CONVOLUTION, AVERAGE SAMPLING, AND A CALDERON RESOLUTION OF THE IDENTITY FOR SHIFT-INVARIANT SPACES

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Abstract. In this paper, we study three interconnected inverse problems in shift invariant spaces: 1) the convolution/deconvolution problem; 2) the uniformly sampled convolution and the reconstruction problem; 3) the sampled convolution followed by sampling on irregular grid and the reconstruction problem. In all three cases, we study both the stable reconstruction as well as ill-posed reconstruction problems. We characterize the convolutors for stable deconvolution as well as those giving rise to ill-posed deconvolution. We also characterize the convolutors that allow stable reconstruction as well as those giving rise to ill-posed reconstruction from uniform sampling. The connection between stable deconvolution, and stable reconstruction from samples after convolution is subtle, as will be demonstrated by several examples and theorems that relate the two problems.

1. Introduction

The problem of sampling and reconstruction was used as a tool for constructing the discrete wavelet bases from the continuous wavelet transform [15, 22, 24]. It was also used in the theoretical development of certain inverse problems such as the moment problem [25]. Furthermore, the theory of bases and frames is intimately related to sampling theory as originally discussed by Duffin and Schaeffer in their seminal paper on non-harmonic Fourier series [17]. Numerical analysis, analog/digital and digital/analog conversions, digital signal and image processing, data compression, transmission and storage are all

Date: August 28, 2004.

1991 *Mathematics Subject Classification.* Primary 41A15,42C40, 46A35, 46E15, 46N99, 47B37.

Key words and phrases. Irregular sampling, Calderon resolution, frame, wavelets, amalgam spaces.

Research of the first author is supported in part by NSF grants DMS-0103104, and DMS-0139740. Research of the third author is supported in part by the Academic Research Fund No. R-146-000-025-112, National University of Singapore.

instances where the problem of sampling and reconstruction plays a fundamental role [3, 7, 9, 12, 13, 16, 19, 23, 29].

For the theory of discrete wavelet bases, the starting point is Calderon's resolution of the identity for $L^2 := L^2(\mathbb{R}^d)$:

$$f(x) = \int_0^\infty \int_{\mathbb{R}^d} \alpha(u,t)\psi_{u,t}(x) du \frac{dt}{t}, \tag{1.1}$$

where $\psi_{u,t}(x) = t^{-d}\psi(\frac{x-u}{t})$, and $\alpha(u,t) = \langle f, \psi_{u,t}\rangle$ [24]. The discrete wavelet bases are then a discretization on dyadic grids of the Calderon reproducing formula. The function $\alpha(u,t)$ can be viewed as a scale dependent convolution, i.e., for fixed scale t, $\alpha(u,t) = f * \eta$, where $\eta(x) = \overline{\psi(\frac{-x}{t})}$. Thus Calderon's resolution of the identity (1.1) can be interpreted as a scale dependent deconvolution.

In a similar fashion, our starting point is a convolution problem

$$f \to f * \psi_l, \quad l = 1, \cdots, s.$$

However, unlike Calderon's resolution of the identity, our underlying space is not the whole space L^2, but as is typical in sampling theory and many applications (see, for instance, [1, 5, 6, 10, 14, 16, 27, 28]), a shift-invariant subspace of L^2 of the form:

$$V^2(\Phi) = \Big\{ \sum_{j\in\mathbb{Z}^d} D(j)^T \Phi(\cdot - j) : D \in (\ell^2)^{(r)} \Big\} \tag{1.2}$$

for some vector function $\Phi = (\phi_1, \ldots, \phi_r)^T \in (L^2)^{(r)}$, where $D = (d_1, \ldots, d_r)^T$ is a vector sequence such that $d_i := \{d_i(j)\}_{j\in\mathbb{Z}^d} \in \ell^2$, i.e., $D \in (\ell^2)^{(r)}$. Thus $\sum_{j\in\mathbb{Z}^d} D(j)^T\Phi(\cdot - j) = \sum_{i=1}^r \sum_{j\in\mathbb{Z}^d} d_i(j)\phi_i(\cdot - j)$.

A prototypical example is the Paley-Wiener space (or space of band-limited functions) for which $r = 1$ and $\phi = \sin(\pi x)/\pi x$. Other prototypical spaces are B-spline spaces, in which $r = 1$ and $\phi = \beta^n$ is the B-spline of degree n. However, in both these two examples, the function ϕ generates a Riesz basis $\{\phi(\cdot - j) : j \in \mathbb{Z}^d\}$ for $V^2(\phi)$. Although the assumption that Φ generates a Riesz basis is reasonable, it is not necessarily satisfied in practice and it may not even be true that ϕ and its shifts generate a frame ([5, 8]). The only assumption that we will require on Φ is that the Gramian

$$G_\Phi(\xi) := \sum_{k\in\mathbb{Z}^d} \widehat{\Phi}(\xi + k)\overline{\widehat{\Phi}(\xi + k)}^T, \tag{1.3}$$

must be bounded:

$$G_\Phi(\xi) \le MI, \ a.e.\ \xi \in \mathbb{R}^d, \tag{1.4}$$

where I is the $r \times r$ identity matrix, M is a positive constant and where we use the notation $\widehat{f}$ to denote the Fourier transform of a tempered distribution f on $\mathbb{R}^d$ and $\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i\xi x}dx$ whenever f is integrable.

Recall that the Gramian matrix-function G_Φ is a semi-positive definitive Hermitian matrix for any $\xi \in \mathbb{R}^d$ so that the inequality $G_\Phi(\xi) \leq MI$ in (1.4) makes sense. An equivalent condition on the Gramian G_Φ is that its components $(G_\Phi)_{i,j},\ i,j = 1,...,r$, belong to L^∞. A simple calculation then shows that

$$\Big\| \sum_{i=1}^{r} \sum_{j\in\mathbb{Z}^d} d_i(j)\phi_i(\cdot - j)\Big\|_2^2 \leq \Big(\sup_{\xi\in\mathbb{R}^d} \sum_{1\leq i,i'\leq r} |(G_\Phi)_{ii'}(\xi)|\Big)\Big(\sum_{i=1}^{r} \|d_i\|_2^2\Big) < \infty$$

which implies that, under the condition (1.4), $V^2(\Phi)$ is a well-defined linear subspace of $L^2(\mathbb{R}^d)$. However, this condition does not imply that $V^2(\Phi)$ is closed since we do not assume that Φ generates a Riesz basis, or equivalently, we do not assume that $G_\Phi(\xi) \geq mI,\ a.e.\ \xi \in \mathbb{R}^d$, for some positive constant $m > 0$. Here we do not even assume that Φ generates a frame, or equivalently, that $mG_\Phi(\xi) \leq G_\Phi^2(\xi),\ a.e.\ \xi \in \mathbb{R}^d$ for some positive constant $m > 0$ (see Theorem A.1). Our weaker assumption in this paper on Φ, not implying that $V^2(\Phi)$ is a closed subspace of L^2, does not pose an inconvenience or limit our theory, as will be clarified in the results below.

The convolution problem we consider consists of convolving a function $f \in V^2(\Phi)$ with s functions $\psi_1, \cdots, \psi_s$, resulting in a vector $(f*\psi_1, \cdots, f*\psi_s)^T \in (L^2)^{(s)}$. We replace the resolution of the identity (1.1) by the requirement that any $f \in V^2(\Phi)$ can be recovered from the vector $(f*\psi_1, \cdots, f*\psi_s)^T$. Our first result is the characterization of the vectors $\Psi = (\psi_1, \cdots, \psi_s)^T$ such that the convolution operator $f \in V^2(\Phi) \to (f*\psi_1, \cdots, f*\psi_s)^T \in (L^2)^{(s)}$ has bounded inverse. We then characterize the vector function $\Psi = (\psi_1, \cdots, \psi_s)^T$ for which the corresponding convolution operators have inverses but not necessarily bounded. These convolution operators are common in signal processing and their study is crucial in deconvolution problems, where they are often called filters.

By sampling the output of a convolution operator on uniform or non-uniform grids $X = \{x_j : j \in J\} \subset \mathbb{R}^d$, where J is a countable index set, we obtain a sequence of numbers $\{f*\psi_l(x_j),\ l = 1, \cdots, s,\ x_j \in X\}$. The next problem is then to reconstruct the function f from the data $\{f*\psi_l(x_j),\ l = 1, \cdots, s,\ x_j \in X\}$, which is the problem of reconstruction from sampled convolutions. The more general problem of reconstruction from averages [1, 5, 28] $\{\langle f, \psi_{l,x_j}\rangle,\ l = 1, \cdots, s,\ x_j \in X\}$ will not be discussed in the context of this paper.

