SAMPLING AND GALERKIN RECONSTRUCTION IN REPRODUCING KERNEL SPACES

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Abstract. In this paper, we consider sampling in a reproducing kernel subspace of L^p. We introduce a pre-reconstruction operator associated with a sampling scheme and propose a Galerkin reconstruction in general Banach space setting. We show that the proposed Galerkin method provides a quasi-optimal approximation, and the corresponding Galerkin equations could be solved by an iterative approximation-projection algorithm. We also present detailed analysis and numerical simulations of the Galerkin method for reconstructing signals with finite rate of innovation.

1. Introduction

The celebrated Whittaker-Shannon-Kotelnikov's sampling theorem states that a bandlimited signal can be recovered from its samples taken at a rate greater than twice the bandwidth [28, 39]. In last two decades, that paradigm has been extended to represent signals in a shift-invariant space [5, 7, 37], signals with finite rate of innovation [11, 24, 27, 32, 33, 38], and signals in a reproducing kernel space [10, 15, 20, 25, 26].

In this paper, we consider signals living in a reproducing kernel space (RKS) of the form

$$V_{K,p} := \{T_0 f :\ f \in L^p\} = \{f \in L^p :\ T_0 f = f\},\ 1 \le p \le \infty, \tag{1.1}$$

where T_0 is an idempotent integral operator with kernel K,

$$T_0 f(x) := \int_{\mathbf{R}^d} K(x, y) f(y) dy, \quad f \in L^p. \tag{1.2}$$

The RKS has rich geometric structure, lots of flexibility and technical suitability for sampling. It has been used for modeling bandlimited signals, wavelet (spline) signals, and signals with finite rate of innovation [5, 25, 26, 32, 37].

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Take a (finite) sampling set Γ and consider the sampling scheme

$$f \longmapsto \{f(\gamma_n), \gamma_n \in \Gamma\}, \quad f \in V_{K,p}.$$

We are interested in finding a quasi-optimal linear approximation Rf, depending completely on the sampling data, in a reconstruction space U for a signal $f \in V_{K,p}$,

$$\|Rf - f\|_p \le C \inf_{h \in U} \|f - h\|_p, \quad f \in V_{K,p}.$$

In this paper, we focus on *pre-reconstruction operators*

$$S_{\Gamma,\delta} f(x) := \sum_{\gamma_n \in \Gamma} |I_n| f(\gamma_n) K(x, \gamma_n), \quad f \in V_{K,p}, \tag{1.3}$$

where $\delta > 0$ and $\{I_n \subset B(\gamma_n, \delta) : \ \gamma_n \in \Gamma\}$ is a disjoint covering of

$$B(\Gamma, \delta) := \cup_{\gamma \in \Gamma} B(\gamma, \delta) = \cup_{\gamma \in \Gamma} \{x : \ |x - \gamma| \le \delta\}.$$

Our crucial observation is that $S_{\Gamma,\delta} f(x)$ is a good approximation to $f(x)$ when δ is sufficiently small and $x \in B(\Gamma, \delta)$ is far away from the complement of $B(\Gamma, \delta)$, see Figure 3 in Section 5.

Associated with the pre-reconstruction operator $S_{\Gamma,\delta}$, we introduce the Garlekin method

$$\langle S_{\Gamma,\delta} Rf, g\rangle = \langle S_{\Gamma,\delta} f, g\rangle, \quad g \in \tilde{U} \subset L^{p/(p-1)} \tag{1.4}$$

to define a quasi-optimal linear approximation Rf in the reconstruction space U, where $\langle \cdot, \cdot\rangle$ is the standard dual product between L^p and $L^{p/(p-1)}$. We recognize that the Galerkin equation (1.4) could be solved by certain iterative approximation-projection algorithm:

$$g_0 \in U \ \text{ and } \ g_{m+1} = g_m - P_{U,\tilde{U}} S_{\Gamma,\delta} g_m + g_0, \ m \ge 0, \tag{1.5}$$

where $P_{U,\tilde{U}}$ is an oblique projection for the trial-test space pair $(U, \tilde{U})$, c.f. [4, 6, 13, 25, 35].

This paper is organized as follows. In Section 2, we introduce the concept of admissibility of pre-reconstruction operators in Banach space setting. We show that (sub-)Galerkin reconstruction provides a quasi-optimal approximation (Theorem 2.3), and such (sub-)Galerkin reconstruction exists whenever the trial and test spaces are finite-dimensional (Theorem 2.4, Corollaries 2.5 and 2.6). In Section 3, we discuss admissibility of the pre-reconstruction operator $S_{\Gamma,\delta}$ in (1.3) (Theorem 3.1). In that section, we also propose to use the iterative approximation-projection algorithm (1.5) to solve the Galerkin equation (1.4) (Theorem 3.6 and Lemma 3.7). Lots of signals with finite rate of innovation live in some reproducing kernel spaces of the form (1.1). In Section 4, we provide detailed analysis for pre-reconstruction operators, and we obtain matrix formulation of Galerkin reconstructions for signals with finite rate of innovation. In last section, we present some numerical simulations to demonstrate our Galerkin method.

2. Sub-Galerkin reconstruction in Banach spaces

In this section, we consider numerical stability and quasi-optimality of a (sub-)Galerkin reconstruction in Banach space setting.

Denote by $\langle \cdot, \cdot \rangle$ the action between elements in a Banach space B and its dual space B^*. First we introduce admissibility of operators for the trial-test space pair.

Definition 2.1. *Let (U, V, B) be a triple of Banach spaces with $U \subset V \subset B$, and let $\tilde{U} \subset B^*$. We say that a bounded linear operator $S : V \to V$ is* admissible *for the trial-test space pair $(U, \tilde{U})$ if there exist positive constants D_1 and D_2 such that*

$$\sup_{g \in \tilde{U}, \|g\| \le 1} |\langle Sf,\ g \rangle| \ge D_1 \|f\| \quad \text{for all} \quad f \in U, \tag{2.1}$$

and

$$\sup_{g \in \tilde{U}, \|g\| \le 1} |\langle Sf,\ g \rangle| \le D_2 \|f\| \quad \text{for all} \quad f \in V. \tag{2.2}$$

An admissible operator S for the trial-test space pair $(U, \tilde{U})$ is bounded below on U,

$$\|Sf\| \ge D_1 \|f\|, \quad f \in U.$$

The performance of our proposed (sub-)Galerkin reconstruction depends on the test space $\tilde{U}$, particularly on the ratio between bounds D_1 and D_2 in (2.1) and (2.2), see Theorem 2.3. In our model for sampling, S is the pre-reconstruction operator $S_{\Gamma,\delta}$ in (1.3), and the triple of Banach spaces contains the reconstruction space U, the reproducing kernel space $V_{K,p}$ and the space L^p.

Next we introduce a general notion of Galerkin reconstruction.

Definition 2.2. *Let $S : V \to V$ be a bounded linear operator, and $(U, \tilde{U})$ be a trial-test space pair. We say that a linear operator $R : V \to U$ is a* Galerkin reconstruction *if*

$$Rh = h, \ \ h \in U \tag{2.3}$$

and

$$\langle SRf, g \rangle = \langle Sf, g \rangle, \ \ f \in V \text{ and } g \in \tilde{U}; \tag{2.4}$$

and a sub-Galerkin reconstruction *if* (2.3) *holds and*

$$\sup_{g\in\tilde U, \|g\|\le 1} |\langle SRf,\ g\rangle| \le D_3 \sup_{g\in\tilde U, \|g\|\le 1} |\langle Sf,\ g\rangle|,\ \ f\in V, \tag{2.5}$$

for some $D_3 > 0$.

In the following theorem, we establish numerical stability and quasi-optimality of (sub-)Galerkin reconstructions associated with admissible operators.

Theorem 2.3. *Let* $V, U, \tilde U$ *be as in Definition 2.1, and* S *be admissible for the pair* $(U, \tilde U)$ *with bounds* D_1 *and* D_2*. If* $R: V\to U$ *is a sub-Galerkin reconstruction with bound* D_3*, then*

(i) R *is numerically stable,*

$$\|Rf\| \le \frac{D_2D_3}{D_1}\|f\|,\ \ f\in V.$$

(ii) R *is quasi-optimal,*

$$\|Rf-f\| \le \frac{D_1+D_2D_3}{D_1}\inf_{h\in U}\|f-h\|,\ \ f\in V.$$

Proof. (i) For $f\in V$, we obtain from (2.1), (2.2) and (2.5) that

$$D_1\|Rf\| \le \sup_{g\in\tilde U, \|g\|\le 1} |\langle SRf,\ g\rangle| \le D_3 \sup_{g\in\tilde U, \|g\|\le 1} |\langle Sf,\ g\rangle| \le D_2D_3\|f\|.$$

This proves numerical stability of the reconstruction operator R.

(ii) For $f\in V$ and $h\in U$,

$$\begin{aligned}\|f-Rf\| &\le \|f-h\|+\|h-Rf\|\\ &= \|f-h\|+\|R(f-h)\| \le \frac{D_1+D_2D_3}{D_1}\|f-h\|,\end{aligned}$$

where we have used the facts that R is a sub-Galerkin reconstruction and has numerical stability. Then quasi-optimality of the reconstruction operator R holds by taking infinimum over $h\in U$. □

By Theorem 2.3, the existence of a quasi-optimal approximation reduces to finding a sub-Galerkin reconstruction. Now we show that such a sub-Galerkin reconstruction always exists when U and $\tilde U$ are finite-dimensional.

