Compactly Supported Refinable Distributions in Triebel-Lizorkin Spaces and Besov Spaces

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Abstract

The aim of this paper is to characterize compactly supported refinable distributions in Triebel-Lizorkin spaces and Besov spaces by projection operators on certain wavelet space and by some operators on a finitely dimensional space.

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1 Introduction

A compactly supported distribution f on $\mathbb{R}^n$ is said to be *refinable* if f satisfies such a *refinement equation*

$$f(x) = \sum_{j\in\mathbb{Z}^n} c_j f(2x-j), \tag{1}$$

where the sequence $\{c_j\}$ has finite support and $\sum_{j\in\mathbb{Z}^n} c_j = 2^n$. Define the *symbol* of the refinement equation (1), or of refinable distribution f, by

$$H(\xi) = 2^{-n} \sum_{j\in\mathbb{Z}^n} c_j e^{-ij\cdot\xi}. \tag{2}$$

Then $H(\xi)$ is a trigonometric polynomial and satisfies $H(0) = 1$.

The solution to the refinement equation (1) is unique up to a multiplying constant. So we only consider the normalized solution to (1), which means $\widehat{f}(0) = 1$. Hereafter the *Fourier transform* $\widehat{f}$ of an integrable function f is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

The Fourier transform of a compactly supported distribution is interpreted as usual.

Refinable function appears in different setting, most notably in subdivision schemes for computer aided design, and in the construction of wavelet bases and multiresolution. The refinable distribution has attracted a lot of attention in recent years and is well studied, including existence, uniqueness and regularity etc. The dependence of the regularity of f on the choice of coefficients c_k in (1) has been studied by many authors (see [1], [3], [5], [6], [7], [8], [20], [21] for Hölder continuous space, [14], [17], [18] for p-integrable space and L^p-Lipschitz space, [9], [13], [16], [24] for Sobolev space, [25] for Besov space, and the survey paper [3]). The results are often formulated in terms of the joint spectral property of operators on a finitely dimensional space, or obtained by the direct estimate for the corresponding symbol $H(\xi)$.

In this paper, we will characterize compactly supported refinable distributions in Triebel-Lizorkin spaces and Besov spaces via projection operators

P_l and Q_l of a multiresolution and via operators B_ϵ on a finitely dimensional space V.

The paper is organized as follows. In Section 2, we fix some notations and state main results. In fact, we give the definitions of Triebel-Lizorkin spaces and Besov spaces, multiresolution, projection operators P_l and Q_l, operators B_ϵ, finitely dimensional space V and $\rho_p(B_\epsilon, V)$, a number similar to p-norm joint spectral radius in [14], and state the main results. Section 3 contains the proof of main theorem. In Section 4, we will give some remarks.

2 Preliminary and Result

The Triebel-Lizorkin spaces and Besov spaces are two important classes of function spaces, which include spaces of all p-integrable functions for $p > 1$, Sobolev spaces and Hardy spaces as well. For the theory of Triebel-Lizorkin spaces and Besov spaces we refer the reader to [23] and [11].

Let ϕ_0 and ψ be functions in the Schwarz class such that $\widehat{\phi}_0$ is supported in $\{\xi; |\xi| \le 2\}$, $\widehat{\psi}$ supported in $\{\xi; 1 \le |\xi| \le 4\}$ and

$$\widehat{\phi}_0(\xi) + \sum_{l \ge 0} \widehat{\psi}(2^{-l}\xi) = 1, \quad \forall \xi \in \mathbb{R}^n.$$

Define the *convolution* $f * g$ of two square integrable functions f and g by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

and the quasi-norm of p-integrable function by $\|f\|_p = (\int_{\mathbb{R}^n} |f(x)|^p dx)^{1/p}$ for $0 < p < \infty$. The convolution of two compactly supported distributions is interpreted as usual.

For $-\infty < \alpha < \infty, 0 < p, q < \infty$, *Triebel-Lizorkin space* $F^\alpha_{p,q}$ is the set of distribution f such that its quasi-norm $\|f\|_{F^\alpha_{p,q}}$ defined by

$$\|f\|_{F^\alpha_{p,q}} = \|\phi_0 * f\|_p + \|(\sum_{l \ge 0} 2^{l\alpha q} |\psi_l * f|^q)^{1/q}\|_p$$

is finite, and *Besov space* $B^\alpha_{p,q}$ is the set of distribution f such that its quasi-norm $\|f\|_{B^\alpha_{p,q}}$ defined by

$$\|f\|_{B^\alpha_{p,q}} = \|\phi_0 * f\|_p + (\sum_{l\geq 0} 2^{l\alpha q}\|\psi_l * f\|_p^q)^{1/q}$$

is finite, where $\psi_l(x) = 2^{ln}\psi(2^l x)$ for $l \geq 0$. The topologies of $F^\alpha_{p,q}$ and $B^\alpha_{p,q}$ are induced by the quasi-norms $\|\cdot\|_{F^\alpha_{p,q}}$ and $\|\cdot\|_{B^\alpha_{p,q}}$ respectively.

A *multiresolution* is a family of closed subspaces $\{V_l\}_{l\in\mathbb{Z}}$ of L^2, the space of square integrable functions, such that

a) $\cap_{l\in\mathbb{Z}}V_l = \{0\}$ and $\cup_{l\in\mathbb{Z}}V_l$ is dense in L^2.

b) $V_l \subset V_{l+1}, \quad \forall\, l \in \mathbb{Z}$.

c) There exists a function ϕ in V_0 such that $\{\phi(\cdot - k); k \in \mathbb{Z}^n\}$ is a Riesz basis of V_0 and V_l is spanned by $\{2^{ln/2}\phi(2^l\cdot - k); k \in \mathbb{Z}^n\}$.

The function ϕ in c) is called a *scaling function* of the multiresolution. The multiresolution was introduced by Mallat and Meyer(see [4], [19]). In one dimension, it is well known that for any integer τ there exists multiresolutions $\{V_l\}$ and $\{\tilde{V}_l\}$ such that the corresponding scaling functions ϕ and $\tilde{\phi}$ are compactly supported, in Hölder class C^τ and biorthogonal (see [5], [2]). Here we denote the *Hölder space* with Hölder exponent τ by C^τ, and we say that ϕ and $\tilde{\phi}$ are *biorthogonal* if

$$\int_{\mathbb{R}^n} \phi(x)\tilde{\phi}(x-j)dx = \begin{cases} 1, & j = 0, \\ 0, & j \neq 0. \end{cases}$$

A compactly supported distribution g is said to be *locally linearly independent* if for any open set A

$$\sum_{j\in\mathbb{Z}^n} d_j g(x-j) = 0, \quad x \in A \quad \textit{implies} \quad d_j = 0, \quad \forall\, j \in K(A),$$

where $j \in K(A)$ means $g(\cdot - j)$ is not identically zero on A. In [22], the second author proved in one dimension case that biorthogonal scaling functions ϕ

and $\tilde{\phi}$ are locally linearly independent. Then for any integer $\tau \geq 1$, we can construct scaling functions ϕ and $\tilde{\phi}$ in higher dimensions by the tensor product method in [19] such that ϕ and $\tilde{\phi}$ are compactly supported, in Hölder space C^τ, biorthogonal and locally linearly independent.