Although the connection between the problem of reconstruction from sampled convolutions and the problem of deconvolution is obvious, the connection between their solutions is subtle. For example, we will see in Example 3.8 that a convolution operator may have a bounded inverse while the reconstruction from sampled convolutions does not. Even more surprisingly, the sampled convolution problem may have a bounded inverse while the reconstruction from convolution may not, see Example 3.9. Under appropriate conditions though, the expected implications are satisfied as we will be developed in Theorem 3.10.

This paper is organized as follows. In Section 2 we introduce the convolution and deconvolution problems. We characterize the convolution operators on $V^2(\Phi)$ that have bounded inverses (Section 2.1). In Section 2.2, we also characterize those that have inverses but that are not necessarily bounded (or so called stable). In Section 2.3, we then show that, under some restriction on the convolution operators and the generator, the existence of an inverse implies its boundedness, and we give an example that shows that the restriction is necessary. Deconvolution formula is given in Section 2.4.

In Section 3, we discuss the problem of critical uniform sampling after convolution. We characterize the convolution operators on $V^2(\Phi)$ such that critical uniform sampling after convolution is sufficient for stable reconstruction. We also characterize those operators for which critical uniform sampling after convolution is sufficient for reconstruction, but without stability in general. We then give conditions on the generator Φ and the convolutor Ψ so that stability is a consequence of the existence of a reconstruction. A reconstruction formula is given in Section 3.3. In Example 3.8, we give an example in which a stable deconvolution operator does not allow a stable reconstruction if it is followed by a critical sampling. More surprisingly, in Example 3.9, we give an example showing that stable reconstruction of critically sampled convolution does not imply stable deconvolution. This peculiarity, however, can be removed by adding some extra conditions on the convolutor Ψ as shown in Theorem 3.10.

Section 4 is devoted to the connection between irregular sampling and reconstruction after convolution and the convolution deconvolution problem. It is proved that under sufficient regularity of the convolutor Ψ, the stability of the reconstruction from sampled convolution is enough for stable deconvolution. Moreover, stable deconvolution implies that reconstruction from samples is stable but only for sufficiently dense samples. The proof of the results are given in Section 5. Finally several new results on shift-invariant spaces that are used in our development are gathered in the Appendices. Appendix A contains a characterization of the closedness of the finitely generated shift-invariant space $V^2(\Phi)$, which plays an important role in the proof of Theorem 2.7. In Appendix B we give conditions under which W^2-L^2 norm equivalence (often used in sampling theory) is satisfied for some finitely generated shift-invariant spaces.

2. Calderon convolutors and deconvolution in shift-invariant spaces

In this section, we study Calderon convolution and deconvolution in shift-invariant spaces.

Definition 2.1. Let V be a shift-invariant subspace of L^2, and let $\psi_1, \dots, \psi_s$ be functions in L^2.

(1) We say that a vector of functions $\Psi = (\psi_1, \dots, \psi_s)^T$ forms a *Calderon convolutor* for V if i) the convolution operators induced by $\psi_1, \dots, \psi_s$ satisfy

$$\sum_{l=1}^{s} \|f * \psi_l\|_2 \le B\|f\|_2, \ \forall\, f \in V,$$

for some $B < \infty$ independent of f, and ii) the only function $f \in V$ satisfying $f * \psi_l = 0$, $0 \le l \le s$, is the zero function.

(2) The vector function $\Psi = (\psi_1, \dots, \psi_s)^T$ is said to form a *stable Calderon convolutor* for V if there exist positive constants A, B such that

$$A\|f\|_2 \le \sum_{l=1}^{s} \|f * \psi_l\|_2 \le B\|f\|_2, \ \forall\, f \in V. \tag{2.1}$$

From the definitions above, we see that stable Calderon convolutors for V induce operators from V to $(L^2)^{(s)}$ that have bounded inverses, while Calderon convolutors induce operators that have inverses that are not necessarily bounded.

2.1. **Stable Calderon convolutor.** For any $\Psi = (\psi_1, \dots, \psi_s)^T$ with $\widehat{\Psi} \in (L^\infty)^{(s)}$ and any $\Phi = (\phi_1, \dots, \phi_r)^T$ with $G_\Phi \in (L^\infty)^{(r\times r)}$, we define

$$G_\Phi^\Psi(\xi) := \sum_{l=1}^{s} \sum_{k\in\mathbb{Z}^d} \widehat{\Phi}(\xi+k)\overline{\widehat{\Phi}(\xi+k)^T}\big|\widehat{\psi}_l(\xi+k)\big|^2. \tag{2.2}$$

The following theorem describes equivalent characterizations of a stable Calderon convolutor Ψ for a finitely generated shift-invariant space $V^2(\Phi)$ and its L^2-closure.

Theorem 2.2. *Let $\Phi = (\phi_1, \ldots, \phi_r)^T$ satisfy $G_\Phi(\xi) \in (L^\infty)^{(r\times r)}$ and $\Psi = (\psi_1, \ldots, \psi_s)^T$ satisfy $\widehat{\Psi} \in (L^\infty)^{(s)}$. Then the following statements are equivalent:*

(i) *Ψ is a stable Calderon convolutor for the L^2-closure of the shift-invariant space $V^2(\Phi)$.*
(ii) *Ψ is a stable Calderon convolutor for $V^2(\Phi)$.*
(iii) *There exists a positive constant m so that*

$$mG_\Phi(\xi) \le G_\Phi^\Psi(\xi) \quad \text{a.e. } \xi \in \mathbb{R}^d. \tag{2.3}$$

Remark 2.3. In Theorem 2.2, the generator Φ and its shifts do not necessarily generate a Riesz basis or a frame. Thus, $V^2(\Phi)$ need not be a closed subspace of L^2. On the other hand, the L^2-closure of the shift-invariant space $V^2(\Phi)$ is always a shift-invariant space $V^2(\Theta)$ engendered by a vector function $\Theta = (\theta_1, \cdots, \theta_r)$ that generates a tight frame (see Lemma 2.9 below). However, in general Θ does not have compact support even if Φ has (see Remark 2.10). More generally, Θ need not be in the Wiener amalgam space W^1 (see (4.2) for its definition) even if Φ belongs to W^1, an assumption often needed in convolution or in sampling theory (see Remark 2.11).

Remark 2.4. Equation (2.3) implies that $\operatorname{Rank} G_\Phi(\xi) \le \operatorname{Rank} G_\Phi^\Psi(\xi)$ for almost all $\xi \in \mathbb{R}^d$. Moreover, in the proof of Theorem 2.5 it is shown that it is always true that $\operatorname{Rank} G_\Phi^\Psi(\xi) \le \operatorname{Rank} G_\Phi(\xi)$. Thus, if Ψ is a stable Calderon convolutor for $V^2(\Phi)$, then $\operatorname{Rank} G_\Phi(\xi) = \operatorname{Rank} G_\Phi^\Psi(\xi)$. Equality of ranks does not imply that Ψ is a stable Calderon convolutor in general, but it implies that Ψ is a Calderon convolutor as stated in the next Theorem.