Theorem 2.4. *Let* $V, U, \tilde U$ *be as in Definition 2.1, and* S *be admissible for the pair* $(U, \tilde U)$*. If* U *and* $\tilde U$ *are finite-dimensional, then there is a sub-Galerkin reconstruction.*

Proof. Let $\{f_i\}_{i=1}^m$ and $\{g_i\}_{i=1}^n$ be bases of U and $\widetilde{U}$ respectively. By the admissibility of S, we may assume that $B := (\langle Sf_i, g_j\rangle)_{1\le i,j\le m}$ is nonsingular. Write $B^{-1} = (b_{ij})$ and define linear operator R by

$$Rf := \sum_{i,j=1}^m \langle Sf, g_i\rangle b_{ij} f_j,\ \ f \in V.$$

Obviously, R satisfies (2.3). Now it remains to show that R satisfies (2.5).

Let $\tilde{U}_*$ be the space spanned by $\{g_j\}_{j=1}^m$. One may verify that Rf solves Galerkin equations

$$\langle SRf, g\rangle = \langle Sf, g\rangle,\ \ g \in \tilde{U}_* \tag{2.6}$$

for any $f \in V$, and

$$C_0\|h\| \le \sup_{g\in\widetilde{U}_*, \|g\|\le 1} |\langle Sh,\ g\rangle|,\ \ h \in U \tag{2.7}$$

for some positive constant C_0. Therefore

$$\begin{aligned}
\sup_{g\in\widetilde{U}, \|g\|\le 1} |\langle SRf,\ g\rangle| &\le D_2\|Rf\| \\
&\le D_2(C_0)^{-1} \sup_{g\in\widetilde{U}_*, \|g\|\le 1} |\langle SRf,\ g\rangle| \\
&= D_2(C_0)^{-1} \sup_{g\in\widetilde{U}_*, \|g\|\le 1} |\langle Sf,\ g\rangle| \\
&\le D_2(C_0)^{-1} \sup_{g\in\widetilde{U}, \|g\|\le 1} |\langle Sf,\ g\rangle|,\ \ f \in V,
\end{aligned}$$

by (2.6), (2.7) and the admissibility of S. □

For the case that U and $\tilde{U}$ have the same dimension, we have

Corollary 2.5. *Let $V, U, \tilde{U}$ be as in Definition 2.1, and S be admissible for the pair $(U, \tilde{U})$. If dimensions of U and $\tilde{U}$ are the same, then for $f \in V$, the unique solution of Galerkin equations*

$$\langle SRf, g\rangle = \langle Sf, g\rangle,\ \ g \in \tilde{U}, \tag{2.8}$$

defines a Galerkin reconstruction.

In Hilbert space setting, we can establish the following result for least squares solutions.

Corollary 2.6. *Let V be a Hilbert space, U and $\tilde{U}$ be linear subspaces of V, and let S be admissible for the pair $(U, \tilde{U})$. If U and $\tilde{U}$ are finite-dimensional, then the least squares solution of Galerkin equations* (2.8)*,*

$$Rf := \mathrm{argmin}_{h\in U} \sup_{g\in\tilde{U}, \|g\|\le 1} |\langle S(h-f), g\rangle|, \quad f \in V,$$

defines a sub-Galerkin reconstruction with bound $D_3 \le 1$.

The above conclusion on least squares solutions with $\tilde U = U$ has been established by Adcock, Gataric and Hansen for non-uniform sampling [1, 2].

3. Sampling and Reconstruction in $V_{K,p}$

To consider sampling and reconstruction in $V_{K,p}$, we *always* assume that the kernel K of the space $V_{K,p}$ in (1.1) satisfies

$$\|K\|_{\mathcal W} := \max\Big\{\sup_{x\in\mathbf R^d}\|K(x,\cdot)\|_1, \sup_{y\in\mathbf R^d}\|K(\cdot,y)\|_1\Big\}<\infty \tag{3.1}$$

and

$$\lim_{\delta\to 0}\|\omega_\delta(K)\|_{\mathcal W}=0, \tag{3.2}$$

where

$$\omega_\delta(K)(x,y) := \sup_{|x'|,|y'|\le\delta}|K(x+x',y+y')-K(x,y)|.$$

Under the above hypothesis, the integral operator T_0 in (1.2) is a bounded operator on L^p,

$$\|T_0 f\|_p \le \|K\|_{\mathcal W}\|f\|_p, \quad f\in L^p.$$

More importantly, its range space $V_{K,p}$ is a reproducing kernel space [25]. In this section, we consider admissibility of the pre-reconstruction operator $S_{\Gamma,\delta}$ in (1.3) and the unique Galerkin reconstruction associated with it.

3.1. **Admissibility, stability and samplability.** To discuss the admissibility, we introduce the *residue* $E(U,F)$ of signals in a linear space $U\subset L^p$ outside a measurable set F,

$$E(U,F) := \sup_{0\ne f\in U}\frac{\|f\|_{L^p(\mathbf R^d\setminus F)}}{\|f\|_p},$$

where $\|\cdot\|_{L^p(E)}$ is the p-norm on a measurable set E. The reader may refer to [1, 21, 22] for some applications of residues of bandlimited signals.

Theorem 3.1. *Let $V_{K,p}$ and $S_{\Gamma,\delta}$ be as in* (1.1) *and* (1.3) *respectively. Assume that $U\subset V_{K,p}$ and $\tilde U\subset L^{p/(p-1)}$. If*

$$\sup_{g\in\tilde U, \|g\|_{p/(p-1)}\le 1}|\langle f,g\rangle| \ge D_4\|f\|_p, \ \ f\in U \tag{3.3}$$

for some constant D_4 satisfying

(3.4)

$$r_0 := D_4^{-1}\big(E(U, B(\Gamma,\delta))\|K\|_{\mathcal W} + \|\omega_\delta(K)\|_{\mathcal W}\big(1+\|K\|_{\mathcal W}+\|\omega_\delta(K)\|_{\mathcal W}\big)\big) < 1,$$

then $S_{\Gamma,\delta}$ is admissible for the pair $(U, \tilde U)$.

Given a sampling set Γ, we say that the sampling scheme

$$U \ni f \longmapsto \{f(\gamma_n), \gamma_n \in \Gamma\} \tag{3.5}$$

has *weighted ℓ^p-stability on U* if there exist positive constants C_1, C_2 and δ such that

$$C_1\|f\|_p \le \Big(\sum_{\gamma_n\in\Gamma} |I_n||f(\gamma_n)|^p\Big)^{1/p} \le C_2\|f\|_p,\ f\in U,$$

if $1 \le p < \infty$, and

$$C_1\|f\|_\infty \le \sup_{\gamma_n\in\Gamma} |f(\gamma_n)| \le C_2\|f\|_\infty,\ f\in U,$$

if $p=\infty$, where $\{I_n \subset B(\gamma_n,\delta), \gamma_n\in\Gamma\}$ is a disjoint covering of the δ-neighborhood $B(\Gamma,\delta)$ of the sampling set Γ. Weighted stability of a sampling scheme implies its unique determination. It is an important concept for robust signal reconstruction, see [5, 6, 9, 12, 25, 33, 34, 35, 37] and references here. The following result connects the weighted ℓ^p-stability of a sampling scheme with the admissibility of a pre-reconstruction operator.

Theorem 3.2. *Let $V_{K,p}$ and $S_{\Gamma,\delta}$ be as in* (1.1) *and* (1.3) *respectively. Assume that $U \subset V_{K,p}$ and $\tilde U \subset L^{p/(p-1)}$. If $S_{\Gamma,\delta}$ is admissible for the pair $(U,\tilde U)$, then the sampling scheme* (3.5) *on Γ has weighted ℓ^p-stability on U.*

By the regularity assumption (3.2) on the reproducing kernel K, the second requirement (3.4) in Theorem 3.1 is satisfied if δ is sufficiently small and $B(\Gamma,\delta)$ is the whole Euclidean space $\mathbf{R}^d$. For the case that $B(\Gamma,\delta)$ contains an open domain F_0 but not necessarily the whole space $\mathbf{R}^d$, we obtain the following samplability result from Theorems 3.1 and 3.2.

Corollary 3.3. *Let $U \subset V_{K,p}$ and D_4 be as in Theorem 3.1. Assume that F_0 is an open domain satisfying $E(U,F_0)\|K\|_{\mathcal W} < D_4$. If Γ is a sampling set with $B(\Gamma,\delta) \supset F_0$ for some sufficiently small $\delta > 0$, then signals in U are uniquely determined by their samples taken on Γ.*

The samplability of various signals is well-studied, see, e.g., [2, 13, 19] for band-limited signals, [5, 37] for signals in a shift-invariant space, [32, 33] for signals with finite rate of innovation, and [20, 25] for signals in a reproducing kernel space.

To prove Theorem 3.1, we need the following lemma.