For these multiresolutions $\{V_l\}_{l\in Z}$ and $\{\tilde{V}_l\}_{l\in Z}$ let wavelet spaces W_l be the biorthogonal complement of V_l in V_{l+1}. Define *projection operators* $P_l, l \geq 0$ to V_l by

$$P_l f(x) = 2^{ln} \sum_{j\in\mathbb{Z}^n} \langle f, \tilde{\phi}(2^l \cdot -j)\rangle \phi(2^l x - j), \tag{3}$$

and projection operators Q_l on W_l by

$$Q_l f = P_{l+1} f - P_l f \tag{4}$$

for square integrable function f. Here for two functions f and g in L^2, their *inner product* is defined by

$$\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx.$$

Now we extend the domain of definitions of P_l and Q_l. Obviously it suffices to extend the domain of definition of inner product. By Parseval identity, we have

$$\langle f, g\rangle = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi$$

Then we may define the inner product of two distributions f and g by the formula above when $\hat{f}(\xi)\overline{\hat{g}(\xi)}$ is integrable.

Denote the class of compactly supported distributions f which satisfy

$$|\hat{f}(\xi)| \leq C(1+|\xi|)^\alpha, \quad \forall \xi \in \mathbb{R}^n$$

by $\hat{\mathcal{D}}^\alpha$. Set $B = \ln \sup_{\xi\in\mathbb{R}^n} |H(\xi)|/\ln 2$. Then $f \in \hat{\mathcal{D}}^B$ for the refinable distribution f in (1). By integration by parts, we have

$$|\hat{f}(\xi)| \leq C(1+|\xi|)^{-\tau}, \quad \forall \xi \in \mathbb{R}^n$$

when $f \in C^\tau$. Thus the inner product between $f \in \hat{\mathcal{D}}^B$ and $g \in C^\tau$ is well defined when $\tau > B+n$. This shows that the inner product between refinable distribution f in (1) and $\tilde{\phi}(\cdot - j)$, the scaling functions of the multiresolution $\{\tilde{V}_j\}$, is well defined when $\tau > \ln \sup_{\xi \in \mathbb{R}^n} |H(\xi)|/\ln 2 + n$ and hence the projection operators P_l and Q_l are well defined.

In this paper, unless otherwise stated we assume that the multiresolutions $\{V_l\}$ and $\{\tilde{V}_l\}$ are chosen such that their corresponding scaling functions ϕ and $\tilde{\phi}$ are compactly supported, biorthogonal, in Hölder space C^τ with $\tau > \ln \sup_{\xi \in \mathbb{R}^n} |H(\xi)|/\ln 2 + n$ and locally linearly independent.

To characterize the refinable distribution, we also need a finitely dimensional space V and operators B_ϵ on V, which are very similar to the transfer operators in [6], [9], [24] etc.

For $\epsilon \in E = \{0,1\}^n$ and the symbol $H(\xi)$ of the refinement equation (1), define operators B_ϵ by

$$B_\epsilon P(\xi) = \sum_{\epsilon' \in E} H(\frac{\xi}{2} + \epsilon' \pi) e^{-i\epsilon \cdot (\frac{\xi}{2} + \epsilon' \pi)} P(\frac{\xi}{2} + \epsilon' \pi) \tag{5}$$

for every trigonometric polynomial P. Let

$$F_0(\xi) = \sum_{j \in \mathbb{Z}^n} \langle f, \tilde{\phi}(\cdot - j) \rangle e^{-ij \cdot \xi},$$

and $R_\epsilon, \epsilon \in E$ be defined by

$$H(\xi) F_0(\xi) - F_0(2\xi) G(\xi) = \sum_{\epsilon \in E} e^{i\epsilon \cdot \xi} R_\epsilon(2\xi), \tag{6}$$

where $G(\xi)$ is the symbol of the scaling function ϕ. Then $R_\epsilon, \epsilon \in E$ are trigonometric polynomials.

Let V be the minimal space containing $R_\epsilon, \epsilon \in E$ such that it is invariant under operators $B_\epsilon, \epsilon \in E$. Then

$$V \quad \text{is spanned by} \quad \{B_{\epsilon_1} \cdots B_{\epsilon_l} \mathrm{R}_\epsilon; \epsilon_j, \epsilon \in E, 1 \le j \le l \quad \text{and} \quad l \ge 0\}.$$

It is easy to see that V is of finite dimension (see [14]). For simplicity, we still denote by B_ϵ the restriction of operators B_ϵ on V.

For $0 < p < \infty$, define the p-quasinorm $\|P\|_p^*$ for trigonometric polynomial $P(\xi) = \sum_{j\in\mathbb{Z}^n} d_j e^{ij\cdot\xi}$ by

$$\|P\|_p^* = (\sum_{j\in\mathbb{Z}^n} |d_j|^p)^{1/p}.$$

Set

$$\rho_p(B_\epsilon, V) = \inf_{l\geq 1} \sup_{\|u\|_p^*=1, u\in V} \Big(2^{-nl} \sum_{\epsilon_1,\cdots,\epsilon_l\in E} (\|B_{\epsilon_1}\cdots B_{\epsilon_l}u\|_p^*)^p\Big)^{\frac{1}{pl}}.$$

The number $\rho_p(B_\epsilon, V)$ is essentially the p-norm joint spectral radius of operators B_ϵ on finite dimensional space V (see [14] and [12]). The authors would thank one anonymous referee to point out this fact to us.

The number $\rho_p(B_\epsilon, V)$ may be also computed by

$$\rho_p(B_\epsilon, V) = \lim_{l\to\infty} \sup_{\|u\|_p^*=1, u\in V} \Big(2^{-nl} \sum_{\epsilon_1,\cdots,\epsilon_l\in E} (\|B_{\epsilon_1}\cdots B_{\epsilon_l}u\|_p^*)^p\Big)^{\frac{1}{pl}}.$$

The assertion above is proved by [14] for $p \geq 1$. Now we give the proof of the assertion above for $0 < p < 1$.

Set

$$D_l = \sup_{\|u\|_p^*=1, u\in V} \Big(2^{-nl} \sum_{\epsilon_1,\cdots,\epsilon_l\in E} (\|B_{\epsilon_1}\cdots B_{\epsilon_l}u\|_p^*)^p\Big)^{\frac{1}{pl}}.$$

Then it suffices to prove that

$$\text{limsup}_{l\to\infty} D_l = \rho(B_\epsilon, V).$$

For any $\delta > 0$, by the definition of $\rho(B_\epsilon, V)$, there exists l_0 such that

$$D_{l_0} \leq \rho(B_\epsilon, V) + \delta.$$

Hence we have

$$\sum_{\epsilon_1,\cdots,\epsilon_{l_0}\in E} (\|B_{\epsilon_1}\cdots B_{\epsilon_{l_0}}u\|_p^*)^p \leq 2^{nl_0}(\rho(B_\epsilon, V) + \delta)^{pl_0}(\|u\|_p^*)^p, \quad \forall\ u \in V$$

