SAMPLING AND RECONSTRUCTION OF SIGNALS IN A REPRODUCING KERNEL SUBSPACE OF $L^p(\mathbb{R}^d)$

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ABSTRACT. In this paper, we consider sampling and reconstruction of signals in a reproducing kernel subspace of $L^p(\mathbb{R}^d), 1 \le p \le \infty$, associated with an idempotent integral operator whose kernel has certain off-diagonal decay and regularity. The space of p-integrable non-uniform splines and the shift-invariant spaces generated by finitely many localized functions are our model examples of such reproducing kernel subspaces of $L^p(\mathbb{R}^d)$. We show that a signal in such reproducing kernel subspaces can be reconstructed in a stable way from its samples taken on a relatively-separated set with sufficiently small gap. We also study the exponential convergence, consistency, and the asymptotic pointwise error estimate of the iterative approximation-projection algorithm and the iterative frame algorithm for reconstructing a signal in those reproducing kernel spaces from its samples with sufficiently small gap.

1. INTRODUCTION

Sampling and reconstruction is a cornerstone of signal processing. The most common form of sampling is the uniform sampling of a bandlimited signal. In this case, perfect reconstruction of the signal from its uniform samples is possible when the samples are taken at a rate greater than twice the bandwidth [28, 39]. Motivated by the intensive research activity taking place around wavelets, the paradigm for sampling and reconstructing band-limited signals has been extended over the past decade to signals in shift-invariant spaces [4, 46]. Recently, the above paradigm has been further extended to representing signals with finite rate of innovation, which are neither band-limited nor living in a shift-invariant space [17, 31, 43, 44, 47]. Here a signal is said to have *finite rate of innovation* if it has finite number of degrees of freedom per unit of time, that is, if it has requires only a finite number of samples per unit of time to specify the signal [47].

In this paper, we consider sampling and reconstruction of signals in a reproducing kernel subspace of $L^p(\mathbb{R}^d), 1 \le p \le \infty$. Here and henceforth $L^p := L^p(\mathbb{R}^d)$ is the space of all p-integrable functions on the d-dimensional Euclidean space $\mathbb{R}^d$ with the standard norm $\|\cdot\|_{L^p(\mathbb{R}^d)}$, or $\|\cdot\|_p$ for short. A *reproducing kernel subspace of* $L^p(\mathbb{R}^d)$ [10] is a closed subspace V of $L^p(\mathbb{R}^d)$

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such that the evaluation functionals on V are continuous, i.e., for any $x \in \mathbb{R}^d$ there exists a positive constant C_x such that

$$|f(x)| \le C_x \|f\|_{L^p(\mathbb{R}^d)} \quad \text{for all } f \in V. \tag{1.1}$$

Let $1 \le p \le \infty$. We say that a bounded linear operator T on $L^p(\mathbb{R}^d)$ is an *idempotent* operator if it satisfies

$$T^2 = T. \tag{1.2}$$

Denote by V the range space of the idempotent operator T on $L^p(\mathbb{R}^d)$, i.e.,

$$V := \big\{ Tf \mid \ f \in L^p(\mathbb{R}^d) \big\}. \tag{1.3}$$

We say that the range space V of the idempotent operator T on $L^p(\mathbb{R}^d)$ is a *reproducing kernel space V associated with the idempotent operator T on* $L^p(\mathbb{R}^d)$ if it is a reproducing kernel subspace of $L^p(\mathbb{R}^d)$.

A trivial example of idempotent linear operators is the identity operator. In this case, the range space is the whole space $L^p(\mathbb{R}^d)$ on which the evaluation functional is not continuous. As pointed out in [34], the whole space $L^2(\mathbb{R}^d)$ is too big to have stable sampling and reconstruction of signals belonging to this space. So it would be reasonable and necessary to have certain additional constraints on the idempotent operator T. In this paper, we further assume that the idempotent operator T is an integral operator

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y) dy, \quad f \in L^p(\mathbb{R}^d), \tag{1.4}$$

whose measurable kernel K has certain off-diagonal decay and regularity, namely,

$$\big\| \sup_{z \in \mathbb{R}^d} |K(\cdot + z, z)| \big\|_{L^1(\mathbb{R}^d)} < \infty, \tag{1.5}$$

and

$$\lim_{\delta \to 0} \big\| \sup_{z \in \mathbb{R}^d} |\omega_\delta(K)(\cdot + z, z)| \big\|_{L^1(\mathbb{R}^d)} = 0 \tag{1.6}$$

[29, 42]. Here the *modulus of continuity* $\omega_\delta(K)$ of a kernel function K on $\mathbb{R}^d \times \mathbb{R}^d$ is defined by

$$\omega_\delta(K)(x,y) = \sup_{x',y' \in [-\delta,\delta]^d} |K(x+x', y+y') - K(x,y)|. \tag{1.7}$$

In this paper, we assume that signals to be sampled and represented live in a reproducing kernel space associated with an idempotent integral operator whose kernel satisfies (1.5) and (1.6). The reason for this setting is three-fold. First, the range space of an idempotent integral operator whose kernel satisfies (1.5) and (1.6) is a reproducing kernel subspace of $L^p(\mathbb{R}^d)$, see Theorem A.1 in the Appendix. Secondly, signals in the range space of an idempotent integral operator whose kernel satisfies (1.5) and (1.6) have finite rate of innovation, see Theorem A.2 in the Appendix. Thirdly, the common model spaces in sampling theory such as the space of p-integrable non-uniform splines of order n satisfying $n-1$ continuity conditions at each knot [38, 48] and the finitely-generated shift-invariant space with its generators having certain regularity and decay at infinity [4, 46], are the range space of some idempotent integral operators whose kernels satisfy (1.5) and (1.6), see Examples A.3 and A.4 in the Appendix.

A discrete subset Γ of $\mathbb{R}^d$ is said to be *relatively-separated* if

$$B_\Gamma(\delta) := \sup_{x\in\mathbb{R}^d} \sum_{\gamma\in\Gamma} \chi_{\gamma+[-\delta/2,\delta/2]^d}(x) < \infty \tag{1.8}$$

for some $\delta > 0$, while a positive number δ is said to be a *gap* of a relatively-separated subset Γ of $\mathbb{R}^d$ if

$$A_\Gamma(\delta) := \inf_{x\in\mathbb{R}^d} \sum_{\gamma\in\Gamma} \chi_{\gamma+[-\delta/2,\delta/2]^d}(x) \ge 1 \tag{1.9}$$

[8]. Note that the set of all positive numbers δ with $A_\Gamma(\delta) \ge 1$ is either an interval or an empty set because $A_\Gamma(\delta)$ is an increasing function of $\delta > 0$. Then for a relatively-separated subset Γ of $\mathbb{R}^d$ having positive gap, we define the smallest positive number δ with $A_\Gamma(\delta) \ge 1$ as its *maximal gap*. One may verify that a bi-infinite increasing sequence $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ of real numbers is relatively-separated if $\inf_{k\in\mathbb{Z}}(\lambda_{k+1}-\lambda_k) > 0$, and that it has maximal gap $\sup_{k\in\mathbb{Z}}(\lambda_{k+1}-\lambda_k)$ if it is finite.

In this paper, we assume that the sample $Y := (f(\gamma))_{\gamma\in\Gamma}$ of a signal f is taken on a relatively-separated subset Γ of $\mathbb{R}^d$ with positive gap.

The samplability is one of most important topics in sampling theory, see for instance [22, 26, 46] for band-limited signals, [4, 43] for signals in a shift-invariant space, [16, 20, 21, 24, 25] for signals in a co-orbit space, and [27, 33] for signals in reproducing kernel Hilbert and Banach spaces. In this paper, we study the *samplability of signals* in a reproducing kernel subspace of $L^p(\mathbb{R}^d)$ associated with an idempotent operator. Particularly, in Section 2, we show that any signal in a reproducing kernel subspace V of $L^p(\mathbb{R}^d)$ associated with an idempotent operator whose kernel satisfies (1.5) and (1.6) can be reconstructed *in a stable way* from its samples taken on a relatively-separated set Γ with sufficiently small gap δ, i.e., there exist positive constants A and B such that

$$A\|f\|_{L^p(\mathbb{R}^d)} \le \|(f(\gamma))_{\gamma\in\Gamma}\|_{\ell^p(\Gamma)} \le B\|f\|_{L^p(\mathbb{R}^d)} \quad \text{for all } f\in V \tag{1.10}$$

(see Theorem 2.1 for the precise statement). Here and henceforth, given a discrete set Γ, $\ell^p := \ell^p(\Gamma), 1 \le p \le \infty$, is the space of all p-summable sequences on Γ with the standard norm $\|\cdot\|_{\ell^p(\Gamma)}$, or $\|\cdot\|_p$ for short.

In this paper, we then study the linear reconstruction of a signal from its samples taken on a relatively-separated set with sufficiently small gap. The iterative approximation-projection reconstruction algorithm is an efficient algorithm to reconstruct a signal from its samples, which was introduced in [22] for reconstructing band-limited signals, and was later generalized to signals in shift-invariant spaces in [2]; see also [4, 7, 23] and the references therein for various generalizations and applications. In Section 3 of this paper, we introduce the *iterative approximation-projection reconstruction algorithm* for reconstructing a signal in a reproducing kernel subspace of $L^p(\mathbb{R}^d)$ from its samples taken on a relatively-separated set with sufficiently small gap, and study its exponential convergence, consistency, and numerical implementation of the above iterative approximation-projection algorithm (see Theorem 3.1, Remark 3.1 and Remark 3.2 for details).

Denote the standard action between functions $f \in L^p(\mathbb{R}^d)$ and $g \in L^{p/(p-1)}(\mathbb{R}^d)$ by

$$\langle f, g\rangle = \int_{\mathbb{R}^d} f(x)g(x)dx. \tag{1.11}$$

Then the stability condition (1.10) can be interpreted as the p-frame property of $\{K(\gamma,\cdot)\}_{\gamma\in\Gamma}$ on the space V. Here for a Banach subspace V of $L^p(\mathbb{R}^d)$, we say that a family $\Phi=\{\psi_\gamma\}_{\gamma\in\Gamma}$ of functions in $L^{p/(p-1)}(\mathbb{R}^d)$ is a *p-frame for V* [6] if there exist positive constants A and B such that

$$A\|f\|_{L^p(\mathbb{R}^d)} \le \big\|(\langle f,\psi_\gamma\rangle)_{\gamma\in\Gamma}\big\|_{\ell^p(\Gamma)} \le B\|f\|_{L^p(\mathbb{R}^d)} \quad \text{for all } f\in V. \tag{1.12}$$

Then a natural linear reconstruction algorithm is the frame reconstruction algorithm; see [11, 49] for reconstructing band-limited signals, [4, 9, 15, 30] for reconstructing signals in shift-invariant spaces, and [35] for reconstructing signals in some reproducing kernel Hilbert spaces. In Section 4, we introduce the *preconditioned frame algorithm* for reconstructing signals in a reproducing kernel space associated with an idempotent integral operator from its samples taken a relatively-separated set Γ with sufficiently small gap, and study its exponential convergence and consistency (see Theorem 4.1 for details).