2.2. **Calderon Convolutor.**

Theorem 2.5. *Let $\Phi = (\phi_1, \ldots, \phi_r)^T$ satisfy $G_\Phi \in (L^\infty)^{(r\times r)}$ and $\Psi = (\psi_1, \ldots, \psi_s)^T$ satisfy $\widehat{\Psi} \in (L^\infty)^{(s)}$. Then the following statements are equivalent to each other.*

(i) *Ψ is a Calderon convolutor for the L^2-closure of $V^2(\Phi)$.*
(ii) *Ψ is a Calderon convolutor for $V^2(\Phi)$.*
(iii) *The matrices $G_\Phi(\xi)$ and $G_\Phi^\Psi(\xi)$ have the same rank for almost all $\xi \in \mathbb{R}^d$.*

As a consequence of Theorem 2.5, we obtain a necessary condition on the support of $\widehat{\Psi}$. Specifically, for a vector-valued function $\Psi =$

$(\psi_1, \ldots, \psi_s)^T$, define the *periodic supporting set* of its Fourier transform by

$$\mathrm{psupp}(\widehat{\Psi}) = \cup_{l=1}^{s} \cup_{k\in\mathbb{Z}^d} (\mathrm{supp}\widehat{\psi_l} + k),$$

where $\mathrm{supp} f$ is the support of a measurable function f. Clearly $\xi \notin \mathrm{psupp}(\widehat{\Psi})$ if and only if $\widehat{\psi_l}(\xi + k) = 0$ for all $k \in \mathbb{Z}^d$ and $1 \le l \le s$. We have the following corollary of Theorem 2.5:

Corollary 2.6. *Let $\Phi = (\phi_1, \ldots, \phi_r)^T$ satisfy $G_\Phi \in (L^\infty)^{(r\times r)}$ and $\Psi = (\psi_1, \ldots, \psi_s)^T$ satisfy $\widehat{\Psi} \in (L^\infty)^{(s)}$. If Ψ is a Calderon convolutor for $V^2(\Phi)$, then*

$$\mathrm{supp}(G_\Phi) \subset \mathrm{psupp}(\widehat{\Psi}).$$

2.3. **Deconvolution and stable deconvolution.** Clearly, stable deconvolution is stronger than simple deconvolution. Thus a stable Calderon convolutor is a Calderon convolutor; however the converse is not true in general. For example, when $r = s = 1, \phi_1 = \psi_1 = \chi_{[0,1]} - \chi_{[1,2]}$, we have $G_{\phi_1}(\xi) = |1 - e^{-2i\pi\xi}|^2$ and $G_{\phi_1}^{\psi_1}(\xi) = \frac{1}{3}|1 - e^{-2i\pi\xi}|^4(2 + \cos 2\pi\xi)$. Thus the ranks of $G_{\phi_1}(\xi)$ and $G_{\phi_1}^{\psi_1}(\xi)$ are the same for all $\xi \in \mathbb{R}$, but there does not exist a positive constant m so that $mG_{\phi_1}(\xi) \le G_{\phi_1}^{\psi_1}(\xi)$ for almost all $\xi \in \mathbb{R}$, since $G_{\phi_1}(\xi) = O(\xi^2)$ while $G_{\phi_1}^{\psi_1}(\xi) = O(\xi^4)$ near $\xi = 0$. Hence ψ_1 is a Calderon convolutor for $V^2(\phi_1)$ but is not a stable Calderon convolutor. However, under additional assumptions, rank equality between the matrices G_Φ and G_Φ^Ψ implies stable recovery of any function $f \in V^2(\Phi)$ from the convolution $f * \psi_l, 1 \le l \le s$:

Theorem 2.7. *Let $\Phi = (\phi_1, \ldots, \phi_r)^T$ satisfy $G_\Phi \in (L^\infty)^{(r\times r)}$ and $\Psi = (\psi_1, \ldots, \psi_s)^T$ satisfy $\widehat{\Psi} \in (L^\infty)^{(s)}$. Assume that G_Φ and G_Φ^Ψ are continuous functions on $\mathbb{R}^d$, and that $V^2(\Phi)$ is a closed subspace of L^2. Then Ψ is a stable Calderon convolutor for $V^2(\Phi)$ if and only if the matrices $G_\Phi(\xi)$ and $G_\Phi^\Psi(\xi)$ have the same rank for all $\xi \in \mathbb{R}^d$.*

Remark 2.8. The extra assumption that $V^2(\Phi)$ is a closed subspace of L^2 is satisfied if, for example, $mI \le G_\Phi \le MI$ for some constants $M, m > 0$ (see also Theorem A).

2.4. **Deconvolution formula.** To establish the deconvolution formula from Calderon convolution, we need the following result about the generators of a shift-invariant space ([11]).

Lemma 2.9. *Let $\Phi = (\phi_1, \ldots, \phi_r)^T$ satisfy $G_\Phi \in (L^\infty)^{(r\times r)}$. Then there exist $\theta_1, \ldots, \theta_r$ in the L^2-closure V of $V^2(\Phi)$ such that $\{\theta_l(\cdot - k)$:*

$1 \le l \le r, k \in \mathbb{Z}^d\}$ is a normalized tight frame of V, that is,

$$f = \sum_{l=1}^{r} \sum_{k \in \mathbb{Z}^d} \langle f, \theta_l(\cdot - k)\rangle \theta_l(\cdot - k), \quad \forall\, f \in V. \tag{2.4}$$

Furthermore, the generators $\theta_1, \ldots, \theta_r$ of the L^2-closure of $V^2(\Phi)$ can be so chosen that

$$\sum_{k \in \mathbb{Z}^d} \widehat{\theta}_i(\xi + k)\overline{\widehat{\theta}_{i'}(\xi + k)} = \delta_{ii'}\chi_{E_i}(\xi), \quad 1 \le i, i' \le r, \tag{2.5}$$

where

$$E_i = \{\xi \in \mathbb{R}^d : \text{ rank of } G_\Phi(\xi) \ge i\}, \quad 1 \le i \le r. \tag{2.6}$$

Remark 2.10. Write

$$G_\Phi(\xi) = A(\xi)\text{diag}(\lambda_1(\xi), \ldots, \lambda_r(\xi))\overline{A(\xi)}^T$$

for some unitary matrix $A(\xi)$, where $\lambda_1(\xi) \ge \ldots \ge \lambda_r(\xi) \ge 0$. Then one may verify that the functions $\theta_1, \ldots, \theta_r$ defined by

$$\widehat{\Theta}(\xi) = \text{diag}(\mu_1(\xi), \ldots, \mu_r(\xi))\overline{A(\xi)}^T \widehat{\Phi}(\xi)^T$$

satisfy all requirement in Lemma 2.9, where $\Theta = (\theta_1, \cdots, \theta_r)^T$ and

$$\mu_i(\xi) = \begin{cases} (\lambda_i(\xi))^{-1/2}, & \text{if } \lambda_i(\xi) \ne 0, \\ 0, & \text{if } \lambda_i(\xi) = 0. \end{cases}$$

The generator Θ in Lemma 2.9 does not necessarily have compact support in general even if Φ has. In fact, in one dimensional case, if the functions $\theta_1, \ldots, \theta_r$ in Lemma 2.9 can be chosen to have compact support, then there exist compactly supported functions $\tilde{\theta}_1, \ldots, \tilde{\theta}_{r'} \in V$ for some $1 \le r' \le r$ so that their shifts form an orthonormal basis of the shift-invariant space V ([20]). We believe that the above result is also true for high dimensions, but we have difficulty in its justification.

Remark 2.11. The generator Θ in Lemma 2.9 need not be in W^1 even if Φ belongs to W^1, an assumption often needed in convolution or in sampling theory. For instance, let $r = 1$ and ϕ_1 be a Schwartz function so that $\widehat{\phi_1}(\xi) > 0$ for all $\xi \in (-1/2, 1/2)$ and $\widehat{\phi_1}(\xi) = 0$ for all $\xi \in \mathbb{R}\backslash(-1/2, 1/2)$. Then ϕ_1 is a continuous function in W^1 and $\phi_1(x) = \sum_{j \in \mathbb{Z}} c(j)\text{sinc}(x - j)$, where the sinc-function is defined by $\text{sinc}(x) = \frac{\sin \pi x}{\pi x}$, and the sequence $\{c(j)\}$ is defined by $\sum_{j \in \mathbb{Z}} c(j)e^{-i2\pi j\xi} = \sum_{k \in \mathbb{Z}} \widehat{\phi}_1(\xi + k)$. Let B_π be the space of all band-limited L^2-functions. Then $V^2(\phi_1)$ has a *smooth* generator and is a *dense* subspace of the space B_π. Suppose, on the contrary, that there exist functions $h_1,\dots,h_s\in W^1$ so that $\{h_l(\cdot-j): j\in\mathbb{Z}, 1\le l\le s\}$ is a tight frame of B_π. By the tight frame property of $h_1,\dots,h_s$, we have

$$\mathrm{sinc}(x)=\sum_{l=1}^s\sum_{j\in\mathbb{Z}}\langle \mathrm{sinc}, h_l(\cdot-j)\rangle h_l(x-j)=\sum_{l=1}^s\sum_{j\in\mathbb{Z}}\overline{h_l(-j)}h_l(\cdot-j).$$

Taking Fourier transform at both sides of the above equation leads to

$$\chi_{[-1/2,1/2]}(\xi)=\sum_{l=1}^s\Big(\sum_{j\in\mathbb{Z}}\overline{h_l(-j)}e^{-ij\xi}\Big)\widehat{h}_l(\xi),$$

which is a contradiction since the left hand side is discontinuous, while the right hand side is continuous.

