A^{-1}\|_{\mathcal{B}(\ell^q_w)}\|Ac\|_{q,w}$ for all $c\in\ell^q_w$, where $\|\cdot\|_{\mathcal{B}(\ell^q_w)}$ is the operator norm on $\mathcal{B}(\ell^q_w)$. Therefore $A$ has $\ell^q_w$-stability. By Lemma 4.5, there exists a sequence $\{g(i)\}_{i\in\mathbb{Z}^d}$ such that (4.2) and (4.3) hold.

Write $A^{-1}:=(b(i,j))_{i,j\in\mathbb{Z}^d}$, set $c_j:=(b(i,j))_{i\in\mathbb{Z}^d}$, and for $l_0\ge 1$ define $c_j^{l_0}:=(b_{l_0}(i,j))_{i\in\mathbb{Z}^d}, j\in\mathbb{Z}^d$, where $b_{l_0}(i,j):=b(i,j)$ if $|i-j|_\infty\le l_0$ and 0 otherwise. Then $c_j^{l_0}\in\ell^\infty\cap\ell^q_w$ and

$$\lim_{l_0\to+\infty}\|c_j^{l_0}-c_j\|_{q,w}=0. \tag{5.4}$$

Applying (4.3) to $c_j^{l_0}$ gives

$$|b_{l_0}(i,j)|\le\sum_{i'\in\mathbb{Z}^d} g(i-i')|(Ac_j^{l_0})(i')|,\qquad i\in\mathbb{Z}^d. \tag{5.5}$$

By (4.2), (5.4), and Theorem 3.2,

$$\begin{aligned}
&\sum_{i'\in\mathbb{Z}^d} g(i-i')|(A(c_j^{l_0}-c_j))(i')|\\
\le\ & w(i)^{-1/q}\Big\|\Big(\sum_{i''\in\mathbb{Z}^d} g(i'-i'')|(A(c_j^{l_0}-c_j))(i'')|\Big)_{i'\in\mathbb{Z}^d}\Big\|_{q,w}\\
\le\ & 2^{4d}3^{2d/q}w(i)^{-1/q}(A_q(w))^{2/q}\|A\|_{\mathcal{B}}\\
&\quad\times\Big\|\Big(\sup_{|j'|_\infty\ge|k|_\infty}|g(j')|\Big)_{k\in\mathbb{Z}^d}\Big\|_1\|c_j^{l_0}-c_j\|_{q,w}\\
\to\ & 0 \quad\text{as } l_0\to+\infty.
\end{aligned}\tag{5.6}$$

Letting $l_0\to+\infty$ in (5.5) and applying (5.6) gives

$$|b(i,j)|\le g(i-j)\quad\text{for all } i,j\in\mathbb{Z}^d. \tag{5.7}$$

Hence the conclusion $A^{-1}\in\mathcal{B}$ follows from (4.2) and (5.7). □

5.2. **Proof of Corollary 1.6.** Write $f(\xi)=\sum_{n\in\mathbb{Z}}a(n)e^{-\sqrt{-1}n\xi}$. Then $A:=(a(i-j))_{i,j\in\mathbb{Z}}$ belongs to $\mathcal{B}$ and has bounded inverse in $\mathcal{B}(\ell^2)$. Moreover, $A^{-1}=(b(i-j))_{i,j\in\mathbb{Z}}$ for the sequence $b:=(b(n))_{n\in\mathbb{Z}}$ determined by $1/f(\xi)=\sum_{n\in\mathbb{Z}}b(n)e^{-\sqrt{-1}n\xi}$. By Theorem 1.4, $A^{-1}\in\mathcal{B}$ which in turn proves the desired conclusion that $1/f\in A^*(\mathbb{T})$. □

5.3. **Proof of Corollary 1.7.** The necessity follows from Theorem 3.2. Now the sufficiency: Let $1\le q<\infty$, $w$ be a discrete $A_q$-weight, and let $A\in\mathcal{B}$ have $\ell^q_w$-stability. Then $A$ has $\ell^2$-stability by Theorem 1.2, i.e., there exists a positive constant $C$ such that

$$C^{-1}\|c\|_2\le\|Ac\|\le C\|c\|_2\quad\text{for all } c\in\ell^2.$$

This implies that $A^*A$ has bounded inverse in $\mathcal{B}(\ell^2)$. On the other hand, $A^*A$ belong to $\mathcal{B}$ by Proposition 2.4 and Theorem 3.2. Therefore

$$(A^*A)^{-1} \in \mathcal{B} \tag{5.8}$$

by Theorem 1.4. Now we prove that $B := (A^*A)^{-1}A^*$ is the desired left inverse of the infinite matrix $A$ in $\mathcal{B}$. The conclusion that $B \in \mathcal{B}$ follows from (5.8), Proposition 2.4 and Theorem 3.2. From the definition of the infinite matrix $B$, it defines a left inverse in $\mathcal{B}(\ell^2)$, it belongs to $\mathcal{B}(\ell^q_w)$ by Theorem 3.2 and $B \in \mathcal{B}$, and the set $\ell^2 \cap \ell^q_w$ is dense in $\ell^q_w$. Therefore the infinite matrix $B$ is a left inverse in $\mathcal{B}(\ell^q_w)$. □

5.4. **Proof of Theorem 5.1.** To prove Theorem 5.1, we need a technical lemma. Similar results are established in [42, 44] for infinite matrices in the Gröchenig-Schur class $\mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ and the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$, see also Remark 3.4.