and for all $l = kl_0 + s, 0 \le s \le l_0$ and $k \ge 1$,

$$\begin{aligned}
& \sum_{\epsilon_1,\cdots,\epsilon_l \in E} (\|B_{\epsilon_1}\cdots B_{\epsilon_l} u\|_p^*)^p \\
= & \sum_{\epsilon_{kl_0+1},\cdots,\epsilon_l \in E} \sum_{\epsilon_{(k-1)l_0+1},\cdots,\epsilon_{kl_0} \in E} \cdots \sum_{\epsilon_1,\cdots,\epsilon_{l_0} \in E} (\|B_{\epsilon_1}\cdots B_{\epsilon_l} u\|_p^*)^p \\
\le & \; 2^{nl_0}(\rho(B_\epsilon, V)+\delta)^{pl_0} \times \\
& \qquad \sum_{\epsilon_{kl_0+1},\cdots,\epsilon_l \in E} \sum_{\epsilon_{(k-1)l_0+1},\cdots,\epsilon_{kl_0} \in E} \cdots \sum_{\epsilon_{l_0+1},\cdots,\epsilon_{2l_0} \in E} (\|B_{\epsilon_{l_0+1}}\cdots B_{\epsilon_l} u\|_p^*)^p \\
\le & \cdots \\
\le & \; 2^{nkl_0}(\rho(B_\epsilon, V)+\delta)^{pkl_0} \sum_{\epsilon_{kl_0+1},\cdots,\epsilon_l \in E} (\|B_{\epsilon_{kl_0+1}}\cdots B_{\epsilon_l} u\|_p^*)^p \\
\le & \; C2^{nl}(\rho(B_\epsilon, V)+\delta)^{pl}(\|u\|_p^*)^p,
\end{aligned}$$

where C is a constant independent of $k \ge 1$. This shows that

$$\text{limsup}_{l\to\infty} D_l \le \rho(B_\epsilon, V) + \delta$$

for any $\delta > 0$. The assertion is proved.

Fix $l_0 \ge 1$. Let $V_{l_0}^*$ be the minimal space invariant under operator $B_\epsilon, \epsilon \in E$ and containing $B_{\epsilon_1}\cdots B_{\epsilon_{l_0}} R_{\epsilon_{l_0+1}}$ with $\epsilon_i \in E, 1 \le i \le l_0+1$. Define

$$\rho_p(B_\epsilon, V_{l_0}^*) = \lim_{l\to\infty} \sup_{\|u\|_p^*=1, u\in V_{l_0}^*} \Big(2^{-nl} \sum_{\epsilon_1,\cdots,\epsilon_l\in E} (\|B_{\epsilon_1}\cdots B_{\epsilon_l} u\|_p^*)^p\Big)^{\frac{1}{pl}}.$$

From the definition of $V_{l_0}^*$, we have $V_{l_0}^* \subset V$ and hence

$$\rho(B_\epsilon, V_{l_0}^*) \le \rho(B_\epsilon, V).$$

Observe that for $l \ge l_0$,

$$\begin{aligned}
& \sup_{\|u\|_p^*\le 1, u\in V} 2^{-nl} \sum_{\epsilon_1,\cdots,\epsilon_l\in E} (\|B_{\epsilon_1}\cdots B_{\epsilon_l} u\|_p^*)^p \\
\le & \; C \sup_{\epsilon_i\in E, l-l_0+1\le i\le l} \sup_{\|u\|_p^*\le 1, u\in V} 2^{-nl} \sum_{\epsilon_1,\cdots,\epsilon_{l-l_0}\in E} (\|B_{\epsilon_1}\cdots B_{\epsilon_{l-l_0}}(B_{\epsilon_{l-l_0+1}}\cdots B_{\epsilon_l} u)\|_p^*)^p \\
\le & \; C \sup_{\|u'\|_p^*\le 1, u'\in V_{l_0}^*} 2^{-n(l-l_0)} \sum_{\epsilon_1,\cdots,\epsilon_{l-l_0}\in E} (\|B_{\epsilon_1}\cdots B_{\epsilon_{l-l_0}} u'\|_p^*)^p,
\end{aligned}$$

where the last inequality follows from the facts that $B_{\epsilon_{l-l_0+1}} \cdots B_{\epsilon_l} u \in V_{l_0}^*$ and $\|B_{\epsilon_{l-l_0+1}} \cdots B_{\epsilon_l} u\| \leq C$ for all u satisfying $\|u\|_p^* \leq 1$. Then we have

$$\rho(B_\epsilon, V) \leq \rho(B_\epsilon, V_{l_0}^*).$$

Hence $\rho(B_\epsilon, V)$ can be computed by $\rho(B_\epsilon, V_{l_0}^*)$.

Now let's state our main results.

Theorem 1 *Let $-\infty < \alpha < +\infty$, $0 < p, q < \infty$ and f be the normalized solution to (1). Set $J = n/\min(p, q, 1)$ and denote the integral part of a real number x by $[x]$. Suppose that $\{V_l\}$ and $\{\tilde{V}_l\}$ are multiresolutions such that their corresponding scaling functions ϕ and $\tilde{\phi}$ are compactly supported, biorthogonal, in Hölder space C^τ and locally linearly independent, where τ is chosen such that*

$$\tau > \max(\ln \sup_{\xi \in \mathbb{R}} |H(\xi)| / \ln 2 + n, [J - n - \alpha], |\alpha|).$$

Then the following statements are equivalent to each other.

1) $f \in F_{p,q}^\alpha$.

2) $f \in B_{p,q}^\alpha$.

3) $2^{l\alpha} \|Q_l f\|_p \to 0$ *as* $l \to \infty$.

4) *There exist constants C and $0 < r < 1$ independent of $l \geq 0$ such that*

$$2^{l\alpha} \|Q_l f\|_p \leq C r^l, \quad \forall\ l \geq 0.$$

5) $\rho_p(B_\epsilon, V) < 2^{-\alpha}$.

From the results above, a compactly supported refinable distribution in Triebel-Lizorkin spaces $F_{p,q}^\alpha$ is also in Besov spaces $B_{p,q}^\alpha$. Comparing with the subdivision scheme in [14], we introduce an appropriate space V, which we use to characterize refinable distributions in Triebel-Lizorkin spaces and Besov spaces via $\rho_p(B_\epsilon, V)$. Comparing with the characterization of p-integrable refinable functions in [17], we use biorthogonal scaling functions ϕ and $\tilde{\phi}$ with higher regularity instead of the characteristic function on $[0,1]$ as the initial, which also makes it possible to consider more general function spaces, Triebel-Lizorkin spaces and Besov spaces instead of p-integrable function spaces.

3 Proof of Theorem 1

We begin with a characterization of Triebel-Lizorkin spaces and Besov spaces.

Lemma 3.1 *Suppose* $0 < p, q < \infty, -\infty < \alpha < +\infty$ *and* $\tau > \max(J - n - \alpha, |\alpha|)$. *Let* P_l *and* Q_l *be defined by* (3) *and* (4) *respectively, and let* f *be a compactly supported distribution. Then* $f \in F^\alpha_{p,q}$ *if and only if*

$$\|P_0 f\|_p + \|(\sum_{l\geq 0} 2^{lq\alpha}|Q_l f|^q)^{1/q}\|_p < \infty,$$

and $f \in B^\alpha_{p,q}$ *if and only if*

$$\|P_0 f\|_p + (\sum_{l\geq 0} 2^{lq\alpha}\|Q_l f\|_p^q)^{1/q} < \infty.$$

A similar result can be found in [10] and [11]. For the perfection of this paper, we include the proof in the appendix. Now we start to prove Theorem 1.

1) $\Rightarrow$ 3): Let $f \in F^\alpha_{p,q}$. Then $(\sum_{l\geq 0} |2^{l\alpha} Q_l f(x)|^q)^{1/q} < \infty$ for almost every $x \in \mathbb{R}^n$ and is p-integrable. Hence $2^{l\alpha} Q_l f(x) \to 0$ for almost every $x \in \mathbb{R}^n$ as $l \to \infty$. By the Lebesgue dominated convergence theorem, we have $2^{l\alpha}\|Q_l f\|_p \to 0$ as $l \to \infty$.

