SOME PROPERTIES OF FIVE-COEFFICIENT REFINEMENT EQUATION

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Archiv Math. 66(1996)

ABSTRACT. In this paper global linear independence, local linear independence and Hölder continuity of the solution of the five-coefficient refinement equation

$$\phi(x)=\sum_{k=0}^{4} c_k\phi(3x-k), \hat\phi(0)=1,$$

are considered. Especially we construct Hölder continuous solution of the above refinement equation which is globally linearly independent but locally linearly dependent. Also multiresolution and explicite construction of compactly supported wavelets are considered.

KEYWORDS. refinement equation, global linear independence, local linear independence, Hölder continuity, multiresolution, wavelet

1. Introduction

The purpose of this paper is to study global linear independence, local linear independence and Hölder continuity of the solution of the following refinement equation

$$\phi(x)=\sum_{k=0}^{4} c_k\phi(3x-k),\ \hat\phi(0)=1, \tag{1}$$

where $c_0c_4\neq 0$ and $\sum_{k=0}^4 c_k=3$, and to construct compactly supported wavelets of the multiresolution generated by ϕ. Here the Fourier transform is defined by

$$\hat f(\xi)=\int_R e^{-ix\xi}f(x)dx$$

for all integrable function f and its form in distribution sense is entended as usual.

The examples of (1) are nondifferential de Rham function with $c_0=c_4=2/3, c_1=c_3=1/3$ and $c_2=1$, and B-spline with $c_0=c_4=1/3, c_1=c_3=2/3$ and $c_2=1$.

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To our purpose, let's introduce some definitions. We say that a compactly supported distribution f is *globally linearly independent* if $\sum_{k\in Z} d_k f(x-k)=0$ on R implies $d_k=0$ for all $k\in Z$. We say that a compactly supported distribution f is *locally linearly independent* if for every open set A conditions $\sum_{k\in Z} d_k f(x-k)=0$ on A and $\mathrm{supp} f(\cdot-k)\cap A\neq\emptyset$ imply $d_k=0$. We say that a function f is *Hölder continuous* if there exists $0<\alpha\leq 1$ such that $|f(x)-f(y)|\leq C|x-y|^\alpha$ holds for all $x,y\in R$. The supermum of α in the above formula is called *Hölder exponent* of f.

To consider local linear independence is reasonable and interesting since local linear independence implies local stability which is very imporant in practice. Here local stablity means equivalence between some norms of the function $\sum_{k\in Z} d_k f(\cdot-k)$ on bounded open set A and its corresponding sequence norm of related d_k. Obviously local linear independence of a distribution implies global linear independence of that distribution, and the converse statement is generally not true. Generally speaking global linear independence is easy, since at least Ron [R] gave a practical characterization to global linear independence by Fourier transform, which says that global linear independence of f is equivalent to existence of a complex number z such that $\hat f(z+2k\pi)=0$ for all $k\in Z$. This arise us to find connections between global and local linear independence. Ben-Artzi and Ron [BR] gave a weak conclusion of local linear independence through dual bases, and the third named author [S1] gave explicitely construction of open set A in one dimension such that $\sum_{k\in Z} d_k f(x-k)=0$ on A and $\mathrm{supp} f(\cdot-k)\cap A\neq\emptyset$ imply $d_k=0$. The most important thing on local and global linear independence seems to find as many examples as possible for which local and global linear independence is equivalent to each other. Until now at least the following examples were found: B-spline in one dimension, box splines in high dimension and distributional solutions of the following type of refinement equations in one dimension

$$\phi(x)=\sum_{k\in Z} c_k\phi(2x-k)$$

with $\sum_{k\in Z} c_k=2$ and c_k being zero for large k. (see [S2], [L] and references therein.) In section 2 and section 3 we will give some practical characterization to global and local linear independence of ϕ. Especially we construct Hölder continuous ϕ which satisfies (1) and is **globally linearly independent but locally linear dependent**.

In recent years, wavelet theory and its application is developed very fast. To consider when ϕ in (1) generates a multiresolution and to construct compactly supported wavelet is elementary in wavelet theory and its application. In a multiresolution or in other practical use, L^2 integrablity even Hölder continuity of ϕ is generally needed. In section 4, we will give some conditions on existence and nonexistence of Hölder continuous and integrable solution ϕ of (1). In last section, we consider the multiresolution generated by ϕ and construct compactly supported wavelets explicitely.

2. Global Linear Independence

In this section we will give a characterization to global linear independence of distributional solution ϕ of (1).