We start to establish the deconvolution formula. Let $\Phi=(\phi_1,\dots,\phi_r)^T$ satisfy $G_\Phi\in(L^\infty)^{(r\times r)}$, and let $\Psi=(\psi_1,\dots,\psi_s)^T$ be a stable Calderon convolutor for the shift-invariant space $V^2(\Phi)$ and satisfy $\widehat{\Psi}\in L^\infty$. By Lemma 2.9, we can select a generator $\Theta=(\theta_1,\dots,\theta_r)^T$ of the L^2-closure of $V^2(\Phi)$ so that

$$G_\Theta(\xi)=\mathrm{diag}(\chi_{E_1}(\xi),\dots,\chi_{E_r}(\xi)) \tag{2.7}$$

for some measurable sets $E_1,\dots,E_r$. By Theorem 2.2, we can find a $r\times r$ matrix $A(\xi)=(a_{ii'}(\xi))_{1\le i,i'\le r}$ with bounded $\mathbb{Z}^d$-periodic entries so that

$$G_\Theta^\Psi(\xi)A(\xi)=G_\Theta(\xi). \tag{2.8}$$

For any $f\in V^2(\Phi)\subset V^2(\Theta)$, we denote $g_l=f*\psi_l, 1\le l\le s$, and write

$$\widehat{f}(\xi)=\sum_{i=1}^r c_i(\xi)\widehat{\theta}_i(\xi)$$

for some square-integrable $\mathbb{Z}^d$-periodic functions $c_i(\xi), 1\le i\le r$. Then the functions $\tilde\psi_{l,i}$, defined by

$$\widehat{\tilde\psi}_{l,i}=\widehat{\psi}_l(\xi)\sum_{i'=1}^r\overline{a_{i'i}(\xi)}\widehat{\theta}_{i'}(\xi),$$

satisfy

$$\begin{aligned}&\sum_{l=1}^s\sum_{k\in\mathbb{Z}^d}\widehat{g}_l(\xi+k)\overline{\widehat{\tilde\psi}_{l,i}(\xi+k)}\\
=&\sum_{i',i''=1}^r c_{i'}(\xi)\Big(\sum_{l=1}^s\sum_{k\in\mathbb{Z}^d}\widehat{\theta}_{i'}(\xi+k)\overline{\widehat{\theta}_{i''}(\xi+k)}|\widehat{\psi}_l(\xi+k)|^2\Big)a_{i''i}(\xi)\\
=& c_i(\xi)\chi_{E_i}(\xi),\end{aligned}$$

where we have used (2.7) and (2.8) to obtain the last equality. This yields the following deconvolution formula from Calderon convolutions:

$$f = \sum_{l=1}^{s} \sum_{j \in \mathbb{Z}^d} \langle f * \psi_l, \tilde{\psi}_{l,i}(\cdot - j)\rangle \theta_i(\cdot - j), \quad f \in V^2(\Phi). \tag{2.9}$$

3. Uniform Average Sampling

In this section, we study the problem of average sampling and the recovery of functions in shift-invariant spaces from a set of average sampled values. We show that in general a stable Calderon convolutor is not a stable uniform averaging sampler. In the opposite direction, we also show that a stable uniform averaging sampler is not necessarily a stable Calderon convolutor except under some appropriate conditions. In addition, we show that for Ψ to be a stable uniform averaging sampler, it often must have "better characteristics" than the generator Φ of V.

Definition 3.1. Let V be a shift-invariant subspace of L^2.

(1) We say that $\Psi = (\psi_1, \dots, \psi_s)^T$ is a *stable uniform averaging sampler* for V if there exist positive constants A, B such that

$$A\|f\|_2 \le \sum_{l=1}^{s} \Big(\sum_{j \in \mathbb{Z}^d} |f * \psi_l(j)|^2 \Big)^{1/2} \le B\|f\|_2 \quad \text{for all } f \in V. \tag{3.1}$$

(2) We say that $\Psi = (\psi_1, \dots, \psi_s)^T$ is a *determining uniform averaging sampler* for V if the only function $f \in V$, satisfying $f * \psi_l(k) = 0$ for all $1 \le l \le s$ and $k \in \mathbb{Z}^d$, is the zero function.

From the definitions above, we see that if Ψ is a stable uniform average sampler for V, then any function $f \in V$ can be recovered in a stable way from the average values $\{\langle f, \overline{\psi_l(j - \cdot)}\rangle : \ j \in \mathbb{Z}^d, 1 \le l \le s\}$, *i.e.*, the average sampling operator has a bounded inverse. The determining averaging sampler can distinguish between two distinct functions f_1, f_2 in V, but the inverse or recovery is not necessarily stable. Thus a stable uniform averaging sampler is also determining but the converse is not true in general.

3.1. Stable uniform averaging sampler.

For any $\Phi = (\phi_1, \dots, \phi_r)^T$ with $G_\Phi \in (L^\infty)^{(r\times r)}$ and $\Psi = (\psi_1, \dots, \psi_s)^T$ with $G_\Psi \in (L^\infty)^{(s\times s)}$, we

define
(3.2)

$$A_\Phi^\Psi(\xi) := \sum_{l=1}^{s} \Big(\sum_{k\in\mathbb{Z}^d} \widehat{\Phi}(\xi+k)\widehat{\psi_l}(\xi+k)\Big) \overline{\Big(\sum_{k'\in\mathbb{Z}^d} \widehat{\Phi}(\xi+k')\widehat{\psi_l}(\xi+k')\Big)}^T.$$

Using the above definition, we obtain equivalent characterizations of a stable uniform averaging sampler Ψ:

Theorem 3.2. *Let $\Phi = (\phi_1,\ldots,\phi_r)^T$ and $\Psi = (\psi_1,\ldots,\psi_s)^T$ satisfy $G_\Phi \in (L^\infty)^{(r\times r)}$ and $G_\Psi \in (L^\infty)^{(s\times s)}$. Then the following statements are equivalent to each other:*

(i) *Ψ is a stable uniform averaging sampler for the L^2-closure of $V^2(\Phi)$.*
(ii) *Ψ is a stable uniform averaging sampler for $V^2(\Phi)$.*
(iii) *There exists a positive constant m such that*

$$m\, G_\Phi(\xi) \le A_\Phi^\Psi(\xi) \quad \text{a.e. } \xi \in \mathbb{R}^d. \tag{3.3}$$

From the definition in (3.2), we see that

$$A_\Phi^\Psi(\xi) = \sum_{l=1}^{s} a_l(\xi)\overline{a_l(\xi)}^T,$$

where $a_l(\xi) = \sum_{k\in\mathbb{Z}^d} \widehat{\Phi}(\xi+k)\widehat{\psi_l}(\xi+k)$. Thus, the rank of the matrix $A_\Phi^\Psi(\xi)$ is at most s for any $\xi\in\mathbb{R}^d$. Let

$$r_{\max} := \max\{l : \text{measure}\{\xi : \text{rank } G_\Phi(\xi) = l\} > 0\},$$

then by Theorem 3.2, we see that the length s of a stable averaging sampler Ψ for the shift-invariant space $V^2(\Phi)$ is at least $r_{\max}$. If a stable averaging sampler Ψ has minimal length, that is, $s = r_{\max}$, then it has a Riesz property as described in the following Theorem:

Theorem 3.3. *Let $\Phi = (\phi_1,\ldots,\phi_r)^T$ and $\Psi = (\psi_1,\ldots,\psi_s)^T$ satisfy $G_\Phi \in (L^\infty)^{(r\times r)}$ and $G_\Psi \in (L^\infty)^{(s\times s)}$. Assume that the rank of the matrix $G_\Phi(\xi)$ is s for almost all $\xi\in\mathbb{R}^d$, and that Ψ is a stable averaging sampler for $V^2(\Phi)$. Then Ψ generates a Riesz basis, that is, $\{\psi_l(\cdot - j) : j\in\mathbb{Z}^d, 1\le l\le s\}$ is a Riesz basis of the shift-invariant space $V^2(\Psi)$.*

As an application of Theorem 3.3, we have the following result about the averaging sampler.