Lemma 3.4. *Let $V_{K,p}$ and $S_{\Gamma,\delta}$ be as in* (1.1) *and* (1.3) *respectively. Then*

$$\|S_{\Gamma,\delta} f\|_p \le \big(\|K\|_{\mathcal W} + \|\omega_\delta(K)\|_{\mathcal W}\big)\big(1+\|\omega_\delta(K)\|_{\mathcal W}\big)\|f\|_p,\ \ f\in V_{K,p}.$$

Proof. Let $\{I_n\}$ be the disjoint covering of $B(\Gamma,\delta)$ in (1.3). For $f\in V_{K,p}$, write

$$\begin{aligned} S_{\Gamma,\delta} f(x) &= \sum_n \int_{I_n}\int_{\mathbf R^d} K(x,\gamma_n)K(\gamma_n,z) f(z)dzdy\\ &= \sum_n \int_{I_n}\int_{\mathbf R^d} \Big\{K(x,y)K(y,z)+(K(x,\gamma_n)-K(x,y))\\ &\quad \times K(y,z)+K(x,y)(K(\gamma_n,z)-K(y,z))\\ &\quad +(K(x,\gamma_n)-K(x,y))(K(\gamma_n,z)-K(y,z))\Big\} f(z)dzdy\\ &=: I+II+III+IV. \end{aligned} \tag{3.6}$$

Observe that

$$\|I\|_p = \Big\|\int_{B(\Gamma,\delta)} K(\cdot,y)f(y)dy\Big\|_p \le \|K\|_{\mathcal W}\|f\|_p,$$

$$\|II\|_p \le \Big\|\int_{\mathbf R^d} \omega_\delta(K)(\cdot,y)|f(y)|dy\Big\|_p \le \|\omega_\delta(K)\|_{\mathcal W}\|f\|_p,$$

$$\begin{aligned}\|III\|_p &\le \Big\|\int_{\mathbf R^d}\int_{\mathbf R^d} |K(\cdot,y)|\omega_\delta(K)(y,z)|f(z)|dzdy\Big\|_p\\ &\le \|K\|_{\mathcal W}\|\omega_\delta(K)\|_{\mathcal W}\|f\|_p,\end{aligned}$$

and

$$\begin{aligned}\|IV\|_p &\le \Big\|\int_{\mathbf R^d}\int_{\mathbf R^d} \omega_\delta(K)(\cdot,y)\omega_\delta(K)(y,z)|f(z)|dzdy\Big\|_p\\ &\le \|\omega_\delta(K)\|_{\mathcal W}^2\|f\|_p.\end{aligned}$$

Combining the above four estimates with (3.6) completes the proof. □

We finish this subsection with proofs of Theorems 3.1 and 3.2.

Proof of Theorem 3.1. The upper bound estimate (2.2) for the operator $S_{\Gamma,\delta}$ follows immediately from Lemma 3.4.

Define

$$T_0^* g(x) := \int_{\mathbf R^d} K(y,x) g(y)dy,\ \ g\in L^{p/(p-1)}.$$

For $f \in U$ and $g \in \tilde U \subset L^{p/(p-1)}$ with $\|g\|_{p/(p-1)} \le 1$, we obtain

$$\begin{aligned}
|\langle S_{\Gamma,\delta} f, g\rangle - \langle f, g\rangle| &\le \Big|\int_{\mathbf{R}^d\setminus B(\Gamma,\delta)} f(x) T_0^* g(x)dx\Big| \\
&\quad + \Big|\sum_n \int_{I_n} f(\gamma_n)(T_0^* g)(\gamma_n) - f(x)(T_0^* g)(x) dx\Big| \\
&\le \|K\|_{\mathcal W} \|f\|_{L^p(\mathbf{R}^d\setminus B(\Gamma,\delta))} \\
&\quad + \|\omega_\delta(K)\|_{\mathcal W}\big(1+\|K\|_{\mathcal W}+\|\omega_\delta(K)\|_{\mathcal W}\big)\|f\|_p, \qquad (3.7)
\end{aligned}$$

where $\{I_n\}$ is the disjoint covering of $B(\Gamma,\delta)$ in (1.3). This together with (3.3) and (3.4) proves the lower bound estimate (2.1) for the operator $S_{\Gamma,\delta}$. □

Proof of Theorem 3.2. Take $f \in V$. Following the argument used in Lemma 3.4, we obtain

$$\big(\|K\|_{\mathcal W}+\|\omega_\delta(K)\|_{\mathcal W}\big)^{-1}\|S_{\Gamma,\delta} f\|_p \le \Big(\sum_n |I_n||f(\omega_n)|^p\Big)^{1/p} \le \big(1+\|\omega_\delta(K)\|_{\mathcal W}\big)\|f\|_p$$

for $1 \le p < \infty$ and

$$\big(\|K\|_{\mathcal W}+\|\omega_\delta(K)\|_{\mathcal W}\big)^{-1}\|S_{\Gamma,\delta} f\|_\infty \le \sup_n |f(\omega_n)| \le \|f\|_\infty$$

for $p=\infty$. The above two estimates together with admissibility of the operator $S_{\Gamma,\delta}$ complete the proof. □

3.2. **Galerkin reconstruction.** To consider Galerkin reconstruction associated with the operator $S_{\Gamma,\delta}$ on the reproducing kernel space $V_{K,p}$, we introduce the oblique projection for a pair $(U,\tilde U)$ of Banach spaces.

Definition 3.5. *Given $U \subset V_{K,p}$ and $\tilde U \subset L^{p/(p-1)}$, a bounded operator $P_{U,\tilde U}: V_{K,p} \to U$ is said to be an* oblique projection *for the pair $(U,\tilde U)$ if*

$$P_{U,\tilde U} h = h,\ \ h \in U, \qquad (3.8)$$

and

$$\langle P_{U,\tilde U} f, g\rangle = \langle f, g\rangle,\ \ f \in V_{K,p}, g \in \tilde U. \qquad (3.9)$$

In Hilbert space setting, an oblique projection $P_{U,\tilde U}$ exists when cosine of the subspace angle between U and $\tilde U^\perp$ is positive [3, 9, 12, 36]. Following the argument used in Theorem 2.4, we can show that if U and $\tilde U$ have the same dimension and satisfy the first requirement (3.3) of Theorem 3.1, then there is an oblique projection $P_{U,\tilde U}$ for the pair $(U,\tilde U)$.

Theorem 3.6. *Let $V_{K,p}$ and $S_{\Gamma,\delta}$ be as in* (1.1) *and* (1.3) *respectively. Assume that $U \subset V_{K,p}$ and $\tilde U \subset L^{p/(p-1)}$ satisfy* (3.3) *and* (3.4)*, and an oblique projection $P_{U,\tilde U}$ associated with the pair $(U, \tilde U)$ exists. Then Galerkin equations*

$$\langle S_{\Gamma,\delta} h, g\rangle = \langle S_{\Gamma,\delta} f, g\rangle, \ g \in \tilde U, \tag{3.10}$$

have a unique solution $h \in U$ for $f \in V_{K,p}$. Moreover, the mapping $f \to h$ defines a Galerkin reconstruction.

To solve Galerkin equations (3.10), we need exponential convergence of the iterative approximation-projection algorithm (1.5). The algorithm (1.5) has been demonstrated to be efficient to reconstruct various signals. The reader may refer to [13, 35] for band-limited signals, [4, 6] for signals in a shift-invariant space, and [25] for signals in a reproducing kernel space.

Lemma 3.7. *Let $V_{K,p}, S_{\Gamma,\delta}, U, \tilde U$ and $P_{U,\tilde U}$ be as in Theorem 3.6, and let $r_0 \in (0, 1)$ be as in* (3.4)*. Then for any $g_0 \in U$, the sequence $g_m, m \ge 0$, in the iterative algorithm* (1.5) *converges to some $g_\infty \in U$,*

$$\|g_m - g_\infty\|_p \le \frac{r_0^{m+1}}{1-r_0}\|g_0\|_p, \quad m \ge 0. \tag{3.11}$$

Moreover, if $g_0 = P_{U,\tilde U} S_{\Gamma,\delta} h + \tilde g$ for some $h, \tilde g \in U$, then

$$\|g_\infty - h\|_p \le \frac{\|\tilde g\|_p}{1-r_0}. \tag{3.12}$$

Proof. Combining (3.3), (3.7) and (3.9), we obtain

$$\begin{aligned} \|P_{U,\tilde U} S_{\Gamma,\delta} f - f\|_p &\le D_4^{-1} \sup_{g\in\tilde U, \|g\|_{p/(p-1)}\le 1} |\langle P_{U,\tilde U} S_{\Gamma,\delta} f - f, g\rangle| \\ &= D_4^{-1} \sup_{g\in\tilde U, \|g\|_{p/(p-1)}\le 1} |\langle S_{\Gamma,\delta} f - f, g\rangle| \\ &\le r_0\|f\|_p, \quad f \in U. \end{aligned} \tag{3.13}$$

Observe from (1.5) that

$$g_{m+1} - g_m = (I - P_{U,\tilde U} S_{\Gamma,\delta})(g_m - g_{m-1}), \ m \ge 1.$$

This together with (3.13) proves (3.11).

Now we prove (3.12). Taking limit in (1.5) leads to the following consistence condition

$$P_{U,\tilde U} S_{\Gamma,\delta} g_\infty = g_0. \tag{3.14}$$

Replacing g_0 in (3.14) by $P_{U,\tilde U} S_{\Gamma,\delta} h + \tilde g$ gives

$$P_{U,\tilde U} S_{\Gamma,\delta}(g_\infty - h) = \tilde g.$$

This together with (3.13) completes the proof. □

Proof of Theorem 3.6. Take $f \in V_{K,p}$, set $g_0 = P_{U,\tilde{U}} S_{\Gamma,\delta} f$, and let $g_\infty \in U$ be the limit of $g_m, m \ge 0$, in the iterative algorithm (1.5). The existence of such a limit follows from Lemma 3.7. Taking limit in (1.5) leads to

$$P_{U,\tilde{U}} S_{\Gamma,\delta} f = P_{U,\tilde{U}} S_{\Gamma,\delta} g_\infty. \tag{3.15}$$

Then for any $g \in \tilde{U}$,

$$\langle S_{\Gamma,\delta} g_\infty, g\rangle = \langle P_{U,\tilde{U}} S_{\Gamma,\delta} g_\infty, g\rangle = \langle P_{U,\tilde{U}} S_{\Gamma,\delta} f, g\rangle = \langle S_{\Gamma,\delta} f, g\rangle \tag{3.16}$$

by (3.9) and (3.15). This proves that g_∞ is a solution of Galerkin equations (3.10).