Reconstructing a function from data corrupted by noise and estimating the reconstruction error are leading problems in sampling theory, however they have not been given as much attention; see [18, 36, 40] for reconstructing bandlimited signals, [5, 18] for reconstructing signals in shift-invariant spaces, and [12, 31, 32] for reconstructing signals with finite rate of innovations. It is observed in [37] that reconstruction from noisy data may introduce spatially-dependent noise in the reconstructed signal (hence spatial dependent artifacts) that are undesirable for sub-pixel signal processing. Thus it is desirable to have an accurate error estimate of the reconstructed signal at each point. In this paper, we show that the reconstruction via the approximation-projection reconstruction algorithm and the frame reconstruction algorithm is unbiased, and we also provide an asymptotic estimate of the variance of the error between the reconstruction from noisy sample of a signal f via these algorithms and the signal f in a reproducing kernel space, see Theorem 5.1 and Remark 5.2.

The range space V of an idempotent operator T on $L^p(\mathbb{R}^d)$ has various properties. For instance, it is complementable and the null space $N(T) := \{g \in L^p(\mathbb{R}^d) \mid Tg = 0\}$ is its algebraic and topological complement. In the appendix, some properties of the range space of an idempotent integral operator on $L^p(\mathbb{R}^d)$ whose kernel satisfies (1.5) and (1.6) are established, such as the reproducing kernel property in Theorem A.1 and the frame property in Theorem A.2.

2. Samplability of signals in a reproducing kernel space

In this section, we consider the samplability of signals in a reproducing kernel subspace V of $L^p(\mathbb{R}^d)$ associated with an idempotent integral operator whose kernel satisfies (1.5) and (1.6), by showing that any signal in V can be reconstructed in a stable way from its samples taken on a relatively-separated set with sufficiently small gap.

Theorem 2.1. *Let $1 \le p \le \infty$, T be an idempotent integral operator whose kernel K satisfies* (1.5) *and* (1.6), *V be the reproducing kernel subspace of $L^p(\mathbb{R}^d)$ associated with the operator T, and $\delta_0 > 0$ be so chosen that*

$$r_0 := \Big\| \sup_{z\in\mathbb{R}^d} |\omega_{\delta_0/2}(K)(\cdot + z, z)|\Big\|_{L^1(\mathbb{R}^d)} < 1. \tag{2.1}$$

Then any signal f in V can be reconstructed in a stable way from its samples $f(\gamma), \gamma \in \Gamma$, taken on a relatively-separated subset Γ of $\mathbb{R}^d$ with gap δ_0. Moreover,

$$\begin{aligned}(2.2)\quad &(1-r_0)\big(\delta_0^{-d}A_\Gamma(\delta_0)\big)^{1/p}\|f\|_{L^p(\mathbb{R}^d)}\\ \le\ &\big\|(f(\gamma))_{\gamma\in\Gamma}\big\|_{\ell^p(\Gamma)} \le (1+r_0)\big(\delta_0^{-d}B_\Gamma(\delta_0)\big)^{1/p}\|f\|_{L^p(\mathbb{R}^d)} \quad \text{for all } f \in V.\end{aligned}$$

Now we apply the above samplability result to signals in a shift-invariant space. Let

$$\mathcal{W} := \Big\{f \big|\ \|f\|_{\mathcal{W}} := \sum_{k\in\mathbb{Z}^d} \sup_{x\in[-1/2,1/2]^d} |f(x+k)| < \infty\Big\} \tag{2.3}$$

be the *Wiener amalgam space* [4, 19]. Let $\phi_1, \ldots, \phi_r \in \mathcal{W}$ be continuous functions on $\mathbb{R}^d$ with the property that $\{\phi_i(\cdot - k) :\ 1 \le i \le r, k \in \mathbb{Z}^d\}$ is an orthonormal subset of $L^2(\mathbb{R}^d)$. Then the integral operator T defined by

$$Tf(x) = \int_{\mathbb{R}^d} \Big(\sum_{i=1}^{r} \sum_{k\in\mathbb{Z}^d} \phi_i(x-k)\phi_i(y-k)\Big) f(y)dy \quad \text{for all } f \in L^2(\mathbb{R}^d) \tag{2.4}$$

is an idempotent operator whose kernel satisfies (1.5) and (1.6). This yields the samplability of signals in a finitely-generated shift-invariant space [2].

Corollary 2.2. *Let $\phi_1, \ldots, \phi_r \in \mathcal{W}$ be continuous functions on $\mathbb{R}^d$ such that $\{\phi_i(\cdot - k) |\ 1 \le i \le r, k \in \mathbb{Z}^d\}$ is an orthonormal subset of $L^2(\mathbb{R}^d)$. Define the finitely-generated shift-invariant space $V_2(\phi_1, \ldots, \phi_r)$ by*

$$V_2(\phi_1, \ldots, \phi_r) = \Big\{ \sum_{i=1}^r \sum_{k\in\mathbb{Z}^d} c_i(k)\phi_i(\cdot - k)\ \Big|\ \sum_{i=1}^r \sum_{k\in\mathbb{Z}^d} |c_i(k)|^2 < \infty \Big\}. \tag{2.5}$$

Then any signal f in $V_2(\phi_1, \ldots, \phi_r)$ can be reconstructed in a stable way from its samples $f(\gamma), \gamma \in \Gamma$, taken on a relatively-separated subset Γ of $\mathbb{R}^d$ with sufficiently small gap δ_0.

The following theorem is a slight generalization of Theorem 2.1.

Theorem 2.3. *Let $1 \le p \le \infty$, T be an idempotent integral operator whose kernel K is continuous and satisfies*

$$\sup_{x\in\mathbb{R}^d} \|K(x, \cdot)\|_{L^1(\mathbb{R}^d)} + \sup_{y\in\mathbb{R}^d} \|K(\cdot, y)\|_{L^1(\mathbb{R}^d)} < \infty, \tag{2.6}$$

V be the reproducing kernel subspace of $L^p(\mathbb{R}^d)$ associated with the operator T, and $\delta_0 > 0$ be so chosen that

$$\begin{aligned} r_0' \ := \ & \Big(\sup_{x\in\mathbb{R}^d} \Big\| \sup_{|t|\le\delta_0/2} |K(x+t, \cdot) - K(x, \cdot)| \Big\|_{L^1(\mathbb{R}^d)}\Big)^{1-1/p} \\ & \times \Big(\sup_{y\in\mathbb{R}^d} \Big\| \sup_{|t|\le\delta_0/2} |K(\cdot+t, y) - K(\cdot, y)| \Big\|_{L^1(\mathbb{R}^d)}\Big)^{1/p} < 1. \end{aligned} \tag{2.7}$$

Then any signal f in V can be reconstructed in a stable way from its samples $f(\gamma), \gamma \in \Gamma$, taken on a relatively-separated subset Γ of $\mathbb{R}^d$ with gap δ_0.

Remark 2.1. The conclusion in Theorem 2.3 is established in [24, Section 7.5] when the kernel K of the idempotent operator T satisfies

$$K(x, y) = \overline{K(y, x)}. \tag{2.8}$$

For $p = 2$, an idempotent operator T with kernel K satisfying (2.8) is a projection operator onto a closed subspace of L^2. Hence the idempotent operator T with its kernel satisfying (2.8) is uniquely determined by its range space V onto L^2. The above conclusion on the idempotent operator does not hold without the assumption (2.8) on its kernel. We leave the above option on the kernel of idempotent operators free for better estimate in the gap δ_0 in Theorem 2.1, and also for our further study on local exact reconstruction (c.f. [3, 41, 45] for signals in shift-invariant spaces). For instance, let us consider samplability of signals in the linear spline space

$$V_1 := \Big\{ \sum_{k\in\mathbb{Z}} c(k)h(x-k) \Big|\ \sup_{k\in\mathbb{Z}} |c(k)| < \infty \Big\},$$

where $h(x) := \max(1 - |x|, 0)$ is the hat function. It is well known [3] that a signal f in the linear spline space V_1 can be reconstructed in a stable way from its samples $f(\gamma_k), k \in \mathbb{Z}$, with maximal gap $\delta_0 := \sup_{k\in\mathbb{Z}}(\gamma_{k+1}-\gamma_k) < 1$. For any integer $N \ge 1$, define

$$K_N(x,y) = \frac{3N^2}{\sqrt{9N^2-6N}} \sum_{k,l\in\mathbb{Z}} h(x-k)h(N(y-l))\big(\sqrt{9N^2-6N}-3N+1\big)^{|k-l|},$$

and let T_N be the integral operator with kernel K_N. One may verify that $T_N, N \ge 1$, are idempotent operators with the same range space V_1 and the kernel K_N satisfies (2.8) **only** when $N = 1$. Recalling that $K_N(x-1,y-1) = K_N(x,y)$ and $K_N(-x,-y) = K_N(x,y)$, we have

$$\begin{aligned} & \sup_{x\in\mathbb{R}} \Big\| \sup_{|t|\le\delta_0/2} |K_N(x+t,\cdot)-K_N(x,\cdot)| \Big\|_1 \\ = & \sup_{x\in[0,1/2]} \Big\| \sup_{|t|\le\delta_0/2} |K_N(x+t,\cdot)-K_N(x,\cdot)| \Big\|_1 \\ \le & \frac{3N^2}{\sqrt{9N^2-6N}} \sum_{s=-\infty}^{\infty} \big(3N-1-\sqrt{9N^2-6N}\big)^{|s|} \\ & \times \sup_{x\in[0,1/2]} \Big\| \sup_{|t|\le\delta_0/2} \sum_{k\in\mathbb{Z}} |h(x-k)-h(x+t-k)| h(N(\cdot-k-s)) \Big\|_1 \\ \le & \frac{9N\delta_0}{6N-4}. \end{aligned}$$

This shows that the inequality (2.7) holds for $K = K_N$ and $p = \infty$ when $\delta_0 < \frac{2}{3} - \frac{4}{9N}$. On the other hand, we have

$$\begin{aligned} & \sup_{x\in\mathbb{R}} \Big\| \sup_{|t|\le\delta_0/2} |K_1(x+t,\cdot)-K_1(x,\cdot)| \Big\|_1 \\ \ge & \|K_1(\delta_0/2,\cdot)-K_1(0,\cdot)\|_1 = \frac{(9-\sqrt{3})\delta_0}{4}, \end{aligned}$$

which implies that the inequality (2.7) does not hold for $K = K_1$ and $p = \infty$ when $\delta_0 \ge \frac{4}{(9-\sqrt{3})} \approx 0.5504$ and so the theorem does not apply.

We conclude this section by providing proofs of Theorems 2.1 and 2.3. To prove Theorem 2.1, we need a technical lemma.