2) $\Rightarrow$ 3): By Lemma 3.1, the sequence $2^{l\alpha}\|Q_l f\|_p$ is q-summable when $f \in B^\alpha_{p,q}$. Hence $2^{l\alpha}\|Q_l f\|_p \to 0$ as $l \to \infty$.

4) $\Rightarrow$ 2): Obviously,

$$\sum_{l\geq 0}(2^{l\alpha}\|Q_l f\|_p)^{q'} \leq C\sum_{l\geq 0} r^{lq'} < \infty.$$

Observe that $P_0 f$ is compactly supported function in C^τ by its definition. Hence $\|P_0 f\|_p < \infty$ and f is in Besov space $B^\alpha_{p,q'}$ for all $0 < q' < \infty$ by Lemma 3.1.

4) $\Rightarrow$ 1): Observe that $F^\alpha_{p,q'} \supset B^\alpha_{p,p}$ when $q' \geq p$. Hence $f \in F^\alpha_{p,q'}$ when $q' \geq p$, since $f \in B^\alpha_{p,p}$. For $0 < q' < p$, we have

$$\begin{aligned}\|(\sum_{l\geq 0}|2^{l\alpha}Q_l f|^{q'})^{1/q'}\|_p^p &\leq C_\delta\|(\sum_{l\geq 0}|2^{l(\alpha+\delta)}Q_l f|^p)^{1/p}\|_p^p \\ &\leq C_\delta\sum_{l\geq 0}(2^\delta r)^{lp} < \infty,\end{aligned}$$

where δ is chosen such that $2^\delta r < 1$, and the second inequality follows from

$$(\sum_{l\geq 0}|2^{l\alpha}Q_l f(x)|^{q'})^{p/q'} \leq (\sum_{l\geq 0}|2^{l(\alpha+\delta)}Q_l f(x)|^p) \times (\sum_{l\geq 0}2^{-lpq'\delta/(p-q')})^{p/(p-q')}.$$

Therefore $f \in F^\alpha_{p,q'}$ and 1) follows.

3) $\Rightarrow$ 4): To prove 4) from 3), we need two lemmas.

Lemma 3.2 *Let ϕ be as in Theorem 1 and $p > 0$. Then there exists a constant C independent of sequence $\{d_j\}$ with finite support such that*

$$C^{-1}(\sum_{j\in\mathbb{Z}^n}|d_j|^p)^{1/p} \leq \|\sum_{j\in\mathbb{Z}^n}d_j\phi(x-j)\|_p \leq C(\sum_{j\in\mathbb{Z}^n}|d_j|^p)^{1/p}. \tag{7}$$

In [15], Jia proved a similar result under weak restriction on ϕ. For the perfection of this paper, we include the proof here.

Proof of Lemma 3.2 The right hand side inequality of (7) follows from the fact that ϕ has compact support and is bounded.

Now we consider the left hand side of (7). By the local linear independence of integer translates of ϕ, we have

$$\int_{[0,1]^n}|\sum_{j\in\mathbb{Z}}d_j\phi(x-j)|^p dx \geq C_1\sum_{j\in K((0,1)^n)}|d_j|^p$$

for any sequence $\{d_j\}$ and a constant C_1 independent of sequence $\{d_j\}$, where for open set A, $j \in K(A)$ means $\phi(\cdot - j)$ is not identically zero on A. Hence

$$\int_{\mathbb{R}^n} |\sum_{j\in\mathbb{Z}} d_j \phi(x-j)|^p dx \geq C_1 \sum_{k\in\mathbb{Z}^n} \sum_{j\in K((0,1)^n+k)} |d_j|^p \geq C_1 \sum_{j\in\mathbb{Z}^n} |d_j|^p$$

and Lemma 3.2 follows. ♠

Lemma 3.3 *Let f satisfy the refinement equation (1) and H be the corresponding symbol. Set*

$$F_l(\xi) = \sum_{j\in\mathbb{Z}^n} \langle f, \tilde{\phi}(2^l \cdot -j)\rangle e^{-ij\cdot\xi}. \tag{8}$$

Then we have

$$F_l(\xi) = H(2^{l-1}\xi)F_{l-1}(\xi), \tag{9}$$
$$H(\xi)P(\xi) = 2^{-n} \sum_{\epsilon\in E} e^{i\epsilon\cdot\xi} B_\epsilon P(2\xi), \tag{10}$$

and

$$\prod_{i=1}^{l} H(2^{i-1}\xi)P(\xi) = 2^{-nl} \sum_{(\epsilon_1,\cdots,\epsilon_l)\in E^l} e^{i\sum_{j=1}^{l} 2^{l-j}\epsilon_j\cdot\xi} B_{\epsilon_1}\cdots B_{\epsilon_l} P(2^l\xi) \tag{11}$$

for any trigonometric polynomial P, where we denote by E^l the l-th Cartesian power of E.

Proof. From (1) and (8), we have

$$\begin{aligned}
F_l(\xi) &= \sum_{j\in\mathbb{Z}^n} \langle f, \tilde{\phi}(2^l \cdot -j)\rangle e^{-ij\cdot\xi} \\
&= \sum_{j\in\mathbb{Z}^n} \sum_{s\in\mathbb{Z}^n} c_s \langle f(2\cdot -s), \tilde{\phi}(2^l \cdot -j)\rangle e^{-ij\cdot\xi} \\
&= 2^{-n} \sum_{j\in\mathbb{Z}^n} \sum_{s\in\mathbb{Z}^n} c_s \langle f, \tilde{\phi}(2^{l-1} \cdot -j + 2^{l-1}s)\rangle e^{-ij\cdot\xi} \\
&= H(2^{l-1}\xi)F_{l-1}(\xi).
\end{aligned}$$

Hence (9) is proved.

Observe that the right hand side of (10) equals to

$$2^{-n}\sum_{\epsilon'\in E} H(\xi+\epsilon'\pi)P(\xi+\epsilon'\pi)\sum_{\epsilon\in E} e^{-i\epsilon\cdot\epsilon'\pi}.$$

Hence (10) follows from the formula above since $\sum_{\epsilon\in E} e^{-i\epsilon\cdot\epsilon'\pi}=0$ when $0\neq\epsilon'\in E$ and the cardinality of E is 2^n.

The formula (11) follows from repeating (10) for l times. ♠

Now let's start to prove 4) from 3). Recall that ϕ is a scaling function. Hence ϕ is refinable. Denote its symbol by

$$G(\xi)=2^{-n}\sum_{j\in\mathbb{Z}^n} g_j e^{ij\cdot\xi}.$$

Then

$$\widehat{\phi}(\xi)=G(\xi/2)\widehat{\phi}(\xi/2) \tag{12}$$

by taking Fourier transform at both sides of (1).

