TOTAL POSITIVITY AND REFINABLE FUNCTIONS WITH GENERAL DILATION

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Abstract. We show that a refinable function ϕ with dilation $M \geq 2$ is a ripplet, i.e., the collocation matrices of its shifts are totally positive, provided that the symbol p of its refinement mask satisfies certain conditions. The main condition is that p (of degree n) satisfies what we term condition (I), which requires that n determinants of the coefficients of p are positive and generalises the conditions of Hurwitz for a polynomial to have all negative zeros. We also generalise a result of Kemperman to show that (I) is equivalent to an M-slanted matrix of the coefficients of p being totally positive. Under condition (I), the ripplet ϕ satisfies a generalisation of the Schoenberg-Whitney conditions provided that n is an integer multiple of $M-1$. Moreover (I) implies that polynomials in a polyphase decomposition of p have interlacing negative zeros, and under these weaker conditions we show that ϕ still enjoys certain total positivity properties.

1. Introduction

Take a polynomial

$$p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \tag{1.1}$$

where, without loss of generality, we take $a_0 > 0$. (For a polynomial it is always assumed that the variable z lies in $\mathbf{C}$). A polynomial is called a *Hurwitz polynomial* if all its zeros have strictly negative real part. It was shown by Hurwitz [15] that p is a Hurwitz polynomial if and only if

$$\det(a_{2j-i} : i, j = 1, \ldots, k) > 0, \; k = 1, \ldots, n, \tag{1.2}$$

where we put $a_j = 0$ for $j < 0$ and $j > n$. It is also shown in [11] that p in (1.1) is a Hurwitz polynomial if and only if the polynomials $\sum_{j\in\mathbf{Z}} a_{2j} z^j$ and $\sum_{j\in\mathbf{Z}} a_{2j+1} z^j$ have interlacing negative zeros (we discuss this more precisely in Section 2).

A third characterisation of Hurwitz polynomials was given by Kemperman who showed in [17] that if p as in (1.1) is a Hurwitz polynomial, then the matrix (a_{2j-i}) is totally positive (i.e., has all its minors non-negative) and any minor is strictly positive if and only if its diagonal elements are strictly positive. For any Hurwitz polynomial p as in (1.1), $a_j > 0, j = 1, \ldots, n$, and this together with Kemperman's result implies (1.2).

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We now turn our attention to refinable functions. It is shown in [13] that if p as in (1.1) is a Hurwitz polynomial with $n \geq 2$ satisfying $p(-1) = 0$ and $p(1) = 2$, such that $p(z)/(z+1)$ has non-negative coefficients, then there is a continuous function ϕ satisfies

$$\phi(x) = \sum_{j=0}^{n} a_j \phi(2x - j), \ x \in \mathbf{R}, \tag{1.3}$$

and

$$\sum_{j \in \mathbf{Z}} \phi(x - j) = 1, \ x \in \mathbf{R}.$$

The equation (1.3) is called a *refinement equation* and ϕ is called a *refinable function.* It is also shown in [13] that ϕ is what is termed there a *ripplet*, i.e., for any $s \geq 1$, $x_1 < \ldots < x_s$, and integer $l_1 < \ldots < l_s$,

$$\det(\phi(x_i - l_j) : i, j = 1, \ldots, s) \geq 0. \tag{1.4}$$

The concept of a ripplet is intermediate between two concepts which have well-known characterisations. The weaker concept is that (1.4) holds when $x_1, \ldots, x_s$ are integers, which is equivalent to the polynomial $\sum_{j \in \mathbf{Z}} \phi(j) z^j$ having negative zeros [1]. The stronger concept is that (1.4) holds when we allow any real numbers $l_1 < \ldots < l_s$. In this case ϕ is called a Pólya frequency function, and such functions have been given two further characterisations, see [16]. If ϕ is a ripplet, then it has properties which are valuable for the construction of curves in computer-aided geometric design, see [12]. These properties were also used in [6] in deriving results about asymptotic normality of refinable functions. It is also shown in [13] that the ripplet ϕ gives strict inequality in (1.4) if and only if $\phi(x_j - l_j) > 0, j = 1, \ldots, s$, which is a generalisation of the Scheonberg-Whitney conditions [18].

Many results on refinable functions extend to refinement equations of the form

$$\phi(x) = \sum_{j=0}^{n} a_j \phi(Mx - j), \ x \in \mathbf{R}, \tag{1.5}$$

for an integer $M \geq 3$. For example all work in [6] is for general M except that which depends on ϕ being a ripplet. However more general dilation factors can allow situations which are not possible for $M = 2$. Thus the symmetric orthogonal wavelets (SOW) and cardinal orthogonal wavelets (COW) for $M \geq 3$, which are not possible for $M = 2$, are constructed (see [7] for SOW with $M = 3$, [14] for SOW with $M = 4$, [2] for SOW with $M \geq 3$, and [3] for COW with $M \geq 3$). Also there are examples of refinable functions whose integer translates are globally but not locally linearly independent ([10, 8] for $M = 3$ and [9] for $M \geq 3$), a property which is again not possible for $M = 2$. (We shall give an example of a refinable ripplet with this property in Section 3.)

In this paper, we investigate analogues for $M \geq 3$ of the results discussed above. It turns out that there is an interesting mixture of complete generalisations, partial generalisations and situations which appear to have no such analogues. In Section 2, we give a natural generalisation of condition (1.2) to n inequalities for $M \geq 3$ which we refer to as (I). We show that (I) implies that $a_j > 0, j = 1, \ldots, n$, and that the matrix $(a_{Mj-i})_{i,j\in\mathbf{Z}}$ is totally positive with any minor strictly positive if and only if its diagonal elements are strictly positive. We also show that (I) implies that the polynomials $\sum_{j\in\mathbf{Z}} a_{Mj+k} z^{r-j}, k = 0, \ldots, M-1$, have interlacing negative zeros, a property we refer to as (II). However, it is not true that (II) implies (I). Moreover there appears to be no expression for (I) and (II) in terms of zeros of p, indeed neither (I) nor (II) is closed under multiplication of polynomials.

In Section 3, we show that if p satisfies (I) with $n \geq M$ and is of the form $p(z) = (z^{M-1} + z^{M-2} + \cdots + 1)q(z)$ where the polynomial q has non-negative coefficients and satisfies $q(1) = 1$, then (1.5) has a solution which is a ripplet. The corresponding generalised Scheonberg-Whitney conditions do not hold in general, but are valid when n is an integer multiple of $M-1$. We also show that if p is as above but satisfying only the weaker condition (II), then for any integer k, the matrix $(\phi(i-j+k/(M-1)))_{i,j\in\mathbf{Z}}$ is totally positive and any minor is strictly positive if and only if its diagonal elements are strictly positive.

2. Zeros and Coefficients of Polynomials

Take an integer $M \geq 2$. For $n \geq 0$, we consider a polynomial of exact degree n,

$$p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \tag{2.1}$$

where we always assume, without loss of generality, $a_0 > 0$. We shall put $a_j = 0$ for $j < 0$ and $j > n$. For $k = 1, \ldots, n$, put $k = (M-1)\alpha + \beta$ for integers α, β with $1 \leq \beta \leq M-1$, and define

$$\begin{aligned} \triangle_k &:= \det(a_{Mj-i+\beta} : i, j = 0, \ldots, \alpha) \\ &= \begin{vmatrix} a_\beta & \cdots & a_{\beta+M\alpha} \\ \vdots & \ddots & \vdots \\ a_{\beta-\alpha} & \cdots & a_k \end{vmatrix}. \end{aligned}$$

We consider the conditions on p:

(I) $\qquad \triangle_k > 0, \quad k = 1, \ldots, n.$

Note that for $k = 1, \ldots, M-1$, we have $\alpha = 0$ and $\beta = k$ and so $\triangle_k = a_k$. Also $\triangle_n = a_n \triangle_{n-M+1}$ and so the condition $\triangle_n > 0$ is equivalent to $a_n > 0$. We note that for $1 \leq n \leq M$, (I) is equivalent to $a_j > 0, j = 0, \ldots, n$.

The following result generalizes work of Kemperman in [17], who proved it for $M = 2$.

Theorem 2.1. *Suppose that p satisfies (I). Then the matrix $A = (a_{Mj-i})_{i,j\in\mathbf{Z}}$ is totally positive, $a_j > 0, j = 0, \ldots, n$, and any minor of A is strictly positive if and only if its diagonal elements are strictly positive.*

Our proof follows closely the work in [17]. We shall use two preliminary results.