Denote the symbol function of (1) by

$$H(\xi) = 1/3 \sum_{k=0}^{4} c_k e^{ik\xi}.$$

Therefore by taking Fourier transform at both sides of (1) we get

$$\hat{\phi}(\xi) = H(\xi/3)\hat{\phi}(\xi/3). \tag{2}$$

Now we give a characterization to global linear independence.

Theorem 1. *Let ϕ be the unique distributional solution of (1) with supp$\phi \not\subset \{0,1,2\}$. Therefore global linear independence of ϕ is equivalent to*
i) $c_2 \neq 0$ or $c_4 c_0 \neq c_1 c_3$;
ii) $H(\pi/3) \neq 0$ or $H(5\pi/3) \neq 0$.

Before we start to prove the above result, we recall a characterization by Ron ([R]).
Lemma 1. *Let f be a compactly supported distribution. Therefore f is globally linearly independent if and only if there does not exist complex number z such that $\hat{f}(z + 2k\pi) = 0$ for all $k \in Z$.*

Proof of Theorem 1. First the necessity. Denote

$$N(\phi) = \{z; \hat{\phi}(z + 2k\pi) = 0, \quad \forall z \in Z\}.$$

Therefore $N(\phi) = \emptyset$ by Lemma 1. Observe that $\hat{\phi}(\pi + 2k\pi) = 0$ for all $k \in Z$ when $H(\pi/3) = H(5\pi/3) = 0$ by (2). Then ii) holds by $N(\phi) = \emptyset$. Also observe that $\hat{\phi}(z_0 + 2k\pi) = H(z_0/3 + 2k\pi/3)\hat{\phi}(z_0/3 + 2k\pi/3) = 0$ for all $k \in Z$ by (2) when $c_2 = 0$ and $c_4 c_0 = c_1 c_3$, where $e^{iz_0} = c_4/c_1$. Then i) holds still by $N(\phi) = \emptyset$. The necessity is proved.

Secondly the sufficiency. Conversely we assume that ϕ is globally linearly dependent. By Lemma 1 and $\hat{\phi}(0) = 1$, we know that there exists $z_0 \notin 2\pi Z$ such that $z_0 \in N(\phi)$, which means that $\hat{\phi}(z_0 + 2k\pi) = 0$ for all $k \in Z$. Therefore $H((z_0 + 2j\pi)/3)\hat{\phi}((z_0 + 2j\pi)/3 + 2k\pi) = 0$ for all $k \in Z$ and $j = 0, 1, 2$ by (2). Observe that $z_0 \in N(\phi)$ is unique in the sense of mudolar $2\pi Z$, since $u, w \in N(\phi)$ with $u - w \notin 2\pi Z$ implies $\phi(x) + e^{-iu}\phi(x + 1) = 0$ and $\phi(x) + e^{-iw}\phi(x + 1) = 0$ on (0,1) which means supp$\phi \subset \{0, 1, 2\}$. Also observe that $H(z_0/3) = H(z_0/3 + 2\pi/3) = H(z_0 + 4\pi/3) = 0$ does not holds by i). Therefore we get $z_0 = (z_0 + 2j\pi)/3 + 2k\pi$ for some $j = 0, 1, 2$ and $k \in Z$, which implies $z_0 \in 2\pi Z$ or $z_0 \in \pi + 2\pi Z$. This contradicts to $z_0 \notin 2\pi Z$ or ii), since ii) implies $\pi \notin N(\phi)$. The sufficency and hence Theorem 1 is proved.

Before we finish this section, let's say a little about supp$\phi \subset \{0,1,2\}$. Obviously we can write $\hat{\phi}(\xi) = \sum_{j=0}^{l} \xi^j T_j(\xi)$ for some $l \geq 0$ when supp$\phi \subset \{0,1,2\}$, where trigonometric polynomials T_j have their degrees at most two and $T_0(0) \neq 0$. Therefore we get $T_j(\xi) = 0$ when $j \geq 1$ by (1) and $H(\xi) = T_0(3\xi)/T_0(\xi)$. Divide $T_0(\xi) = C(e^{i\xi} - z_1)(e^{i\xi} - z_2)$ for $z_1, z_2 \neq 0, 1$. Therefore after some simple reduction we get $H(\xi) = 1 - 2e^{i\xi} + e^{2i\xi} - 2e^{3i\xi} + e^{4i\xi}$, or $H(\xi) = 1 - e^{i(\xi+\pi/3)} - e^{i(3\xi+5\pi/3)} + e^{4i\xi}$ or $H(\xi) = 1 - e^{i(\xi+5\pi/3)} - e^{i(3\xi+\pi/3)} + e^{4i\xi}$. On the other hand, we know that when $H(\xi)$ has one of the above three form the support of ϕ is contained in $\{0,1,2\}$. Furthermore we can write $\phi(x) = a_0\delta(x) + a_1\delta(x-1) + a_2\delta(x-2)$, where we denote $\delta(x)$ be the delta distribution. Therefore $\sum_{k\in Z} z^k \phi(x-k) = 0$ for all z with $a_0 z^2 + a_1 z + a_2 = 0$, and ϕ is globally linearly dependent when supp$\phi \subset \{0,1,2\}$ and ϕ satisfies (1).