From the definitions of P_l, F_l and (9), we obtain

$$(\widehat{P_l f})(\xi)=F_l(2^{-l}\xi)\widehat{\phi}(2^{-l}\xi)$$

and

$$\begin{aligned}(\widehat{Q_l f})(\xi) &= (F_{l+1}(2^{-l-1}\xi)-F_l(2^{-l}\xi)G(2^{-l-1}\xi))\widehat{\phi}(2^{-l-1}\xi)\\ &= \prod_{i=1}^{l} H(2^{-i}\xi)\times(H(2^{-l-1}\xi)F_0(2^{-l-1}\xi)-F_0(2^{-l}\xi)G(2^{-l-1}\xi))\widehat{\phi}(2^{-l-1}\xi).\end{aligned}$$

Recall that

$$H(\xi)F_0(\xi)-F_0(2\xi)G(\xi)=\sum_{\epsilon\in E} e^{i\epsilon\cdot\xi}R_\epsilon(2\xi).$$

Then by Lemma 3.2, (10) and (11), we conclude that

$$\begin{aligned}\|Q_l f\|_p^p &\geq C2^{l(p-1)n}\Big(\|\prod_{i=1}^{l} H(2^i\xi)\times(H(\xi)F_0(\xi)-F_0(2\xi)G(\xi))\|_p^*\Big)^p\\ &= C2^{-ln}(\|\sum_{\epsilon\in E}\sum_{(\epsilon_1,\cdots,\epsilon_l)\in E^l} e^{i(\epsilon\cdot\xi+\sum_{j=1}^{l}2^{l-j+1}\epsilon_j\cdot\xi)}B_{\epsilon_1}\cdots B_{\epsilon_l}R_\epsilon(2^{l+1}\xi)\|_p^*)^p\\ &= C2^{-nl}\sum_{\epsilon\in E}\sum_{(\epsilon_1,\cdots,\epsilon_l)\in E^l}(\|B_{\epsilon_1}\cdots B_{\epsilon_l}R_\epsilon\|_p^*)^p.\end{aligned}$$

Hence

$$2^{l(\alpha p-n)}\sum_{\epsilon\in E}\sum_{(\epsilon_1,\cdots,\epsilon_l)\in E^l}(\|B_{\epsilon_1}\cdots B_{\epsilon_l}R_\epsilon\|_p^*)^p\to 0$$

as $l\to\infty$ because $2^{l\alpha}\|Q_l f\|_p\to 0$ as $l\to\infty$.

Furthermore we have

Lemma 3.4 *Let R_ϵ, V be defined by (5) and (6), and let V be the minimal space invariant under the operators B_ϵ and containing $R_\epsilon, \epsilon\in E$. If*

$$\lim_{l\to\infty}2^{l(\alpha p-n)}\sum_{\epsilon\in E}\sum_{(\epsilon_1,\cdots,\epsilon_l)\in E^l}(\|B_{\epsilon_1}\cdots B_{\epsilon_l}R_\epsilon\|_p^*)^p=0,$$

then there exists an integer l_0 such that

$$2^{l_0(\alpha p-n)}\sum_{(\epsilon_1,\cdots,\epsilon_{l_0})\in E^{l_0}}(\|B_{\epsilon_1}\cdots B_{\epsilon_{l_0}}u\|_p^*)^p\le\frac{1}{2}(\|u\|_p^*)^p,\quad\forall\ u\in V. \tag{13}$$

Proof. Set

$$V^*=\{R_\epsilon, B_{\epsilon_1}\cdots B_{\epsilon_k}R_\epsilon;\ \epsilon,\epsilon_1,\cdots,\epsilon_k\in E, k=1,2,\cdots\}.$$

It is easy to see that V is the finite dimensional space spanned by V^*. Thus there exists finite elements $e_1,\cdots,e_l\in V^*$ such that $e_1,\cdots,e_l$ is a basis of V, and for any $u\in V$ there exist real numbers $u_1,\cdots,u_l$ uniquely satisfying

$$u=u_1e_1+\cdots+u_le_l.$$

Obviously,

$$C^{-1}(|u_1|^p+\cdots+|u_l|^p)\le(\|u\|_p^*)^p\le C(|u_1|^p+\cdots+|u_l|^p)$$

when $p<1$ and

$$C^{-1}(|u_1|+\cdots+|u_l|)\le\|u\|_p^*\le C(|u_1|+\cdots+|u_l|)$$

when $p\ge 1$ hold for some constant C independent of $u\in V$.

Let $e_i = B_{\epsilon_1^i} \cdots B_{\epsilon_{k_i}^i} R_{\epsilon_0^i}$, in which $e_i = R_{\epsilon_0^i}$ when $k_i = 0$. Then

$$\begin{aligned} & 2^{l(\alpha p-n)} \sum_{(\epsilon_1,\cdots,\epsilon_l)\in E^l} (\|B_{\epsilon_1}\cdots B_{\epsilon_l} e_i\|_p^*)^p \\ \le\ & 2^{(l+k_i)(\alpha p-n)} \sum_{\epsilon\in E} \sum_{(\epsilon_1,\cdots,\epsilon_{l+k_i})\in E^{l+k_i}} (\|B_{\epsilon_1}\cdots B_{\epsilon_{l+k_i}} R_\epsilon\|_p^*)^p 2^{-k_i(\alpha-n)} \to 0 \end{aligned}$$

as l tends to infinity. Hence there exists an integer l_0 such that

$$2^{l_0(\alpha p-n)} \sum_{(\epsilon_1,\cdots,\epsilon_{l_0})\in E^{l_0}} (\|B_{\epsilon_1}\cdots B_{\epsilon_{l_0}} e_i\|_p^*)^p \le \frac{1}{2} C^{-\max(1,p)}, \quad 1 \le i \le l.$$

For any $u = u_1 e_1 + \cdots + u_l e_l \in V$, we have

$$\begin{aligned} & 2^{l_0(\alpha p-n)} \sum_{(\epsilon_1,\cdots,\epsilon_{l_0})\in E^{l_0}} (\|B_{\epsilon_1}\cdots B_{\epsilon_{l_0}} u\|_p^*)^p \\ \le\ & 2^{l_0(\alpha p-n)} \sum_{i=1}^{l} |u_i|^p \sum_{(\epsilon_1,\cdots,\epsilon_{l_0})\in E^{l_0}} (\|B_{\epsilon_1}\cdots B_{\epsilon_{l_0}} e_i\|_p^*)^p \\ \le\ & \frac{1}{2} C^{-1} \sum_{i=1}^{l} |u_i|^p \le \frac{1}{2} (\|u\|_p^*)^p \end{aligned}$$

when $p < 1$ and

$$\begin{aligned} & 2^{l_0(\alpha p-n)} \sum_{(\epsilon_1,\cdots,\epsilon_{l_0})\in E^{l_0}} (\|B_{\epsilon_1}\cdots B_{\epsilon_{l_0}} u\|_p^*)^p \\ \le\ & \Big(\sum_{i=1}^{l} |u_i| (2^{l_0(\alpha p-n)} \sum_{(\epsilon_1,\cdots,\epsilon_{l_0})\in E^{l_0}} (\|B_{\epsilon_1}\cdots B_{\epsilon_{l_0}} e_i\|_p^*)^p)^{1/p} \Big)^p \\ \le\ & \frac{1}{2} (C^{-1} \sum_{i=1}^{l} |u_i|)^p \le \frac{1}{2} (\|u\|_p^*)^p \end{aligned}$$

when $p \ge 1$. Hence (13) and Lemma 3.4 is proved. ♠

Set

$$A_l = 2^{l(\alpha p-n)} \sum_{\epsilon\in E} \sum_{(\epsilon_1,\cdots,\epsilon_l)\in E^l} (\|B_{\epsilon_1}\cdots B_{\epsilon_l} R_\epsilon\|_p^*)^p.$$

Then by Lemma 3.4, we have

$$\begin{aligned} A_l &= 2^{l(\alpha p-n)} \sum_{(\epsilon_1,\cdots,\epsilon_{l_0})\in E^{l_0}} \sum_{\epsilon\in E} \sum_{(\epsilon_{l_0+1},\cdots,\epsilon_l)\in E^{l-l_0}} (\|B_{\epsilon_1}\cdots B_{\epsilon_{l_0}} B_{\epsilon_{l_0+1}}\cdots B_{\epsilon_l} R_\epsilon\|_p^*)^p \\ &\le \frac{1}{2} 2^{(l-l_0)(\alpha p-n)} \sum_{\epsilon\in E} \sum_{(\epsilon_{l_0+1},\cdots,\epsilon_l)\in E^{l-l_0}} (\|B_{\epsilon_{l_0+1}}\cdots B_{\epsilon_l} R_\epsilon\|_p^*)^p = \frac{1}{2} A_{l-l_0} \end{aligned}$$

when $l \ge l_0$. Hence by Lemma 3.2, we obtain

$$A_l \le C 2^{-l/l_0}$$

and

$$2^{l\alpha}\|Q_l f\|_p^p \le C A_l \le C 2^{-l/l_0}.$$

This is the desired result.