Lemma 2.2. *If p has exact degree $n \geq 1$ and satisfies (I), then there is a unique polynomial q of exact degree $n-1$,*

$$q(z) = b_0 z^{n-1} + b_1 z^{n-2} + \cdots + b_{n-1}, \tag{2.2}$$

satisfying (I) and for $j \in \mathbf{Z}$,

$$a_{Mj+k} = b_{Mj+k-1}, \ k = 1, \ldots, M-1, \tag{2.3}$$

$$a_{Mj} = b_{Mj-1} + \frac{a_0}{a_1} b_{Mj}, \tag{2.4}$$

where $b_j = 0$ for $j < 0$ and $j \geq n$.

Proof. Let $c = a_0/a_1$. Clearly (2.3) and (2.4) are equivalent to

$$\begin{aligned} b_{Mj+k} &= a_{Mj+k+1}, \ k = 0, \ldots, M-2, \\ b_{Mj-1} &= a_{Mj} - c a_{Mj+1}. \end{aligned}$$

Note that $b_{-1} = a_0 - c a_1 = 0$ and so $b_j = 0$ for $j < 0$ and $j \geq n$. Also $b_0 = a_1 > 0$.

It remains only to show that q satisfies (I). For $k = 1, \ldots, n-1$, let $\triangle'_k = \det(b_{Mj-i+\beta'}, i, j = 0, \ldots, \alpha')$, where α', β' are defined as for (I). In evaluating $\triangle_k$ as in (I), for any row with $i = \beta + Ml, l \in \mathbf{Z}$, we subtract c times the previous row from the i-th row to give, for $k = 2, \ldots, n$,

$$\triangle_k = \begin{cases} \triangle'_{k-1}, & 2 \leq \beta \leq M-1, \\ a_1 \triangle'_{k-1}, & \beta = 1, \end{cases}$$

where β is defined as for (I). Thus $\triangle'_k > 0, \ k = 1, \ldots, n-1$, i.e., q satisfies (I). □

Lemma 2.3. *Suppose that $A = (a_{ij})_{i,j \in \mathbf{Z}}$ and $B = (b_{ij})_{i,j \in \mathbf{Z}}$ are matrices and for some $l \in \mathbf{Z}, c > 0$, we have for all integers j,*

$$\begin{aligned} a_{ij} &= b_{ij}, \ i \neq l, \\ a_{lj} &= b_{lj} + c b_{l-1,j}. \end{aligned}$$

If B is totally positive and satisfies the condition that any minor is strictly positive if and only if its diagonal elements are strictly positive, then the same holds for A.

Proof. We use the usual notation that $A \begin{pmatrix} i_1 & \cdots & i_r \\ j_1 & \cdots & j_r \end{pmatrix}$ denotes the determinant of the matrix comprising rows $i_1, \ldots, i_r$ and columns $j_1, \ldots, j_r$ of A.

Suppose that A and B satisfy the conditions of the Lemma and $r \geq 1$, $i_1 < \cdots < i_r, j_1 < \cdots < j_r$. If $\{i_1, \ldots, i_r\}$ does not contain l, or it contains both l and $l-1$, then

$$A \begin{pmatrix} i_1 & \cdots & i_r \\ j_1 & \cdots & j_r \end{pmatrix} = B \begin{pmatrix} i_1 & \cdots & i_r \\ j_1 & \cdots & j_r \end{pmatrix}.$$

Suppose that for some $m, 1 \leq m \leq r$, $i_m = l, i_{m-1} \neq l-1$. Then

$$A \begin{pmatrix} i_1 & \cdots & i_r \\ j_1 & \cdots & j_r \end{pmatrix} = B \begin{pmatrix} i_1 & \cdots & i_r \\ j_1 & \cdots & j_r \end{pmatrix} + cB \begin{pmatrix} i'_1 & \cdots & i'_r \\ j_1 & \cdots & j_r \end{pmatrix},$$

where $(i'_1, \ldots, i'_r)$ denotes $(i_1, \ldots, i_r)$ with i_m replaced by $l-1$. Then we have $A\begin{pmatrix} i_1 & \cdots & i_r \\ j_1 & \cdots & j_r \end{pmatrix} \geq 0$ with strict inequality if and only if $B\begin{pmatrix} i_1 & \cdots & i_r \\ j_1 & \cdots & j_r \end{pmatrix} > 0$ or $B\begin{pmatrix} i'_1 & \cdots & i'_r \\ j_1 & \cdots & j_r \end{pmatrix} > 0$ if and only if $b_{i_s,j_s} > 0, 1 \leq s \leq r, s \neq m$ and either $b_{l,j_m} > 0$ or $b_{l-1,j_m} > 0$, which is equivalent to $a_{i_s,j_s} > 0, 1 \leq s \leq r$. The result follows. □

Proof of Theorem 2.1. The proof is by induction on n. If $n \leq M-1$, then (I) gives $a_j > 0, j = 0, \ldots, n$, and since every row of A contains at most one non-zero element, the result is clearly true.

Suppose that $n \geq 1$ and the result holds when n is replaced by $n-1$. By Lemma 2.2, $a_j > 0, j = 0, \ldots, n$. Also by Lemma 2.2 and successive application of Lemma 2.3 to the matrices $A = (a_{Mj-i+1})$ and $B = (b_{Mj-i})$, we see that A is totally positive and any minor of A is strictly positive if and only if its diagonal elements are strictly positive. □

It is well-known, see [11], that for $M = 2$, p satisfies (I) if and only if it is a Hurwitz polynomial, i.e., its zeros have strictly negative real part. There appears to be no corresponding characterisation for $M \geq 3$.

To see this, we consider an example for $M = 3$. Take $n = 4$ and for $\alpha, \beta \in [0, \pi]$. let

$$\begin{aligned} p(z) &= (z + e^{i\alpha})(z + e^{-i\alpha})(z + e^{-i\beta})(z + e^{i\beta}) \\ &= z^4 + 2(\cos\alpha + \cos\beta)z^3 + 2(1 + 2\cos\alpha\cos\beta)z^2 + 2(\cos\alpha + \cos\beta)z + 1. \end{aligned}$$

Then the condition (I) becomes

$$1 + 2\cos\alpha\cos\beta > 0 \quad \text{and} \quad \cos\alpha + \cos\beta > 1/2.$$

But we may assume that $\cos\alpha > 0$, and then $\cos\alpha + \cos\beta > 1/2$ implies that

$$-1 - 2\cos\alpha\cos\beta < -1 + 2\cos\alpha(\cos\alpha - 1/2) < 0.$$

So (I) is satisfied if and only if $\cos\alpha + \cos\beta > 1/2$. Thus there are polynomials of the above form which are Hurwitz polynomials but do not satisfies (I), and other such polynomials which satisfies (I) but which are not Hurwitz polynomials. Moreover, since all quadratic polynomials with positive coefficients satisfies (I), we see that the set of polynomials satisfying (I) is not closed under multiplication.

Note, however, that for any integer $l \geq 2$, the matrix $(a_{Mlj-i})_{i,j\in\mathbf{Z}}$ is a submatrix of $(a_{Mj-i})_{i,j\in\mathbf{Z}}$ and so, from Theorem 2.1, if p satisfies (I), then p also satisfies (I) with M replaced by Ml. In particular, if p is a Hurwitz polynomial, then p satisfies (I) for all even M. We also have the following result.

Theorem 2.4. *Suppose that p has degree $n \leq (M-1)m + 1$ for some integer $m \geq 1$. If p has its roots in the sector $\{-re^{-iu} : r > 0, |u| < \pi/(m+1)\}$, then p satisfies (I).*

Proof. Suppose that p has its roots in the above sector. Then by a result of Schoenberg ([16, p. 415]), the matrix (a_{j-i}) has all minors up to order m non-negative, and they are strictly positive if the diagonal elements are strictly positive. If $n-1 \le (M-1)m$, then in condition (I) the order of any determinant $\triangle_k$ is at most m. Since p is a Hurwitz polynomial, we have $a_j > 0, j = 0, \ldots, n$ and so (I) is satisfied. □

It is shown in [11] that condition (I) for $M = 2$ is satisfied by p if and only if p_e and p_o have interlacing negative zeros, where $p_e(z^2) = p(z) + p(-z), zp_o(z^2) = p(z) - p(-z)$. We shall give a partial generalisation of this result to general M. First we define what we mean by interlacing negative zeros.

Let $p_0, \ldots, p_m$ be polynomials of exact degree $r \ge 1$. By canceling any common power of z we may assume that at least one of the polynomials is non-zero at $z = 0$. We say that $p_0, \ldots, p_m$ have *interlacing negative zeros* if for $k = 0, \ldots, m$, p_k has zeros $\alpha_1^k < \ldots < \alpha_r^k \le 0$ such that

$$\alpha_j^m < \alpha_{j+1}^0,\ j = 1, \ldots, r-1,$$

$$\alpha_j^k \le \alpha_j^{k+1},\ k = 0, \ldots, m-1,\ j = 1, \ldots, r,$$

with equality only if $j = r, k \ge 1$ and $\alpha_r^k = 0$.