Lemma 2.4. *Let $1 \le p \le \infty$, $\delta_0 \in (0,\infty)$, $r \in (0,1)$, and Γ be a discrete subset of $\mathbb{R}^d$ with the property that*

$$1 \le A_\Gamma(\delta_0) \le B_\Gamma(\delta_0) < \infty. \tag{2.9}$$

Assume that $f \in L^p(\mathbb{R}^d)$ satisfies

$$\|\omega_{\delta_0/2}(f)\|_{L^p(\mathbb{R}^d)} \le r\|f\|_{L^p(\mathbb{R}^d)}, \tag{2.10}$$

and $U := \{u_\gamma\}_{\gamma\in\Gamma}$ is a bounded uniform partition of unity (BUPU) associated with the covering $\{\gamma + [-\delta_0/2, \delta_0/2]^d\}_{\gamma\in\Gamma}$ of $\mathbb{R}^d$, i.e.,

$$\begin{cases} 0 \le u_\gamma(x) \le 1 \text{ for all } x \in \mathbb{R}^d \text{ and } \gamma \in \Gamma, \\ u_\gamma \text{ is supported in } \gamma + [-\delta_0/2, \delta_0/2]^d \text{ for each } \gamma \in \Gamma, \text{and} \\ \sum_{\gamma\in\Gamma} u_\gamma(x) \equiv 1 \text{ for all } x \in \mathbb{R}^d. \end{cases} \tag{2.11}$$

Then

$$(1-r)\|f\|_{L^p(\mathbb{R}^d)} \le \big\|\big(f(\gamma)\|u_\gamma\|_{L^1(\mathbb{R}^d)}^{1/p}\big)_{\gamma\in\Gamma}\big\|_{\ell^p(\Gamma)} \le (1+r)\|f\|_{L^p(\mathbb{R}^d)}. \tag{2.12}$$

Proof. By the definition of the modulus of continuity,

$$|f(x)| - |\omega_{\delta_0/2}(f)(x)| \le |f(\gamma)| \le |f(x)| + |\omega_{\delta_0/2}(f)(x)| \tag{2.13}$$

for all $x \in \gamma + [-\delta_0/2, \delta_0/2]^d$ and $\gamma \in \Gamma$. This together with (2.9) and (2.10) proves (2.12).

For $1 \le p < \infty$, it follows from (2.10), (2.11), and (2.13) that

$$\begin{aligned} \|f\|_p &= \Big(\sum_{\gamma\in\Gamma}\int_{\mathbb{R}^d} |f(x)|^p u_\gamma(x)dx\Big)^{1/p} \\ &\le \Big(\sum_{\gamma\in\Gamma}\int_{\mathbb{R}^d} |f(\gamma)|^p u_\gamma(x)dx\Big)^{1/p} + \Big(\sum_{\gamma\in\Gamma}\int_{\mathbb{R}^d} |\omega_{\delta_0/2}(f)(x)|^p u_\gamma(x)dx\Big)^{1/p} \\ &\le \Big(\sum_{\gamma\in\Gamma} |f(\gamma)|^p \|u_\gamma\|_1\Big)^{1/p} + r\|f\|_p, \end{aligned}$$

and

$$\begin{aligned} \Big(\sum_{\gamma\in\Gamma} |f(\gamma)|^p \|u_\gamma\|_1\Big)^{1/p} &\le \Big(\sum_{\gamma\in\Gamma}\int_{\mathbb{R}^d} \big||f(x)| + \omega_{\delta_0/2}(f)(x)\big|^p u_\gamma(x)dx\Big)^{1/p} \\ &\le (1+r)\|f\|_p. \end{aligned}$$

Then (2.12) for $1 \le p < \infty$ is proved. □

Remark 2.2. Two popular examples of bounded uniform partitions of unity (BUPU) associated with the covering $\{\gamma+[-\delta_0/2, \delta_0/2]^d\}_{\gamma\in\Gamma}$ of $\mathbb{R}^d$ are given by

$$u_\gamma(x) = \frac{\chi_{\gamma+[-\delta_0/2,\delta_0/2]^d}(x)}{\sum_{\gamma'\in\Gamma} \chi_{\gamma'+[-\delta_0/2,\delta_0/2]^d}(x)},\ \gamma \in \Gamma, \tag{2.14}$$

and

$$u_\gamma(x) = \chi_{V_\gamma}(x),\ \gamma \in \Gamma, \tag{2.15}$$

where V_γ is the Voronoi polygon whose interior consists of all points in $\mathbb{R}^d$ being closer to γ than any other point $\gamma' \in \Gamma$.

Given a continuously differentiable function f on the real line, its modulus of continuity $\omega_\delta(f)(x)$ is dominated by the integral of its derivative f' on $x+[-\delta,\delta]$, i.e.,

$$\omega_\delta(f)(x) \le \int_{-\delta}^{\delta} |f'(x+t)|dt \quad \text{for all } x\in\mathbb{R}.$$

Then the following result (which is well known for band-limited signals [22]) follows easily from Lemma 2.4.

Corollary 2.5. *Let $1\le p\le\infty$, f be a time signal satisfying*

$$\|f'\|_{L^p(\mathbb{R})} \le B_0\|f\|_{L^p(\mathbb{R})} \tag{2.16}$$

for some positive constant B_0, and $\Gamma=\{\gamma_k\}_{k\in\mathbb{Z}}$ be a relatively-separated subset of $\mathbb{R}$ with maximal gap $\delta_0 < 1/B_0$. Then there exists a positive constant C (that depends on $B_0, B_\Gamma(\delta_0)$ and $A_\Gamma(\delta_0)$ only) such that

$$C^{-1}\|f\|_{L^p(\mathbb{R})} \le \big\|\big(f(\gamma)\|u_\gamma\|_{L^1(\mathbb{R}^d)}^{1/p}\big)_{\gamma\in\Gamma}\big\|_{\ell^p(\Gamma)} \le C\|f\|_{L^p(\mathbb{R})}. \tag{2.17}$$

Now we prove Theorem 2.1.

Proof of Theorem 2.1. For any $f\in V$,

$$\begin{aligned}\|\omega_{\delta_0/2}(f)\|_p &= \|\omega_{\delta_0/2}(Tf)\|_p \le \Big\|\int_{\mathbb{R}^d}\omega_{\delta_0/2}(K)(\cdot,y)|f(y)|dy\Big\|_p \\ &\le \big\|\sup_{z\in\mathbb{R}^d}|\omega_{\delta_0/2}(K)(\cdot+z,z)|\big\|_1\|f\|_p = r_0\|f\|_p.\end{aligned} \tag{2.18}$$

For any discrete set Γ with $1\le A_\Gamma(\delta_0)\le B_\Gamma(\delta_0)<\infty$, we define $\{u_\gamma\}_{\gamma\in\Gamma}$ as in (2.14). Then

$$\frac{\delta_0^d}{B_\Gamma(\delta_0)} \le \|u_\gamma\|_1 \le \frac{\delta_0^d}{A_\Gamma(\delta_0)} \quad \text{for all } \gamma\in\Gamma. \tag{2.19}$$

From (2.1), (2.18) and Lemma 2.4, we obtain the estimates in (2.2) for $p=\infty$. On the other hand, from (2.1), (2.18), (2.19) and Lemma 2.4, we get the following estimate for $1\le p<\infty$:

$$\begin{aligned}\Big(\sum_{\gamma\in\Gamma}|f(\gamma)|^p\Big)^{1/p} &\le (\delta_0^{-d}B_\Gamma(\delta_0))^{1/p}\Big(\sum_{\gamma\in\Gamma}|f(\gamma)|^p\|u_\gamma\|_1\Big)^{1/p} \\ &\le (\delta_0^{-d}B_\Gamma(\delta_0))^{1/p}(1+r_0)\|f\|_p\end{aligned}$$

and

$$\begin{aligned}\Big(\sum_{\gamma\in\Gamma}|f(\gamma)|^p\Big)^{1/p} &\ge (\delta_0^{-d}A_\Gamma(\delta_0))^{1/p}\Big(\sum_{\gamma\in\Gamma}|f(\gamma)|^p\|u_\gamma\|_1\Big)^{1/p} \\ &\ge (\delta_0^{-d}A_\Gamma(\delta_0))^{1/p}(1-r_0)\|f\|_p.\end{aligned}$$

This proves (2.2) for $1\le p<\infty$. □

Proof of Theorem 2.3. Similar argument used in the proof of Theorem 2.1 can be applied to prove Theorem 2.3. We leave the detailed proof for the interested readers. □

3. Iterative approximation-projection reconstruction algorithm

In this section, we show that signals in a reproducing kernel subspace of $L^p(\mathbb{R}^d)$ associated with an idempotent integral operator can be reconstructed, via an iterative approximation-projection reconstruction algorithm, from its samples taken on a relatively-separated set with sufficiently small gap.

Theorem 3.1. *Let $1 \le p \le \infty$, T be an idempotent integral operator whose kernel K satisfies* (1.5) *and* (1.6)*, V be the reproducing kernel subspace of $L^p(\mathbb{R}^d)$ associated with the operator T, and $\delta_0 > 0$ be so chosen that* (2.1) *holds. Set*

$$r_0 := \Big\| \sup_{z\in\mathbb{R}^d} |\omega_{\delta_0/2}(K)(\cdot+z,z)|\Big\|_{L^1(\mathbb{R}^d)}.$$

Then for any relatively-separated subset Γ with gap δ_0 and $c_0 = (c_0(\gamma))_{\gamma\in\Gamma} \in \ell^p(\Gamma)$, the sequence $\{f_n\}_{n=0}^\infty$ of signals in V defined by

$$\left\{\begin{array}{l} f_0(x) = \sum_{\gamma\in\Gamma} c_0(\gamma)\ Tu_\gamma(x),\\ f_n(x) = f_0(x) + f_{n-1}(x) - \sum_{\gamma\in\Gamma} f_{n-1}(\gamma)\ Tu_\gamma(x) \quad \text{for } n\ge 1, \end{array}\right. \tag{3.1}$$

converges exponentially, precisely

$$\|f_n - f_\infty\|_{L^p(\mathbb{R}^d)} \le \|T\| \|f_0\|_{L^p(\mathbb{R}^d)} r_0^{n+1}/(1-r_0) \qquad \text{for some } f_\infty \in V, \tag{3.2}$$

where $U := \{u_\gamma\}_{\gamma\in\Gamma}$ is a BUPU in (2.11)*. The sample of the limit signal f_∞ and the given initial data c_0 are related by*

$$\sum_{\gamma\in\Gamma} \big(c_0(\gamma) - f_\infty(\gamma)\big) Tu_\gamma(x) \equiv 0. \tag{3.3}$$

Furthermore the iterative algorithm (3.1) *is consistent, i.e., if the given initial data $c_0 = (g(\gamma))_{\gamma\in\Gamma}$ is obtained by sampling a signal $g \in V$ then the sequence $\{f_n\}_{n=0}^\infty$ in the iterative algorithm* (3.1) *converges to g.*

Proof. Define a bounded operator $Q_{\Gamma,U}$ on L^p by

$$\begin{aligned} Q_{\Gamma,U}f(x) &:= \sum_{\gamma\in\Gamma} (Tf)(\gamma) u_\gamma(x) - (Tf)(x) \\ &= \int_{\mathbb{R}^d} \Big(\sum_{\gamma\in\Gamma} u_\gamma(x) K(\gamma,y) - K(x,y)\Big) f(y)dy,\ f\in L^p. \end{aligned} \tag{3.4}$$

Then

$$Q_{\Gamma,U}T = Q_{\Gamma,U} \tag{3.5}$$

by (1.2), and

$$\|Q_{\Gamma,U} f\|_p \le r_0 \|f\|_p \quad \text{for all } \ f\in L^p \tag{3.6}$$

by the following estimate for the integral kernel of the operator $Q_{\Gamma,U}$:

$$\Big|\sum_{\gamma\in\Gamma} u_\gamma(x)K(\gamma,y) - K(x,y)\Big| \le \sup_{z'\in\mathbb{R}^d} \big|\omega_{\delta_0/2}(K)(x-y+z',z')\big|. \tag{3.7}$$

Define the approximation-projection operator $P_{\Gamma,U}$ by

$$P_{\Gamma,U} = TQ_{\Gamma,U} + T. \tag{3.8}$$

Then it follows from (1.2), (3.5) and (3.6) that

$$P_{\Gamma,U}T = TP_{\Gamma,U} = P_{\Gamma,U}, \tag{3.9}$$

$$(T - P_{\Gamma,U})^n = (-1)^n TQ_{\Gamma,U}^n \quad \text{for all } n \geq 1, \tag{3.10}$$

and

$$\|(T - P_{\Gamma,U})^n\| \leq \|T\| r_0^n \quad \text{for all } n \geq 1. \tag{3.11}$$

By (3.1), (3.4) and (3.8),

$$\begin{aligned} f_{n+1} - f_n &= (T - P_{\Gamma,U})(f_n - f_{n-1}) \\ &= \cdots \\ &= (T - P_{\Gamma,U})^n (f_1 - f_0) \\ &= (T - P_{\Gamma,U})^{n+1} f_0, \quad n \geq 0. \end{aligned} \tag{3.12}$$

This together with (3.11) proves the exponential convergence of $f_n, n \geq 0$, and the estimate (3.2).