4) ⇒ 5): From the proof of 4) from 3), we obtain

$$C^{-1} 2^{-nl} \sum_{\epsilon\in E} \sum_{(\epsilon_1,\cdots,\epsilon_l)\in E^l} (\|B_{\epsilon_1}\cdots B_{\epsilon_l} R_\epsilon\|_p^*)^p \le \|Q_l f\|_p^p.$$

Recall that V is the minimal space invariant under B_ϵ and containing R_ϵ for all $\epsilon \in E$. Then by (13) there exists an integer l_0 such that

$$2^{l_0\alpha}\Big(2^{-nl_0} \sum_{(\epsilon_1,\cdots,\epsilon_{l_0})\in E^{l_0}} (\|B_{\epsilon_1}\cdots B_{\epsilon_{l_0}} u\|_p^*)^p\Big)^{1/p} \le \frac{1}{2}\|u\|_p^*, \quad u \in V.$$

Hence 5) follows by the definition of $\rho_p(B_\epsilon, V)$.

5) ⇒ 4): By the definition of $\rho_p(B_\epsilon, V)$, there exists an integer l_0 such that

$$2^{-nl_0} \sum_{\epsilon\in E} \sum_{(\epsilon_1,\cdots,\epsilon_l)\in E^l} (\|B_{\epsilon_1}\cdots B_{\epsilon_l} u\|_p^*)^p \le 2^{-p(\alpha l_0+1)}(\|u\|_p^*)^p, \quad u \in V.$$

Hence we may prove (4) by the estimate of $Q_l f$ in the procedure used in the proof of 4) from 3).

Theorem 1 is proved. ♠

4 Remarks

The finitely dimensional space V is very important to compute $\rho_p(B_\epsilon, V)$. From the definition of V, we see that it needs to compute $\langle f, \tilde{\phi}(\cdot - j)\rangle$ for all $j \in \mathbb{Z}^n$ at first. Our first remark is whether the space V can be replaced by a finitely dimensional space which is easy to compute.

Theorem 2. *Let $\alpha, p, q, \phi, \tilde{\phi}, B_\epsilon$ be as in Theorem 1, and let G and $\tilde{G}$ be the corresponding symbol of biorthogonal scaling functions ϕ and $\tilde{\phi}$. Then the refinement equation (1) has a solution in $B^\alpha_{p,q}$ or $F^\alpha_{p,q}$ if and only if there exists a trigonometric polynomial $\tilde{F}_0$ such that $\tilde{F}_0(0) \neq 0$,*

$$\tilde{F}_0(\xi) = \sum_{\epsilon \in E} H(\frac{\xi}{2} + \epsilon\pi)\tilde{G}(-\frac{\xi}{2} - \epsilon\pi)\tilde{F}_0(\frac{\xi}{2} + \epsilon\pi) \tag{14}$$

and $\rho_p(B_\epsilon, \tilde{V}) < 2^{-\alpha}$, where $\tilde{V}$ is the minimal space invariant under B_ϵ and containing $\tilde{R}_\epsilon$, and where $\tilde{R}_\epsilon$ is defined by

$$H(\xi)\tilde{F}_0(\xi) - \tilde{F}_0(2\xi)G(\xi) = \sum_{\epsilon \in E} e^{i\epsilon\cdot\xi}\tilde{R}_\epsilon(2\xi).$$

Proof. From the proof of Theorem 1, the necessity reduces to F_0 satisfying (14) and $F_0(0) \neq 0$. Write

$$\tilde{G}(\xi) = 2^{-n} \sum_{j \in \mathbb{Z}^n} \tilde{g}_j e^{-ij\xi}.$$

By the definition of $F_0(\xi)$, we have

$$\begin{aligned}
&\sum_{\epsilon \in E} H(\frac{\xi}{2} + \epsilon\pi)\tilde{G}(-\frac{\xi}{2} - \epsilon\pi)F_0(\frac{\xi}{2} + \epsilon\pi) \\
= \;& 2^{-2n} \sum_{j_1 \in \mathbb{Z}^n} \sum_{j_2 \in \mathbb{Z}^n} \sum_{j \in \mathbb{Z}^n} c_{j_1}\tilde{g}_{j_2}\langle f(\cdot), \tilde{\phi}(\cdot - j)\rangle \sum_{\epsilon \in E} e^{-i(\xi/2+\epsilon\pi)\cdot(j+j_1-j_2)} \\
= \;& \sum_{j_1 \in \mathbb{Z}^n} \sum_{j_2 \in \mathbb{Z}^n} \sum_{j \in \mathbb{Z}^n} c_{j_1}\tilde{g}_{j_2}\langle f(2\cdot - j_1), \tilde{\phi}(2\cdot - 2j - j_2)\rangle e^{-ij\cdot\xi} \\
= \;& \sum_{j \in \mathbb{Z}^n} \langle f(\cdot), \tilde{\phi}(\cdot - j)\rangle e^{-ij\cdot\xi} = F_0(\xi).
\end{aligned}$$

On the other hand, $F_0(0) = \widehat{f}(0) = 1$ by the facts that $\widehat{f}(\xi) = \prod_{l=1}^{\infty} H(\xi/2^l)$ and $H(0) = 1$. The necessity is proved.

To prove the sufficiency, we define $g_l(x)$ in terms of Fourier transform by

$$\widehat{g_l}(\xi) = \prod_{s=1}^{l} H(2^{-s}\xi) \times \tilde{F}_0(2^{-l}\xi)\widehat{\phi}(2^{-l}\xi).$$

Then $\widehat{g}_l(\xi)$ converges to $\prod_{l=1}^{\infty} H(\xi/2^l)\tilde{F}_0(0)$, which we denote by $\widehat{f}$. Therefore

$$\widehat{f}(\xi) = H(\frac{\xi}{2})\widehat{f}(\frac{\xi}{2})$$

and f, the inverse Fourier transform of $\widehat{f}$, is a compactly supported solution of the refinement equation (1).