Corollary 3.4. *Let Φ and Ψ be scalar-valued compactly supported L^2 functions. If Ψ is a stable averaging sampler for $V^2(\Phi)$, then $\{\Psi(\cdot - j) : j\in\mathbb{Z}^d\}$ is a Riesz basis of $V^2(\Psi)$.*

Remark 3.5. Equation (3.3) implies that $\mathrm{Rank}\, G_\Phi(\xi) \le \mathrm{Rank}\, A_\Phi^\Psi(\xi)$ for almost all $\xi \in \mathbb{R}^d$. Moreover, it is not difficult to show that $\mathrm{Rank}\, A_\Phi^\Psi(\xi) \le \mathrm{Rank}\, G_\Phi(\xi)$. Thus, if Ψ is a stable uniform averaging sampler for $V^2(\Phi)$, then $\mathrm{Rank}\, G_\Phi(\xi) = \mathrm{Rank}\, A_\Phi^\Psi(\xi)$. Equality of ranks does not imply that Ψ is a stable uniform averaging sampler in general. However, it does imply that Ψ is a determining uniform averaging sampler as in the next Theorem.

3.2. Determining uniform average sampler.

Theorem 3.6. *Let $\Phi = (\phi_1, \dots, \phi_r)^T$ and $\Psi = (\psi_1, \dots, \psi_s)^T$ satisfy $G_\Phi \in (L^\infty)^{(r\times r)}$ and $G_\Psi \in (L^\infty)^{(s\times s)}$. Then the following three statements are equivalent:*

(i) *Ψ is a determining uniform average sampler for the L^2-closure of $V^2(\Phi)$.*
(ii) *Ψ is a determining uniform averaging sampler for $V^2(\Phi)$.*
(iii) *The matrices $G_\Phi(\xi)$ and $A_\Phi^\Psi(\xi)$ have the same rank for almost all $\xi \in \mathbb{R}^d$.*

Similar to the situation of a stable Calderon convolutor and Calderon convolutor, a stable averaging sampler is a determining averaging sampler, but the converse is not true in general. For instance, when $r = s = 1, \phi_1 = \chi_{[0,1]} - \chi_{[1,2]}$ and $\psi_1 = \phi_1(-\cdot)$, $G_{\phi_1}(\xi) = |1 - e^{-i2\pi\xi}|^2$ and $A_{\phi_1}^{\psi_1}(\xi) = |1 - e^{-2i\pi\xi}|^4$. For this case, the rank of $G_{\phi_1}(\xi)$ and $A_{\phi_1}^{\psi_1}(\xi)$ are equal for all $\xi \in \mathbb{R}$. However there does not exist a positive constant m so that $mG_{\phi_1}(\xi) \le A_{\phi_1}^{\psi_1}(\xi)$ for all $\xi \in \mathbb{R}$, since $G_{\phi_1}(\xi) = O(\xi^2)$ while $A_{\phi_1}^{\psi_1}(\xi) = O(\xi^4)$ near the origin. Thus ψ_1 is a determining sampler for $V^2(\phi_1)$, but not a stable sampler for $V^2(\phi_1)$ by Theorems 3.2 and 3.6. Parallel to Theorem 2.7, we show in the following theorem that under additional assumptions, equality in ranks of the matrices G_Φ and A_Φ^Ψ implies stable recovery from average sampling:

Theorem 3.7. *Let $\Phi = (\phi_1, \dots, \phi_r)^T$ and $\Psi = (\psi_1, \dots, \psi_s)^T$ satisfy $G_\Phi \in (L^\infty)^{(r\times r)}$ and $G_\Psi \in (L^\infty)^{(s\times s)}$. Assume that G_Φ and A_Φ^Ψ are continuous functions on $\mathbb{R}^d$, and that $V^2(\Phi)$ is a closed subspace of L^2. Then Ψ is a stable average sampler for $V^2(\Phi)$ if and only if the matrices $G_\Phi(\xi)$ and $A_\Phi^\Psi(\xi)$ have the same rank for all $\xi \in \mathbb{R}^d$.*

3.3. **Reconstruction Formula.** Let $\Phi = (\phi_1, \dots, \phi_r)^T$ satisfy $G_\Phi \in (L^\infty)^{(r\times r)}$, and let $\Psi = (\psi_1, \dots, \psi_s)^T$ be a stable averaging sampler for the shift-invariant space $V^2(\Phi)$ and satisfy $G_\Psi \in (L^\infty)^{(s\times s)}$. Let $\Theta = (\theta_1, \dots, \theta_r)$ be the generators of the L^2-closure of the shift-invariant space $V^2(\Phi)$ so that

$$G_\Theta(\xi) = \mathrm{diag}(\chi_{E_1}(\xi), \dots, \chi_{E_r}(\xi))$$

for some measurable sets $E_1, \ldots, E_r$, and let $B(\xi) = (b_{ii'}(\xi))_{1\le i,i'\le r}$ with bounded measurable entries be so chosen that

$$A_\Theta^\Psi(\xi)B(\xi) = G_\Theta(\xi).$$

The existences of such a generator Θ and matrix $B(\xi)$ follow from Lemma 2.9 and Theorem 3.2 respectively. Define

$$d_{li}(\xi) = \sum_{i'=1}^r \Big(\overline{\sum_{k\in\mathbb{Z}^d} \widehat{\theta}_{i'}(\xi+k)\widehat{\psi}_l(\xi+k)}\Big) b_{i'i}(\xi), \quad 1\le l\le s, 1\le i\le r.$$

Then $d_{li}, 1\le l\le s, 1\le i\le r$, are bounded measurable $\mathbb{Z}^d$-periodic functions. Moreover, recalling that any function f in $V^2(\Phi)$ has the following expression in the Fourier domain,

$$\widehat{f}(\xi) = \sum_{i=1}^r c_i(\xi)\widehat{\theta}_i(\xi)$$

for some square-integrable $\mathbb{Z}^d$-periodic functions $c_1, \ldots, c_r$, we have

$$\begin{aligned} &\sum_{l=1}^s \mathcal{F}(\{(f*\psi_l)(j)\})(\xi) d_{li}(\xi) \\ = &\sum_{i',i''=1}^r c_{i'}(\xi)\Big(\sum_{l=1}^s \Big(\sum_{k\in\mathbb{Z}^d} \widehat{\theta}_{i'}(\xi+k)\widehat{\psi}_l(\xi+k)\Big) \\ &\qquad \times \overline{\Big(\sum_{k'\in\mathbb{Z}^d} \widehat{\theta}_{i''}(\xi+k')\widehat{\psi}_l(\xi+k')\Big)}\Big) b_{i''i}(\xi) \\ = & \ c_i(\xi)\chi_{E_i}, \ 1\le i\le r. \end{aligned}$$

Here for summable sequence $c = \{c(j)\}$, $\mathcal{F}(c)$ denotes its Fourier series, which is defined by $\mathcal{F}(c) = \sum_{j\in\mathbb{Z}^d} c(j)e^{-ij\xi}$. Multiplying $\widehat{\theta}_i(\xi)$ at both sides of the above equation, and summing over i from 1 to r leads to

$$\widehat{f}(\xi) = \sum_{i=1}^r \sum_{l=1}^s \mathcal{F}(\{(f*\psi_l)(j)\})(\xi) d_{li}(\xi)\widehat{\theta}_i(\xi).$$

Finally, taking inverse Fourier transform of the above equation and letting $\{d_{li}(j)\}$ be the Fourier coefficients of the square-integrable $\mathbb{Z}^d$-periodic function $d_{li}(\xi), 1\le l\le s, 1\le i\le r$, we obtain the following reconstruction formula from the average sampling values,

$$f = \sum_{i=1}^r \sum_{l=1}^s \sum_{j\in\mathbb{Z}^d} c_{li}(j)\theta_i(\cdot - j), \ f\in V^2(\Phi), \tag{3.4}$$

where the sequence $\{c_{li}(j)\}$ is the convolution between the sequences $\{(f*\psi_l)(j)\}$ and $\{d_{li}(j)\}$.