Now take p as in (2.1) and let $n = Mr + s, 0 \le s \le M-1$. For $k = 0, \ldots, M-1$, we define the polyphase decomposition of p by

$$A_k p(z) = \sum_{j \in \mathbf{Z}} a_{k+Mj} z^{r-j}, \tag{2.5}$$

recalling that $a_j = 0$ for $j < 0$ and $j > n$. We note that for $\omega = e^{2\pi i/M}$,

$$A_k p(z^M) = \frac{1}{M} z^{k-s} \sum_{l=0}^{M-1} \omega^{(k-s)l} p(\omega^l z). \tag{2.6}$$

We now consider the conditions on p:

(II) $\quad A_0 p, \cdots, A_{M-1} p$ have interlacing negative zeros.

Theorem 2.5. *If p satisfies (I), then p satisfies (II).*

In order to prove Theorem 2.5 we first recall Lemma 2.2 and show the following.

Lemma 2.6. *Suppose that p is given by (2.1) with $n \ge M+1$, $a_0, \ldots, a_{M-1} > 0$, and q is given by (2.2), (2.3), (2.4) (where $b_j = 0$ for $j < 0$ and $j \ge n$). If $A_0 q, \ldots, A_{M-1} q$ have interlacing negative zeros then so do $A_0 p, \ldots, A_{M-1} p$.*

Proof. As before we write $n = Mr + s, 0 \le s \le M-1$. Then $n - 1 = Mr' + s'$, where $r' = r - \alpha$ and

$$\alpha = \begin{cases} 1, & s = 0, \\ 0, & 1 \le s \le M-1. \end{cases}$$

Now for $k = 1, \dots, M-1$,

$$A_k p(z) = \sum_{j \in \mathbf{Z}} a_{k+Mj} z^{r-j} = \sum_{j \in \mathbf{Z}} b_{k-1+Mj} z^{r'+\alpha-j} = z^\alpha A_{k-1} q(z)$$

Also for $c = a_0/a_1$,

$$\begin{aligned} A_0 p(z) &= \sum_{j \in \mathbf{Z}} a_{Mj} z^{r-j} = \sum_{j \in \mathbf{Z}} b_{Mj-1} z^{r'+\alpha-j} + c \sum_{j \in \mathbf{Z}} b_{Mj} z^{r'+\alpha-j} \\ &= z^{\alpha-1} A_{M-1} q(z) + c z^\alpha A_0 q(z). \end{aligned}$$

Suppose that $A_0 q, \dots, A_{M-1} q$ have interlacing negative zeros. For $k = 0, \dots, M-1$, let $A_k q$ have zeros $\beta_1^k < \dots < \beta_{r'}^k$. First suppose $\alpha = 0$. For $k = 1, \dots, M-1$, $A_k p = A_{k-1} q$ and so $A_k p$ has zeros $\alpha_j^k = \beta_j^{k-1}, j = 1, \dots, r$. Now for $j = 1, \dots, r$,

$$(-1)^{j+r} A_0 p(\beta_j^0) = (-1)^{j+r} (\beta_j^0)^{-1} A_{M-1} q(\beta_j^0) > 0.$$

Also for $j = 2, \dots, r$,

$$(-1)^{j+r} A_0 p(\beta_{j-1}^{M-2}) = (-1)^{j+r} (\beta_{j-1}^{M-2})^{-1} A_{M-1} q(\beta_{j-1}^{M-2}) + (-1)^{j+r} c A_0 q(\beta_{j-1}^{M-2}) < 0$$

since

$$(-1)^{j+r} A_{M-1} q(\beta_{j-1}^{M-2}) > 0, \; (-1)^{j+r} A_0 q(\beta_{j-1}^{M-2}) < 0.$$

So $A_0 p$ has a zero α_j^0 in $(\beta_{j-1}^{M-2}, \beta_j^0) = (\alpha_{j-1}^{M-1}, \alpha_j^1), j = 2, \dots, r$. Also we have $\lim_{x \to -\infty} (-1)^r A_0 p(x) > 0$ and so $A_0 p$ has a zero α_1^0 in $(-\infty, \beta_1^0) = (-\infty, \alpha_1^1)$. So $A_0 p, \dots, A_{M-1} p$ have interlacing negative zeros.

Next take $\alpha = 1$. For $k = 1, \dots, M-1$, $A_k p(z) = z A_{k-1} q(z)$ and so $A_k p$ has zeros $\alpha_j^k = \beta_j^{k+1}, j = 1, \dots, r-1, \alpha_r^k = 0$. For $j = 2, \dots, r$,

$$(-1)^{j+r} A_0 p(\beta_j^0) = (-1)^{j+r} A_{M-1} q(\beta_j^0) > 0,$$

(where we put $\beta_r^0 = 0$) and

$$(-1)^{j+r} A_0 p(\beta_{j-1}^{M-1}) = (-1)^{j+r} c \beta_{j-1}^{M-1} A_0 q(\beta_{j-1}^{M-1}) < 0,$$

and so $A_0 p$ has a zero α_j^0 in $(\beta_{j-1}^{M-1}, \beta_j^0) \subset (\beta_{j-1}^{M-2}, \beta_j^0) = (\alpha_{j-1}^{M-1}, \alpha_j^1)$. Also, as before, $A_0 p$ has a zero α_1^0 in $(-\infty, \beta_1^0) = (-\infty, \alpha_1^1)$. So again $A_0 p, A_1 p, \dots, A_{M-1} p$ have interlacing negative zeros. □

Proof of Theorem 2.5. Suppose that p satisfies (I). For $n \leq M-1$, $A_k p, k = 1, \dots, M-1$, have degree 0 and (II) follows trivially. For $n = M$, $A_0 p(z) = a_0 z + a_M$, $A_k p(z) = a_k z, k = 1, \dots, M-1$, and since $a_j > 0, j = 0, \dots, M$, (II) holds.

We now prove the result by induction on n. Take $n \geq M+1$ and suppose the result is true with n replaced by $n-1$. By Lemma 2.2, q satisfies (I) and so, by our inductive hypothesis, $A_0 q, \dots, A_{M-1} q$ have interlacing negative zeros. So, by Lemma 2.6, $A_0 p, \dots, A_{M-1} p$ have interlacing negative zeros, i.e., p satisfies (II). □

The converse of Theorem 2.5 is not true in general for $M \geq 3$, as the following example shows.

Take $M = 3$ and $p(z) = z^6 + z^5 + 2z^4 + 3z^3 + 2z^2 + z + 1$. Then p does not satisfies (I) since

$$\begin{vmatrix} a_1 & a_4 & 0 \\ a_0 & a_3 & a_6 \\ 0 & a_2 & a_5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 0 & 2 & 1 \end{vmatrix} = -1.$$

However

$$A_0 p(z) = z^2 + 3z + 1, \ A_1 p(z) = z^2 + 2z, \ A_2 p(z) = 2z^2 + z,$$

and a simple calculation shows that these have interlacing negative zeros.

By symmetry we can see that if p as in (2.1) satisfies (I), respectively (II), then the polynomial $q(z) = z^n p(z^{-1})$ also satisfies (I), respectively (II). Our final two results give further information about which polynomials satisfies (I) or (II).

Theorem 2.7. *Take $\lambda > 0$ and let $q(z) = p(\lambda z)$ and $Q(z) = (z + \lambda)p(z)$. If p satisfies (I), then q and Q satisfy (I). If p satisfies (II), then q and Q satisfy (II).*

Proof. Suppose that p satisfies (I), i.e., $\triangle_k > 0, k = 1, \ldots, n$. The determinant corresponding to $\triangle_k$ for q is $\tilde{\triangle}_k := \det(\lambda^{n-Mj+i-\beta} a_{Mj-i+\beta} : i, j = 0, \ldots, \alpha)$ and so $\tilde{\triangle}_k > 0, k = 1, \ldots, n$, i.e., q satisfies (I).

Now for

$$Q(z) = \sum_{j \in \mathbf{Z}} c_j z^{n+1-j},$$

where

$$c_j = a_j + \lambda a_{j-1}, \quad j \in \mathbf{Z}.$$

Let $A = (a_{ij})_{i,j \in \mathbf{Z}}$, $C = (c_{ij})_{i,j \in \mathbf{Z}}$, where

$$a_{ij} = a_{Mj-i}, \quad c_{ij} = c_{Mj-i}, \quad i, j \in \mathbf{Z}.$$

By Theorem 2.1, A is totally positive, $a_j > 0$, $j = 0, \ldots, n$, and any minor of A is strictly positive if and only if its diagonal elements are strictly positive. Thus $c_j > 0$, $j = 0, \ldots, n+1$, and since

$$c_{ij} = a_{ij} + \lambda a_{i+1,j}, \quad i, j \in \mathbf{Z},$$

it follows as in Lemma 2.3 that C is totally positive, $c_j > 0$, $j = 0, \ldots, n$, and any minor of C is strictly positive if and only if its diagonal elements are strictly positive. Thus Q satisfies (I).