Hence the sufficiency of Theorem 2 is reduced to proving that $P_l f = g_l$ for all $l \geq 0$ by the proof of Theorem 1 and

$$\|g_{l+1} - g_l\|_p^p \leq C2^{-nl} \sum_{\epsilon\in E} \sum_{(\epsilon_1,\cdots,\epsilon_l)\in E^l} (\|B_{\epsilon_1}\cdots B_{\epsilon_l}\tilde{R}_\epsilon\|_p^*)^p.$$

By (14), we have for $k \in \mathbb{Z}^n$

$$\begin{aligned}
&\langle g_{l+1} - g_l, \tilde{\phi}(2^l \cdot -k)\rangle \\
= &\int_{\mathbb{R}^n} \prod_{s=1}^{l} H(2^{-s}\xi) \times (H(2^{-l-1}\xi)\tilde{F}_0(2^{-l-1}\xi) - \tilde{F}_0(2^{-l}\xi)G(2^{-l-1}\xi)) \\
&\times\widehat{\phi}(2^{-l-1}\xi)\overline{\widehat{\tilde{\phi}}(2^{-l}\xi)}e^{-i2^{-l}k\cdot\xi}d\xi \\
= &\ 2^{-ln}\int_{\mathbb{R}^n} \prod_{s=1}^{l} H(2^{l-s}\xi) \times (H(\frac{\xi}{2})\tilde{F}_0(\frac{\xi}{2})\tilde{G}(-\frac{\xi}{2}) - \tilde{F}_0(\xi)G(\frac{\xi}{2})\tilde{G}(-\frac{\xi}{2})) \\
&\times\widehat{\phi}(\frac{\xi}{2})\widehat{\tilde{\phi}}(-\frac{\xi}{2})e^{-ik\cdot\xi}d\xi \\
= &\ 2^{-ln}\int_{[-\pi,\pi]^n} \prod_{s=1}^{l} H(2^{l-s}\xi) \times \Big(\sum_{\epsilon\in E} H(\frac{\xi}{2}+\epsilon\pi)\tilde{F}_0(\frac{\xi}{2}+\epsilon\pi)\tilde{G}(-\frac{\xi}{2}-\epsilon\pi) \\
&-\tilde{F}_0(\xi)\sum_{\epsilon\in E} G(\frac{\xi}{2}+\epsilon\pi)\tilde{G}(-\frac{\xi}{2}-\epsilon\pi)\Big)e^{-ik\cdot\xi}d\xi \\
= &\ 0,
\end{aligned}$$

where the third equality follows from

$$\sum_{k\in\mathbb{Z}^n} \widehat{\phi}(\xi+2k\pi)\widehat{\tilde{\phi}}(-\xi-2k\pi)=1, \quad \forall \xi\in\mathbb{R}^n$$

by the biorthogonality between ϕ and $\tilde{\phi}$, and the last equality from

$$\sum_{\epsilon\in E} G(\xi+\epsilon\pi)\tilde{G}(-\xi-\epsilon\pi)=1$$

and (14). Thus $g_{l+1}-g_l$ is in the wavelet space W_l, the orthogonal complement of V_l in V_{l+1}, and $P_l g_s = g_l$ when $s\ge l$. Hence $P_l f = g_l$ and Theorem 2 is proved. ♠

Our second remark is on the index α of Triebel-Lizorkin spaces and Besov spaces of refinable distribution. From 1) and 2) of Theorem 1, we show that $f\in F^{\alpha}_{p,q}$ and $f\in B^{\alpha}_{p,q}$ are equivalent. From 4) of Theorem 1 we can get more.

Theorem 3. *Suppose $-\infty<\alpha<+\infty$ and $0<p,q<\infty$. For an arbitrary compactly supported refinable distribution ϕ in a Triebel-Lizorkin space $F^{\alpha}_{p,q}$ or $B^{\alpha}_{p,q}$, there exists a positive number $\delta>0$ such that the refinable distribution ϕ is also in Triebel-Lizorkin space $F^{\alpha+\delta}_{p,q'}$ and Besov space $B^{\alpha+\delta}_{p,q'}$ for all $0<q'<\infty$.*

The last remark is on integrable function space L^1. In Theorem 1, we characterize refinable function in the local Hardy space $F^0_{1,2}$, a subspace of integrable function space. In [17], Lau and Wang characterized the refinable function in integrable function space L^1. For an integrable function f, it is easy to see that $\|Q_l f\|_1\to 0$ as $l\to\infty$. Then f is in $F^0_{1,2}$ when f is integrable and refinable by Theorem 1.

Define the Riesz transform $R_j, 1\le j\le n$, in terms of Fourier transform by

$$\widehat{R_j f}(\xi)=\frac{i\xi_j}{|\xi|}\widehat{f}(\xi),$$

where ξ_j denotes the j-th component of $\xi\in\mathbb{R}^n$. Let Hardy space H^1 be the set of all integrable functions f such that $R_j f$ are still integrable for all $1\le j\le n$. Then Hardy space H^1 is a subspace of local Hardy space $F^0_{1,2}$ (see [23]).

Theorem 4. *If a function f is compactly supported, refinable and integrable, then there exist a compactly supported bounded function g and a function h in Hardy space H^1 such that*

$$f = g + h.$$

Proof. Let f be compactly supported, refinable and integrable. Observe that $\|P_l f - f\|_1 \to 0$ as $l \to \infty$. Then $\|Q_l f\|_1 \to 0$ as $l \to \infty$. By Theorem 1 there exist a constant C and a positive number $0 < r < 1$ such that

$$\|Q_l f\|_1 \le C r^l.$$

On the other hand, it is easy to check that

$$\|R_j Q_l f\|_1 \le C r^l, \quad \forall \quad 1 \le j \le n.$$

Hence $R_j(\sum_{l \ge 0} Q_l f)$ is integrable. By the definition of Hardy space H^1, we obtain

$$f - P_0 f = \sum_{l \ge 0} Q_l f \in H^1.$$

Hence Theorem 4 follows when we let $g = P_0 f$ and $h = \sum_{l \ge 0} Q_l f$. ♠

Appendix: Proof of Lemma 3.1

Because the assertion for Besov spaces can be proved by similar procedure as the one of Triebel-Lizorkin spaces, we only give the proof for Triebel-Lizorkin spaces here.

Let ψ_s and $\tilde{\psi}_s, s = 1, \cdots, 2^n - 1$ be the compactly supported biorthogonal mother wavelets corresponding to the multiresolutions $\{V_l\}$ and $\{\tilde{V}_l\}$ in Section 2. Then for $0 \le |\beta| \le \max([J - n - \alpha], [\alpha])$ and $s = 1, \cdots, 2^n - 1$, we have

$$\int_{\mathbb{R}^n} x^\beta \psi_s(x) dx = \int_{\mathbb{R}^n} x^\beta \tilde{\psi}_s(x) dx = 0.$$

Therefore

$$\{\psi_s(2^l\cdot -j)2^{ln/2};\ s=0,\cdots,2^n-1, l\geq 0, j\in \mathbb{Z}^n\}\cup\{\phi(\cdot-j);\ j\in\mathbb{Z}^n\}$$

and

$$\{\tilde{\psi}_s(2^l\cdot -j)2^{ln/2};\ s=0,\cdots,2^n-1, l\geq 0, j\in \mathbb{Z}^n\}\cup\{\tilde{\phi}(\cdot-j);\ j\in\mathbb{Z}^n\}$$

are inhomogeneous smooth molecules of $F^{\alpha}_{p,q}$ (see [10, p. 56 and p. 132]). Recall that

$$\begin{aligned} f &= P_0 f+\sum_{l=0}^{\infty} Q_l f \\ &= \sum_{j\in\mathbb{Z}^n} b_j\phi(\cdot-j)+\sum_{s=0}^{2^n-1}\sum_{l\geq 0} 2^{ln/2}\sum_{j\in\mathbb{Z}^n} a_{l,j,s}\psi_s(2^l\cdot -j) \end{aligned}$$

in distributional sense, where $b_j=\langle f,\tilde{\phi}(\cdot-j)\rangle$ and $a_{j,l,s}=2^{ln/2}\langle f,\tilde{\psi}_s(2^l\cdot-j)\rangle$. Then by the inhomogeneous analogue of Theorem 3.5 in [10, p.132] we have