3.4. **A stable Calderon convolutor which is not a stable averaging sampler.** It is not surprising that a stable Calderon convolutor Ψ is not necessarily a stable averaging sampler. A more surprising result is that there exists a stable Calderon convolutor ψ for a shift-invariant space $V^2(\phi)$ such that $\psi(\cdot - x_0)$ is not a stable averaging sampler for *any* $x_0 \in \mathbb{R}$, as illustrated in the following example:

Example 3.8. Let ϕ and ψ be defined by $\widehat{\phi}(\xi) = \chi_{[-3/2,3/2]}$ and

$$\widehat{\psi}(\xi) = \begin{cases} g(\xi), & \xi \in [-1/2, 1/2], \\ 1, & \xi \in [-3/2, -1/2] \cup [1/2, 3/2], \\ 0, & \text{otherwise}, \end{cases}$$

where g is a symmetric continuous function on $[-1/2, 1/2]$ so that $\max g = 3$, $\min g = -3$ and $g(-1/2) = g(1/2) = 1$. For any $x_0 \in [0, 1]$, let $\xi_0 \in (-1/2, 1/2)$ be so chosen that $g(\xi_0) + 2\cos 2\pi x_0 = 0$. The existence of ξ_0 follows from the definition of g. Note that

$$\sum_{k\in\mathbb{Z}} \widehat{\phi}(\xi_0 + k)\widehat{\psi}(\xi_0 + k)e^{-i2\pi x_0(\xi_0+k)} = e^{-i2\pi x_0\xi_0}(2\cos 2\pi x_0 + g(\xi_0)) = 0.$$

Then $A_\phi^\psi(\xi_0) = 0$ and $A(\xi)$ is continuous ξ_0, while $G_\phi(\xi) \geq 1$ for all $\xi \in \mathbb{R}$. Thus, $\psi(\cdot - x_0)$ is not a stable average sampler for any $x_0 \in \mathbb{R}$ by Theorem 3.2. On the other hand,

$$G_\phi^\psi(\xi) = \sum_{k\in\mathbb{Z}} |\widehat{\phi}(\xi + k)|^2 |\widehat{\psi}(\xi + k)|^2 \geq 1, \quad \xi \in \mathbb{R},$$

and $G_\phi(\xi) \leq 2$ for all $\xi \in \mathbb{R}$. It follows that ψ is a stable Calderon convolutor for $V^2(\phi)$ by condition (iii) of Theorem 2.2.

3.5. **A stable averaging sampler which is not a stable Calderon convolutor.** One would expect that a stable averaging sampler Ψ for $V^2(\Phi)$ is a stable Calderon convolutor, since we can always sample the image of a Calderon convolution and use the samples for the recovery. Surprisingly, this is not the case as demonstrated by the following example:

Example 3.9. Let $E_n, n \geq 1$, be a partition of $[-1/2, 1/2)$ with $E_n = -E_n$ and $|E_n| > 0$ for all $n \geq 1$. Define ϕ by

$$\widehat{\phi}(\xi) = \begin{cases} (2n+1)^{-1/2}, & \text{if } \xi \in \cup_{j=-n}^{n}(E_n + j), \\ 0, & \text{otherwise}, \end{cases}$$

and $\psi = \phi(-\cdot)$. Then ϕ has orthonormal shifts, since $\cup_{n\geq 1} E_n = [-1/2, 1/2]$ and $\sum_{k\in\mathbb{Z}} |\widehat{\phi}(\xi + k)|^2 = 1$ for all $\xi \in E_n + \mathbb{Z}$. Hence

$$f = \sum_{j\in\mathbb{Z}} (f * \psi)(j)\phi(\cdot - j) \quad f \in V^2(\phi),$$

and $\|f\|_2 = \|\{(f*\psi)(j)\}_{j\in\mathbb{Z}}\|_{\ell^2}$. Thus ψ is a stable averaging sampler. On the other hand, consider the sequence of functions $f_n \in V^2(\phi)$ defined by $\widehat{f_n}(\xi) = \widehat{c_n}(\xi)\widehat{\phi}(\xi)$, with $\widehat{c_n}$ supported on $E_n+\mathbb{Z}$ and $\|c_n\|_{\ell^2} = 1$. Then

(3.5)
$$\|\widehat{f_n}\widehat{\psi}\|_2^2 = \int_{\mathbb{R}} |\widehat{c_n}(\xi)|^2|\widehat{\phi}(\xi)|^4 d\xi = \frac{1}{2n+1}\int_{E_n} |\widehat{c_n}(\xi)|^2 d\xi = \frac{1}{2n+1}.$$

Hence ψ is not a stable Calderon convolutor for $V^2(\phi)$ since $\|f_n\|_2 = 1$ while $\|\widehat{f_n}\widehat{\psi}\|_2 = (\frac{1}{2n+1})^{-1/2}$, and n can be chosen to be arbitrarily large.

Although this example demonstrates that a stable averaging sampler is not necessarily a stable Calderon convolutor we still expect that a stable averaging sampler is a stable Calderon convolutor in most cases. This intuition is confirmed by the following theorem:

Theorem 3.10. *Let $\Phi = (\phi_1, \dots, \phi_r)^T$ and $\Psi = (\psi_1, \dots, \psi_s)^T$ satisfy $G_\Phi \in (L^\infty)^{(r\times r)}$ and $G_\Psi \in (L^\infty)^{(s\times s)}$, and assume that*

$$\lim_{N\to\infty} \sup_{\xi\in[0,1]^d} \sum_{l=1}^{s} \sum_{|k|\geq N} |\widehat{\psi_l}(\xi+k)|^2 \to 0. \tag{3.6}$$

If Ψ is a stable uniform averaging sampler for $V^2(\Phi)$, then Ψ is a stable convolutor for $V^2(\Phi)$.

Thus, by removing some pathological situations as described in the Theorem above, our intuition is validated.

4. Non-uniform average sampling

In this section, we study the problem of non-uniform average sampling with sufficiently large density for finitely generated shift-invariant spaces. We give conditions under which a stable Calderon convolutor is a stable non-uniform averaging sampler for sufficiently small gaps. We also show that under appropriate conditions the converse is also true. We start with the following definitions about sampling sets:

Definition 4.1. Let X be a countable subset of $\mathbb{R}^d$.

(1) We say that X is a *sampling set* with *maximal gap* δ if

$$\sum_{x_j\in X} \chi_{B(x_j,\delta)}(x) \geq 1, \qquad x \in \mathbb{R}^d,$$

where $B(x_j, \delta)$ is the ball centered at x_j and with radius δ.

(2) A sampling set X is said to be a *relatively separated sampling set* if it satisfies

$$\sum_{x_j \in X} \chi_{B(x_j,1)}(x) \le D, \quad x \in \mathbb{R}^d,$$

for some positive integer $D \ge 1$. The constant D is said to be the *gap bound* of the sampling set X.

(3) A sampling set X is said to be *separated* with separateness ϵ if it satisfies

$$\sum_{x_j \in X} \chi_{B(x_j,\epsilon)} \le 1, \quad x \in \mathbb{R}^d.$$

Remark 4.2. The maximal gap measures the density of the set of points X in $\mathbb{R}^d$ and it is sometimes referred to as δ-density [1, 3]. The relatively separateness and the separateness of a sampling set are related as follows: a relatively separated sampling set X with gap bound D can always be written as union of sampling sets $X_1, \ldots, X_J$ with separateness at least 1 for some positive integer $J \le 2^d D$, while, conversely, a separated sampling set X with separateness ϵ is a relatively separated sampling set X with gap bound D being approximately $2^d\epsilon^{-1} + 1$.

We also introduce the following definition about an averaging sampler Ψ:

Definition 4.3. For a shift-invariant subspace V of L^2, we say that $\Psi = (\psi_1, \ldots, \psi_s)^T$ is a *stable non-uniform averaging sampler with maximal gap* δ for V if for any relatively separated sampling set X with maximal gap δ, there exist positive constants C_1 and C_2 (dependent only on the space V, the function Ψ, the maximal gap δ, and the gap bound D of the sampling set X only) such that

$$C_1\|f\|_2^2 \le \sum_{l=1}^{s} \sum_{x_j \in X} |f * \psi_l(x_j)|^2 \le C_2\|f\|_2^2, \quad \forall\ f \in V. \tag{4.1}$$

Our next result shows that a stable Calderon convolutor is a stable non-uniform averaging sampler with sufficiently small gap provided that the convolutors are continuous function in the space W^1. Here for $1 \le p < \infty$, we say that a measurable function f on $\mathbb{R}^d$ belongs to the Wiener amalgam space W^p if it satisfies

$$\|f\|_{W^p} = \Big(\sum_{j \in \mathbb{Z}^d} \text{ess sup}\{|f(x+j)|^p : x \in [0,1]^d\}\Big)^{1/p} < \infty. \tag{4.2}$$

Theorem 4.4. *Let V be a shift-invariant subspace of L^2, and let $\Psi = (\psi_1, \ldots, \psi_s)^T$ be a continuous vector-valued function in W^1. If Ψ is* *a stable Calderon convolutor for V, then Ψ is a stable non-uniform averaging sampler for V for all sufficiently small maximal gaps δ.*

Although the previous theorem may not seem surprising, we have seen from the previous section on uniform averaging sampler and examples 3.8 and 3.9, that the issues are delicate and not straight forward. A more surprising result is a converse:

Theorem 4.5. *Let V be a shift-invariant subspace of L^2 and let $\Psi = (\psi_1, \ldots, \psi_s)^T$ be a continuous vector-valued function in W^1. If Ψ satisfies condition (4.1) for some relatively separated sampling set X with positive maximal gap, then it is a stable Calderon convolutor for V.*

Remark 4.6. Note that if $f \in V^2(\Phi)$, then $f * \psi_l \in V^2(\Phi * \psi_l)$. Moreover, since $f*\psi_l \in W^2$, we have that $\|f*\psi_l(X)\|_{\ell^2} \leq K\|f*\psi_l\|_{W^2}$ for some positive constant K (see [3]). Therefore, to prove Theorem 4.5, it would be sufficient to prove that

$$\|g\|_{W^2} \leq C\|g\|_2 \tag{4.3}$$

for all $g \in V^2(\Phi * \psi_l)$, for some constant C. However, the inequality (4.3) is not true in general, as seen from the following example.