3. Local Linear Independece

In this section we will consider local linear independence of distributional solution of (1).

Define transform matrices B_i (i=1, 2, 3) by

$$B_i = \begin{pmatrix} c_i & c_{i-1} \\ c_{i+3} & c_{i+2} \end{pmatrix},$$

where we define $c_{-1} = c_5 = 0$ and define distribution $\Phi(x)$ on (0,1) by

$$\Phi(x) = \begin{pmatrix} \phi(x) \\ \phi(x+1) \end{pmatrix}, x \in (0,1).$$

Therefore the following important formula holds by (1)

$$B_i \Phi(x) = \Phi(\frac{x+i}{3}),\ i = 0,1,2.$$

Theorem 2. *Let ϕ be the unique distributional solution of (1). If B_i, $i = 0,1,2$, are nonsingular, then local and global linear independence of ϕ are equivalent to each other.*

Before we give its proof, we state two lemmas for which similar ones can be founded in [S2].

Lemma 2. *Let $A \subset (0,1)$ be open and let B_i, $i = 0,1,2$, be nonsingular. If there exists nonzero vector d with $d\Phi(x) = 0$ on A and $\Phi \not\equiv 0$ on A, then $d'\Phi(x) = 0$ on (0,1) for some nonzero d'.*

Lemma 3. *Let ϕ be the unique distributional solution of (1). If there exists nonzero complex number z_0 with $\sum_{k\in Z} z_0^k \phi(x-k) = 0$ on $R \setminus Z$, then $\sum_{k\in Z} z_1^k \phi(x-k) = 0$ on R for some nonzero z_1.*

The proofs of the above lemmas can follow line by line as in [S2].

Lemma 4. *Let ϕ be the unique distributional solution of (1). If* $\text{supp}\phi \subset \{0\} \cup [\epsilon, 2]$ *or* $\text{supp}\phi \subset \{2\} \cup [0, 2-\epsilon]$ *for some* $0 < \epsilon < 1$*, then* $\text{supp}\phi \subset \{0,1,2\}$*.*

The above lemma follows by comparing ϕ's support at both sides of (1).

Proof of Theorem 2. Obviously local linear independence of ϕ implies global linear independence of ϕ. Therefore it suffices to prove local linear independence of ϕ when ϕ is globally linearly independent. Recall that ϕ is globally linearly dependent when $\text{supp}\phi \subset \{0,1,2\}$. Therefore we can assume $\text{supp}\phi \not\subset \{0,1,2\}$ in the following proof.

Conversely we assume that ϕ are locally linearly dependent. Then there exists an open set $A \subset (0,1)$ and nonzero vector d such that $d\Phi(x) = 0$ on A and $\Phi \not\equiv 0$ on A by Lemma 4. By Lemma 1, we know that there exists nonzero vector $d' = (d_1', d_2')$ such that $d'\Phi(x) = 0$ on (0,1). If $d_1'd_2' \neq 0$, then $\sum_{k\in Z}(d_2'/d_1')^k\phi(x+k) = 0$ on $R \setminus Z$ and $\sum_{k\in Z} z_0^k \phi(x+k) = 0$ on R for some nonzero z_0 furthermore by Lemma 2, which contradicts to global linear independence of ϕ. Now the matter reduces d' having one component zero, but in the above case we know that $\text{supp}\phi \subset \{0,1,2\} \cup (0,1)$ or $\text{supp}\phi \subset \{0,1,2\} \cup (1,2)$, which contradicts to $\text{supp}\phi \not\subset \{0,1,2\}$ by Lemma 4. Hence Theorem 2 is proved.

Remark 1. When B_i $(i = 0,1,2)$ are nonsingular matrices, we know that i) in Theorem 1 holds.

Theorem 3. *Let ϕ be the unique distributional solution of (1). If B_i is singular for some* $i = 0,1,2$*, then ϕ is locally linearly dependent except* $c_1 = c_2 = c_3 = 0$*.*

Proof. Recall that ϕ is globally linearly dependent, hence locally linearly dependent when $\text{supp}\phi \subset \{0,1,2\}$. Therefore we assume $\text{supp}\phi \not\subset \{0,1,2\}$ in the following proof. We divide two cases to prove Theorem 3.