$$\|f\|_{F^{\alpha}_{p,q}}\leq C(\sum_{j\in\mathbb{Z}^n}|b_j|^p)^{1/p}+C\sum_{s=0}^{2^n-1}\Big\|\Big(\sum_{l\geq 0}\sum_{j\in\mathbb{Z}^n}(|a_{l,j,s}|2^{l(\alpha+n/2)}\chi_{[0,1]^n}(2^l\cdot-j))^q\Big)^{1/q}\Big\|_p$$

and by the inhomogeneous analogue of Theorem 3.7 in [10, p.132] we obtain

$$\|f\|_{F^{\alpha}_{p,q}}\geq C(\sum_{j\in\mathbb{Z}^n}|b_j|^p)^{1/p}+C\sum_{s=0}^{2^n-1}\Big\|\Big(\sum_{l\geq 0}\sum_{j\in\mathbb{Z}^n}(|a_{l,j,s}|2^{l(\alpha+n/2)}\chi_{[0,1]^n}(2^l\cdot-j))^q\Big)^{1/q}\Big\|_p,$$

where $\chi_{[0,1]^n}$ denotes the characteristic function on the unit cube $[0,1]^n$. Hence it remains to prove that there exist constants C_1 and C_2 such that

$$\begin{aligned} & C_1\sum_{s=0}^{2^n-1}\Big\|\Big(\sum_{l\geq 0}\sum_{j\in\mathbb{Z}^n}(|a_{l,j,s}|2^{l(\alpha+n/2)}\chi_{[0,1]^n}(2^l\cdot-j))^q\Big)^{1/q}\Big\|_p \\ \leq\ & \|(\sum_{l\geq 0}2^{lq\alpha}|Q_l f|^q)^{1/q}\|_p \\ \leq\ & C_2\sum_{s=0}^{2^n-1}\Big\|\Big(\sum_{l\geq 0}\sum_{j\in\mathbb{Z}^n}(|a_{l,j,s}|2^{l(\alpha+n/2)}\chi_{[0,1]^n}(2^l\cdot-j))^q\Big)^{1/q}\Big\|_p \end{aligned}$$

Recall that $\psi_s, s = 1, \cdots, 2^n - 1$ are compactly supported and bounded. Hence

$$
\begin{aligned}
|Q_l f| &\leq C \sum_{j \in \mathbb{Z}^n} \sum_{s=0}^{2^n-1} |a_{l,j,s}| 2^{ln/2} \chi_{[-C_3, C_3]^n}(2^l \cdot -j) \\
&\leq C \sum_{s=0}^{2^n-1} (\sum_{j \in \mathbb{Z}^n} |a_{l,j,s}|^q 2^{lnq/2} \chi_{[-C_3, C_3]^n}(2^l \cdot -j))^{1/q},
\end{aligned}
$$

where C_3 is chosen such that $\psi_s, s = 1, \cdots, 2^n - 1$ are supported in $[-C_3, C_3]$ and the second inequality follows from the equivalence between the different quasi-norms on finite dimensional space.

Fix $0 < A < \min(p, q, 1)$. Let $M_A f$ be the Hardy-Littlewood maximal operator defined by

$$
M_A f(x) = \sup_{x \in Q} (\frac{1}{|Q|} \int_Q |f(y)|^A dy)^{1/A},
$$

where the supremum is taken oven all cubes Q containing x. Hence

$$
(\sum_{l \geq 0} 2^{lq\alpha} |Q_l f|^q)^{1/q} \leq C \sum_{s=0}^{2^n-1} \Big(\sum_{l \geq 0} \Big(M_A (\sum_{j \in \mathbb{Z}^n} |a_{l,j,s}| 2^{l(\alpha + n/2)} \chi_{[0,1]^n}(2^l \cdot -j)) \Big)^q \Big)^{1/q}
$$

and

$$
\begin{aligned}
& \|(\sum_{l \geq 0} 2^{lq\alpha} |Q_l f|^q)^{1/q}\|_p \\
\leq\ & C \sum_{s=0}^{2^n-1} \Big\| \Big(\sum_{l \geq 0} \Big(M_A (\sum_{j \in \mathbb{Z}^n} |a_{l,j,s}| 2^{l(\alpha + n/2)} \chi_{[0,1]^n}(2^l \cdot -j)) \Big)^q \Big)^{1/q} \Big\|_p \\
\leq\ & C \sum_{s=0}^{2^n-1} \Big\| \Big(\sum_{l \geq 0} \sum_{j \in \mathbb{Z}^n} (|a_{l,j,s}| 2^{l(\alpha + n/2)} \chi_{[0,1]^n}(2^l \cdot -j))^q \Big)^{1/q} \Big\|_p
\end{aligned}
$$

by Theorem A.1 in [10, p. 141].

For $s = 1, \cdots, 2^n - 1$, let K_s be the set of $j \in \mathbb{Z}^n$ such that $\psi_s \not\equiv 0$ on $j + [0,1]^n$. Then

$$
\int_{[0,1]^n} | \sum_{s=1}^{2^n-1} \sum_{j \in K_s} d_{j,s} \psi_s(y - j)|^A dy)^{1/A}
$$

is a quasi-norm on the finitely dimensional space $\mathbb{R}^{\#K}$, where $\#K$ is the sum of the cardinality of K_s over $s = 1, 2, \cdots, 2^n - 1$ (see Lemma 3.2). By the equivalence between different quasi-norm on $\mathbb{R}^{\#K}$ there exists a constant C such that

$$(\int_{[0,1]^n} |\sum_{s=1}^{2^n-1} \sum_{j\in K_s} d_{j,s}\psi_s(y-j)|^A dy)^{1/A} \geq C \sum_{s=1}^{2^n-1} (\sum_{j\in K_s} |d_{j,s}|^q)^{1/q}.$$

Let D be an integer chosen that $\psi_s \not\equiv 0$ on $[-D, D]^n$ for all $s = 1, \cdots, 2^n - 1$. Then we have

$$(\int_{x+[-D-1,D+1]^n} |\sum_{s=1}^{2^n-1} \sum_{j\in \mathbb{Z}^n} d_{j,s}\psi_s(y-j)|^A dy)^{1/A} \geq C \sum_{s=1}^{2^n-1} (\sum_{j\in \mathbb{Z}^n} |d_{j,s}|^q \chi_{[0,1]^n}(x-j))^{1/q}.$$

Therefore

$$\sum_{s=1}^{2^n-1} \Big(\sum_{l\geq 0} \sum_{j\in \mathbb{Z}^n} (|a_{l,j,s}| 2^{l(\alpha+n/2)} \chi_{[0,1]^n}(2^l x-j))^q\Big)^{1/q} \leq \Big(\sum_{l\geq 0} 2^{l\alpha q}(M_A(|Q_l f|)(x))^q\Big)^{1/q}$$

and

$$\begin{aligned} &\sum_{s=1}^{2^n-1} \Big\|\Big(\sum_{l\geq 0} \sum_{j\in \mathbb{Z}^n} (|a_{l,j,s}| 2^{l(\alpha+n/2)} \chi_{[0,1]^n}(2^l \cdot -j))^q\Big)^{1/q}\Big\|_p \\ \leq\ & C\|(\sum_{l\geq 0} 2^{l\alpha q}(M_A(|Q_l f|))^q)^{1/q}\|_p \leq C\|(\sum_{l\geq 0} 2^{l\alpha q}|Q_l f|^q)^{1/q}\|_p \end{aligned}$$

by Theorem A.1 in [10, p.147]. ♠

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