Example 4.7. Let h be a C^∞ function so that h is supported in $[0,1]$, $\|h\|_2 = 1, \max_{x\in[0,1]} h(x) = 2$, and $\min_{x\in[0,1]} h(x) = 0$, and let $E_i = [a_i, b_i], i \geq 1$, be subintervals of $[0,1]$ so that they are mutually disjoint and $|b_i - a_i| = 2^{-i}$, and define

$$\phi(x) = \begin{cases} 2^{-i/2} h\Big(\frac{x-a_i}{b_i-a_i}\Big), & x \in E_i \text{ for some } i \geq 1, \\ -2^{-i/2} h\Big(\frac{x-2^i-a_i}{b_i-a_i}\Big), & x \in E_i + 2^i \text{ for some } i \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

By direct computation,

$$\sup_{x\in n+[0,1]} |\phi(x)| = \begin{cases} \sqrt{2}, & n = 0, \\ 2^{-i/2+1}, & n = 2^i \quad \text{for some } 1 \leq i \in \mathbb{Z}, \\ 0, & \text{otherwise,} \end{cases} \tag{4.4}$$

and

$$\|\phi\|_{W^1} \leq \sqrt{2} + 2\sum_{i=1}^{\infty} 2^{-i/2} = 3\sqrt{2} + 2 < \infty.$$

Hence ϕ is a continuous function in W^1. Set

$$g = \begin{cases} 2^{-i/2} h\Big(\frac{x-a_i-j}{b_i-a_i}\Big), & x \in E_i + j \text{ for some } 0 \leq j \leq 2^i - 1, i \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly we have

$$g(x)-g(x-1)=\phi(x)$$

and

$$\|g\|_2^2=\sum_{i=1}^\infty\int_{E_i+\mathbb{Z}}|g(x)|^2dx=\sum_{i=1}^\infty|E_i|2^{-i}2^i\|h\|_2^2=\sum_{i=1}^\infty 2^{-i}=1<\infty.$$

Note that

$$\sup_{x\in n+[0,1]}|g(x)|=\begin{cases}\sqrt{2}, & n=0,1,\\ 2^{-(j+1)/2+1}, & 2^j\le n\le 2^{j+1}-1 \text{ for some } 0\le j\in\mathbb{Z},\\ 0, & \text{otherwise,}\end{cases}$$

and hence

$$\|g\|_{W^2}^2=4+4\sum_{j=1}^\infty 2^{-j-1}2^j=+\infty.$$

Assume, on the contrary, that there exists a positive constant C so that $\|f\|_{W^2}\le C\|f\|_2$ for all $f\in V^2(\phi)$. Define $g_n\in V^2(\phi), n\ge 1$, by

$$\widehat{g_n}(\xi)=\widehat{\phi}(\xi)(1-e^{-i\xi})^{-1}\chi_{|1-e^{-i\xi}|\ge 1/n}(\xi).$$

Then g_n tends to g in L^2, since $\widehat{g_n}(\xi)-\widehat{g}(\xi)=\chi_{|1-e^{-i\xi}|\le 1/n}(\xi)\widehat{g}(\xi)$ and $g\in L^2$. This together with the norm equivalence implies that $g_n, n\ge 1$, is a Cauchy sequence in W^2. Thus g_n has a limit g_∞ in W^2. Recall that g_n tends to g in L^2. Therefore $g=g_\infty$, and hence $g\in W^2$, which is a contradiction.

From the above example, we see that the non-uniform sampling operator on $V^2(\Phi)$ is **not** a bounded operator. The fact that the inequality (4.3) is not true in general also shows that results on sampling after convolution cannot be deduced from results on ideal sampling such as those in [1, 3, 6, 12, 13, 27]. To reduce Theorem 4.5 to previous sampling results, extra conditions on Φ and Ψ must be imposed such that (4.3) is satisfied, for example, such conditions in Theorem B.1 in the appendix.

5. PROOFS

In this section, we give the proofs of Theorems 2.2, 2.5, 2.7, 3.2, 3.3, 4.4 and 4.5.

5.1. **Proofs of Theorem 2.2.** The fact that (i)$\Longrightarrow$ (ii) is trivial. To show that (ii)$\Longrightarrow$(i), we first note that $\sum_{l=1}^{s}\|f*\psi_l\|_2 \le B\|f\|_2$ is satisfied for any $f\in L^2$ because $\widehat{\psi}_i\in L^\infty$, $i=1,\dots,s$. Let f be in the L^2-closure of $V^2(\Phi)$ and choose a sequence $f_n\in V^2(\Phi)$ such that $f_n\to f$. We get

$$A\|f\|_2 = A\lim_{n\to\infty}\|f_n\|_2 \le \lim_{n\to\infty}\sum_{l=1}^{s}\|f_n*\psi_l\|_2 = \sum_{l=1}^{s}\|f*\psi_l\|_2 \le B\|f\|_2.$$

For any $f\in V^2(\Phi)$, there exists a square-integrable $\mathbb{Z}^d$-periodic function $C(\xi)=(c_1(\xi),\dots,c_r(\xi))^T$ so that $\widehat{f}(\xi)=C(\xi)^T\widehat{\Phi}(\xi)$. This together Parseval identity yields

$$\|f\|_2^2 = \int_{[0,1)^d} C(\xi)^T G_\Phi(\xi)\overline{C(\xi)}d\xi, \tag{5.1}$$

and

$$\begin{aligned}\sum_{l=1}^{s}\|f*\psi_l\|_2^2 &= \sum_{l=1}^{s}\int_{\mathbb{R}^d}|\widehat{f}(\xi)\widehat{\psi}_l(\xi)|^2 d\xi\\ &= \sum_{l=1}^{s}\int_{\mathbb{R}^d} C(\xi)^T\widehat{\Phi}(\xi)\overline{\widehat{\Phi}(\xi)}^T\,\overline{C(\xi)}|\widehat{\psi}_l(\xi)|^2 d\xi\\ &= \int_{[0,1)^d} C(\xi)^T G_\Phi^\Psi(\xi)\overline{C(\xi)}d\xi. \qquad (5.2)\end{aligned}$$

From the definition of $f\in V^2(\Phi)$, the space $\widehat{V^2(\Phi)}:=\{\widehat{f}(\xi): f\in V^2(\Phi)\}$ is characterized by

$$\widehat{V^2(\Phi)} = \Big\{C(\xi)^T\widehat{\Phi}(\xi):\ C(\xi)\in(L_p^2)^{(r)}\Big\}, \tag{5.3}$$

where $(L_p^2)^{(r)}$ is r copies of the space of all square-integrable $\mathbb{Z}^d$-periodic functions. Using (5.1), (5.2) and (5.3), we note that

$$\|f\|_2^2 \le A^{-1}\sum_{l=1}^{s}\|f*\psi_l\|_2^2 \quad \forall\ f\in V^2(\Phi)$$

if and only if

$$\int_{[0,1)^d} C(\xi)^T G_\Phi(\xi)\overline{C(\xi)}d\xi \le A^{-1}\int_{[0,1)^d} C(\xi)^T G_\Phi^\Psi(\xi)\overline{C(\xi)}d\xi.$$

Since $G_\Phi(\xi)$ and $G_\Phi^\Psi(\xi)$ are non-negative, self-adjoint $a.e.\,\xi\in\mathbb{R}^d$, and since $C\in(L_p^2)^{(r)}$ can be chosen to be an arbitrary $\mathbb{Z}^d$-periodic measurable vector function, the last inequality is satisfied if and only if $G_\Phi(\xi)\le A^{-1}G_\Phi^\Psi(\xi)$, $a.e.\,\xi\in\mathbb{R}^d$. Hence (ii) and (iii) are equivalent, since, as before, $\|f * \psi_l\|_2 \le B\|f\|_2$ is satisfied for any $f \in L^2$ because $\widehat{\psi}_i \in L^\infty,\ i = 1, \dots, s$, thus also for $V^2(\Phi) \subset L^2$.