Case 1. B_1 is singular.

Recall that $c_0c_4 \neq 0$. Therefore $c_1c_3 \neq 0$ and $c_1/c_4 = c_0/c_3$. If $c_1\phi(x) + c_0\phi(x+1) = 0$ on $(0,1)$, then $\text{supp}\phi \not\subset \{0,1,2\}$ and $c_1c_0 \neq 0$ imply locally linearly dependent. If $c_1\phi(x) + c_0\phi(x+1) \not\equiv 0$ on $(0,1)$, then $-c_4\phi(x) + c_1\phi(x+1) = 0$ on $(1/3, 2/3)$ and ϕ is locally linearly dependent since $\phi(x) = c_1\phi(3x-1) + c_0\phi(3x) \neq 0$ on $(1/3,2/3)$ and $\phi(x+1) = c_4/c_1\phi(x)$ on $(1/3,2/3)$.

Case 2. B_0 and B_2 are singular.

Recall that $c_0c_4 \neq 0$. Therefore singularity of B_0 and B_2 is equivalent to $c_2 = 0$. If $c_3 \neq 0$, then $-c_3\phi(x) + c_0\phi(x+1) = 0$ on $(0,1/3)$ and ϕ is locally linearly dependent since $\phi(x) = c_0\phi(3x)$ and $\phi(x+1) = c_3/c_0\phi(x)$ on $(0,1/3)$, and $\text{supp}\phi \cap (0,1) \neq \emptyset$ by Lemma 4. Similarly if $c_1 \neq 0$, then $-c_4\phi(x) + c_1\phi(x+1) = 0$ on $(2/3,1)$ and ϕ is locally linearly dependent since $\phi(x) = c_1\phi(3x-1)$, $\phi(x+1) = c_4/c_1\phi(x)$ on $(2/3,1)$ and $\text{supp}\phi \cap (1,2) \neq \emptyset$ by Lemma 4. Theorem 3 is proved.

Theorem 4. *Let ϕ be the unique distributional solution of (1) with* $c_1 = c_2 = c_3 = 0$*. Then ϕ is locally linearly independent.*

Proof. By Lemma 4, it suffices to proving the statement

$$\alpha_1\phi(x)+\alpha_2\phi(x+1)=0 \text{ on } A \text{ and } \mathrm{supp}\phi(\cdot+k)\cap A\neq\emptyset \text{ implies } \alpha_k=0 \tag{3}$$

holds for $A=(a,b)\subset(0,1)$. Recall that $B_1\Phi(x)=\Phi(x/3+1/3)$ and $(\alpha_1,\alpha_2)B_1=(c_4\alpha_2,c_0\alpha_1)$. Observe that (3) holds for A with $A\subset(1/3,2/3)$ if and only if (3) holds for $3A-1$. Therefore the matter reduces to the following cases: $1/3\in A, 2/3\in A, A\subset(0,1/3)$, and $A\subset(2/3,1)$. If $1/3\in A$, then $c_4\alpha_2\phi(x)+c_0\alpha_1\phi(x+1)=0$ on $(0,\min(3b-1,1))$. Observe that $\phi(x)=0$ on $(1,\ 4/3)$. Therefore $\alpha_2=0$ by Lemma 4 and (3) holds for these A with $1/3\in A$. Similarly we get $\alpha_2=0$ when $A\subset(0,1/3)$ and $\mathrm{supp}\phi\cap A\neq\emptyset$. If $2/3\in A$, $c_4\alpha_2\phi(x)+c_0\alpha_1\phi(x+1)=0$ on $(\min(3a-1,1)$. Observe that $\phi(x)=0$ on (2/3,1). Therefore $\alpha_1=0$ by Lemma 4 and (3) holds for these A with $2/3\in A$. Similarly (3) holds for these A with $A\subset(2/3,1)$. Theorem 4 is proved.

Remark 2. By Theorem 1 and 2 we know that the de Rham nondifferential function with $c_0=c_4=2/3, c_1=c_3=1/3$ and $c_2=1$ in (1) is locally linearly independent.