Next, we show that g_∞ is the unique solution of Galerkin equations (3.10). Let $h \in U$ be another solution. Then

$$\langle P_{U,\tilde{U}} S_{\Gamma,\delta}(h - g_\infty), g\rangle = \langle S_{\Gamma,\delta}(h - g_\infty), g\rangle = 0.$$

This together with (3.3) implies that

$$P_{U,\tilde{U}} S_{\Gamma,\delta}(h - g_\infty) = 0.$$

Recall from (3.13) that $P_{U,\tilde{U}} S_{\Gamma,\delta}$ is invertible on U. Then $h = g_\infty$ and the uniqueness follows.

Observe that any $f \in U$ satisfies Galerkin equations (3.10). This together with (3.16) proves that the unique solution of Galerkin equations (3.10) defines a Galerkin reconstruction. □

We finish this section with a remark on the iterative approximation-projection algorithm (1.5).

Remark 3.8. Given $\delta > 0$, a sampling set Γ and probability measures μ_n supported on I_n, we define

$$\tilde{S}_{\Gamma,\delta} f(x) = \sum_{\gamma_n \in \Gamma} |I_n| f(\gamma_n) \int_{I_n} K(x,y) d\mu_n(y), \quad f \in V_{K,p},$$

where $\{I_n \subset B(\gamma,\delta),\ \gamma_n \in \Gamma\}$ is a disjoint covering of $B(\Gamma,\delta)$. The operator $\tilde{S}_{\Gamma,\delta}$ just defined becomes the sampling operator $S_{\Gamma,\delta}$ in (1.3) when μ_n are point measures supported on γ_n, and the sampling operator

$$S_{\Gamma,\delta} f(x) = \sum_{\omega_n \in \Gamma} f(\gamma_n) \int_{I_n} K(x,y) dy, \quad f \in V_{K,p}$$

when μ_n are normalized Lebsegue measure supported on I_n. Following the argument used in Theorem 3.1 and Lemma 3.7, we can show that the approximation-projection algorithm (1.5) with $S_{\Gamma,\delta}$ replaced by $\tilde S_{\Gamma,\delta}$ has exponential convergence if

$$D_4^{-1}\big(E(U,B(\Gamma,\delta))\|K\|_{\mathcal W}+\|\omega_{2\delta}(K)\|_{\mathcal W}\big(1+\|K\|_{\mathcal W}+\|\omega_{2\delta}(K)\|_{\mathcal W}\big)\big)<1,$$

c.f., the second requirement (3.4) in Theorem 3.1.

4. Sampling signals with finite rate of innovation

A signal with *finite rate of innovation* (FRI) has finitely many degrees of freedom per unit of time [11, 24, 27, 32, 33, 38]. Define the *Wiener amalgam space* by

$$\mathcal W^1:=\Big\{\phi,\ \|\phi\|_{\mathcal W^1}:=\sum_{k\in\mathbf Z}\sup_{0\le x\le 1}|\phi(x+k)|<\infty\Big\}.$$

It is observed in [32] that lots of FRI signals live in a space of the form

$$V_2(\Phi):=\Big\{\sum_{i\in\mathbf Z}c_i\phi_i(\cdot-i),\ \sum_{i\in\mathbf Z}|c_i|^2<\infty\Big\}, \tag{4.1}$$

where the generator $\Phi:=(\phi_i)_{i\in\mathbf Z}$ satisfies

$$\|\Phi\|_{\mathcal W^1}:=\big\|\sup_{i\in\mathbf Z}|\phi_i|\big\|_{\mathcal W^1}<\infty\ \text{ and }\ \lim_{\delta\to 0}\big\|\sup_{i\in\mathbf Z}\omega_\delta(\phi_i)\big\|_{\mathcal W^1}=0. \tag{4.2}$$

In this section, we consider Galerkin reconstruction of signals in finite-dimensional spaces

$$V_{2,L}(\Phi)=\Big\{\sum_{i=-L}^{L}c_i\phi_i(\cdot-i),\ \sum_{i=-L}^{L}|c_i|^2<\infty\Big\},\ L\ge 1. \tag{4.3}$$

4.1. Reproducing kernel spaces.

For $\Phi:=(\phi_i)_{i\in\mathbf Z}$ and $\tilde\Phi:=(\tilde\phi_j)_{j\in\mathbf Z}$ satisfying (4.2), define their correlation matrix by

$$A_{\Phi,\tilde\Phi}:=\big(\langle\phi_i(\cdot-i),\tilde\phi_j(\cdot-j)\rangle\big)_{i,j\in\mathbf Z}.$$

In this subsection, we consider when $V_2(\Phi)$ and $V_2(\tilde\Phi)$ in (4.1) are range spaces of some idempotent integral operators with kernels satisfying (3.1) and (3.2).

Theorem 4.1. *Let Φ and $\tilde\Phi$ satisfy* (4.2)*. If the correlation matrix $A_{\Phi,\tilde\Phi}$ has bounded inverse on ℓ^2, then*

$$V_2(\Phi)=V_{K,2}\quad\text{and}\quad V_2(\tilde\Phi)=V_{K^*,2}$$

for some kernel K satisfying (3.1) *and* (3.2)*, where*

$$K^*(x,y):=K(y,x),\ x,y\in\mathbf R.$$

Let $\mathcal{C}_1$ contain all infinite matrices $A := \big(a_{ij}\big)_{i,j\in\mathbf{Z}}$ with

$$\|A\|_{\mathcal{C}_1} := \sum_{k\in\mathbf{Z}} \Big(\sup_{i-j=k} |a_{ij}|\Big) < \infty.$$

To prove Theorem 4.1, we recall Wiener's lemma for the *Baskakov-Gohberg-Sjöstrand class* $\mathcal{C}_1$, see [8, 16, 18, 29, 30, 31] and references therein.

Lemma 4.2. *If $A \in \mathcal{C}_1$ has bounded inverse on ℓ^2, then its inverse A^{-1} belongs to $\mathcal{C}_1$ too.*

Proof of Theorem 4.1. By direct calculation, we have

$$\|A_{\Phi,\tilde{\Phi}}\|_{\mathcal{C}_1} \le \|\Phi\|_{\mathcal{W}^1} \|\tilde{\Phi}\|_{\mathcal{W}^1}.$$

Thus the inverse of the correlation matrix $A_{\Phi,\tilde{\Phi}}$ belongs to the Baskakov-Gohberg-Sjöstrand class by Lemma 4.2. Write $(A_{\Phi,\tilde{\Phi}})^{-1} = (b_{ij})_{i,j\in\mathbf{Z}}$. One may verify that the kernel defined by

$$K_{\Phi,\tilde{\Phi}}(x,y) := \sum_{i,j\in\mathbf{Z}} \phi_i(x-i) b_{ji} \tilde{\phi}_j(y-j) \tag{4.4}$$

satisfies all requirements of the theorem. □

4.2. **Admissibility and Galerkin reconstruction.** Given a sampling set $\Gamma = \{\gamma_n\}_{n=1}^N$ ordered as $\gamma_1 < \gamma_2 < \cdots < \gamma_N$, define

$$S_{\Phi,\tilde{\Phi},\Gamma} f(x) := \sum_{n=1}^N \frac{\gamma_{n+1} - \gamma_{n-1}}{2} f(\gamma_n) K_{\Phi,\tilde{\Phi}}(x,\gamma_n),\ \ f \in V_2(\Phi), \tag{4.5}$$

and

$$A_{\Phi,\tilde{\Phi},\Gamma} := \Big(\sum_{n=1}^N \frac{\gamma_{n+1} - \gamma_{n-1}}{2} \phi_i(\gamma_n - i)\tilde{\phi}_j(\gamma_n - j)\Big)_{-L\le i,j\le L},\ L \ge 1, \tag{4.6}$$

where $\gamma_0 = \gamma_1, \gamma_{N+1} = \gamma_N$, and the kernel $K_{\Phi,\tilde{\Phi}}$ is given in (4.4). In this subsection, we investigate admissibility of the operator $S_{\Phi,\tilde{\Phi},\Gamma}$ and its corresponding Galerkin reconstruction, c.f. Corollary 2.5, and Theorems 3.1 and 3.6.

Theorem 4.3. *Let Φ and $\tilde{\Phi}$ satisfy* (4.2)*. Assume that the correlation matrix $A_{\Phi,\tilde{\Phi}}$ has bounded inverse on ℓ^2. Then the following statements are equivalent:*

(i) *The $L \times L$ matrix $A_{\Phi,\tilde{\Phi},\Gamma}$ in* (4.6) *is nonsingular.*
(ii) *$S_{\Phi,\tilde{\Phi},\Gamma}$ is admissible for the pair $(V_{2,L}(\Phi), V_{2,L}(\tilde{\Phi}))$.*

(iii) *For any $f \in V_2(\Phi)$, Galerkin equations*

$$\langle S_{\Phi,\tilde\Phi,\Gamma} h, g\rangle = \langle S_{\Phi,\tilde\Phi,\Gamma} f, g\rangle,\ \ g\in V_{2,L}(\tilde\Phi) \tag{4.7}$$

have a unique solution h in $V_{2,L}(\Phi)$.