Next suppose that p satisfies (II). For $k = 0, \ldots, M-1$,

$$A_k q(z) \ = \ \lambda^{n-k-Mr} A_k p(\lambda^M z).$$

Thus $A_0q, \cdots, A_{M-1}q$ have interlacing negative zeros, and hence q satisfies (II). Now

$$Q(z) = \sum_{j=0}^{n+1} (a_j + \lambda a_{j-1}) z^{n+1-j}.$$

Note that $n+1 = Mr' + s', 0 \le s' \le M-1$, where $r' = r + \alpha$,

$$\alpha = \begin{cases} 0, & 0 \le s \le M-2, \\ 1, & s = M-1. \end{cases}$$

Then for $k = 0, \ldots, M-1$,

$$\begin{aligned} A_kQ(z) &= \sum_{j\in\mathbf{Z}} (a_{k+Mj} + \lambda a_{k+Mj-1}) z^{r'-j} \\ &= z^\alpha \sum_{j\in\mathbf{Z}} a_{k+Mj} z^{r-j} + \lambda z^\alpha \sum_{j\in\mathbf{Z}} a_{k-1+Mj} z^{r-j}, \end{aligned}$$

and so

$$\begin{aligned} A_kQ(z) &= z^\alpha A_kp(z) + \lambda z^\alpha A_{k-1}p(z),\ k = 1, \ldots, M-1, \\ A_0Q(z) &= z^\alpha A_0p(z) + \lambda z^{\alpha-1} A_{M-1}p(z). \end{aligned}$$

First suppose $\alpha = 0$. Then

$$A_kQ(0) \begin{cases} > 0, & 0 \le k \le s+1, \\ = 0, & s+2 \le k \le M-1. \end{cases}$$

For $k = 1, \ldots, M-1$, suppose that A_kp has zeros $\alpha_1^k < \ldots < \alpha_r^k$. Take $1 \le j \le r, 1 \le k \le M$ with $\alpha_j^{k-1} < \alpha_j^k$. Now

$$\begin{aligned} (-1)^{j+r} A_kQ(\alpha_j^{k-1}) &= (-1)^{j+r} A_kp(\alpha_j^{k-1}) < 0, \\ (-1)^{j+r} A_kQ(\alpha_j^k) &= (-1)^{j+r} \lambda A_{k-1}p(\alpha_j^k) > 0, \end{aligned}$$

and so A_kQ has a zero β_j^k in $(\alpha_j^{k-1}, \alpha_j^k)$. Also for $2 \le j \le r$,

$$\begin{aligned} (-1)^{j+r} A_0Q(\alpha_{j-1}^{M-1}) &= (-1)^{j+r} A_0p(\alpha_{j-1}^{M-1}) < 0, \\ (-1)^{j+r} A_0Q(\alpha_j^0) &= (-1)^{j+r} \lambda (\alpha_j^0)^{-1} A_{M-1}p(\alpha_j^0) > 0, \end{aligned}$$

and so A_0q has a zero β_j^0 in $(\alpha_{j-1}^{M-1}, \alpha_j^0)$. Since

$$\begin{aligned} &(-1)^{1+r} A_0Q(\alpha_1^0) = (-1)^{1+r} \lambda (\alpha_1^0)^{-1} A_{M-1}p(\alpha_1^0) > 0, \\ &\lim_{x\to-\infty} (-1)^r A_0Q(x) > 0, \end{aligned}$$

A_0Q also has a zero β_1^0 in $(-\infty, \alpha_1^0)$. Thus $A_0Q, \ldots, A_{M-1}Q$ have interlacing negative zeros for the case $\alpha = 0$.

Next take $\alpha = 1$. Then

$$A_kQ(0) \begin{cases} > 0, & k = 0, \\ = 0, & 1 \le k \le M-1, \end{cases}$$

and so for $k = 1, \ldots, M-1$, A_kQ has a zero $\beta^k_{r+1} = 0$. Also $A_0Q(\alpha_r^{M-1}) = \alpha_r^{M-1}A_0p(\alpha_r^{M-1}) < 0$ and so A_0Q has a zero β^0_{r+1} in $(\alpha_r^{M-1}, 0)$. As before we see that for $1 \le j \le r, 1 \le k \le M$, A_kQ has a zero β_j^k in $(\alpha_j^{k-1}, \alpha_j^k)$, while for $2 \le j \le r$, A_0Q has a zero β_j^0 in $(\alpha_{j-1}^{M-1}, \alpha_j^0)$, and A_0Q has a zero β_1^0 in $(-\infty, \alpha_1^0)$. So $A_0Q, \ldots, A_{M-1}Q$ have interlacing negative zeros for the case $\alpha = 1$.

Thus Q satisfies (II). □

Theorem 2.8. *If p satisfies (II), then the polynomial $(z^{M-1} + z^{M-2} + \cdots + 1)p(z)$ satisfies (II).*

Proof. Suppose that p as in (1.1) satisfies (II). Let

$$q(z) = (z^{M-1} + \cdots + 1)p(z) = \sum_{j\in\mathbf{Z}} b_j z^{n+M-1-j},$$

where

$$b_j = \sum_{l=0}^{M-1} a_{j-l},\ j \in \mathbf{Z}.$$

As before we write $n = Mr + s, 0 \le s \le M-1$. Then $n + M - 1 = Mr' + s', 0 \le s' \le M-1$, where $r' = r + 1 - \alpha$ with

$$\alpha = \begin{cases} 1, & s = 0, \\ 0, & 1 \le s \le M-1. \end{cases}$$

For $k = 0, \ldots, M-1$,

$$\begin{aligned} A_kq(z) &= \sum_{j\in\mathbf{Z}} b_{k+Mj} z^{r+1-\alpha-j} = \sum_{l=0}^{M-1}\sum_{j\in\mathbf{Z}} a_{Mj+k-l} z^{r+1-\alpha-j} \\ &= z^{1-\alpha}\sum_{l=0}^{k} A_{k-l}p(z) + z^{-\alpha}\sum_{l=k+1}^{M-1} A_{M+k-l}p(z) \\ &= z^{1-\alpha}\sum_{l=0}^{k} A_l p(z) + z^{-\alpha}\sum_{l=k+1}^{M-1} A_l p(z). \end{aligned} \tag{2.7}$$

First suppose $n \le M-1$. Then $\alpha = 0$ and $A_kp(z) = a_k, k = 0, \ldots, M-1$. So for $k = 0, \ldots, M-1$,

$$A_kq(z) = z\sum_{l=0}^{k} a_l + \sum_{l=k+1}^{n} a_l,$$

which has zero $\beta_1^k = -\sum_{l=k+1}^n a_l / \sum_{l=0}^k a_l$. So $\beta_1^k \le \beta_1^{k+1}, k = 0, \ldots, n-1$, and $\beta_1^k = 0, n \le k \le M-1$. Thus $A_0q, \ldots, A_{M-1}q$ have interlacing negative zeros and so q satisfies (II).

Now take $n \geq M$ and for $k = 0, \dots, M-1$, suppose that $A_k p$ has zeros $\alpha_1^k < \dots < \alpha_r^k$. First suppose $\alpha = 0$. Then $s' = s-1$ and

$$A_k q(0) \begin{cases} > 0, & 0 \leq k \leq s-1, \\ = 0, & s \leq k \leq M-1. \end{cases}$$

So for $s \leq k \leq M-1$, $A_k q$ has a zero $\beta_{r+1}^k = 0$. We shall show by induction that for all other cases, $0 \leq k \leq M-2, 1 \leq j \leq r$, $A_k q$ has a zero β_{j+1}^k in $(\alpha_j^{k+1}, \alpha_{j+1}^k)$, and $A_{M-1} q$ has a zero β_j^{M-1} in $(\alpha_j^0, \alpha_j^{M-1})$, where $\beta_{j+1}^k < \beta_{j+1}^{k+1}, k = 0, \dots, M-2$, and $\beta_j^{M-1} < \beta_{j+1}^0$. Here we put $\alpha_{r+1}^k = 0$. Now by (2.7),

$$A_{s-1} q(\alpha_r^s) = \alpha_r^s \sum_{l=0}^{s-1} A_l p(\alpha_r^s) + \sum_{l=s+1}^{M-1} A_l p(\alpha_r^s) < 0,$$

since $A_l p(\alpha_r^s) > 0, l = 0, \dots, s-1$, and $A_l p(\alpha_r^s) < 0, l = s+1, \dots, M-1$. Since $A_{s-1} q(0) > 0$, $A_{s-1} q$ has a zero β_{r+1}^{s-1} in $(\alpha_r^s, 0)$.