The equation (3.3) follows easily by taking limit on both sides of (3.1) and applying (2.2).

Define

$$R_{\mathrm{AP}} := T + \sum_{n=1}^{\infty} (T - P_{\Gamma,U})^n. \tag{3.13}$$

Then it follows from (3.9) and (3.11) that R_{AP} is a bounded operator on L^p and a pseudo-inverse of the operator $P_{T,U}$, i.e.,

$$R_{\mathrm{AP}} P_{\Gamma,U} = P_{\Gamma,U} R_{\mathrm{AP}} = T, \tag{3.14}$$

and moreover it satisfies

$$R_{\mathrm{AP}} T = T R_{\mathrm{AP}} = R_{\mathrm{AP}}.$$

Applying (3.12) iteratively leads to

$$f_n = \Big(T + \sum_{k=1}^{n} (T - P_{\Gamma,U})^k \Big) f_0 \quad \text{for all } n \geq 1, \tag{3.15}$$

which together with (3.13) implies that

$$f_\infty = \lim_{n \to \infty} f_n = R_{\mathrm{AP}} f_0. \tag{3.16}$$

In the case that the initial data c_0 is the sample of a signal $g \in V$, the initial signal f_0 in the iterative algorithm (3.1) and the signal g are related by

$$f_0 = P_{\Gamma,U} g. \tag{3.17}$$

Combining (3.14), (3.16) and (3.17) proves the consistency of the iterative algorithm (3.1). □

From the proof of Theorem 3.1, we have the following result for the operator $R_{\rm AP}$ in (3.13).

Corollary 3.2. *Let $1\le p\le\infty$, T be an idempotent integral operator whose kernel K satisfies* (1.5) *and* (1.6)*, V be the reproducing kernel subspace of $L^p(\mathbb{R}^d)$ associated with the operator T, $\delta_0>0$ be so chosen that* (2.1) *holds, Γ be a relatively-separated subset with gap δ_0, $U:=\{u_\gamma\}_{\gamma\in\Gamma}$ is a BUPU in* (2.11)*, and $R_{\rm AP}$ be as in* (3.13)*. Then $R_{\rm AP}$ is a bounded integral operator on $L^p(\mathbb{R}^d)$ and its kernel $K_{\rm AP}$ satisfies* (1.5)*,* (1.6)*, and*

(3.18)

$$K_{\rm AP}(x,y)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K(x,z_1)K_{\rm AP}(z_1,z_2)K(z_2,y)dz_1dz_2\qquad \text{for all } x,y\in\mathbb{R}^d.$$

Remark 3.1. If the initial sample c_0 in the iterative approximation-projection reconstruction algorithm (3.1) is the corrupted sample of a signal $g\in V$, i.e.,

$$c_0=(g(\gamma)+\epsilon(\gamma))_{\gamma\in\Gamma}$$

for some noise $\epsilon=(\epsilon(\gamma))_{\gamma\in\Gamma}$, then the L^p norm of the original signal g and the recovered signal f_∞ via the iterative approximation-projection reconstruction algorithm (3.1) is bounded by the ℓ^p norm of the noise ϵ. More precisely, from (3.11) and (3.12) we obtain

$$\begin{aligned}
(3.19)\qquad & \|f_n-g\|_p\\
\le\ & \sum_{k=n+1}^\infty \|TQ_{\Gamma,U}^k(f_0-h_0)\|_p+\sum_{k=0}^n\|TQ_{\Gamma,U}^k h_0\|_p\\
\le\ & \|T\|\sum_{k=n+1}^\infty r_0^k\|f_0-h_0\|_p+\|T\|\sum_{k=0}^n r_0^k\|h_0\|_p\\
\le\ & \|T\|(1-r_0)^{-1}(\|f_0\|_p r_0^{n+1}+\|h_0\|_p)\\
\le\ & \|T\|^2(1-r_0)^{-1}\big(\sup_{\gamma\in\Gamma}\|u_\gamma\|_1\big)^{1/p}(\|c_0\|_p r_0^{n+1}+\|\epsilon\|_p)
\end{aligned}$$

and

$$\begin{aligned}
(3.20)\qquad \|f_\infty-g\|_p\ \le\ & \|T\|(1-r_0)^{-1}\|h_0\|_p\\
\le\ & \|T\|^2(1-r_0)^{-1}\big(\sup_{\gamma\in\Gamma}\|u_\gamma\|_1\big)^{1/p}\|\epsilon\|_p,
\end{aligned}$$

where $h_0=\sum_{\gamma\in\Gamma}\epsilon(\gamma)Tu_\gamma$ and $f_n, n\ge 0$, are given in the approximation-projection reconstruction algorithm (3.1). Define the sample-to-noise ratio in the logarithmic decibel scale, a term for the power ratio between a sample and the background noise, by

$$\text{(3.21)}\qquad {\rm SNR(dB)}=20\log_{10}\frac{\|c_0\|_p}{\|\epsilon\|_p}.$$

The estimate in (3.19) suggests that the stopping step n_0 for the iterative approximation-projection reconstruction algorithm (3.1) is

$$n_0 = \left[\frac{\mathrm{SNR(dB)}}{20\ln_{10}(1/r_0)}\right], \tag{3.22}$$

where $[x]$ denotes the integral part of a real number x. In this case,

$$\|f_{n_0} - g\|_p \le 2\|T\|^2(1-r_0)^{-1}\big(\sup_{\gamma\in\Gamma}\|u_\gamma\|_1\big)^{1/p}\|\epsilon\|_p, \tag{3.23}$$

and the error between the resulting signal f_{n_0} and the original signal g is about twice the error due to the noise in the initial sample data.

Remark 3.2. Given the initial data $c_0 = (c_0(\gamma))_{\gamma\in\Gamma}$, define

$$F_n = (f_n(\gamma))_{\gamma\in\Gamma}, \quad n \ge 0, \tag{3.24}$$

and

$$A = \big((Tu_{\gamma'})(\gamma)\big)_{\gamma,\gamma'\in\Gamma}, \tag{3.25}$$

where $f_n, n \ge 0$, is given in the iterative approximation-projection reconstruction algorithm (3.1). This leads to the *discrete version of the iterative approximation-projection reconstruction algorithm* (3.1):

$$\begin{cases} F_0 = Ac_0, \\ F_n = F_0 + (I-A)F_{n-1}, \quad n \ge 1. \end{cases} \tag{3.26}$$

Exponential convergence: Now let us consider the exponential convergence of the sequence $F_n, n \ge 0$, when (1.5), (1.6) and (2.1) hold. By (3.26), we have

$$F_n - F_{n-1} = (I-A)^n F_0 = (I-A)^n Ac_0, \quad n \ge 1. \tag{3.27}$$

Define

$$\|c\|_{p,U} = \Big\|\sum_{\gamma\in\Gamma}|c(\gamma)|u_\gamma\Big\|_p \quad \text{for } c = (c(\gamma))_{\gamma\in\Gamma}, \tag{3.28}$$

where $1 \le p \le \infty$. For $c = (c(\gamma))_{\gamma\in\Gamma}$ with $\|c\|_{p,U} < \infty$, write $(I-A)^n Ac = (d_n(\gamma))_{\gamma\in\Gamma}$ and define $c_{\Gamma,U}(x) = \sum_{\gamma\in\Gamma} c(\gamma)u_\gamma(x)$. Similar to the equation (3.11) we have

$$d_n(\gamma) = (-1)^n (TQ^n_{\Gamma,U}c_{\Gamma,U})(\gamma). \tag{3.29}$$

This together with (3.6) implies that

$$\begin{aligned} &\|(I-A)^n Ac\|_{p,U} \\ \le\ & \Big\|\sum_{\gamma\in\Gamma} u_\gamma(\cdot)\int_{\mathbb{R}^d}|K(\gamma,z)||(Q^n_{\Gamma,U}c_{\Gamma,U})(z)|dz\Big\|_p \\ \le\ & \Big\|\int_{\mathbb{R}^d}\big(|K(\cdot,z)| + |\omega_{\delta_0/2}(K)(\cdot,z)|\big)|(Q^n_{\Gamma,U}c_{\Gamma,U})(z)|dz\Big\|_p \\ \le\ & C_0 r_0^n\|c\|_{p,U} \end{aligned} \tag{3.30}$$

where

$$C_0 = \Big\| \sup_{z\in\mathbb{R}^d} |K(\cdot + z, z)|\Big\|_1 + \Big\| \sup_{z\in\mathbb{R}^d} \omega_{\delta_0/2}(K)(\cdot + z, z)\Big\|_1. \tag{3.31}$$

Hence the exponential convergence of the sequence F_n in the $\|\cdot\|_{p,U}$ norm follows from (3.27) and (3.30).

Numerical stability and stopping rule: Next let us consider the numerical stability of the iterative algorithm (3.26). Assume that the numerical error in n-th iterative step in the iterative algorithm (3.26) is $\epsilon_n, n \ge 0$, i.e.,

$$\begin{cases} \tilde F_0 = Ac_0 + \epsilon_0 \\ \tilde F_n = \tilde F_0 + (I-A)\tilde F_{n-1} + \epsilon_n, \quad n \ge 1. \end{cases} \tag{3.32}$$

Let $F_n = (f_n(\gamma))_{\gamma\in\Gamma}, n \ge 0$, where $f_n, n \ge 0$, are given in the iterative approximation-projection reconstruction algorithm (3.1) with initial data c_0. By induction, we obtain

$$\tilde F_n - F_n = -\sum_{k=0}^{n-1} (I-A)^{n-1-k} A\tilde\epsilon_k + \tilde\epsilon_n, \tag{3.33}$$

where $\tilde\epsilon_0 = \epsilon_0$ and $\tilde\epsilon_k = (k+1)\epsilon_0 + \epsilon_1 + \cdots + \epsilon_k$ for $k \ge 1$. Therefore

$$\begin{aligned} & \|\tilde F_n - F_n\|_{p,U} \\ \le\ & \sum_{k=0}^{n-1} \|(I-A)^{n-1-k} A\tilde\epsilon_k\|_{p,U} + \|\tilde\epsilon_n\|_{p,U} \\ \le\ & \sum_{k=0}^{n-1} C_0 r_0^{n-1-k} \|\tilde\epsilon_k\|_{p,U} + \|\tilde\epsilon_n\|_{p,U} \\ \le\ & C_0 \sum_{k=0}^{n-1} r_0^{n-1-k}\Big((k+1)\|\epsilon_0\|_{p,U} + \sum_{j=1}^{k} \|\epsilon_j\|_{p,U}\Big) \\ & +(n+1)\|\epsilon_0\|_{p,U} + \sum_{j=1}^{n} \|\epsilon_j\|_{p,U} \\ \le\ & \frac{1-r_0+C_0}{1-r_0}\Big((n+1)\|\epsilon_0\|_{p,U} + \sum_{j=1}^{n} \|\epsilon_j\|_{p,U}\Big). \end{aligned} \tag{3.34}$$