(iv) *For any $g \in V_2(\tilde\Phi)$, dual Galerkin equations*

$$\langle S_{\Phi,\tilde\Phi,\Gamma} f, \tilde h\rangle = \langle S_{\Phi,\tilde\Phi,\Gamma} f, g\rangle,\ \ f\in V_{2,L}(\Phi)$$

have a unique solution $\tilde h$ in $V_{2,L}(\tilde\Phi)$.

Proof. For $h=\sum_{i=-L}^L c_i\phi_i(\cdot-i)\in V_{2,L}(\Phi)$ and $g=\sum_{j=-L}^L d_j\tilde\phi_j(\cdot-j)\in V_{2,L}(\tilde\Phi)$, we obtain

$$\begin{aligned}\langle S_{\Phi,\tilde\Phi,\Gamma} h, g\rangle &= \sum_{i,j=-L}^{L}\Big(\sum_{n=1}^N \frac{\gamma_{n+1}-\gamma_{n-1}}{2}\phi_i(\gamma_n-i)\langle K_{\Phi,\tilde\Phi}(t,\gamma_n), \tilde\phi_j(t-j)\rangle\Big)c_i d_j\\ &= \sum_{i,j=-L}^{L}\Big(\sum_{n=1}^N \frac{\gamma_{n+1}-\gamma_{n-1}}{2}\phi_i(\gamma_n-i)\tilde\phi_j(\gamma_n-j)\Big)c_i d_j\\ &= c^T A_{\Phi,\tilde\Phi,\Gamma} d, \qquad (4.8)\end{aligned}$$

where $c=(c_i)_{-L\le i\le L}$ and $d=(d_j)_{-L\le j\le L}$. By the invertibility assumption on $A_{\Phi,\tilde\Phi}$, $\{\phi_i(\cdot-i), -L\le i\le L\}$ and $\{\tilde\phi_i(\cdot-i), -L\le i\le L\}$ are Riesz bases of $V_{2,L}(\Phi)$ and $V_{2,L}(\tilde\Phi)$ respectively. This together with (4.8) proves the desired equivalent statements. □

4.3. **Oblique Projection and iterative approximation-projection algorithm.** In this subsection, we first discuss existence and uniqueness of oblique projection for the pair $(V_{2,L}(\Phi), V_{2,L}(\tilde\Phi))$.

Theorem 4.4. *Let $L\ge 1$, and let Φ and $\tilde\Phi$ satisfy* (4.2)*. Assume that the correlation matrix $A_{\Phi,\tilde\Phi}$ has bounded inverse on ℓ^2. Then the principal submatrix*

$$A_{\Phi,\tilde\Phi,L} := \big(\langle \phi_i(\cdot-i), \tilde\phi_j(\cdot-j)\rangle\big)_{-L\le i,j\le L} \tag{4.9}$$

of the correlation matrix $A_{\Phi,\tilde\Phi}$ is nonsingular if and only if there exists a unique oblique projection for the pair $(V_{2,L}(\Phi), V_{2,L}(\tilde\Phi))$. Moreover, the oblique projection could be defined by

$$P_{\Phi,\tilde\Phi,L} f := \sum_{-L\le i,j\le L} \langle f, \tilde\phi_i(\cdot-i)\rangle \tilde b_{ij}\phi_j(\cdot-j),\ \ f\in V_2(\Phi), \tag{4.10}$$

where $(A_{\Phi,\tilde\Phi,L})^{-1}=(\tilde b_{ij})_{-L\le i,j\le L}$.

Proof. The sufficiency is obvious. Now we prove the necessity. Suppose, to the contrary, that $A_{\Phi,\tilde\Phi,L}$ in (4.9) is singular. Take a nonzero vector $e=(e_i)_{-L\le i\le L}$ in the null space $N((A_{\Phi,\tilde\Phi,L})^T)$ and a nonzero linear functional $\mathcal{J}$ on $V_2(\Phi)$ such that $\mathcal{J}(h)=0$ for all $h\in V_{2,L}(\Phi)$. Define

$$Q(f):=\mathcal{J}(f)\sum_{-L\le i\le L} e_i\phi_i(\cdot-i),\ f\in V_2(\Phi).$$

Then Q is a nonzero linear operator from $V_2(\Phi)$ to $V_{2,L}(\Phi)$,

$$Qh=0,\quad h\in V_{2,L}(\Phi)$$

and

$$\langle Qf,g\rangle=\mathcal{J}(f)\sum_{-L\le i,j\le L} e_i\langle\phi_i(\cdot-i),\tilde\phi_j(\cdot-j)\rangle d_j=0,$$

where $g=\sum_{-L\le j\le L} d_j\tilde\phi_j(\cdot-j)\in V_{2,L}(\tilde\Phi)$. This contradicts to the uniqueness of oblique projections. □

In this subsection, we then examine exponential convergence of an iterative algorithm for the recovery of signals with finite rate of innovation. Replacing $P_{U,\tilde U}$ and $S_{\Gamma,\delta}$ in the iterative algorithm (1.5) by $P_{\Phi,\tilde\Phi,L}$ and $S_{\Phi,\tilde\Phi,\Gamma}$ respectively, it becomes

(4.11)
$$g_{m+1}=g_m-\sum_{n=1}^N\sum_{i,j=-L}^L\frac{\gamma_{n+1}-\gamma_{n-1}}{2}g_m(\gamma_n)\tilde\phi_i(\gamma_n-i)\tilde b_{ij}\phi_j(\cdot-j)+g_0,\ m\ge 0,$$

with $g_0\in V_{2,L}(\Phi)$.

Theorem 4.5. *Let Φ and $\tilde\Phi$ satisfy* (4.2). *Assume that $A_{\Phi,\tilde\Phi,L}$ is nonsingular. If*

$$\|A_{\Phi,\tilde\Phi,\Gamma}(A_{\Phi,\tilde\Phi,L})^{-1}-I\|<1, \tag{4.12}$$

then the iterative algorithm (4.11) *has exponential convergence. Moreover, it recovers the original signal $h\in V_{2,L}(\Phi)$ when*

$$g_0=\sum_{n=1}^N\sum_{i,j=-L}^L\frac{\gamma_{n+1}-\gamma_{n-1}}{2}h(\gamma_n)\tilde\phi_i(\gamma_n-i)\tilde b_{ij}\phi_j(\cdot-j).$$

Proof. Write $g_m=\sum_{-L\le i\le L} c_m(i)\phi_i(\cdot-i)$ and set $c_m=(c_m(i))_{-L\le i\le L}$. Then we can reformulate the iterative algorithm (4.11) as

$$c_{m+1}^T=c_m^T-c_m^TA_{\Phi,\tilde\Phi,\Gamma}(A_{\Phi,\tilde\Phi,L})^{-1}+c_0^T,\ m\ge 0.$$

This together with (4.12) proves the desired conclusions. □

5. Numerical Simulation

In this section, we present several examples to illustrate our Galerkin reconstruction of signals with finite rate of innovation.

Let $\Theta := \{\theta_i\}$ be either $\Theta_O := \{0\}$ (the identical zero set), or Θ_I with θ_i being randomly selected in $[-0.2, 0.2]$. Set

$$\Phi_0 = \{\phi_0(\cdot - \theta_i)\}_{i\in\mathbf{Z}},$$

where the generating function ϕ_0 is either (i) the sinc function $\text{sinc}(t) := \frac{\sin \pi t}{\pi t}$, or (ii) the Gaussian function $\text{gauss}(t) := \exp(-3t^2/2)$, or (iii) the cubic B-spline spline(t), see Figure 1 for examples of signals in $V_2(\Phi_0)$. In our numerical simulations, reconstructed signals live in the space

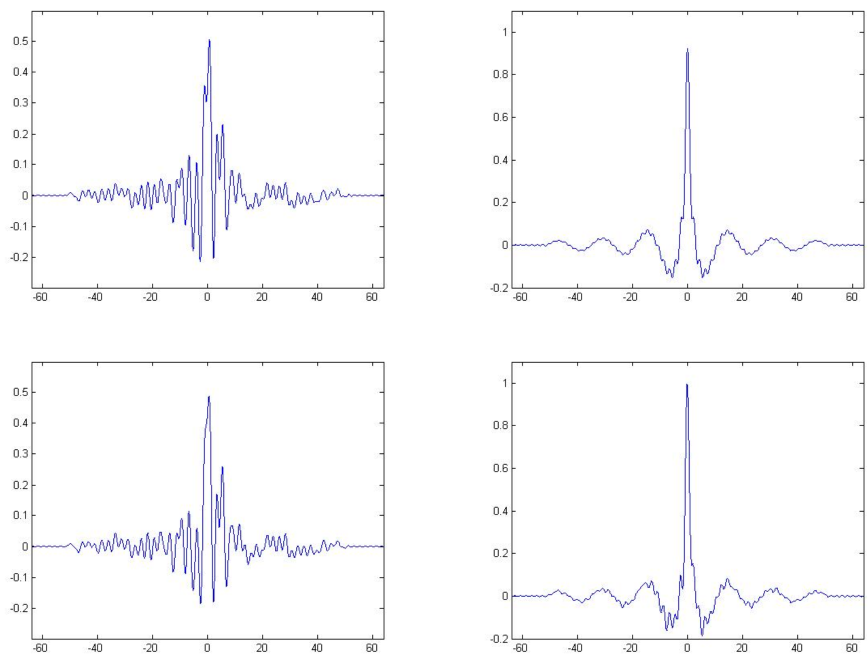
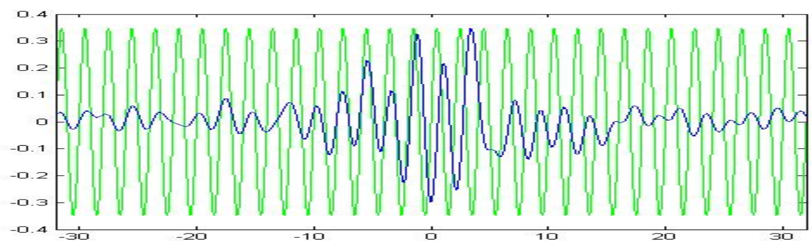