Example 1. Now let's construct Hölder continuous solution ϕ of (1) which is globally linearly independent but locally linearly dependent. Let $c_0+c_3=c_1+c_4=c_2=1$. Observe that B_1 is singular only if $c_0=c_1$. Therefore the solution ϕ of the following refinement equation

$$\phi(x)=c_0\phi(3x)+c_0\phi(3x-1)+\phi(3x-2)+(1-c_0)\phi(3x-3)+(1-c_0)\phi(3x-4) \tag{4}$$

is locally linearly dependent by Theorem 2. Observe that the symbol function $H(\xi)$ does not equal zero at $\xi=\pi/3$ when c_0 is real. Therefore the solution ϕ of (4) is **globally linearly independent but locally linearly dependent** when $0<c_0<1$. In particular, we know that ϕ is Hölder continuous with Hölder exponent $|\mathrm{ln}\max(c_0,1-c_0)|/\mathrm{ln}3$ when $0<c_0<1$.

4. Hölder Continuity and Integrablity

In this section, we will consider Hölder continuity and integrablity of solution of (1).

The following is a result on Hölder continuity of ϕ.

Theorem 5. *Let ϕ be the unique distributional solution of (1). Then ϕ is continuous if and only if*

i) $c_0+c_3=c_1+c_4=c_2=1$;

ii) $|c_0|<1, |c_4|<1$ *and* $|1-c_4-c_0|<1$.

Proof. First the necessity. Observe that $\Phi(0)\neq 0$, otherwise $\phi\equiv 0$ by continuity of ϕ. Therefore $c_2=1$ since $B_0\Phi(0)=\Phi(0)$. Observe that $B_0^k=\begin{pmatrix} c_1^k & 0\\ c_4(1-c_1^k)/(1-c_1) & 1\end{pmatrix}$ and $B_2^k=\begin{pmatrix} 1 & c_1(1-c_4^k)/(1-c_4)\\ 0 & c_4^k\end{pmatrix}$. Therefore we get $|c_4|<1$ and $c_1+c_4=1$ by

$B_2^k\Phi(0) \to \Phi(0)$ as $k \to +\infty$, and we get $|c_0| < 1$ and $c_0 + c_3 = 1$ by $B_0^k\Phi(1) \to \Phi(0)$ and $B_0^k B_1\Phi(1) \to \Phi(0)$ as $k \to +\infty$. Now we prove $|1-c_4-c_0| < 1$. Write $B_1 = B^{-1}AB$, where B is nonsingular and $A = \begin{pmatrix} 1-c_4-c_0 & 0 \\ 0 & 1 \end{pmatrix}$ or $A = \begin{pmatrix} 1-c_4-c_0 & 1 \\ 0 & 1 \end{pmatrix}$ when $1-c_4-c_0 = 1$. Therefore $B_1^k = B^{-1}\begin{pmatrix} (1-c_4-c_0)^k & 0 \\ 0 & 1 \end{pmatrix} B$ or $B_1^k = B^{-1}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} B$. Observe that $B\Phi(0) \neq c\Phi(0)$ when $1-c_4-c_0 \neq 1$ or $B\Phi(0) = c\Phi(1)$ when $1-c_4-c_0 = 1$ since otherwise $B_1\Phi(0) = \Phi(0)$. Recall that $B_1^k\Phi(0) \to \Phi(1/2)$. Then $|1-c_4-c_0| < 1$. The necessity is proved.

Secondly the sufficiency. Observe that $B_0(1,-1)^T = c_0(1,-1)^T, B_1(1,-1)^T = (1-c_4-c_0)(1,-1)^T$ and $B_2(1,-1)^T = c_4(1,-1)^T$, where A^T denotes the transpose of a vector A. Therefore by the method used in [DL], we know that ϕ is continuous. The sufficiency and then Theorem 5 is proved.

By the procedure in the proof of Theorem 5, we know that

$$|\phi(x) - \phi(y)| \leq C|x-y|^\alpha,$$

where $\alpha = |\ln\max(|c_0|, |c_4|, |1-c_4-c_0|)|/\ln 3$. Denote

$$W = \text{span}\{\Phi(z) - \Phi(0), z = l/3^k \text{ for some } l \text{ and } k\}.$$

Therefore W is spanned by $(1,-1)$. Hence we know that α is Hölder exponent of ϕ by the procedure used in [HL].

Now we give a result on integrablity of ϕ.