Denote the limit of F_n as n tends to infinity by F_∞. By (3.27) and (3.30) we have

$$\|F_n - F_\infty\|_{p,U} \le \sum_{k=n}^{\infty} C_0 r_0^{k+1} \big\|c_0\big\|_{p,U} \le \frac{C_0 r_0}{1-r_0} r_0^n \big\|c_0\big\|_{p,U}. \tag{3.35}$$

Define the sample-to-numerical-error ratio (SNER) of the iterative algorithm (3.32) in the logarithmic decibel scale by

$$\mathrm{SNER(dB)} = 20 \inf_{n\ge1} \log_{10} \frac{n\|c_0\|_{p,U}}{n\|\epsilon_0\|_{p,U} + \sum_{j=1}^n \|\epsilon_j\|_{p,U}}. \tag{3.36}$$

Then

$$\|\tilde{F}_n - F_n\|_{p,U} \le \frac{1-r_0+C_0}{1-r_0}(n+1)10^{-\mathrm{SNER(dB)}/20}\|c_0\|_{p,U}, \tag{3.37}$$

which together with (3.35) implies that

$$\|\tilde{F}_n - F_\infty\|_{p,U} \le \frac{1-r_0+C}{1-r_0}\Big(r_0^{n+1} + (n+1)10^{-\mathrm{SNER(dB)}/20}\Big)\|c_0\|_{p,U}. \tag{3.38}$$

This suggests that a reasonable stopping step n_1 in the iterative algorithm (3.26) is

$$n_1 = \Big[\frac{\mathrm{SNER(dB)}}{20\log_{10}(1/r_0)} - \frac{\log_{10}(\ln(1/r_0))}{\log_{10} 1/r_0} - 1\Big], \tag{3.39}$$

as the function $f(y) = r_0^y + y10^{-\mathrm{SNER(dB)}/20}$ attains the absolute minimum at

$$y_0 := \frac{\mathrm{SNER(dB)}}{20\log_{10}(1/r_0)} - \frac{\log_{10}(\ln(1/r_0))}{\log_{10} 1/r_0}. \tag{3.40}$$

4. Iterative frame reconstruction algorithm

In this section, we study the convergence and consistency of the iterative frame algorithm for reconstructing a signal in the reproducing kernel subspace of $L^p(\mathbb{R}^d)$ associated with an idempotent integral operator from its samples taken a relatively-separated set with sufficient small gap. The readers may refer to [13, 14] for an introduction to frame theory, and [4, 9, 11, 15, 30, 35, 49] for various frame algorithms to reconstruct a signal from its samples.

Theorem 4.1. *Let $1 \le p \le \infty$, T be an idempotent integral operator whose kernel K satisfies* (1.5) *and* (1.6), *V be the reproducing kernel subspace of $L^p(\mathbb{R}^d)$ associated with the operator T, and $\delta_1 > 0$ be so chosen that*

$$r_2 := (2r_1 + r_0)r_0 < 1, \tag{4.1}$$

where

$$r_0 := \Big\| \sup_{z\in\mathbb{R}^d} |\omega_{\delta_1/2}(K)(\cdot+z,z)|\Big\|_{L^1(\mathbb{R}^d)}$$

and

$$r_1 := \Big\| \sup_{z\in\mathbb{R}^d} |K(\cdot+z,z)|\Big\|_{L^1(\mathbb{R}^d)}.$$

Let Γ be a relatively-separated subset of $\mathbb{R}^d$ with gap δ_1, $U = \{u_\gamma\}_{\gamma\in\Gamma}$ be a BUPU associated with the covering $\{\gamma + [-\delta_1/2, \delta_1/2]^d\}_{\gamma\in\Gamma}$, and

$$S_{\Gamma,U} f(x) := \sum_{\gamma\in\Gamma} (Tf)(\gamma)\|u_\gamma\|_{L^1(\mathbb{R}^d)} K(x,\gamma), \qquad f \in L^p(\mathbb{R}^d) \tag{4.2}$$

be the preconditioned frame operator on $L^p(\mathbb{R}^d)$*. Given a sequence* $c_0 = (c_0(\gamma))_{\gamma\in\Gamma} \in \ell^p(\Gamma)$*, we define the iterative frame reconstruction algorithm by*

$$\begin{cases} f_0 = \sum_{\gamma\in\Gamma} c_0(\gamma)\|u_\gamma\|_{L^1(\mathbb{R}^d)} K(\cdot,\gamma), \\ f_n = f_0 + f_{n-1} - S_{\Gamma,U} f_{n-1}, \quad n \ge 1. \end{cases} \tag{4.3}$$

Then the iterative algorithm (4.3) *converges to* f_∞ *exponentially and is consistent. Moreover,*

$$f_\infty = R_{\mathrm{F}} f_0, \tag{4.4}$$

where

$$R_{\mathrm{F}} := T + \sum_{n=1}^{\infty} (T - S_{\Gamma,U})^n \tag{4.5}$$

defines a bounded integral operator on $L^p(\mathbb{R}^d)$ *and is a pseudo-inverse of the preconditioned frame operator* $S_{\Gamma,U}$*, i.e.,*

$$R_{\mathrm{F}} T = T R_{\mathrm{F}} = R_{\mathrm{F}} \quad \text{and} \quad R_{\mathrm{F}} S_{\Gamma,U} = S_{\Gamma,U} R_{\mathrm{F}} = T. \tag{4.6}$$

Furthermore, the kernel $K_F(x,y)$ *of the integral operator* R_{F} *satisfies* (1.5), (1.6)*, and*

$$K_F(x,y) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,z_1) K_F(z_1,z_2) K(z_2,y)\, dz_1 dz_2 \quad \text{for all } x,y \in \mathbb{R}^d. \tag{4.7}$$

Proof. Define an integral operator $C_{\Gamma,U}$ by

$$\begin{aligned} C_{\Gamma,U} f(x) \;=\; & \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Big(\sum_{\gamma\in\Gamma} \big(K(x,\gamma) - K(x,z)\big) u_\gamma(z) \\ & \times \big(K(\gamma,y) - K(z,y)\big)\Big) f(y)\, dy dz \quad \text{for all } f \in L^p, \end{aligned} \tag{4.8}$$

and let $Q^*_{\Gamma,U}$ be the adjoint of the integral operator $Q_{\Gamma,U}$ in (3.4), i.e.,

$$Q^*_{\Gamma,U} f(x) = \int_{\mathbb{R}^d} \Big(\sum_{\gamma\in\Gamma} \big(K(\gamma,x) - K(y,x)\big) u_\gamma(y)\Big) f(y)\, dy \quad \text{for all } f \in L^p. \tag{4.9}$$

Then

$$S_{\Gamma,U} - T = T Q_{\Gamma,U} + Q^*_{\Gamma,U} T + C_{\Gamma,U}, \tag{4.10}$$

which implies that

$$
\begin{aligned}
(4.11) \qquad & \|S_{\Gamma,U}f - Tf\|_p \\
\le\ & \|T\|\|Q_{\Gamma,U}f\|_p + \|Q^*_{\Gamma,U}Tf\|_p + \|C_{\Gamma,U}f\|_p \\
\le\ & \|T\|\Big\|\int_{\mathbb{R}^d} h_{\delta_1/2}(\cdot - y)|f(y)|dy\Big\|_p \\
& +\Big\|\int_{\mathbb{R}^d} h_{\delta_1/2}(z-\cdot)|Tf(z)|dz\Big\|_p \\
& +\Big\|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h_{\delta_1/2}(\cdot - z)h_{\delta_1/2}(z-y)|f(y)|dydz\Big\|_p \\
\le\ & r_2\|f\|_p \quad \text{for all } f \in V,
\end{aligned}
$$

where $h_\delta = \sup_{z'\in\mathbb{R}^d}\omega_\delta(K)(\cdot + z', z')$.

By the iterative algorithm (4.3),

$$
f_n = f_0 + \sum_{k=1}^{n}(T - S_{\Gamma,U})^k f_0 \quad \text{for all } n \ge 1. \tag{4.12}
$$

This together with (4.11) proves the exponential convergence of $f_n, n \ge 0$, and the limit function f_∞ is given by (4.4).

By (1.2), (4.2) and Theorem A.1 in the Appendix, we have

$$
S_{\Gamma,U}T = TS_{\Gamma,U} = S_{\Gamma,U}. \tag{4.13}
$$

This together with the exponential convergence of the right hand side of the equation (4.5) establishes that R_{F} is a bounded operator and satisfies (4.6), and hence it is the pseudo-inverse of $S_{\Gamma,U}$.

The consistency of the frame iterative algorithm (4.3) follows from (4.4) and the fact that $f_0 = S_{\Gamma,U}g$ if the initial data $c_0 = (g(\gamma))_{\gamma\in\Gamma}$ is the sample of $g \in V$ taken on the set Γ.

From (1.5), (4.1), (4.8), (4.9) and (4.10), it follows that

$$
\Big\|\sup_{z'\in\mathbb{R}^d}|K_F(\cdot + z', z')|\Big\|_1 \le \Big\|\sup_{z'\in\mathbb{R}^d}|K(\cdot + z', z')|\Big\|_1 + \sum_{n=1}^{\infty}(r_2)^n < \infty.
$$

Hence K_F satisfies the off-diagonal decay property (1.5). The reproducing equality (4.7) follows from

$$
TR_{\mathrm{F}}T = R_{\mathrm{F}}
$$

by (4.6). The regularity property (1.6) for the kernel K_F holds because of the off-diagonal decay property (1.5) for the kernel F, the regularity property

(1.6) for the kernel K of the idempotent operator T, and the following estimate

$$\begin{aligned}\omega_\delta(K_F)(x,y) \le & \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\omega_\delta(K)(x,z_1)|K_F(z_1,z_2)| \\ & \times\big(|K(z_2,y)|+\omega_\delta(K)(z_2,y)\big)dz_1dz_2 \\ & +\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|K(x,z_1)||K_F(z_1,z_2)||\omega_\delta(K)(z_2,y)|dz_1dz_2\end{aligned}$$

by (4.7). □

5. Asymptotic pointwise error estimates for reconstruction algorithms

In this section, we discuss the asymptotic pointwise error estimate for reconstructing a signal from its samples corrupted by white noises, as the maximal gap of the sampling set tends to zero.