Next suppose that the induction hypothesis is true for some k and $j, 1 \leq k \leq M-1, 1 \leq j \leq r$. As above (2.7) gives

$$(-1)^{r+j} A_{k-1} q(\alpha_j^k) = (-1)^{r+j} \alpha_j^k \sum_{l=0}^{k-1} A_l p(\alpha_j^k) + (-1)^{r+j} \sum_{l=k+1}^{M-1} A_l p(\alpha_j^k) < 0, \tag{2.8}$$

(2.9)

$$(-1)^{r+j+1} A_{k-1} q(\alpha_{j+1}^{k-1}) = (-1)^{r+j+1} \alpha_{j+1}^{k-1} \sum_{l=0}^{k-2} A_l p(\alpha_{j+1}^{k-1}) + (-1)^{r+j+1} \sum_{l=k}^{M-1} A_l p(\alpha_{j+1}^{k-1}) < 0.$$

Also by (2.7),

$$(-1)^{r+j} (A_k q(z) - A_{k-1} q(z)) = (-1)^{r+j} (z-1) A_k p(z) < 0$$

on $(\alpha_j^k, \alpha_{j+1}^k)$ and so

$$(-1)^{r+j} A_{k-1} q(\beta_{j+1}^k) > (-1)^{r+j} A_k q(\beta_{j+1}^k) = 0. \tag{2.10}$$

Thus by (2.8), (2.9) and (2.10), $A_{k-1} q$ has a zero β_{j+1}^k in $(\alpha_j^k, \alpha_{j+1}^{k-1})$ with $\beta_{j+1}^{k-1} < \beta_{j+1}^k$.

Now we suppose the inductive hypothesis is true for $k = 0$ and some $j, 1 \leq j \leq r$. By (2.7),

$$(-1)^{r+j} A_{M-1} q(\alpha_j^0) = (-1)^{r+j} \alpha_j^0 \sum_{l=0}^{M-1} A_l p(\alpha_j^0) > 0,$$

$$(-1)^{r+j} A_{M-1} q(\alpha_j^{M-1}) = (-1)^{r+j} \alpha_j^{M-1} \sum_{l=0}^{M-2} A_l p(\alpha_j^{M-1}) < 0.$$

Also by (2.7),

$$(-1)^{j+r} (A_{M-1} q(z) - z A_0 q(z)) = (-1)^{j+r} z(1-z) A_0 p(z) < 0$$

on $(\alpha_j^0, \alpha_{j+1}^0)$ and so

$$(-1)^{j+r}A_{M-1}q(\beta_{j+1}^0) < (-1)^{j+r}\beta_{j+1}^0 A_0(\beta_{j+1}^0) = 0.$$

Thus $A_{M-1}q$ has a zero β_j^{M-1} in $(\alpha_j^0, \alpha_j^{M-1})$ with $\beta_j^{M-1} < \beta_{j+1}^0$. So our induction hypothesis is established.

We have shown that $A_{M-1}q$ has a zero β_1^{M-1} in $(\alpha_1^0, \alpha_1^{M-1})$. We now show by induction that for $k = 0, \dots, M-2$, $A_k q$ has a zero β_1^k with $\beta_1^k < \alpha_1^k$, $\beta_1^k < \beta_1^{k+1}$. Take $1 \le k \le M-2$, and suppose $A_{k+1}q$ has a zero $\beta_1^{k+1} < \alpha_1^{k+1}$. Now

$$\lim_{x\to-\infty} (-1)^{r+1}A_k q(z) > 0$$

and by (2.7),

$$(-1)^r A_k q(\alpha_1^k) = (-1)^r \alpha_1^k \sum_{l=0}^{k-1} A_l p(\alpha_1^k) + (-1)^r \sum_{l=k+1}^{M-1} A_l p(\alpha_1^k) > 0.$$

Also

$$(-1)^r (A_{k+1}q(z) - A_k q(z)) = (-1)^r (z-1) A_{k+1} p(z) < 0$$

for $z < \alpha_1^{k+1}$, and so

$$(-1)^r A_k q(\beta_1^{k+1}) > (-1)^r A_{k+1} q(\beta_1^{k+1}) = 0.$$

So $A_k q$ has a zero β_1^k with $\beta_1^k < \alpha_1^k$ and $\beta_1^k < \beta_1^{k+1}$.

Thus we have shown that $A_0 q, \dots, A_{M-1}q$ have interlacing negative zeros and so q satisfies (II).

For the case $\alpha = 1$, when $r' = r$, we can similarly show that for $k = 0, \dots, M-1$, $A_k q$ has zeros $\beta_1^k < \dots < \beta_r^k < 0$, where for $1 \le j \le r$, β_j^k is in $(\alpha_{j-1}^{k+1}, \alpha_j^k)$, $0 \le k \le M-2$, and β_j^{M-1} is in $(\alpha_j^0, \alpha_j^{M-1})$, where $\alpha_0^k = -\infty, k = 1, \dots, M-1$. As before, the zeros of $A_0 q, \dots, A_{M-1}q$ interlace and so q satisfies (II). ☐

The factor $z^{M-1} + z^{M-2} + \cdots + 1$, as in Theorem 2.8, will play an important role in the next section. The following result will also be used in the next section.

Lemma 2.9. *If a polynomial p has a factor of the form $q(z^M)$ for a polynomial q of degree ≥ 1, then p does not satisfies (II).*

Proof. Let $p(z) = q(z^M)r(z)$ for polynomials q, r. Then it is easily seen that for $k = 0, \dots, M-1$, $A_k p(z) = q(z) A_k r(z)$. Thus if q has degree ≥ 1, $A_0 p, \dots, A_{M-1}p$ have a common zero and so cannot have interlacing zeros. ☐

3. Refinable Functions

As before we take an integer $M \ge 2$. We first give a basic result on refinable functions for dilation M. For $M = 2$ this is part of work in [13] and our proof also follows this work.

Theorem 3.1. *Let p as in (2.1) be a polynomial of the form*

$$p(z) = (z^{M-1} + z^{M-2} + \cdots + 1)q(z),$$

where $q(1) = 1$, $q(z) = \sum_{j\in\mathbf{Z}} b_j z^{m-j}, b_j \geq 0, j \in \mathbf{Z}, b_j = 0$ for $j < 0$ and $j > m$, $b_0 > 0$ and $\sum_{j\in\mathbf{Z}} b_{jM} < 1$. Then there is a continuous, non-negative function ϕ such that

$$\phi(x) = \sum_{j=0}^{n} a_j \phi(Mx - j),\ x \in \mathbf{R}, \tag{3.1}$$

$$\sum_{j\in\mathbf{Z}} \phi(x - j) = 1,\ x \in \mathbf{R}. \tag{3.2}$$

Moreover ϕ has support in $[0, n/(M-1)]$ and if $a_j > 0, j = 0, \ldots, n$, then $\phi(x) > 0$ for $0 < x < n/(M-1)$.

Proof. Define $T_p : C(\mathbf{R}) \to C(\mathbf{R})$ by

$$T_p f(x) = \sum_{j\in\mathbf{Z}} a_j f(Mx - j),\ x \in \mathbf{R},$$

where as before we put $a_j = 0$ for $j < 0$ and $j > n$. Now for $\lambda \in \ell_1(\mathbf{Z})$, $f \in C(\mathbf{R}), x \in \mathbf{R}$,

$$\sum_{k\in\mathbf{Z}} \lambda_k (T_p f)(x - k) = \sum_{j\in\mathbf{Z}} (S_p\lambda)_j f(Mx - j),$$

where

$$(S_p\lambda)_j = \sum_{k\in\mathbf{Z}} a_{j-Mk}\lambda_k,\ j \in \mathbf{Z}.$$

So by induction, for any $1 \leq m \in \mathbf{Z}$,

$$\sum_{k\in\mathbf{Z}} \lambda_k (T_p^m f)(x - k) = \sum_{j\in\mathbf{Z}} (S_p^m\lambda)_j f(M^m x - j),\ x \in \mathbf{R}.$$

We choose f to be the B-spline N given by

$$N(x) = \begin{cases} x, & 0 \leq x \leq 1, \\ 2 - x, & 1 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

We choose $\lambda = \delta$, where $\delta_0 = 1$ and $\delta_k = 0$ for $k \neq 0$. Then putting $f_m = T_p^m N, m = 1, 2, \ldots,$

$$f_m(x) = \sum_{j\in\mathbf{Z}} (S_p^m\delta)_j N(M^m x - j),\ x \in \mathbf{R}.$$

It is well-known that

$$N(x) = \sum_{k\in\mathbf{Z}} c_k N(Mx - k),\ x \in \mathbf{R},$$

where

$$Q(z) := \sum_{k\in\mathbf{Z}} c_k z^k = \frac{1}{M}(z^{M-1} + z^{M-2} + \cdots + 1)^2.$$