Theorem 6. *Let ϕ be the unique distributional solution of (1) and let c_k, $0 \leq k \leq 4$ in (1) be real and satisfy $c_0 + c_3 = c_1 + c_4 = c_2 = 1$. Then $|\hat{\phi}(\xi)| \leq C(1+|\xi|)^{-\alpha}$ for some $\alpha > 1/2$ and hence ϕ is square integrable provided one of the following holds:*

i) $\frac{-\sqrt{3}+1}{2} < c_4 + c_0 < \frac{\sqrt{3}+1}{2}$ *and* $c_0c_4 > 0$.

ii) $\frac{-\sqrt{3}+1}{2} < c_4 + c_0 < \frac{\sqrt{3}+1}{2}$, $c_0c_4 < 0$ *and* $(c_0+c_4)(1-c_0-c_4) \geq -4c_0c_4$;

iii) $\frac{-\sqrt{3}+1}{2} < c_4 + c_0 < \frac{\sqrt{3}+1}{2}$, $c_0c_4 < 0$ *and* $(c_0+c_4)(1-c_0-c_4) \leq 4c_0c_4$;

iv) $-(c_0+c_4)(1-c_0-c_4)^2 + 4c_0c_4(1-2c_0-2c_4+2c_0^2+2c_4^2) - 12c_0c_4 > 0, c_0c_4 < 0$ *and* $4c_0c_4 < (c_0+c_4)(1-c_0-c_4) < -4c_0c_4$.

Furthermore ϕ is not integrable when $|2c_4+2c_0-1| \geq 3$, except $(c_0,c_4) = (1,1)$, and ϕ is not square integrable when $1 + c_0 + c_4 - c_0c_4 - c_0^2 - c_4^2 \leq 0$ except $(c_0,c_4) = (1,1)$

Proof. Observe that $H(\xi) = (c_0 + (1-c_4-c_0)e^{i\xi} + c_4e^{2i\xi})(1+e^{i\xi}+e^{2i\xi})/3$. Denote $B = \sup_{\xi\in R}|R(\xi)|^2 = \sup_{\xi\in R}|c_0 + (1-c_4-c_0)e^{i\xi} + c_4e^{2i\xi}|^2$. Therefore we know that

$$|\hat{\phi}(\xi)| \leq C(1+|\xi|)^{-1+\ln B/2\ln 3}$$

by usual procedure. Observe that $|R(\xi)|^2 = 4c_0c_4\cos^2\xi + 2(1-c_4-c_0)(c_4+c_0)\cos\xi + 2c_0^2 + 2c_4^2 + 1 - 2c_4 - 2c_0$. Therefore $B = \max(|R(0)|^2, |R(\pi)|^2) = max(1, |2c_4+2c_0-1|^2)$ when $c_0c_4 \geq 0$ or $c_0c_4 < 0$ but $|(1-c_4-c_0)(c_4+c_0)/(4c_0c_4)| \geq 1$ and $B = |R(\xi)|^2$ with $\cos\xi = -(c_4+c_0)(1-c_4-c_0)/(4c_0c_4)$ when $c_0c_4 < 0$ and $|(1-c_4-c_0)(c_4+c_0)/(4c_0c_4)| < 1$. Under conditions in Theorem 6, we know that $B < 3$. Therefore the first part of Theorem 6 is proved.

Now we prove the unintegrablity of ϕ. Observe that $\hat{\phi}(\xi) \to 0$ as $\xi \to \infty$ when ϕ is integrable. By (2) we know that $\hat{\phi}(3^l(2k+1)\pi) = H(\pi)^l\hat{\phi}((2k+1)\pi)$ for all $l \geq 1$ and $k \in Z$. It is easy to see that $\hat{\phi}(\pi + 2k\pi) \neq 0$ for some $k \in Z$ when c_0, c_4 are real and $(c_0, c_4) \neq (1,1)$. Therefore $|H(\pi)|$ must be less than one, i.e., $|2c_4+2c_0-1|/3 < 1$, when ϕ is integrable.

For L^2 function ϕ we let $B(\xi) = \sum_{k\in Z}\int \phi(x)\phi(x-k)dx e^{ik\xi}$. Since $B(3\xi) = |H(\xi)|^2B(\xi) + |H(\xi+2\pi/3)|^2B(\xi+2\pi/3) + |H(\xi+4\pi/3)|^2B(\xi+4\pi/3)$, we get $3\int|\phi(x)|^2dx = (1-c_0c_4)/(1+c_0+c_4-c_0c_4-c_0^2-c_4^2) > 0$. Observe that $-1 < c_0+c_4 < 2$ when $(c_0,c_4) \neq (1,1)$ since ϕ is integrable. Therefore $1-c_0c_4 > 0$ except $(c_0,c_4) = (1,1)$ and $1+c_0+c_4-c_0c_4-c_0^2-c_4^2 \leq 0$ implies that ϕ is not square integrable. Theorem 6 is proved.

5. Multiresolution and Wavelets

In this section will consider multiresolution and construction semi-orthogonal wavelets with its scaling function ϕ satisfies (1).