Theorem 5.1. *Let $1\le p\le\infty$, T be an idempotent integral operator whose kernel K satisfies* (1.5) *and* (1.6), *and V be the reproducing kernel subspace of $L^p(\mathbb{R}^d)$ associated with the operator T. Let Γ be a relatively-separated subset of $\mathbb{R}^d$ with gap δ, $U:=\{u_\gamma\}_{\gamma\in\Gamma}$ be a BUPU associated with the covering $\{\gamma+[-\delta/2,\delta/2]^d\}_{\gamma\in\Gamma}$, and $R:=\{R_\gamma(x)\}_{\gamma\in\Gamma}$ be either the displayer $\{(\|u_\gamma\|_{L^1(\mathbb{R}^d)})^{-1}R_{\rm AP}u_\gamma\}_{\gamma\in\Gamma}$ in the approximation-projection reconstruction algorithm or the displayer $\{R_{\rm F}K(\cdot,\gamma)\}_{\gamma\in\Gamma}$ in the frame reconstruction algorithm where the operators $R_{\rm AP}$ and $R_{\rm F}$ are defined in* (3.13) *and* (4.5) *respectively. Assume that $\epsilon(\gamma),\gamma\in\Gamma$, are bounded i.i.d. noises with zero mean and σ^2 variance, i.e.,*

$$\epsilon(\gamma)\in[-B,B],\ E(\epsilon(\gamma))=0,\quad\text{and}\quad \mathrm{Var}(\epsilon(\gamma))=\sigma^2 \tag{5.1}$$

for some positive constant B, and that the initial data c_0 is the sample of a signal $g\in V$ taken on Γ corrupted by random noise $\epsilon:=(\epsilon(\gamma))_{\gamma\in\Gamma}$, i.e.,

$$c_0=(g(\gamma)+\epsilon(\gamma))_{\gamma\in\Gamma}. \tag{5.2}$$

Then for any $x\in\mathbb{R}^d$

$$E\big(g(x)-Rc_0(x)\big)=0 \tag{5.3}$$

and

$$\begin{aligned}\mathrm{Var}\big(g(x)-Rc_0(x)\big) &= \sum_{\gamma\in\Gamma}\|u_\gamma\|^2_{L^1(\mathbb{R}^d)}|R_\gamma(x)|^2 \\ &\le \sigma^2\sup_{\gamma\in\Gamma}\|u_\gamma\|_{L^1(\mathbb{R}^d)}\Big(\int_{\mathbb{R}^d}|K(x,z)|^2dz+o(1)\Big)\quad\text{as }\delta\to0,\end{aligned} \tag{5.4}$$

where

$$Rc_0(x)=\sum_{\gamma\in\Gamma}c_0(\gamma)\|u_\gamma\|_{L^1(\mathbb{R}^d)}R_\gamma(x)\quad\text{for all }c_0=(c_0(\gamma))_{\gamma\in\Gamma}\in\ell^\infty(\Gamma). \tag{5.5}$$

Furthermore if

$$\|u_\gamma\|_{L^1(\mathbb{R}^d)} = \alpha(\delta)(1+o(1)) \quad \text{as } \delta \to 0 \tag{5.6}$$

for some positive numbers $\alpha(\delta)$ *independent of* γ, *then the inequality in* (5.4) *becomes an equality, i.e.,*

$$\mathrm{Var}\big(g(x) - Rc_0(x)\big) = \alpha(\delta)\sigma^2\Big(\int_{\mathbb{R}^d} |K(x,z)|^2 dz + o(1)\Big) \tag{5.7}$$

as δ *tends to zero.*

Remark 5.1. The error estimate (5.7) is established in [5] for reconstructing signals in a finitely-generated shift-invariant subspace of $L^2(\mathbb{R}^d)$ from corrupted uniform sampling data via the frame reconstruction algorithm. More precisely, $\Gamma = \delta\mathbb{Z}^d$, $u_\gamma(x) = \chi_{[-\delta/2,\delta/2]^d}(x-\gamma)$ for $\gamma \in \Gamma$, the idempotent operator T is defined in (2.4), and the range space associated with the idempotent operator T is the shift-invariant space $V_2(\phi_1,\ldots,\phi_r)$ in (2.5).

Remark 5.2. By the definition of a BUPU associated with the covering $\{\gamma + [-\delta/2,\delta/2]^d\}_{\gamma\in\Gamma}$ of $\mathbb{R}^d$, we have

$$\|u_\gamma\|_{L^1(\mathbb{R}^d)} \le \delta^d. \tag{5.8}$$

The above inequality becomes an equality when $\Gamma = \delta\mathbb{Z}^d$ and $u_\gamma = \chi_{[-\delta/2,\delta/2]^d}$. It is expensive to find the operators R_{AP} and R_{F} when the sampling set has very small gap δ. As noticed in the proof of Theorem 5.1, both operators are close to the idempotent operator T when the sampling set has very small gap. Then a natural replacement of the displayer R_γ in (5.5) is either $(\|u_\gamma\|_{L^1(\mathbb{R}^d)})^{-1}Tu_\gamma$ or $K(\cdot,\gamma)$. In both cases, the variance estimates in (5.4) and (5.7) still hold, but the unbiased condition (5.4) does not.

To prove Theorem 5.1, we need several technical lemmas. The first lemma is a slight generalization of Theorem 5.1.

Lemma 5.2. *Let the operator* T, *the kernel* K, *the reproducing kernel space* V, *the sampling set* Γ, *the bounded uniform partition of unity* $U = \{u_\gamma\}_{\gamma\in\Gamma}$, *the random noise* ϵ, *and the variance* σ *of the noise* ϵ *be as in Theorem 5.1, and let the displayer* $R := \{R_\gamma(x)\}_{\gamma\in\Gamma}$ *satisfy*

$$g(x) = \sum_{\gamma\in\Gamma} g(\gamma)\|u_\gamma\|_{L^1(\mathbb{R}^d)} R_\gamma(x) \quad \text{for all } g \in V, \tag{5.9}$$

and

$$\lim_{\delta\to 0} \Big\| \sup_{\gamma\in\Gamma} \sup_{z\in\gamma+[-\delta/2,\delta/2]^d} |R_\gamma(\cdot + z) - K(\cdot+z,z)| \Big\|_{L^1(\mathbb{R}^d)} = 0. \tag{5.10}$$

Then (5.3), (5.4) *and* (5.7) *hold.*

Proof. Set

$$h_\delta(x) = \sup_{\gamma\in\Gamma} \sup_{z\in\gamma+[-\delta/2,\delta/2]^d} |R_\gamma(x+z) - K(x+z,z)|. \tag{5.11}$$

By (1.5), (5.10) and (5.11), we have

$$\begin{aligned}\sum_{\gamma\in\Gamma}\|u_\gamma\|_1|R_\gamma(x)| &\le \int_{\mathbb{R}^d}\sum_{\gamma\in\Gamma}u_\gamma(z)\big(|K(x,z)|+h_\delta(x-z)\big)dz\\ &\le \Big\|\sup_{z\in\mathbb{R}^d}|K(\cdot+z,z)|\Big\|_1+\|h_\delta\|_1<\infty.\end{aligned}\tag{5.12}$$

This together with (5.1) and (5.9) leads to

$$\begin{aligned}E\big(g(x)-Rc_0(x)\big) &= E\Big(\sum_{\gamma\in\Gamma}\epsilon(\gamma)\|u_\gamma\|_1R_\gamma(x)\Big)\\ &= \sum_{\gamma\in\Gamma}E(\epsilon(\gamma))\|u_\gamma\|_1R_\gamma(x)=0,\end{aligned}\tag{5.13}$$

and the unbiased property (5.3) for the reconstruction process in (5.5) follows.

By (5.1), (5.3) and (5.12), we obtain

$$\begin{aligned}\mathrm{Var}\big(g(x)-Rc_0(x)\big) &= E\Big(\sum_{\gamma\in\Gamma}\epsilon(\gamma)\|u_\gamma\|_1R_\gamma(x)\Big)^2\\ &= \sigma^2\sum_{\gamma\in\Gamma}\|u_\gamma\|_1^2|R_\gamma(x)|^2.\end{aligned}$$

Therefore

$$\begin{aligned}&\mathrm{Var}\big(g(x)-Rc_0(x)\big)\\ \le\ &\sigma^2\big(\sup_{\gamma\in\Gamma}\|u_\gamma\|_1\big)\Big(\sum_{\gamma\in\Gamma}\|u_\gamma\|_1|R_\gamma(x)|^2\Big)\\ \le\ &\sigma^2\big(\sup_{\gamma\in\Gamma}\|u_\gamma\|_1\big)\Big(\int_{\mathbb{R}^d}\big(|K(x,z)|+|h_\delta(x-z)|\big)^2dz\Big)\\ \le\ &\sigma^2\big(\sup_{\gamma\in\Gamma}\|u_\gamma\|_1\big)\Big(\int_{\mathbb{R}^d}|K(x,z)|^2dz+o(1)\Big),\end{aligned}\tag{5.14}$$

where we have used (5.10) and (5.11) to obtain the last two estimates. Hence the variance estimate (5.4) for the reconstruction process in (5.5) is established.

By (5.6), (5.10) and (5.14), we get

$$\begin{aligned}&\mathrm{Var}\big(g(x)-Rc_0(x)\big)\\ =\ &\sigma^2\big(\alpha(\delta)+o(1)\big)\Big(\sum_{\gamma\in\Gamma}\|u_\gamma\|_1|R_\gamma(x)|^2\Big)\\ =\ &\sigma^2\big(\alpha(\delta)+o(1)\big)\Big(\int_{\mathbb{R}^d}\big(K(x,z)+O(h_\delta(x-z))\big)^2dz\Big)\\ =\ &\sigma^2\alpha(\delta)\Big(\int_{\mathbb{R}^d}|K(x,z)|^2dz+o(1)\Big),\end{aligned}\tag{5.15}$$

and hence (5.7) is proved. □

Lemma 5.3. *Let the operator T, the kernel K, the reproducing kernel space V, the sampling set Γ, the bounded uniform partition of unity $U=\{u_\gamma\}_{\gamma\in\Gamma}$, the random noise ϵ, and the variance σ of the noise ϵ be as in Theorem 5.1, and let the displayer $R=\{R_\gamma\}_{\gamma\in\Gamma}$ be defined by*

$$R_\gamma = (\|u_\gamma\|_1)^{-1} R_{\rm AP} u_\gamma, \gamma\in\Gamma \tag{5.16}$$

where $R_{\rm AP}$ is given in (3.13). *Then the above displayer R satisfies* (5.9) *and* (5.10).

Proof. By (3.13), (3.16) and (3.17), the reconstruction formula (5.9) holds for the displayer R in (5.16).

Denote the kernel of the integral operators $R_{\rm AP}-T$ by $\tilde K_{\rm AP}$. By (1.2), (3.7), (3.10), (3.13) and (3.18), we have

$$\tilde K_{AP}(x,y) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,z_1)\tilde K_{\rm AP}(z_1,z_2)K(z_2,y)dz_1dz_2, \tag{5.17}$$

and

$$\begin{aligned} & \big\| \sup_{z'\in\mathbb{R}^d} |\tilde K_{\rm AP}(\cdot+z',z')|\big\|_1 \\ \le & \sum_{n=1}^\infty \big\|\sup_{z'\in\mathbb{R}^d}|K(\cdot+z',z')|\big\|_1\Big(\big\|\sup_{z'\in\mathbb{R}^d}|\omega_{\delta/2}(K)(\cdot+z',z')|\big\|_1\Big)^n \\ \to & \ 0 \quad \text{as } \delta\to 0. \end{aligned} \tag{5.18}$$

This together with (1.5) and (1.6) implies that

$$\begin{aligned} & \Big\|\sup_{\gamma\in\Gamma}\sup_{z'\in\gamma+[-\delta/2,\delta/2]^d}\big|(\|u_\gamma\|_1)^{-1}R_{\rm AP}u_\gamma(\cdot+z')-K(\cdot+z',z')\big|\Big\|_1 \\ \le & \Big\|\sup_{z'\in\mathbb{R}^d}\omega_\delta(K)(\cdot+z',z')\Big\|_1+\Big\|\sup_{z\in\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|K(\cdot+z,z_1)| \\ & \quad \times|\tilde K_{\rm AP}(z_1,z_2)|\big(|K(z_2,z)|+|\omega_\delta(K)(z_2,z)|\big)dz_1dz_2\Big\|_1 \\ \to & \ 0 \quad \text{as } \delta\to 0. \end{aligned} \tag{5.19}$$

Hence (5.10) follows. □

Lemma 5.4. *Let the operator T, the kernel K, the reproducing kernel space V, the sampling set Γ, the bounded uniform partition of unity $U=\{u_\gamma\}_{\gamma\in\Gamma}$, the random noise ϵ, and the variance σ of the noise ϵ be as in Theorem 5.1, and let the displayer $R=\{R_\gamma\}_{\gamma\in\Gamma}$ be defined by*

$$R_\gamma = R_{\rm F}K(\cdot,\gamma), \quad \gamma\in\Gamma \tag{5.20}$$

where $R_{\rm F}$ is given in (4.5). *Then the above displayer R satisfies* (5.9) *and* (5.10).