Then for $x \in \mathbf{R}, m = 1, 2, \dots,$

$$\begin{aligned} f_m(x) &= \sum_{j\in\mathbf{Z}} (S_p^m \delta)_j \sum_{k\in\mathbf{Z}} c_k N(M^{m+1}x - Mj - k) \\ &= \sum_{k\in\mathbf{Z}} \sum_{j\in\mathbf{Z}} (S_p^m \delta)_j c_{k-Mj} N(M^{m+1}x - k). \end{aligned}$$

Also for $x \in \mathbf{R}$,

$$\begin{aligned} f_{m+1}(x) &= \sum_{k\in\mathbf{Z}} (S_p^{m+1}\delta)_k N(M^{m+1}x - k) \\ &= \sum_{k\in\mathbf{Z}} \sum_{j\in\mathbf{Z}} a_{k-Mj} (S_p^m \delta)_j N(M^{m+1}x - k). \end{aligned}$$

Thus

$$f_{m+1}(x) - f_m(x) = \sum_{k\in\mathbf{Z}} A_k N(M^{m+1}x - k),\ x \in \mathbf{R}, \tag{3.3}$$

where

$$A_k = \sum_{j\in\mathbf{Z}} (S_p^m \delta)_j (a_{k-Mj} - c_{k-Mj}),\ k \in \mathbf{Z}. \tag{3.4}$$

Now

$$\begin{aligned} &\sum_{j\in\mathbf{Z}} (a_j - c_j) z^j = z^n p(z^{-1}) - Q(z) \\ = &\ (z^{M-1} + z^{M-2} + \cdot + 1)\left\{ z^m q(z^{-1}) - \frac{1}{M}(z^{M-1} + z^{M-2} + \cdot + 1)\right\} \\ = &\ (z^M - 1)R(z), \end{aligned}$$

for a polynomial $R(z) := \sum_{j\in\mathbf{Z}} d_j z^j$, since $q(1) = 1$. Thus

$$a_j - c_j = d_{j-M} - d_j,\ j \in \mathbf{Z},$$

for a finitely supported sequence $\{d_j\}$. So for $k \in \mathbf{Z}$, from (3.4),

$$A_k = \sum_{j\in\mathbf{Z}} (S_p^m \delta)_j (d_{k-M(j+1)} - d_{k-Mj}) = -\sum_{j\in\mathbf{Z}} (\triangle S_p^m \delta)_j d_{k-Mj},$$

where for a sequence λ, $(\triangle\lambda)_j = \lambda_j - \lambda_{j-1}, j \in \mathbf{Z}$. Hence there is a constant K such that

$$|A_k| \le K \|\triangle S_p^m \delta\|_\infty,$$

and by (3.3),

$$\|f_{m+1} - f_m\|_\infty \le K \|\triangle S_p^m \delta\|_\infty, \tag{3.5}$$

since $\sum_{k\in\mathbf{Z}} N(\cdot - k) = 1$. Now $(z-1)p(z) = (z^M - 1)q(z)$ and so

$$a_{j-1} - a_j = b_{j-M} - b_j,\ j \in \mathbf{Z}.$$

Thus for any sequence λ,

$$\begin{aligned}(\triangle S_p\lambda)_j &= \sum_{k\in\mathbf{Z}}(a_{j-Mk} - a_{j-1-Mk})\lambda_k \\ &= \sum_{k\in\mathbf{Z}}(b_{j-Mk} - b_{j-Mk-M})\lambda_k = \sum_{k\in\mathbf{Z}} b_{j-Mk}(\lambda_k - \lambda_{k-1}).\end{aligned}$$

Now for $k = 0, \ldots, M-1$, $\sum_{j\in\mathbf{Z}} b_{jM+k} < \sum_{j\in\mathbf{Z}} b_j = 1$. Put

$$\rho = \max\left\{\sum_{j\in\mathbf{Z}} b_{jM+k} : k = 0, \ldots, M-1\right\} < 1.$$

Then for $\lambda \in \ell_\infty(\mathbf{Z})$,

$$\|\triangle S_p\lambda\|_\infty \le \rho\|\triangle\lambda\|_\infty,$$

and so we see from (3.5) by induction that

$$\|f_{m+1} - f_m\|_\infty \le K\rho^m,\ m = 1, 2, \cdots.$$

Thus (f_m) is a Cauchy sequence in $C(\mathbf{R})$ and so there is a function ϕ in $C(\mathbf{R})$ with

$$\lim_{m\to\infty} f_m(x) = \phi(x)$$

uniformly on $\mathbf{R}$. Since $f_{m+1} = T_p f_m$, we have $T_p\phi = \phi$, i.e., ϕ satisfies (3.1).

Also for $j \in \mathbf{Z}$, $a_j = \sum_{l=0}^{M-1} b_{j-l}$ and so $\sum_{k\in\mathbf{Z}} a_{j+Mk} = \sum_{j\in\mathbf{Z}} b_j = 1$. Thus if f has compact support and $\sum_{k\in\mathbf{Z}} f(\cdot - k) = 1$, then for $x \in \mathbf{R}$,

$$\begin{aligned}\sum_{k\in\mathbf{Z}} T_p f(x-k) &= \sum_{j,k\in\mathbf{Z}} a_j f(Mx - Mk - j) \\ &= \sum_{j,k\in\mathbf{Z}} a_{j-Mk} f(Mx - j) = 1.\end{aligned}$$

Since $\sum_{k\in\mathbf{Z}} N(\cdot - k) = 1$, we see by induction that $\sum_{k\in\mathbf{Z}} f_m(\cdot - k) = 1, m = 1, 2, \ldots$, and so $\sum_{k\in\mathbf{Z}} \phi(\cdot - k) = 1$.

Now if f_m has support in $[0, b_m]$ for some $b_m > 0$, then $f_{m+1} = T_p f_m$ has support in $[0, (b_m + n)/M]$. With $b_0 = 2$, (b_m) forms a monotone sequence which converges to $n/(M-1)$. Thus ϕ has support in $[0, n/(M-1)]$.

Since $a_j \ge 0, j \in \mathbf{Z}$, f_{m+1} will be non-negative provided that f_m is non-negative. Thus ϕ is non-negative. Now suppose

$$\alpha := \min\{a_j : j = 0, \ldots, n\} > 0.$$

For $n/(M-1) - 1 \le x \le n + 1$,

$$\sum_{j=0}^{n} \phi(x - j) = \sum_{j\in\mathbf{Z}} \phi(x - j) = 1,$$

and so from (3.1)

$$\phi(x/M) = \sum_{j=0}^{n} a_j \phi(x-j) \geq \alpha \sum_{j=0}^{n} \phi(x-j) = \alpha.$$

Thus $\phi(x) \geq \alpha > 0$ on a closed interval I_0 with length

$$\frac{1}{M}\left(n+1-\left(\frac{n}{M-1}-1\right)\right) = \frac{2(M-1)+n(M-2)}{M(M-1)} \geq 1,$$

since $n \geq M-1$. Now if $\phi(x) > 0$ on an interval $I_m = [a_m, b_m]$ of length ≥ 1, then from (3.1), $\phi(x) > 0$ for all x in the interval $I_{m+1} = [a_{m+1}, b_{m+1}]$, where $a_{m+1} = a_m/M$ and $b_{m+1} = (b_m+n)/M$. Since $\lim_{m\to\infty} a_m = 0$ and $\lim_{m\to\infty} b_m = n/(M-1)$, we see by induction that $\phi(x) > 0$ for $0 < x < n/(M-1)$. □

If p as in Theorem 3.1 also satisfies condition (II) of Section 2, we can deduce some total positivity properties of ϕ.

Theorem 3.2. *Let p as in (2.1) be a polynomial which satisfies (II) and is of the form*

$$p(z) = (z^{M-1} + z^{M-2} + \cdots + 1)q(z),$$

where $q(1) = 1, q(z) = \sum_{j\in\mathbf{Z}} b_j z^{m-j}, b_j \geq 0, j \in \mathbf{Z}$, $b_j = 0$ for $j < 0$ and $j > m, m \geq 1$, $b_0 > 0$. Then there is a continuous, non-negative function ϕ satisfying (3.1) and (3.2). Moreover $\phi(x) > 0$ if and only if $0 < x < n/(M-1)$, and for $k \in \mathbf{Z}$, the matrix $A := (\phi(i-j+k/(M-1))_{i,j\in\mathbf{Z}}$ is totally positive and any minor of A is strictly positive if and only if its diagonal elements are positive.

Proof. By Lemma 2.9, $\sum_{j\in\mathbf{Z}} b_{jM} < \sum_{j\in\mathbf{Z}} b_j = 1$. So we can apply Theorem 3.1 to give (3.1) and (3.2). By (II), $a_j > 0, j = 0,\ldots,n$ and so Theorem 3.1 also gives $\phi(x) > 0$ if and only if $0 < x < n/(M-1)$.