Figure 1. Above are bandlimited signals $x(\text{sinc}, 0) = \sum_i \alpha_i \text{sinc}(t-i)$ with $(1+|i|)\alpha_i \in [-1, 1]$ randomly selected (left), and $x(\text{sinc}, 1) = \sum_i \beta_i \text{sinc}(t-i)$ with $\beta_i = (1+|i|)^{-1}\cos(\pi i/8)$ (right). Below are signals $x(\text{sinc}, 2) = \sum_i \alpha_i \text{sinc}(t-i-\theta_i)$ (left) and $x(\text{sinc}, 3) = \sum_i \beta_i \text{sinc}(t-i-\theta_i)$ with $\theta_i \in [-0.2, 0.2]$ randomly selected (right).

$$V_{2,L}(\Phi_0) = \Big\{ \sum_{i=-L}^{L} c_i\phi_0(t-i-\theta_i) : \sum_{i=-L}^{L} |c_i|^2 < \infty \Big\},\ L \geq 1,$$

and sampling schemes are

- Nonuniform sampling on $\Gamma_N := \{\gamma_k, |k| \le L+2\}$, where $\gamma_{-L-3} = -L-2$ and $\gamma_k - \gamma_{k-1} \in [0.9, 1.1], |k| \le L+2$, are randomly selected.
- Jittered sampling on $\Gamma_J := \{\gamma_k := k + \delta_k, |k| \le L+2\}$, where $\delta_k \in [-0.1, 0.1]$ are randomly selected.
- Adaptive sampling on $\Gamma_C := \{\gamma_k \in [-L-2, L+2]\}$ of a bounded signal $x \in V_2(\Phi)$ via crossing time encoding machine (C-TEM), where $x(t) \ne \|x\|_\infty \sin(\pi t)$ for all $t \in [-L-2, L+2]$ except $t = \gamma_k$ for some k, see Figure 2 [14, 17, 23].



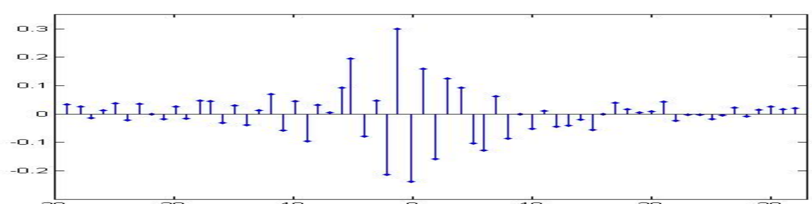


FIGURE 2. Above is the signal $x(\mathrm{sinc}, 0)$ in Figure 1 and the crossing signal $\|x(\mathrm{sinc}, 0)\|_\infty \sin \pi t$ on $[-L-2, L+2]$, and below is the sampling data of $x(\mathrm{sinc}, 0)$ on the sampling set $\Gamma_C \subset [-L-2, L+2]$, where $L = 30$.

To reconstruct signals via Galerkin method, we take

$$\tilde{\Phi}_0 = \{\tilde{\phi}_0\} \quad \text{with} \quad \tilde{\phi}_0 = \chi_{[-1/2,1/2)}.$$

Then the equation (4.7) to determine the Galerkin reconstruction

$$G_{\Phi_0, \tilde{\Phi}_0, \Gamma} f := \sum_{i=-L}^{L} c_i \phi_0(\cdot - i - \theta_i) \in V_{2,L}(\Phi_0)$$

can be reformulated as follows:

$$\sum_{i=-L}^{L}\Big(\sum_{n=1}^{N}\frac{\gamma_{n+1}-\gamma_{n-1}}{2}\phi_0(\gamma_n-i-\theta_i)\tilde\phi_0(\gamma_n-j)\Big)c_i$$
$$(5.1)\qquad = \sum_{n=1}^{N}\frac{\gamma_{n+1}-\gamma_{n-1}}{2}f(\gamma_n)\tilde\phi_0(\gamma_n-j), -L\le j\le L,$$

where $f\in V_2(\Phi_0)$ and $\Gamma:=\{\gamma_n\}_{n=1}^N$ is either the nonuniform sampling set Γ_N, or the jittered sampling set Γ_J, or the adaptive C-TEM sampling set Γ_C. Considering the bandlimited signal $x(\mathrm{sinc},0)$ described in Figure 1, we present some numerical results for its pre-reconstruction in $V_2(\Phi_0)$ and Galerkin reconstruction in $V_{2,L}(\Phi_0)$ in Figure 3. We see that a pre-reconstruction may provide a reasonable approximation, while a Galerkin reconstruction could recover the original signal almost perfectly in the sampling interval.

For $\Phi_0=\{\phi_0(\cdot-\theta_i)\}$, let signals $x(\phi_0,l)\in V_2(\Phi_0), 0\le l\le 3$, be as $x(\mathrm{sinc},l)$ in Figure 1 with the sinc function replaced by the function ϕ_0. In Figure 4, we illustrate their best approximation in $V_{2,L}(\Phi_0)$ and solutions of the Galerkin system (5.1) with f replaced by $x(\phi_0,l), 0\le l\le 3$, respectively. We observe that given a signal in $V_2(\Phi_0)$, its Galerkin reconstruction in $V_{2,L}(\Phi_0)$ could almost match its best approximation in $V_{2,L}(\Phi_0)$, except near the boundary of the sampling interval. The boundary effect is viewable especially when ϕ_0 has slow decay at infinity.

Given signals $x(\phi_0,l), 0\le l\le 3$, let $y_L(\phi_0,l)$ be their best approximators in $V_{2,L}(\Phi_0)$, and denote by

$$e(\phi_0,l)=\|x(\phi_0,l)-y_L(\phi_0,l)\|$$

their best approximation error in $V_{2,L}(\Phi_0)$. For $\Gamma=\Gamma_N$ or Γ_J or Γ_C, set

$$\epsilon_\Gamma(\phi_0,l)=\|z_L(\Gamma,\phi_0,l)-y_L(\phi_0,l)\|,$$

where $z_L(\Gamma,\phi_0,l)$ is obtained from solving Galerkin system (5.1) with f replaced by $x(\phi_0,l)$. For signals $x(\phi_0,l), 0\le l\le 3$, and sampling sets $\Gamma=\Gamma_N,\Gamma_J$ and Γ_C, Galerkin reconstruction (5.1) provides quasi-optimal approximation in $V_{2,L}(\Phi_0)$, and the quasi-optimal constant in Theorem 2.3 is well behaved,

$$\frac{\|z_L(\Gamma,\phi_0,l)-x(\phi_0,l)\|}{\|y_L(\phi_0,l)-x(\phi_0,l)\|}\le 1+\frac{\epsilon_\Gamma(\phi_0,l)}{e(\phi_0,l)}\le\frac{3}{2},$$

see Table 1 for numerical results with abbrievated notations.