**Lemma 5.3.** *Let* $1 \le p \le \infty$. *If the weight matrix* $u$ *satisfies* (3.1), (3.2), (3.3), (3.9) *and* (5.1) *for some positive constants* $D \in (0, \infty)$ *and* $\theta \in (0, 1)$, *then*

$$\|A^2\|_{\mathcal{B}_{p,u}} \le 2^{2+2/p} 5^{(d-1)/p} D \|A\|^{1+\theta}_{\mathcal{B}_{p,u}} \|A\|^{1-\theta}_{\mathcal{B}(\ell^2)} \quad \text{for all } A \in \mathcal{B}_{p,u}. \tag{5.9}$$

*Proof.* Let $A := (a(i,j))_{i,j\in\mathbb{Z}^d} \in \mathcal{B}_{p,u}$, and let $A_N$ and $B_N(p)$ be as in (5.2) and (5.3) respectively. Recall that $|a(i,j)| \le \|A\|_{\mathcal{B}(\ell^2)}$ for all $i, j \in \mathbb{Z}^d$. Then for $1 < p < \infty$,

$$\begin{aligned}
&\sum_{k'\in\mathbb{Z}^d} |a(i,k')|u(i,k')|a(k',j)|v(k',j)\\
\le\ & \inf_{N\ge 1} \Big\{ \|A\|_{\mathcal{B}(\ell^2)} \sum_{|k'-j|_\infty \le N} |a(i,k')|u(i,k')v(k',j)\\
&\quad + \sum_{|k'-j|_\infty > N} |a(i,k')|u(i,k')|a(k',j)|v(k',j) \Big\}\\
\le\ & \inf_{N\ge 1} \Big\{ \|A\|_{\mathcal{B}(\ell^2)} \Big( \sum_{|k''-j|_\infty \le N} v(k'',j) \Big)^{(p-1)/p}\\
&\qquad \times \Big( \sum_{|k'-j|_\infty \le N} (|a(i,k')|u(i,k'))^p v(k',j) \Big)^{1/p}\\
&\quad + \Big( \sum_{|k''-j|_\infty > N} |a(k'',j)|v(k'',j) \Big)^{(p-1)/p}\\
&\qquad \times \Big( \sum_{|k'-j|_\infty > N} (|a(i,k')|u(i,k'))^p |a(k',j)|v(k',j) \Big)^{1/p} \Big\}.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
& \Big\{\sum_{k\in\mathbb{Z}^d}\sup_{|i-j|_\infty\ge|k|_\infty}\Big(\sum_{k'\in\mathbb{Z}^d}|a(i,k')|u(i,k')|a(k',j)|v(k',j)\Big)^p\Big\}^{1/p}\\
\le\ & \inf_{N\ge1}\Big\{\|A\|_{\mathcal{B}(\ell^2)}(A_N)^{(p-1)/p}\\
&\qquad\times\Big(A_N\sum_{k\in\mathbb{Z}^d}\sup_{|i'-j'|_\infty\ge|k|_\infty/2}(|a(i',j')|u(i',j'))^p\\
&\qquad+\|A\|_{\mathcal{B}_{p,u}}^p\sum_{k\in\mathbb{Z}^d}\sup_{|k|_\infty/2\le|i'-j'|_\infty\le N}v(i',j')\Big)^{1/p}\\
&\quad+\|A\|_{\mathcal{B}_{p,u}}^{(p-1)/p}(B_N(p))^{(p-1)/p}\\
&\qquad\times\Big(\|A\|_{\mathcal{B}_{p,u}}B_N(p)\sum_{k\in\mathbb{Z}^d}\sup_{|i'-j'|_\infty\ge|k|_\infty/2}(|a(i',j')|u(i,j'))^p\\
&\qquad+\|A\|_{\mathcal{B}_{p,u}}^{p+1}\Big(\sum_{k\in\mathbb{Z}^d}\sup_{|i'-j'|_\infty\ge\max(|k|_\infty/2,N)}\\
&\qquad\Big(\frac{v(i',j')}{u(i',j')}\Big)^{p/(p-1)}\Big)^{(p-1)/p}\Big)^{1/p}\Big\}\\
\le\ & 2^{1+2/p}5^{(d-1)/p}\|A\|_{\mathcal{B}_{p,u}}\inf_{N\ge1}\Big(\|A\|_{\mathcal{B}(\ell^2)}A_N+B_N(p)\|A\|_{\mathcal{B}_{p,u}}\Big)\\
\le\ & 2^{1+2/p}5^{(d-1)/p}D\|A\|_{\mathcal{B}_{p,u}}^{1+\theta}\|A\|_{\mathcal{B}(\ell^2)}^{1-\theta}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \Big\{\sum_{k\in\mathbb{Z}^d}\sup_{|i-j|_\infty\ge|k|_\infty}\Big(\sum_{k'\in\mathbb{Z}^d}|a(i,k')|v(i,k')|a(k',j)|u(k',j)\Big)^p\Big\}^{1/p}\\
\le\ & 2^{1+2/p}5^{(d-1)/p}D\|A\|_{\mathcal{B}_{p,u}}^{1+\theta}\|A\|_{\mathcal{B}(\ell^2)}^{1-\theta}.
\end{aligned}$$