Let $\{c_k\}_{k=0}^4$ be real numbers such that $c_0+c_3 = c_1+c_4 = c_2 = 1$ and $c_0c_4 \neq 0$, and let ϕ be L^2 solution of the refinement equation

$$\phi(x) = \sum_{k=0}^{4} c_k\phi(3x-k),\ \hat{\phi}(0) = 1.$$

Define subspaces V_j $(j \in Z)$ of L^2 by

$$V_j = \{\sum_{k\in Z} d_k\phi(2^j\cdot -k),\ \sum_{k\in Z}|d_k|^2 < \infty\}.$$

Observe that ϕ is globally linearly independent by Theorem 1 when $(c_0,c_4) \neq (1,1)$. Therefore by standard method we know that $\{V_j\}_{j\in Z}$ is a multiresolution of L^2, i.e.,

i) $V_j \subset V_{j+1}$;

ii) $\cap_{j\in Z}V_j = \{0\}$ and $\overline{\cup_{j\in Z}V_j} = L^2$;

iii) there exists $0 < A \leq B < \infty$ such that

$$A\sum_{k\in Z}|d_k|^2 \leq \int_R |\sum_{k\in Z} d_k\phi(x-k)|^2dx \leq B\sum_{k\in Z}|d_k|^2,$$

provided $(c_0, c_4) \neq (1,1)$.

Let the wavelet space W_0 be the orthogonal complement of V_0 in V_1. Now we begin to construct compactly supported wavelets ψ_1 and ψ_2 explicitely such that $W_0 = W_0^1 \oplus +W_0^2$, where

$$W_0^s = \{\sum_{k\in Z} d_k \psi_s(\cdot - k),\ \sum_{k\in Z} |d_k|^2 < \infty\}.$$

Hereafter we assume $(c_0, c_4) \neq (1,1)$.

Denote

$$B(\xi) = \sum_{k\in Z} \int \phi(x)\phi(x-k)dx e^{ik\xi} = be^{-i\xi} + a + be^{i\xi}.$$

Then $B(\xi)$ satisfies

$$B(3\xi) = P(B(\xi)|H(\xi)|^2)$$

by (1) and furthermore we get

$$\begin{aligned} B(\xi) =& \frac{1}{2}(1 + c_0 + c_4 - c_0c_4 - c_0^2 - c_4^2)^{-1} \\ & \big((c_0 + c_4 - c_0^2 - c_4^2)e^{-i\xi} + 2(1 - c_0c_4) + (c_0 + c_4 - c_0^2 - c_4^2)e^{i\xi}\big), \end{aligned}$$

where we define the operator P by

$$P(f)(\xi) = f(\xi) + f(\xi + 2\pi/3) + f(\xi + 4\pi/3)$$

for all 2π-perodic function f.

Let ψ be any L^2 function in W_0. Then $\hat{\psi}(\xi) = A(\xi/3)\hat{\phi}(\xi/3)$ with $A \in L^2[0, 2\pi]$ and $P(A(\xi)\overline{H(\xi)}B(\xi)) = 0$. Recall that $H(2\pi/3) = H(4\pi/3) = 0$ since $c_0 + c_3 = c_1 + c_4 = c_2 = 1$. Therefore $\tilde{A}(\xi) = A(\xi)(1 - e^{i\xi})^{-1} \in L^2[0, 2\pi]$. Denote

$$\tilde{H}(\xi) = c_0 e^{-i\xi} + (1 - c_0 - c_4) + c_4 e^{i\xi}$$

and

$$\alpha(\xi) = 2(1 + c_0 + c_4 - c_0c_4 - c_0^2 - c_4^2)\overline{\tilde{H}(\xi)}B(\xi) = \sum_{k=-2}^{2} \alpha_k e^{ik\xi}.$$

Define

$$\begin{cases} e_1(\xi) =& -\alpha_{-1} + \alpha_0 e^{i\xi} - \alpha_2 e^{3i\xi} \\ e_2(\xi) =& -\alpha_{-2} + \alpha_0 e^{2i\xi} - \alpha_1 e^{3i\xi} \end{cases}$$

when $\alpha_0 \neq 0$ and define

$$\begin{cases} e_1(\xi) =& 1 \\ e_2(\xi) =& -\alpha_{-2}e^{-2i\xi} + \alpha_{-1}e^{-i\xi} - \alpha_1 e^{i\xi} + \alpha_2 e^{2i\xi} \end{cases}$$

when $\alpha_0 = 0$. Then we get

$$\tilde{A}(\xi) = R_1(3\xi)e_1(\xi) + R_2(3\xi)e_2(\xi)$$

for some $R_s \in L^2[0, 2\pi]$ $(s = 1, 2)$.