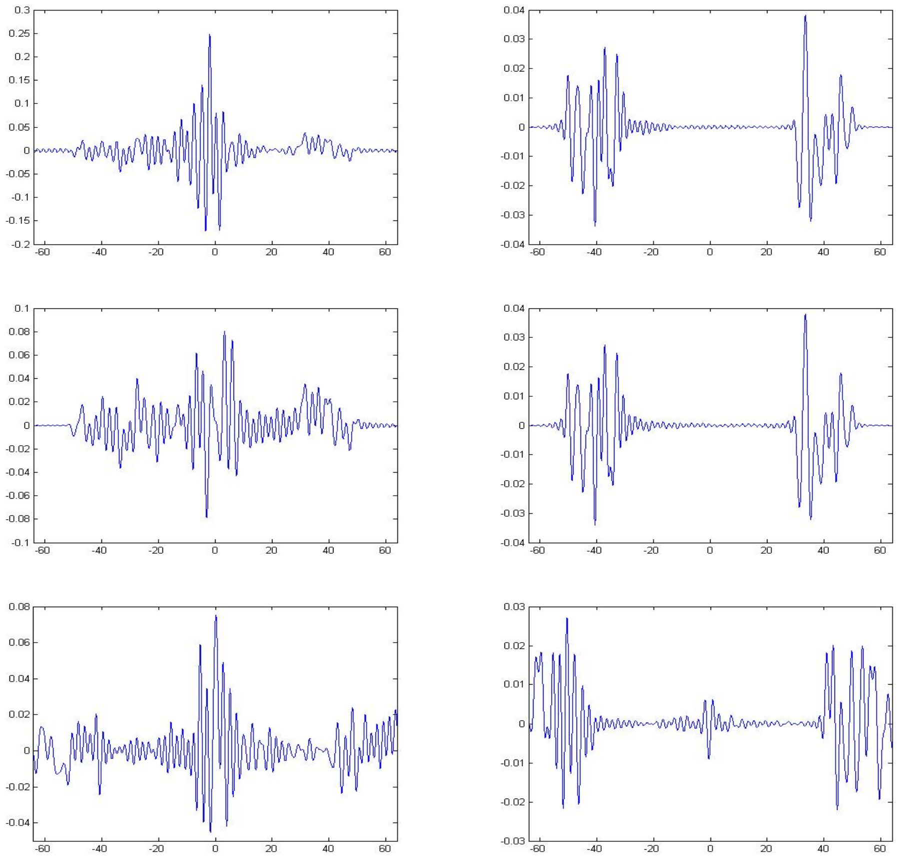


FIGURE 3. On the top left is the difference between the signal $x(\mathrm{sinc}, 0)$ in Figure 1 and its pre-reconstructed signal $S_{\Phi_0,\tilde{\Phi}_0,\Gamma_N}x(\mathrm{sinc}, 0)$, while on the top right is the difference between $x(\mathrm{sinc}, 0)$ and its Galerkin reconstruction $G_{\Phi_0,\tilde{\Phi}_0,\Gamma_N}x(\mathrm{sinc}, 0)$. The middle are differences $x(\mathrm{sinc}, 0) - S_{\Phi_0,\tilde{\Phi}_0,\Gamma_J}x(\mathrm{sinc}, 0)$ (left) and $x(\mathrm{sinc}, 0) - G_{\Phi_0,\tilde{\Phi}_0,\Gamma_J}x(\mathrm{sinc}, 0)$ (right) associated with jittered sampling. Listed below are differences $x(\mathrm{sinc}, 0) - S_{\Phi_0,\tilde{\Phi}_0,\Gamma_C}x(\mathrm{sinc}, 0)$ (left) and $x(\mathrm{sinc}, 0) - G_{\Phi_0,\tilde{\Phi}_0,\Gamma_C}x(\mathrm{sinc}, 0)$ (right) associated with adaptive C-TEM sampling.

Numerical stability of Galerkin reconstruction (5.1) could be reflected by the condition number $\mathrm{cond}_{\Gamma,\Theta}(\phi_0)$ of the square matrix

$$A_{\Phi_0,\tilde{\Phi}_0,\Gamma} = \Big(\sum_{n=1}^{N} \frac{\gamma_{n+1} - \gamma_{n-1}}{2} \phi_0(\gamma_n - i - \theta_i) \tilde{\phi}_0(\gamma_n - j) \Big)_{-L\le i,j\le L}.$$

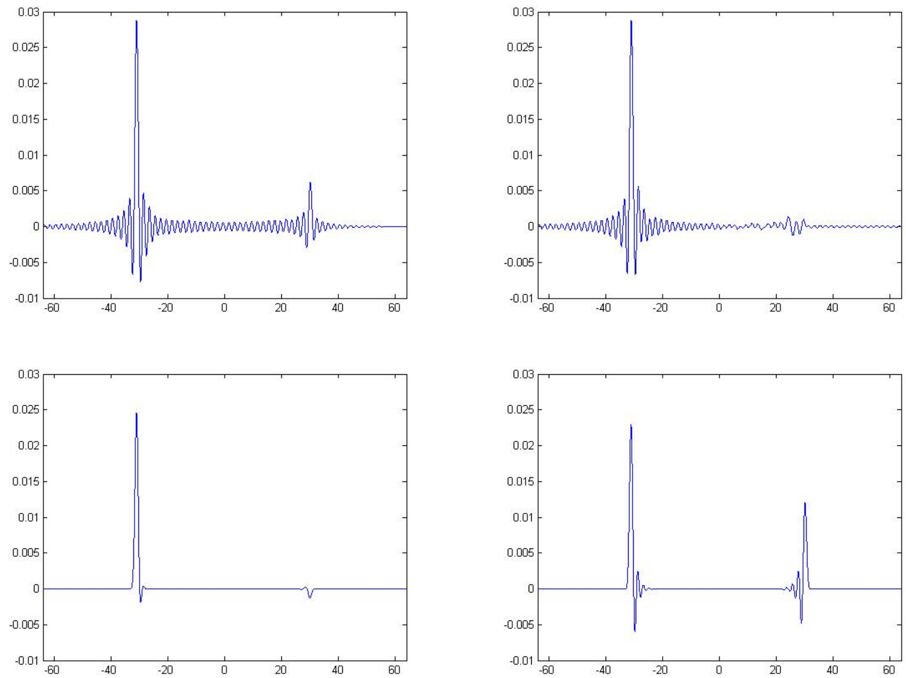


Figure 4. Listed are differences between best approximations of signals $x(\phi_0, 0)$ in $V_{2,30}(\Phi_0)$ and their Galerkin reconstructions associated with operators $S_{\Phi_0,\tilde{\Phi}_0,\Gamma}$, where on the above, $\phi_0 = \text{sinc}$, $\Gamma = \Gamma_N$(left) and $\Gamma = \Gamma_J$ (right), while on the bottom $\Gamma = \Gamma_N$, $\phi_0 = \text{gauss}$ (left) and $\phi_0 = \text{spline}$ (right).

Some numerical results of condition numbers $\text{cond}_{\Gamma,\Theta}(\phi_0)$ with $\Gamma = \Gamma_N$ or Γ_J, and $\Theta = \Theta_O$ or Θ_I, are presented in Table 2 with abbreviated notations. For the robust (sub-)Galerkin reconstruction, the generating function $\tilde{\phi}_0$ of the test space $V_{2,L}(\tilde{\Phi}_0)$ should be so chosen that the corresponding matrice $A_{\Phi_0,\tilde{\Phi}_0,\Gamma}$ is well-conditioned, c.f. Theorem 2.3.

We conclude this section with two more remarks.

Remark 5.1. The iterative approximation-projection algorithm (4.11) could have better performance on solving Galerkin equations (5.1), especially while matrices $A_{\Phi_0,\tilde{\Phi}_0,\Gamma}$ have large condition number, which is the case when the sampling set Γ and/or the shifting set Θ are not chosen appropriately.

Remark 5.2. For the admissibility of the pre-reconstruction operator $S_{\Gamma,\delta}$, the test space $\tilde{U}$ must have its dimension larger than or equal

TABLE 1. Quasi-optimality of Galerkin reconstructions for bandlimited/Gauss/spline signals

L	10	15	20	25	30
$e(\mathrm{sinc}, 0)$	0.2176	0.1711	0.1388	0.1166	0.1024
$\epsilon_N(\mathrm{sinc}, 0)$	0.0795	0.0668	0.0197	0.0201	0.0294
$\epsilon_J(\mathrm{sinc}, 0)$	0.0770	0.0668	0.0201	0.0214	0.0290
$\epsilon_C(\mathrm{sinc}, 0)$	0.0789	0.0715	0.0239	0.0263	0.0325
$e(\mathrm{sinc}, 1)$	0.2600	0.2124	0.1816	0.1457	0.1303
$\epsilon_N(\mathrm{sinc}, 1)$	0.0344	0.0809	0.0370	0.0294	0.0431
$\epsilon_J(\mathrm{sinc}, 1)$	0.0353	0.0806	0.0372	0.0301	0.0433
$\epsilon_C(\mathrm{sinc}, 1)$	0.0363	0.0831	0.0379	0.0319	0.0442
$e(\mathrm{sinc}, 2)$	0.2095	0.1703	0.1365	0.1167	0.1007
$\epsilon_N(\mathrm{sinc}, 2)$	0.0619	0.0618	0.0256	0.0163	0.0281
$\epsilon_J(\mathrm{sinc}, 2)$	0.0596	0.0618	0.0260	0.0177	0.0275
$\epsilon_C(\mathrm{sinc}, 2)$	0.0608	0.0664	0.0284	0.0226	0.0308
$e(\mathrm{sinc}, 3)$	0.2655	0.2180	0.1863	0.1477	0.1322
$\epsilon_N(\mathrm{sinc}, 3)$	0.0461	0.0810	0.0374	0.0258	0.0406
$\epsilon_J(\mathrm{sinc}, 3)$	0.0446	0.0809	0.0375	0.0265	0.0401
$\epsilon_C(\mathrm{sinc}, 3)$	0.0474	0.0837	0.0392	0.0298	0.0418
$e(\mathrm{gauss}, 0)$	0.2055	0.1682	0.1398	0.1250	0.1086
$\epsilon_N(\mathrm{gauss}, 0)$	0.0437	0.0515	0.0270	0.0158	0.0093
$\epsilon_J(\mathrm{gauss}, 0)$	0.0439	0.0523	0.0259	0.0160	0.0096
$\epsilon_C(\mathrm{gauss}, 0)$	0.0433	0.0527	0.0270	0.0181	0.0108
$e(\mathrm{spline}, 0)$	0.1482	0.1325	0.1110	0.0924	0.0664
$\epsilon_N(\mathrm{spline}, 0)$	0.0405	0.0298	0.0204	0.0266	0.0176
$\epsilon_J(\mathrm{spline}, 0)$	0.0403	0.0299	0.0204	0.0281	0.0184
$\epsilon_C(\mathrm{spline}, 0)$	0.0407	0.0292	0.0209	0.0279	0.0181

to the one of the reconstruction space U. For $U = V_{2,L}(\Phi_0)$ and $\tilde{U} = V_{2,\tilde{L}}(\tilde{\Phi}_0)$ with $\tilde{L} \geq L$, least square solutions of the linear system (5.1) with $-L \leq j \leq L$ replaced by $-\tilde{L} \leq j \leq \tilde{L}$ defines a sub-Galerkin reconstruction $\sum_{i=-L}^{L} c_i\phi_0(\cdot - i - \theta_i) \in V_{2,L}(\Phi_0)$ by Corollary 2.6, where $f \in V_2(\Phi_0)$ and $\Gamma := \Gamma_N, \Gamma_J, \Gamma_C$. Our numerical simulations show that the above sub-Galerkin reconstructions for different $\tilde{L} \geq L$ have comparable approximation errors.