Combining the above two estimates and applying (3.13) with $B=A$, we then get the desire conclusion (5.9) for $1<p<\infty$.

The conclusion (5.9) for $p=1$ and for $p=\infty$ can be established similarly. We omit the details here. □

Having the above technical lemma, we can combine the arguments in [42, 44] and Wiener's lemma for $\mathcal{B}$ to prove Theorem 5.1.

*Proof of Theorem 5.1.* Let $A\in\mathcal{B}_{p,u}$ and $A^{-1}\in\mathcal{B}(\ell^p_w)$. Then $A^{-1}\in\mathcal{B}\subset\mathcal{B}(\ell^2)$ by Theorems 1.4 and 3.2. This implies that $C_1I\le A^*A\le C_2I$ for some positive constants $C_1$ and $C_2$, where $A^*$ is the conjugate transpose of the matrix $A$ and $I$ is the identity matrix. Now set

$$B := I - \frac{2}{C_1 + C_2} A^* A.$$

Then

$$\|B\|_{\mathcal{B}(\ell^2)} \le \frac{C_2 - C_1}{C_2 + C_1} := r_0 < 1. \tag{5.10}$$

On the other hand, $A^*A \in \mathcal{B}_{p,u}$ by Proposition 2.4 and Theorem 3.2, and $I \in \mathcal{B}_{p,u}$ by (3.9). This shows that

$$\|B\|_{\mathcal{B}_{p,u}} < \infty. \tag{5.11}$$

Given any integer $n \ge 1$, write $n = \sum_{l=0}^{l_0} \epsilon_l 2^l$ with $\epsilon_l \in \{0, 1\}$. Applying Theorem 3.2 and Lemma 5.3 iteratively gives

$$\begin{aligned}
\|B^n\|_{\mathcal{B}_{p,u}} &\le (C\|B\|_{\mathcal{B}_{p,u}})^{\epsilon_0} \|B^{n-\epsilon_0}\|_{\mathcal{B}_{p,u}} \\
&\le C(C\|B\|_{\mathcal{B}_{p,u}})^{\epsilon_0} (\|B\|_{\mathcal{B}(\ell^2)})^{(1-\theta)\sum_{l=0}^{l_0-1} \epsilon_{l+1} 2^l} \\
&\qquad \times (\|B^{\sum_{l=0}^{l_0-1} \epsilon_{l+1} 2^l}\|_{\mathcal{B}_{p,u}})^{(1+\theta)} \\
&\le \cdots \le C^{l_0} (C\|B\|_{\mathcal{B}_{p,u}})^{\sum_{l=0}^{l_0} \epsilon_l (1+\theta)^l} (\|B\|_{\mathcal{B}(\ell^2)})^{\sum_{l=0}^{l_0} \epsilon_l (2^l - (1+\theta)^l)} \\
&\le C^{\log_2 n} \big(C r_0^{-1} \|B\|_{\mathcal{B}_{p,u}}\big)^{n^{\log_2(1+\theta)}} r_0^n,
\end{aligned}$$

where $C = \max(2^{2+2/p} 5^{(d-1)/p} D, 2^{1+2/p} 5^{(d-1)/p} M_p(u))$. This together with (5.10) and (5.11) shows that

$$\begin{aligned}
\|A^{-1}\|_{\mathcal{B}_{p,u}} &= \|(A^*A)^{-1} A^*\|_{\mathcal{B}_{p,u}} \\
&= \frac{C_1 + C_2}{2} \Big\| A^* + \Big( \sum_{n=1}^{\infty} B^n \Big) A^* \Big\|_{\mathcal{B}_{p,u}} \\
&\le \frac{C_1 + C_2}{2} \Big\{ \|A^*\|_{\mathcal{B}_{p,u}} + C\|A^*\|_{\mathcal{B}_{p,u}} \\
&\qquad \times \Big( \sum_{n=1}^{\infty} C^{\log_2 n} \big(C r_0^{-1} \|B\|_{\mathcal{B}_{p,u}}\big)^{n^{\log_2(1+\theta)}} r_0^n \Big) \Big\} < \infty.
\end{aligned}$$

Hence the conclusion $A^{-1} \in \mathcal{B}_{p,u}$ is proved. □

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