Define

$$\Phi_{s1}(3\xi) = P(e_s(\xi)\overline{e_1(\xi)}(1 - e^{-i\xi})(1 - e^{ix})B(\xi)),\ \ s = 1, 2.$$

Let $D(\xi)$ be the common divisor of $\Phi_{11}(\xi)$ and $\Phi_{21}(\xi)$.

Define compactly supported wavelets ψ_1 and ψ_2 with help of Fourier transform by

$$\hat{\psi}_1(\xi) = e_1(\xi/3)(1 - e^{i\xi/3})\hat{\phi}(\xi/3) \tag{5}$$

and

$$\hat{\psi}_2(\xi) = D(\xi)^{-1}(1 - e^{i\xi/3})(-\Phi_{21}(\xi)e_1(\xi) + \Phi_{11}(\xi)e_2(\xi))\hat{\phi}(\xi/3). \tag{6}$$

Therefore $\int \psi_s(x)\phi(x - k)dx = 0$ $(s = 1, 2)$ and $\int \psi_1(x)\psi_2(x - k)dx = 0$ for all $k \in Z$. Furthermore $\{\psi_s(\cdot - k), s = 1, 2, k \in Z\}$ is a Riesz basis of W_0 by $\Phi_{11}(\xi) \neq 0$ and our construction of wavelets ψ_1 and ψ_2.

Now let's consider two special cases. First the case $\{\phi(\cdot - k), k \in Z\}$ is orthonormal base of V_0. By usual procedure we know that $\{\phi(\cdot - k)\}_{k\in Z}$ is orthonormal bases of V_0 if and only if

$$|H(\xi)|^2 + |H(\xi + 2\pi/3)|^2 + |H(\xi + 4\pi/3)|^2 = 1 \tag{7}$$

and

$$\hat{\phi}(\xi + 2k\pi) \not\equiv 0 \ \text{ on } k \in Z \tag{8}$$

for all $\xi \in R$. By simple computation, we get (7) holds only if

$$(c_0 - 1/2)^2 + (c_4 - 1/2)^2 = 1/2$$

when c_0 and c_4 are real. On the other hand we know that (8) holds when $(c_0, c_4) \neq (1, 1)$. Furthermore $\alpha_0 \neq 0$ and $B(\xi) = 1$. Then

$$\begin{cases} e_1(\xi) = -c_4 + (1 - c_0 - c_4)e^{i\xi} \\ e_2(\xi) = (1 - c_0 - c_4)e^{2i\xi} - c_0 e^{3i\xi} \end{cases}$$

and $\int \psi_s(x)\psi_t(x - k)dx = 0$ when $k \neq 0$ and $s, t = 1, 2$. Then $\{\psi_s^*(\cdot - k),\ s = 1, 2,\ k \in Z\}$ is an orthonormal basis of W_0, where we define $\psi_s^*(x) = (\int |\psi_s(y)|^2 dy)^{-1/2}\psi_s(x), , s = 1, 2$.

Secondly the case ϕ is symmetric. Let $\phi(x) = \phi(c - x)$ for some c. Then $c = 2$ by comparing support of both sides of (1). Furthermore we get $c_0 = c_4$, $c_1 = c_3$ in (1). Observe that $\alpha_0 \neq 0$ and $e_1(\xi) = \overline{e_2(\xi)e^{-3i\xi}}$. Define $\tilde{e}_1(\xi) = e_1(\xi) + e_2(\xi)e^{-3i\xi}$ and $\tilde{e}_2(\xi) = e_1(\xi) - e_2(\xi)e^{-3i\xi}$. Define $\tilde{\Phi}_{s1}(\xi) = P(\tilde{e}_s(\xi)\overline{\tilde{e}_1(\xi)}(1 - e^{-i\xi})(1 - e^{i\xi})B(\xi))$. Then the common divisor $D(\xi)$ of $\tilde{\Phi}_{11}(\xi)$ and $\tilde{\Phi}_{21}(\xi)$ is one. Let $\tilde{\psi}_1$ and $\tilde{\psi}_2$ be defined as (5) and (6) with e_s replaced by $\tilde{e}_s$ and $\Phi_{s1}(\xi)$ replaced by $\tilde{\Phi}_{s1}(\xi), s = 1, 2$. Then $\tilde{\psi}_1$ ($\tilde{\psi}_2$) constructed by above algorithm is antisymmetric (symmetric).

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