Proof. The reconstruction formula (5.9) follows from Theorem 4.1.

Denote the integral kernel of the integral operator $R_{\rm F}-T$ by $\tilde K_F$. Then

$$\tilde K_F(x,y)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,z_1)\tilde K_F(z_1,z_2)K(z_2,y)dz_1dz_2, \tag{5.21}$$

and

$$\begin{aligned}
&\Big\|\sup_{z\in\mathbb{R}^d}|\tilde K_F(\cdot+z,z)|\Big\|_1 \\
\le\ & \sum_{n=1}^\infty\Big(2\Big\|\sup_{z\in\mathbb{R}^d}|K(\cdot+z,z)|\Big\|_1+\Big\|\sup_{z\in\mathbb{R}^d}|\omega_\delta(K)(\cdot+z,z)|\Big\|_1\Big)^n\\
&\quad\times\Big(\Big\|\sup_{z\in\mathbb{R}^d}|\omega_\delta(K)(\cdot+z,z)|\Big\|_1^2\Big)^n\to 0 \quad \text{as } \delta\to 0
\end{aligned} \tag{5.22}$$

by (1.6), (4.5), and (4.10). Therefore

$$\begin{aligned}
&\int_{\mathbb{R}^d}\sup_{\gamma\in\Gamma}\sup_{z\in\gamma+[-\delta/2,\delta/2]^d}|R_FK(\cdot,\gamma))(x+z)-K(x+z,z)|dx\\
\le\ & \int_{\mathbb{R}^d}\sup_{\gamma\in\Gamma}\sup_{z\in\gamma+[-\delta/2,\delta/2]^d}|K(x+z,\gamma)-K(x+z,z)|dx\\
&+\int_{\mathbb{R}^d}\sup_{\gamma\in\Gamma}\sup_{z\in\gamma+[-\delta/2,\delta/2]^d}\\
&\qquad\Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K(x+z,z_1)\tilde K_F(z_1,z_2)K(z_2,\gamma)dz_1d_2\Big|dx\\
\le\ & \int_{\mathbb{R}^d}\sup_{z'\in\mathbb{R}^d}|\omega_{\delta/2}(K)(x+z',z')|dx\\
&+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big(\sup_{z'\in\mathbb{R}^d}|K(x-z_1+z',z')|\big)\big(\sup_{z'\in\mathbb{R}^d}|\tilde K_F(z_1-z_2+z',z')|\big)\\
&\qquad\times\big(\sup_{z'\in\mathbb{R}^d}|K(z_2+z',z')|+\sup_{z'\in\mathbb{R}^d}\omega_{\delta/2}(K)(z_2+z',z')|\big)dz_1d_2dx\\
\to\ & 0 \quad \text{as } \delta\to 0.
\end{aligned}$$

Then (5.10) is established for the displayer R in (5.20). □

Proof of Theorem 5.1. The conclusions in Theorem 5.1 follows directly from Lemmas 5.2, 5.3 and 5.4. □

Appendix A. Reproducing kernel subspaces of $L^p(\mathbb{R}^d)$ associated with idempotent integral operators

The range space associated with an idempotent operator T on $L^p(\mathbb{R}^d)$ whose kernel satisfies (1.5) and (1.6) include the space of all p-integrable non-uniform splines of order n satisfying $n-1$ continuity conditions at each knot (Example A.3), and the space introduced in [43] for modeling signals with finite rate of innovation (Example A.4). In this appendix, we establish some properties of such range spaces, particularly the reproducing kernel property in Theorem A.1 and the frame property in Theorem A.2.

A.1. **Reproducing kernel property.** In this subsection, we show that the range space of an idempotent operator on $L^p(\mathbb{R}^d)$ whose kernel satisfies (1.5) and (1.6) has some properties similar to the ones for a reproducing kernel Hilbert subspace of $L^2(\mathbb{R}^d)$.

Theorem A.1. *Let T be an idempotent integral operator on $L^p(\mathbb{R}^d)$ whose kernel K satisfies* (1.5) *and* (1.6)*, and V be the range space of the operator T. Set*

$$a_\delta(q) = \delta^{-d+d/q}\Big(\big\|\sup_{z\in\mathbb{R}^d}|K(\cdot+z,z)|\big\|_{L^1(\mathbb{R}^d)}\Big)^{1/q} \times\Big(\big\|\sup_{z\in\mathbb{R}^d}|K(\cdot+z,z)|\big\|_{L^1(\mathbb{R}^d)}+\big\|\sup_{z\in\mathbb{R}^d}|\omega_\delta(K)(\cdot+z,z)|\big\|_{L^1(\mathbb{R}^d)}\Big)^{1-1/q}$$

and

$$b_\delta(q) = (6^d+1)^{1-1/q}\delta^{-d+d/q}\big\|\sup_{z\in\mathbb{R}^d}|\omega_\delta(K)(\cdot+z,z)|\big\|_{L^1(\mathbb{R}^d)}$$

for $\delta>0$ and $1\le q\le\infty$. Then

(i) *V is a reproducing kernel subspace of $L^p(\mathbb{R}^d)$. Moreover,*

$$|f(x)| \le a_\delta(p/(p-1))\|f\|_{L^p(\mathbb{R}^d)}$$

for any $f\in V$ and $\delta>0$.

(ii) *The kernel K satisfies the "reproducing kernel property":*

$$\int_{\mathbb{R}^d} K(x,z)K(z,y)dz = K(x,y) \quad \text{for all } x,y\in\mathbb{R}^d. \tag{A.1}$$

(iii) *$K(\cdot,y)\in V$ for any $y\in\mathbb{R}^d$.*

(iv) *The functions $K(x,\cdot)$, $K(\cdot,y)$, $\omega_\delta(K)(x,\cdot)$ and $\omega_\delta(K)(\cdot,y)$ belong to $L^q(\mathbb{R}^d)$ for all $x,y\in\mathbb{R}^d$ and $1\le q\le\infty$, and their $L^q(\mathbb{R}^d)$-norms are uniformly bounded. Moreover,*

$$\max\Big(\sup_{x\in\mathbb{R}^d}\|K(x,\cdot)\|_{L^q(\mathbb{R}^d)},\ \sup_{y\in\mathbb{R}^d}\|K(\cdot,y)\|_{L^q(\mathbb{R}^d)}\Big)\le a_\delta(q) \tag{A.2}$$

and

$$\max\Big(\sup_{x\in\mathbb{R}^d}\|\omega_\delta(K)(x,\cdot)\|_{L^q(\mathbb{R}^d)},\ \sup_{y\in\mathbb{R}^d}\|\omega_\delta(K)(\cdot,y)\|_{L^q(\mathbb{R}^d)}\Big)\le b_\delta(q). \tag{A.3}$$

Proof. (iv): By the definition of the modulus of continuity,

$$|K(x,y)|\le\delta^{-d}\int_{k\delta+[-\delta/2,\delta/2]^d}\big(|K(x,z)|+|\omega_\delta(K)(x,z)|\big)dz \tag{A.4}$$

where $y,z\in k\delta+[-\delta/2,\delta/2]^d$ and $x\in\mathbb{R}^d$. Thus

$$\sup_{x\in\mathbb{R}^d}\|K(x,\cdot)\|_\infty \le \delta^{-d}\Big(\big\|\sup_{z\in\mathbb{R}^d}|K(\cdot+z,z)|\big\|_1+\big\|\sup_{z\in\mathbb{R}^d}|\omega_\delta(K)(\cdot+z,z)|\big\|_1\Big)$$

and

$$\sup_{x\in\mathbb{R}^d} \|K(x,\cdot)\|_1 \le \Big\| \sup_{z\in\mathbb{R}^d} |K(\cdot+z,z)|\Big\|_1.$$

Interpolating the above estimates for the L^1 and L^∞ norms of $K(x,\cdot)$ yields $\sup_{x\in\mathbb{R}^d} \|K(x,\cdot)\|_q \le a_\delta(q)$. Similarly, we have that $\sup_{y\in\mathbb{R}^d} \|K(\cdot,y)\|_q \le a_\delta(q)$. Therefore (A.2) follows.

The estimate (A.3) for $\omega_\delta(K)$ can be established by similar argument used in the proof of the estimate (A.2) except replacing (A.4) by the following two inequalities:

$$\omega_\delta(K)(x,y) \ \le\ \delta^{-d}\int_{k\delta+[-\delta/2,\delta/2]^d} \big(\omega_\delta(K)(x,z)+\omega_{2\delta}(K)(x,z)\big)dz$$

for any $x\in\mathbb{R}^d, y\in k\delta+[-\delta/2,\delta/2]^d$ and $k\in\mathbb{Z}^d$, and

$$\omega_{2\delta}(K)(x,y) \le \sum_{\epsilon,\epsilon'\in\{-1,0,1\}^d} \omega_\delta(K)(x+\epsilon\delta, y+\epsilon'\delta) \tag{A.5}$$

for all $x,y\in\mathbb{R}^d$.

(i): By (1.4) and (A.2), we have that $|f(x)|\le \|K(x,\cdot)\|_{p/(p-1)}\|f\|_p \le a_\delta(p/(p-1))\|f\|_p$ for all $x\in\mathbb{R}^d$ and $f\in V$. Then (A.1) holds and V is a reproducing kernel subspace of L^p.

(ii): Noting that

$$\int_{\mathbb{R}^d} \sup_{z\in\mathbb{R}^d} \Big|\int_{\mathbb{R}^d} K(x+z,y)K(y,z)dy\Big|dx \le \Big(\int_{\mathbb{R}^d}\big(\sup_{z\in\mathbb{R}^d}|K(x+z,z)|\big)dx\Big)^2<\infty,$$

we then have that the kernel $A(x,y):=\int_{\mathbb{R}^d} K(x+z,y)K(y,z)dy - K(x,y)$ of the linear operator T^2-T satisfies $\|\sup_{z\in\mathbb{R}^d}|A(\cdot+z,z)|\|_1<\infty$. This together with (1.2) proves (A.1).