Now take $k \in \mathbf{Z}$ and for $m = 0, 1, 2, \ldots$, define f_m as in the proof of Theorem 3.1 and define the polynomial

$$p_m(z) = \sum_{j\in\mathbf{Z}} f_m(j + k/(M-1))z^j.$$

Then for $m = 0, 1, 2, \ldots,$

$$\begin{aligned} p_{m+1}(z) &= \sum_{j\in\mathbf{Z}} T_p f_m(j+k/(M-1))z^j \\ &= \sum_{j\in\mathbf{Z}}\sum_{r\in\mathbf{Z}} a_r f_m(Mj+k+k/(M-1)-r)z^j. \end{aligned} \tag{3.6}$$

Now for $\tilde{p}(z) = \sum_{j\in\mathbf{Z}} a_j z^j$,

$$\tilde{p}(z)p_m(z) = \sum_{j\in\mathbf{Z}} d_j z^j, \tag{3.7}$$

where for $j \in \mathbf{Z}$,

$$d_j = \sum_{r \in \mathbf{Z}} a_r f_m(j - r + k/(M-1)).$$

From (3.6),

$$p_{m+1}(z) = \sum_{j \in \mathbf{Z}} d_{k+mj} z^j. \tag{3.8}$$

Now if p_m has all negative zeros, then by Theorem 2.7, $\tilde{p}p_m$ satisfies (II) and so by (3.7) and (3.8), p_{m+1} has all distinct negative zeros. Since p_0 has all negative zeros, we see by induction that for all $m = 1, 2, \ldots$, p_m has all distinct negative zeros. Since (p_m) converges to $p(z) := \sum_{j \in \mathbf{Z}} \phi(j + k/(M-1))z^j$, it follows that p has all negative zeros. The required result then follows from a result in [1]. □

We recall from Theorem 2.8 that p as in Theorem 3.2 satisfies (II) provided that q satisfies (II). If we assume that p satisfies the stronger condition (I), then we can deduce a stronger total positivity property. As in Theorem 3.1 we follow here the work of [13], where the result is proved for $M = 2$.

Theorem 3.3. *Suppose that p as in (2.1) is a polynomial which satisfies (I) and is of the same form as in Theorem 3.2. Then the function ϕ as in Theorem 3.2 satisfies the property that for any $s \geq 1$, $x_1 < \cdots < x_s$, and integers $l_1 < \cdots < l_s$,*

$$\det(\phi(x_i - l_j) : i, j = 1, \ldots, s) \geq 0. \tag{3.9}$$

Proof. For $m = 0, 1, \ldots$, we define f_m as in the proof of Theorem 3.1, so that for $i \in \mathbf{Z}, x \in \mathbf{R}$,

$$\begin{aligned} f_{m+1}(x - i) &= \sum_{j \in \mathbf{Z}} a_j f_m(Mx - Mi - j) \\ &= \sum_{j \in \mathbf{Z}} a_{j-Mi} f_m(Mx - j). \end{aligned}$$

Letting $B = (a_{j-Mi})_{i,j \in \mathbf{Z}}$, we apply the Cauchy-Binet formula [16, p.1] to give for $x_1 < \cdots < x_s, l_1 < \cdots < l_s$,

$$\begin{aligned} &\det(f_{m+1}(x_i - l_j) : i, j = 1, \ldots, s) \\ &\quad = \sum_{k_1 < \cdots < k_s} B \begin{pmatrix} l_1 & \cdots & l_s \\ k_1 & \cdots & k_s \end{pmatrix} \det(f_m(Mx_i - k_j) : i, j = 1, \ldots, s). \end{aligned}$$

By Theorem 2.1, the terms $B \begin{pmatrix} l_1 & \cdots & l_s \\ k_1 & \cdots & k_s \end{pmatrix}$ are all non-negative. Also $f_0 = N$, which satisfies (3.9) with ϕ replaced by N. So by induction (3.9) is satisfied with ϕ replaced by $f_m, m = 1, 2, \cdots$. Since (f_m) converges to ϕ, (3.9) holds. □

We remark that in [13] a function satisfying the conclusion of Theorem 3.3 is called a *ripplet*. In [13] it is also shown that for $M = 2$ there is strict inequality in (3.9) if and only if the diagonal elements of the matrix concerned are strictly positive, i.e.,

$$0 < x_j - l_j < \frac{n}{M-1},\ j = 1, \ldots, s. \tag{3.10}$$

This result is a generalisation of results in [4] and [5] for B-splines, which are in turn a stronger form of the Schoenberg-Whitney Theorem [18]. It is not true in general for $M \geq 3$.

To see this, take $M = 3$ and let

$$p(z) = \frac{1}{8}(z^2 + z + 1)(z + 1)^3 = \sum_{j=0}^{5} a_j z^{5-j}.$$

Since the polynomial z^2+z+1 satisfies (I), the polynomial p satisfies (I) by Theorem 2.7. Now let ϕ be the corresponding refinable function as in Theorem 3.3, which has support in $[0, 5/2]$. For $7/6 \leq x \leq 4/3$, we have $3x - 4 \leq 0, 3x - 1 \geq 5/2$, and so the refinement equation (3.1) gives

$$\begin{aligned} \phi(x) &= a_2\phi(3x - 2) + a_3\phi(3x - 3) \\ &= \frac{7}{8}\phi(3x - 2) + \frac{7}{8}\phi(3x - 3) = \frac{7}{8}, \end{aligned}$$

by (3.2). Now by (3.2),

$$\phi(x + 1) + \phi(x) + \phi(x - 1) = 1,\ 1 \leq x \leq 3/2,$$

and so

$$\phi(x + 1) + \phi(x - 1) = \frac{1}{8} = \frac{1}{7}\phi(x),\ 7/6 \leq x \leq 4/3.$$

Thus $\phi(\cdot + 1), \phi, \phi(\cdot - 1)$ are linearly dependent on $[7/6, 4/3]$. So for any $7/6 \leq x_{-1} < x_0 < x_1 \leq 4/3$,

$$\det(\phi(x_i - j) : i, j = -1, 0, 1) = 0.$$

Thus we may have equality in (3.9) although (3.10) is satisfied.

We also note that while the functions $\phi(\cdot - j), j \in \mathbf{Z}$, are locally linearly dependent, as shown above, they are globally linearly independent. To see this, we note that (3.2) implies

$$\phi(x + 1) + \phi(x) = 1,\ 1/2 \leq x \leq 1.$$

It is easily checked from (3.1) that ϕ is not constant on $[1/2, 1]$. Thus $\phi(x+1)$ and ϕ are linearly independent on $[1/2, 1]$. Suppose that $\sum_{j\in\mathbf{Z}} c_j\phi(x - j) = 0, x \in \mathbf{R}$. Then $c_{-1}\phi(x+1) + c_0\phi(x) = 0, 1/2 \leq x \leq 1$, and so $c_{-1} = c_0 = 0$. Similarly $c_j = 0$ for all integers.

We shall now show that when $n/(M - 1)$ is an integer, then it is true that there is strict inequality in (3.9) if and only if (3.10) holds.

Theorem 3.4. *If p and ϕ are as in Theorem 3.3 and n is an integer multiple of $M - 1$, then there is strict inequality in (3.9) if and only if (3.10) holds.*

Again we follow the work of [13]. We shall need a generalization of Theorem 2.1. Let p as in (2.1) satisfy (I) and let B denote the matrix $(a_{j-Mi})_{i,j\in\mathbf{Z}}$. Then from Theorem 2.1 we see that for $s \geq 1$ and $i_1 < \ldots < i_s, j_1 < \ldots < j_s$,

$$B\begin{pmatrix} i_1 & \cdots & i_s \\ j_1 & \cdots & j_s \end{pmatrix} \geq 0$$

with strict inequality if and only if

$$0 \leq j_l - Mi_l \leq n,\ l = 1, \ldots, s.$$

Proposition 3.5. *If n is an integer multiple of $M-1$, then for integers $r, s \geq 1$, $i_1 < \ldots < i_s, j_1 < \ldots < j_s$,*

$$B^r\begin{pmatrix} i_1 & \cdots & i_s \\ j_1 & \cdots & j_s \end{pmatrix} \geq 0, \tag{3.11}$$

with strict inequality if and only if

$$0 \leq j_l - M^r i_l \leq \frac{(M^r-1)n}{M-1},\ l = 1, \ldots, s. \tag{3.12}$$

Proof. The proof is by induction on r. We assume the result is true for some $r \geq 1$ and use the Cauchy-Binet formula to give

$$B^{r+1}\begin{pmatrix} i_1 & \cdots & i_s \\ j_1 & \cdots & j_s \end{pmatrix} = \sum_{k_1<k_2<\ldots<k_s} B\begin{pmatrix} i_1 & \cdots & i_s \\ k_1 & \cdots & k_s \end{pmatrix} B^r\begin{pmatrix} k_1 & \cdots & k_s \\ j_1 & \cdots & j_s \end{pmatrix}.$$

Then (3.11) holds with r replaced by $r+1$. Note that (3.12) with r replaced by $r+1$ may be written as, for $l = 1, \ldots, s$,

$$i_l \leq M^{-r-1} j_l, \tag{3.13}$$

$$M^{-r-1}((M-1)j_l + n) \leq (M-1)i_l + n. \tag{3.14}$$

Now (3.11), with r replaced by $r+1$, holds with strict inequality if and only if there are integers $k_1 < \ldots < k_s$ with, for $l = 1, \ldots, s$,

$$i_l \leq M^{-1}k_l \leq M^{-r-1}j_l, \tag{3.15}$$

$$M^{-r-1}((M-1)j_l + n) \leq M^{-1}((M-1)k_l + n) \leq (M-1)i_l + n. \tag{3.16}$$

So if (3.11), with r replaced by $r+1$, holds with strict inequality, then for $l = 1, \ldots, s$, (3.15) and (3.16) are true for some $k_1 < \ldots < k_s$, which implies (3.13) and (3.14) for $l = 1, \ldots, s$.