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TABLE 2. Stability of Galerkin reconstructions for nonuniform/jittered sampling

L	10	15	20	25	30
$\text{cond}_{N,O}$(sinc)	1.2059	1.2367	1.3458	1.4273	1.2904
$\text{cond}_{N,I}$(sinc)	1.9190	1.8946	1.9828	2.0635	2.0421
$\text{cond}_{N,O}$(gauss)	3.0162	2.7000	2.7908	3.3314	2.8362
$\text{cond}_{N,I}$(gauss)	3.2850	3.1447	3.1421	4.0283	3.4391
$\text{cond}_{N,O}$(spline)	3.7677	3.7534	3.0534	3.1400	4.1708
$\text{cond}_{N,I}$(spline)	4.4768	5.2417	3.3507	3.5354	5.0292
$\text{cond}_{J,O}$(sinc)	1.3737	1.4164	1.4105	1.4149	1.3763
$\text{cond}_{J,I}$(sinc)	1.9723	1.9351	2.3328	2.2037	2.1744
$\text{cond}_{J,O}$(gauss)	2.7066	2.7074	2.6936	2.6957	2.7190
$\text{cond}_{J,I}$(gauss)	3.0847	3.1591	3.0696	3.0197	3.0878
$\text{cond}_{J,O}$(spline)	3.1052	3.2109	3.2218	3.3257	3.2331
$\text{cond}_{J,I}$(spline)	3.5570	3.7388	3.7140	3.9172	4.1830

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References

[1] B. Adcock, M. Gataric and A. C. Hansen, On stable reconstruction from univariate nonuniform Fourier measurements, arXiv1310.7820

[2] B. Adcock, M. Gataric and A. C. Hansen, Weighted frames of exponentials and stable recovery of multidimensional functions from nonuniform Fourier samples, arXiv:1405.3111

[3] B. Adcock, A. C. Hansen and C. Poon, Beyond consistent reconstructions: optimality and sharp bounds for generalized sampling, and application to the uniform resampling problem, *SIAM J. Math. Anal.*, **45**(2013), 3114–3131.

[4] A. Aldroubi and H. Feichtinger, Exact iterative reconstruction algorithm for multivariate irregularly sampled functions in spline-like spaces: the L_p theory. *Proc. Amer. Math. Soc.*, **126**(1998), 2677–2686.

[5] A. Aldroubi and K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant spaces, *SIAM Rev.*, **43**(2001), 585–620.

[6] A. Aldroubi, Q. Sun and W.-S. Tang, Non-uniform average sampling and reconstruction in multiply generated shift-invariant spaces, *Constr. Approx.*, **20**(2004), 173–189.

[7] A. Aldroubi, Q. Sun and W.-S. Tang, Convolution, average sampling, and a Calderon resolution of the identity for shift-invariant spaces, *J. Fourier Anal. Appl.*, **11**(2005), 215–244.

[8] A. G. Baskakov, Wiener's theorem and asymptotic estimates for elements of inverse matrices, *Funktsional Anal i Prilozhen*, **24**(1990), 64–65; translation in *Funct. Anal. Appl.*, **24**(1990), 222–224.

[9] P. Berger and K. Gröchenig, Sampling and reconstruction in different subspaces by using oblique projections, arXiv 1312.1717
[10] J. G. Christensen, Sampling in reproducing kernel Banach spacecs in Lie group, *J. Approx. Theor.*, **164**(2012), 179–203.
[11] P. L. Dragotti, M. Vetterli and T. Blu, Sampling moments and reconstructing signals of finite rate of innovation: Shannon meets Strans-Fix, *IEEE Trans. Signal Process.*, **55**(2007), 1741–1757.
[12] Y. C. Eldar and T. Werther, General framework for consistent sampling in Hilbert spaces, *Int. J. Wavelets Multiresolution Inf. Process.*, **3**(2005), 347–359.
[13] H. G. Feichtinger and K. Gröchenig, Iterative reconstruction of multivariate band-limited functions from irregular sampling values, *SIAM J. Math. Anal.*, **231**(1992), 244–261.
[14] H. G. Feichtinger, J. C. Principe, J. L. Romero, A. A. Singh, and G. A. Alexander, Approximate reconstruction of bandlimited functions for the integrate and fire sampler, *Adv. Comput. Math.*, **36**(2012), 67–78.
[15] A. G. Garcia and A. Portal, Sampling in reproducing kernel Banach spaces, *Mediterr. J. Math.*, **103**(2013), 1401–1417.
[16] I. Gohberg, M. A. Kaashoek and H. J. Woerdeman, The band method for positive and strictly contractive extension problems: an alternative version and new applications, *Integral Equation Oper. Theory*, **12**(1989), 343–382.
[17] D. Gontier and M. Vetterli, Sampling based on timing: time encoding machines on shift-invariant subspaces, *Appl. Computat. Harmonic Anal.*, **36**(2014), 63–78.
[18] K. Gröchenig, Wiener's lemma: theme and variations, an introduction to spectral invariance and its applications, In: *Four Short Courses on Harmonic Analysis: Wavelets, Frames, Time-Frequency Methods, and Applications to Signal and Image Analysis*, editor by P. Massopust and B. Forster, Birkhauser, Boston, 2010.
[19] K. Gröchenig, Reconstructing algorithms in irregular sampling, *Math. Comput.*, **59**(1992), 181–194.
[20] D. Han, M. Z. Nashed and Q. Sun, Sampling expansions in reproducing kernel Hilbert and Banach spaces, *Numer. Funct. Anal. Optim.*, **30**(2009), 971–987.
[21] J. A. Hogan and J. D. Lakey, *Duration and Bandwidth Limiting: Prolate Functions, Sampling, and Applications*, Birkhäuser, 2012.
[22] P. Jaming, A. Karoui, R. Kerman and S. Spektor, Approximation of almost time and band limited functions I: Hermite expansion, Arxiv 1407.1293
[23] A. A. Lazar and L. T. Toth, Perfect recovery and sensitivity analysis of time encoded bandlimited signals, *IEEE Trans. Circuits System*, **51**(2004), 2060–2073.
[24] M. Mishali, Y. C. Eldar and A. J. Elron, Xampling: signal acquisition and processing in union of subspaces, *IEEE Trans. Signal Process.*, **59**(2011), 4719–4734.
[25] M. Z. Nashed and Q. Sun, Sampling and reconstruction of signals in a reproducing kernel subspace of $L^p(R^d)$, *J. Funct. Anal.*, **258**(2010), 2422–2452.
[26] M. Z. Nashed and G. G. Walter, General sampling theorems for functions in reproducing kernel Hilbert spaces, *Math. Control Signals Systems*, **4**(1991), 363–390.

[27] H. Pan, T. Blu and P. L. Dragotti, Sampling curves with finite rate of innovation, *IEEE Trans. Signal Process.*, **62**(2014), 458–471.

[28] C. E. Shannon, Communication in the presence of noise, *Proc. IRE*, **37**(1949), 10–21.

[29] J. Sjöstrand, Wiener type algebra of pseudodifferential operators, Cent. Math., Ecole Polytechnique, Palaiseau France, Seminaire 1994, 1995, December 1994.

[30] Q. Sun, Wiener's lemma for infinite matrices, *Trans. Amer. Math. Soc.*, **359**(2007), 3099–3123.

[31] Q. Sun, Wiener's lemma for infinite matrices II, *Constr. Approx.*, **34**(2011), 209–235.

[32] Q. Sun, Frames in spaces with finite rate of innovation, *Adv. Comput. Math.*, **28**(2008), 301–329.

[33] Q. Sun, Non-uniform average sampling and reconstruction for signals with finite rate of innovations, *SIAM J. Math. Anal.*, **38**(2006), 1389–1422.

[34] Q. Sun and J. Xian, Rate of innovation for (non-)periodic signals and optimal lower stability bound for filtering, *J. Fourier Anal. Appl.*, **20**(2014), 119–134.

[35] W. Sun and X. Zhou, Reconstruction of bandlimited signals from local averages, *IEEE Trans. Inf. Theory*, **48**(2002), 2955–2963.

[36] W.-S. Tang, Oblique projections, biorthogonal Riesz bases and multiwavelets in Hilbert spaces, *Proc. Amer. Math. Soc.*, **128**(1999), 463–473..

[37] M. Unser, Sampling – 50 years after Shannon, *Proc. IEEE*, **88**(2000), 569–587.

[38] M. Vetterli, P. Marziliano and T. Blu, Sampling signals with finite rate of innovation, *IEEE Trans. Signal Process.*, **50**(2002), 1417–1428.

[39] J. M. Whittaker, *Interpolating Function Theory*, Cambridge University Press, London, 1935.

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