(iii): The conclusion that $K(\cdot,y)\in V$ for any $y\in\mathbb{R}^d$ follows from (A.1) and (A.2). □

A.2. **Frame property.** In this subsection, we show that the range space of an idempotent integral operator whose kernel satisfies (1.5) and (1.6) has localized frames. Let $1\le p\le\infty$, $V\subset L^p$ and $W\subset L^{p/(p-1)}$. We say that the p-frame $\tilde\Phi=\{\tilde\phi_\lambda\}_{\lambda\in\Lambda}\subset W$ for V and the $p/(p-1)$-frame $\Phi=\{\phi_\lambda\}_{\lambda\in\Lambda}\subset V$ for W form a *dual pair* if the following reconstruction formulae hold:

$$f=\sum_{\lambda\in\Lambda}\langle f,\tilde\phi_\lambda\rangle\phi_\lambda \quad \text{for all } f\in V, \tag{A.6}$$

and

$$g=\sum_{\lambda\in\Lambda}\langle g,\phi_\lambda\rangle\tilde\phi_\lambda \quad \text{for all } g\in W. \tag{A.7}$$

Here we denote by $\langle f,g\rangle$ the standard action (1.11) between a function $f\in L^p$ and a function $g\in L^{p/(p-1)}$.

Theorem A.2. *Let $1 \le p \le \infty$, T be an idempotent integral operator on $L^p(\mathbb{R}^d)$ whose kernel K satisfies* (1.5) *and* (1.6)*, T^* be the adjoint of the idempotent operator T, i.e.,*

$$T^*g(x) = \int_{\mathbb{R}^d} K(y,x)g(y)dy \quad \text{for all } g \in L^{p/(p-1)}(\mathbb{R}^d), \tag{A.8}$$

and let V and V^* be the range spaces of the operator T on $L^p(\mathbb{R}^d)$ and the operator T^* on $L^{p/(p-1)}(\mathbb{R}^d)$ respectively. Then there exist a relatively-separated subset Λ, and two families $\Phi := \{\phi_\lambda\}_{\lambda\in\Lambda}$ of functions $\phi_\lambda \in V$ and $\tilde\Phi := \{\tilde\phi_\lambda\}_{\lambda\in\Lambda}$ of functions $\tilde\phi_\lambda \in V^*$ such that

(i) *Both Φ and $\tilde\Phi$ are localized in the sense that*

$$\begin{cases} |\phi_\lambda(x)| + |\tilde\phi_\lambda(x)| \le h(x-\lambda) \\ |\omega_\delta(\phi_\lambda)(x) + \omega_\delta(\tilde\phi_\lambda)(x)| \le h_\delta(x-\lambda) \quad \text{for all } \lambda \in \Lambda \text{ and } x \in \mathbb{R}^d, \end{cases} \tag{A.9}$$

where h and h_δ are integrable functions with $\lim_{\delta\to 0}\|h_\delta\|_1 = 0$.

(ii) *$\tilde\Phi$ is a p-frame for V and Φ is a $p/(p-1)$-frame for V^*.*

(iii) *Φ and $\tilde\Phi$ form a dual pair.*

(iv) *Both V and V^* are generated by Φ and $\tilde\Phi$ respectively in the sense that*

$$V = V_p(\Phi) := \Big\{\sum_{\lambda\in\Lambda} c(\lambda)\phi_\lambda \Big| (c(\lambda))_{\lambda\in\Lambda} \in \ell^p(\Lambda)\Big\}, \tag{A.10}$$

and

$$V^* = V_{p/(p-1)}(\tilde\Phi) := \Big\{\sum_{\lambda\in\Lambda} \tilde c(\lambda)\tilde\phi_\lambda \Big| (\tilde c(\lambda))_{\lambda\in\Lambda} \in \ell^{p/(p-1)}(\Lambda)\Big\}. \tag{A.11}$$

Remark A.1. The space $V_p(\Phi)$ was introduced in [43] to model signals with finite rate of innovations. From Theorem A.2, we see that signals in a reproducing kernel subspace associated with an idempotent operator on $L^p(\mathbb{R}^d)$ with its kernel satisfying (1.5) and (1.6) have finite rate of innovation.

Proof of Theorem A.2. Let $\delta_0 > 0$ be a sufficiently small positive number chosen later. Define the operator T_{δ_0} by

$$T_{\delta_0} f(x) = \int_{\mathbb{R}^d} K_{\delta_0}(x,y)f(y)dy \quad f \in L^p(\mathbb{R}^d), \tag{A.12}$$

where

$$K_{\delta_0}(x,y) = \delta_0^{-d}\int_{[-\delta_0/2,\delta_0/2]^d}\int_{[-\delta_0/2,\delta_0/2]^d}\sum_{\lambda\in\delta_0\mathbb{Z}^d} K(x,\lambda+z_1)K(\lambda+z_2,y)dz_1dz_2. \tag{A.13}$$

Then

$$T_{\delta_0}T = TT_{\delta_0} = T_{\delta_0} \tag{A.14}$$

by (1.2), and

$$|K_{\delta_0}(x,y)-K(x,y)| \le \int_{\mathbb{R}^d} |K(x,z)||\omega_{\delta_0}(K)(z,y)|dz \tag{A.15}$$

by Theorem A.1. Therefore

$$\|T_{\delta_0}f-Tf\|_p \le r_1(\delta_0)\|f\|_p \quad \text{for all } f\in L^p, \tag{A.16}$$

where $r_1(\delta_0) = \|\sup_{z\in\mathbb{R}^d}|K(\cdot+z,z)|\|_1\|\sup_{z\in\mathbb{R}^d}|\omega_{\delta_0}(K)(\cdot+z,z)\|_1$. Let $\delta_0>0$ be so chosen that $r_1(\delta_0)<1$. The existence of such a positive number follows from (1.5) and (1.6). Then it follows from (A.14), (A.15) and (A.16) that the operator $T^\dagger_{\delta_0} := T+\sum_{n=1}^\infty (T-T_{\delta_0})^n$ is a bounded integral operator with the property that $T^\dagger_{\delta_0}T_{\delta_0} = T_{\delta_0}T^\dagger_{\delta_0}=T$ and that the kernel K_{D,δ_0} of the operator $T^\dagger_{\delta_0}$ satisfies $\|\sup_{z\in\mathbb{R}^d}|K_{D,\delta_0}(\cdot+z,z)|\|_1<\infty$ and $\lim_{\delta\to 0}\|\sup_{z\in\mathbb{R}^d}|\omega_\delta(K_{D,\delta_0})(\cdot+z,z)|\|_1=0$. Define

$$\begin{cases} \phi_\lambda(x)=\delta_0^{-d/p}\int_{\mathbb{R}^d}\int_{[-\delta_0/2,\delta_0/2]^d} K_{D,\delta_0}(x,z_1)K(z_1,\lambda+z_2)dz_2dz_1\\ \tilde\phi_\lambda(x)=\delta_0^{-d+d/p}\int_{[-\delta_0/2,\delta_0/2]^d} K(\lambda+z,x)dz\end{cases} \tag{A.17}$$

for all $\lambda\in\delta_0\mathbb{Z}^d$, and set $\Phi=\{\phi_\lambda\}_{\lambda\in\delta_0\mathbb{Z}^d}$ and $\tilde\Phi=\{\tilde\phi_\lambda\}_{\lambda\in\delta_0\mathbb{Z}^d}$. Then one may verify that the above two families Φ and $\tilde\Phi$ of functions satisfy all required properties. We leave the detailed verification for the interested readers. □

A.3. **Examples.** In this subsection, we present two examples of a reproducing kernel space associated with an idempotent integral operator on L^p.

Example A.3. [38] Let $n\ge 1$, $\Lambda=\{\lambda_k\}_{k\in\mathbb{Z}}$ be a bi-infinite increasing sequence of real numbers with $0<\inf_{k\in\mathbb{Z}}(\lambda_{k+1}-\lambda_k)\le\sup_{k\in\mathbb{Z}}(\lambda_{k+1}-\lambda_k)<\infty$, and

$$S_n^{n-1}(\Lambda) = \Big\{f\in C^{n-1}(\mathbb{R}):\ f|_{[\lambda_k,\lambda_{k+1}]} \text{ is a polynomial having degree at most } n \text{ for each } k\in\mathbb{Z}\Big\}. \tag{A.18}$$

Let B_i be the normalized B-spline associated with the knots $\lambda_i,\ldots,\lambda_{i+n+1}$, and define its autocorrelation matrix $A=\big(\langle B_i,B_j\rangle\big)_{i,j\in\mathbb{Z}}$. Then the infinite matrix A is invertible and its inverse $B=(b_{ij})_{i,j\in\mathbb{Z}}$ has exponential off-diagonal decay, that is, there exist constants C and ϵ such that $|b_{ij}|\le C\exp(-\epsilon|i-j|)$ for all $i,j\in\mathbb{Z}$. Define

$$K(x,y)=\sum_{i,j\in\mathbb{Z}} B_i(x)b_{ij}B_j(y)$$

and

$$Tf(x)=\int_{\mathbb{R}} K(x,y)f(y)dy.$$

Then one may verify that the above integral operator T is an idempotent operator on $L^p(\mathbb{R})$, the kernel K of the operator T satisfies (1.5) and (1.6), and $S_n^{n-1}(\Lambda) \cap L^p(\mathbb{R})$ is the range of the operator T on $L^p(\mathbb{R})$. The spline model has many practical advantages over the band-limited model in Shannon's sampling theory, and has been well-studied (see [44, 46, 48] and the references therein).

Example A.4. [43] Let Λ be a relatively-separated subset of $\mathbb{R}^d$ with positive gap, $\Phi = \{\phi_\lambda\}_{\lambda\in\Lambda}$ and $\tilde{\Phi} = \{\tilde{\phi}_\lambda\}_{\lambda\in\Lambda}$ be two families of functions such that

$$|\phi_\lambda(x)| + |\tilde{\phi}_\lambda(x)| \le h(x-\lambda),\ x \in \mathbb{R}^d,$$

and

$$|\omega_\delta(\phi_\lambda)(x)| + |\omega_\delta(\tilde{\phi}_\lambda)(x)| \le h_\delta(x-\lambda),\ x \in \mathbb{R}^d,$$

hold for all $\lambda \in \Lambda$ and $\delta > 0$, where h and h_δ are functions in the Wiener amalgam space $\mathcal{W}$ with $\lim_{\delta\to 0} \|h_\delta\|_{\mathcal{W}} = 0$. Then one may verify that the kernel function

$$K(x,y) := \sum_{\lambda\in\Lambda} \phi_\lambda(x)\tilde{\phi}_\lambda(y) \tag{A.19}$$

satisfies (1.5) and (1.6). If we further assume that Φ and $\tilde{\Phi}$ satisfy

$$\int_{\mathbb{R}^d} \phi_\lambda(x)\tilde{\phi}_{\lambda'}(x)dx = \delta_{\lambda,\lambda'} \text{ for all } \lambda, \lambda' \in \Lambda,$$

where $\delta_{\lambda,\lambda'}$ stands for the Kronecker symbol, then the operator T with the kernel K in (A.19) is an idempotent operator on L^2. In this case,

$$V_2(\Phi) := \Big\{ \sum_{\lambda\in\Lambda} c(\lambda)\phi_\lambda(x) \ \Big|\ \sum_{\lambda\in\Lambda} |c(\lambda)|^2 < \infty \Big\} \tag{A.20}$$

is the range space of the operator T on L^2 and hence a reproducing kernel subspace of L^2. A special case of the above space $V_2(\Phi)$ is the finitely-generated shift-invariant space $V_2(\phi_1, \ldots, \phi_r)$ in (2.5), see [1, 4, 8, 30] and references therein.

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