The converse is more difficult. Suppose (3.13) and (3.14) hold for $l = 1, \ldots, s$. We must show that there are integers $k_1 < \ldots < k_s$ such that (3.15) an (3.16) hold for $l = 1, \ldots, s$. We shall prove this by induction on s. Take $s \geq 1$ and suppose that the result is true for s replaced by $s-1$. For $l = 1, \ldots, s$, let k_l be the smallest integer satisfying

$$i_l \leq M^{-1}k_l \tag{3.17}$$

$$M^{-r}((M-1)j_l+n) \le (M-1)k_l+n, \tag{3.18}$$

and $k_l > k_{l-1}$, (which this last condition is omitted for $l=1$). We know that (3.15) and (3.16) hold for $l=1,\dots,s-1$, and it remains only to prove

$$k_s \le M^{-r}j_s \tag{3.19}$$

$$k_s \le Mi_s+n. \tag{3.20}$$

Note that if $k_{m+1} \ge k_m+2$ for some $m, 1 \le m \le s-1$, then the condition $k_l > k_{l-1}$ is not enforced for $l=m+1$. Thus we can apply our inductive hypothesis for $l=m+1,\dots,s$, to deduce (3.19) and (3.20). So we may assume

$$k_{l+1} = k_l+1,\ l=1,\dots,s-1. \tag{3.21}$$

If $s \ge 2$, then

$$\begin{aligned} k_s &= k_{s-1}+1 \le Mi_{s-1}+n+1 \\ &\le Mi_s - M+n+1 \le Mi_s+n \end{aligned}$$

and so (3.20) holds. If $s=1$ and $k_1 > Mi_1+n$, then

$$k_1-1 \ge Mi_1+n \ge Mi_1$$

and

$$\begin{aligned} (M-1)(k_1-1)+n &\ge (M-1)(Mi_1+n)+n \\ &\ge M((M-1)i_1+n) \ge M^{-r}((M-1)j_1+n) \end{aligned}$$

by (3.14). This contradicts the definition of k_1 being the smallest integer satisfying (3.17) and (3.18). So again (3.20) holds.

It remains to prove (3.19). Suppose (3.19) is not true, then by (3.21),

$$k_1 > M^{-r}j_s - s+1. \tag{3.22}$$

Now by (3.13),

$$k_1 > Mi_s - s+1 \ge Mi_1 + (M-1)(s-1)$$

and hence

$$k_1-1 \ge Mi_1+(M-1)(s-1) \ge Mi_1.$$

So by definition, k_1 is the smallest integer satisfying (3.18). Thus

$$(M-1)(k_1-1)+n < M^{-r}((M-1)j_1+n), \tag{3.23}$$

which gives

$$\begin{aligned} k_1-1 &< M^{-r}(j_1-(M^r-1)n/(M-1)) \\ &\le M^{-r}(j_s-s+1-(M^r-1)n/(M-1)) \\ &= M^{-r}j_s+M^{-r}-s+(1-M^{-r})(s-n/(M-1)). \end{aligned}$$

If $s \le n/(M-1)$, then $k_1 - 1 < M^{-r}j_s + M^{-r} - s$ and so $k_1 - 1 \le M^{-r}j_s - s$, which contradicts (3.22), and so (3.19) is true. So we may assume $s \ge n/(M-1)+1$. Now by (3.16),

$$\begin{aligned} k_s = k_1 + s - 1 &\le Mi_1 + n + s - 1 \\ &\le M(i_s - s + 1) + n + s - 1 \\ &= Mi_s + n - (M-1)(s-1) \\ &\le M^{-r}j_s + n - (M-1)(s-1) \end{aligned}$$

by (3.13). Since $s - 1 \ge n/(M-1)$, then $k_s \le M^{-r}j_s$, i.e., (3.19) holds. □

Proposition 3.5 is not true in general if n is not an integer multiple of $M-1$. To see this, first note that for the matrix B as before, $B^2 = (\tilde{a}_{j-M^2 i})_{i,j\in\mathbf{Z}}$, where

$$\sum_{j=0}^{(M+1)n} \tilde{a}_j z^{(M+1)n-j} = p(z)p(z^M).$$

Take $M = n = 3$ and $p(z) = z^3 + z^2 + z + 1$. Then p satisfies (I) but

$$B^2 \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{vmatrix} \tilde{a}_{10} & \tilde{a}_1 \\ \tilde{a}_{11} & \tilde{a}_2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0.$$

Proof of Theorem 3.4. Suppose that there is strict inequality in (3.9). If $x_t - l_t \le 0$ for some $t, 1 \le t \le s$, then $\phi(x_i - l_j) = 0$ for $i = 1,\dots,t, j = t,\dots,s$. So the first t rows of the matrix in (3.9) are linearly dependent, which contradicts the determinant being strictly positive. Similarly, if $x_t - l_t \ge n/(M-1)$ for some $t, 1 \le t \le s$, then the last $s-t+1$ rows of the determinant are linearly dependent and we again get a contradiction. Thus (3.10) must be satisfied. We note that this argument does not depend on $n/(M-1)$ being an integer.

We now assume that (3.10) is satisfied and shall deduce strict inequality in (3.9). By the refinement equation (3.1), for $i \in \mathbf{Z}, x \in \mathbf{R}$,

$$\phi(x-i) = \sum_{j\in\mathbf{Z}} a_j \phi(Mx - Mi - j) = \sum_{j\in\mathbf{Z}} B_{ij}\phi(Mx - j),$$

and repeating this procedure gives for any integer $r \ge 1$,

$$\phi(x-i) = \sum_{j\in\mathbf{Z}} B^r_{ij}\phi(M^r x - j).$$

So by the Cauchy-Binet formula,

$$\begin{aligned} &\det(\phi(x_i - l_j) : i,j = 1,\dots,s) \\ &= \sum_{k_1<\dots<k_s} B^r \begin{pmatrix} l_1 & \cdots & l_s \\ k_1 & \cdots & k_s \end{pmatrix} \det(\phi(M^r x_i - k_j) : i,j = 1,\dots,s) \end{aligned} \tag{3.24}$$

For $r \ge 1$ and $i = 1,\dots,s$, choose an integer $k_{i,r}$ with $0 < M^r x_i - k_{i,r} \le 1$. Since $\phi(x) > 0$ for $0 < x < n/(M-1)$ and $n/(M-1) > 1$, the diagonal terms of the matrix $(\phi(M^r x_i - k_{j,r}) : i, j = 1, \ldots, s)$ are positive. Also the off-diagonal terms of this matrix are zero for large enough r since for $i \neq j$,

$$|M^r x_i - k_{j,r}| = |M^r(x_i - x_j) + M^r x_j - k_{j,r}| \to \infty$$

as $r \to \infty$. Thus for large enough r, we have $k_{1,r} < \ldots < k_{s,r}$ and

$$\det(\phi(M^r x_i - k_{j,r}) : i, j = 1, \ldots, s) > 0. \tag{3.25}$$

Now $\lim_{r\to\infty} M^{-r} k_{j,r} = x_j, j = 1, \ldots, s$, and so by (3.10)

$$0 < M^{-r} k_{j,r} - l_j < (1 - M^{-r})n/(M - 1),\ j = 1, \ldots, s,$$

for large enough r. Then by Proposition 3.5,

$$B^r \begin{pmatrix} l_1 & \cdots & l_s \\ k_{1,r} & \cdots & k_{s,r} \end{pmatrix} > 0$$

for all large enough r. So by (3.24) and (3.25) there is strict inequality in (3.9). □

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