5.2. **Proof of Theorem 2.5.** The implication (i)$\Longrightarrow$(ii) is obvious. Then it remains to prove (ii)$\Longrightarrow$(iii) and (iii)$\Longrightarrow$(i).

First we prove (ii)$\Longrightarrow$(iii). Note that rank $G_\Phi^\Psi(\xi) \le$ rank $G_\Phi(\xi)$ for almost all $\xi \in \mathbb{R}^d$. To see this, let v be any vector in $\mathbb{R}^d$ such that $v^T G_\Phi(\xi) = 0$, then

$$0 = v^T G_\Phi(\xi)\overline{v} = \sum_{k\in\mathbb{Z}^d} \|v^T \widehat{\Phi}(\xi + k)\|^2.$$

So $v^T\widehat{\Phi}(\xi + k) = 0$ for all $k \in \mathbb{Z}^d$, which implies $v^T G_\Phi^\Psi(\xi) = 0$. Hence the null space of $G_\Phi(\xi)$ is contained in the null space of $G_\Phi^\Psi(\xi)$. It follows that the rank$G_\Phi^\Psi(\xi) \le$ rank$G_\Phi(\xi), a.e.\, \xi \in \mathbb{R}^d$. Now suppose that (ii) holds but not (iii), then there exists a measurable set E with positive measure such that rank$G_\Phi^\Psi(\xi)$ is strictly less than rank$G_\Phi(\xi)$ for almost all $\xi \in E \subset [0, 1]^d$. This fact together with the fact that $v^T G_\Phi(\xi) = 0 \Longrightarrow v^T G_\Phi^\Psi(\xi) = 0$ for any $v \in \mathbb{R}^d$ imply that there exists a nonzero vector function v such that its components are measurable, supported in $E + \mathbb{Z}^d$, with $\|v\|_{L^2([0,1]^d)} = 1$, and such that

$$v(\xi)^T G_\Phi(\xi)\overline{v(\xi)} > 0 \quad \text{a.e. } \xi \in E, \tag{5.4}$$

and

$$v(\xi)^T G_\Phi^\Psi(\xi)\overline{v(\xi)} \equiv 0 \quad \text{a.e. } \xi \in E. \tag{5.5}$$

Then the function f defined by $\widehat{f}(\xi) = v(\xi)^T\widehat{\Phi}(\xi)$ is a nonzero function in $V^2(\Phi)$ by (5.1) and (5.4), but $f * \psi_l = 0$ for all $1 \le l \le s$ since $\widehat{f * \psi_l}(\xi) = v(\xi)^T\widehat{\Phi}(\xi)\widehat{\psi}_l(\xi) = 0$ by (5.2) and (5.5), which contradicts our assumption that (ii) holds.

Finally we prove (iii)$\Longrightarrow$(i). Suppose not, then there exists $f \in V$ such that $f \ne 0$ but $f * \psi_l \equiv 0$ for $1 \le l \le s$, where we denote the L^2-closure of $V^2(\Phi)$ by V. Write

$$\widehat{f}(\xi) = C(\xi)^T\widehat{\Phi}(\xi), \tag{5.6}$$

where C is a $\mathbb{Z}^d$-periodic measurable function (not square-integrable in general) satisfying

$$|C(\xi)| < \infty \quad \text{a.e. } \xi \in \mathbb{R}^d \tag{5.7}$$

(see [10]). Then

$$C(\xi)^T G_\Phi(\xi)\overline{C(\xi)} = \sum_{k\in\mathbb{Z}^d} |\widehat{f}(\xi + k)|^2 \not\equiv 0 \tag{5.8}$$

by (5.6) and the assumption $f \neq 0$, and

$$C(\xi)^T G_\Phi^\Psi(\xi)\overline{C(\xi)} = \sum_{l=1}^{s}\sum_{k\in\mathbb{Z}^d} |\widehat{f*\psi_l}(\xi+k)|^2 \equiv 0 \tag{5.9}$$

by (5.6) and the assumption $f*\psi_l = 0$ for all $1 \le l \le s$. Combining (5.7), (5.8) and (5.9), we see that the rank of G_Φ is strictly larger than the one of G_Φ^Ψ on the support of C, which contradicts the assumption (iii).

5.3. **Proof of Theorem 2.7.** By Theorem 2.2, there exists a positive constant m such that $mG_\Phi(\xi) \le G_\Phi^\Psi(\xi)$ for almost all $\xi \in \mathbb{R}^d$. This together with the continuity of G_Φ and G_Φ^Ψ implies that the above inequality holds for all $\xi \in \mathbb{R}^d$. Recall (from the proof of Theorem 2.5) that the rank of $G_\Phi^\Psi(\xi)$ is no larger than the rank of $G_\Phi(\xi)$ for all $\xi \in \mathbb{R}^d$. Therefore the ranks of $G_\Phi(\xi)$ and $G_\Phi^\Psi(\xi)$ are the same for all $\xi \in \mathbb{R}^d$.

Now we prove the sufficiency. Let $\lambda_l(\xi), 1 \le l \le r$, be the eigenvalues of the matrix $G_\Phi(\xi), \xi \in \mathbb{R}^d$, which are ordered so that $\lambda_1(\xi) \ge \lambda_2(\xi) \ge \cdots \ge \lambda_r(\xi)$. Then $\lambda_k(\xi), 1 \le k \le r$, are continuous functions of ξ by the continuity assumption on G_Φ. From the closedness of the shift-invariant space $V^2(\Phi)$, we have from Theorem A.1 that

$$G_\Phi^2(\xi) \ge mG_\Phi(\xi) \quad \text{a.e. } \xi \in \mathbb{R}^d,$$

for some positive constant m. Therefore

$$\lambda_k(\xi)^2 \ge m\lambda_k(\xi) \quad \text{a.e. } \xi \in \mathbb{R}^d.$$

This together with the continuity of $\lambda_k(\xi)$ imply that either $\lambda_k(\xi) \ge m$ for all $\xi \in \mathbb{R}^d$, or $\lambda_k(\xi) = 0$ for all $\xi \in \mathbb{R}^d$. Thus there exists $1 \le k_0 \le r$ such that

$$\lambda_1(\xi) \ge \ldots \ge \lambda_{k_0}(\xi) \ge m > 0 = \lambda_{k_0+1}(\xi) = \ldots = \lambda_r(\xi) \tag{5.10}$$

for all $\xi \in \mathbb{R}^d$. This also implies that the rank of the matrix $G_\Phi(\xi)$ is always k_0 for any $\xi \in \mathbb{R}^d$.

Let $\mu_1(\xi), \ldots, \mu_r(\xi)$ be eigenvalues of the matrix $G_\Phi^\Psi(\xi)$ ordered such that $\mu_1(\xi) \ge \cdots \ge \mu_r(\xi)$. Recall that the rank of $G_\Phi(\xi)$ is k_0 for all $\xi \in \mathbb{R}^d$ by the assumption on G_Φ and G_Φ^Ψ. Thus $\mu_1(\xi) \ge \cdots \ge \mu_{k_0}(\xi) > 0$ and $\mu_{k_0+1}(\xi) = \ldots = \mu_r(\xi) = 0$ for all $\xi \in \mathbb{R}^d$. Note that $\mu_k(\xi)$ are continuous function about ξ and also are $\mathbb{Z}^d$-periodic. Therefore there exists a positive constant m_1 so that

$$\mu_1(\xi) \ge \cdots \ge \mu_{k_0}(\xi) \ge m_1 > 0 = \mu_{k_0+1}(\xi) = \cdots = \mu_r(\xi) \tag{5.11}$$

for all $\xi \in \mathbb{R}^d$. Recall (form the proof of Theorem 2.5) that the null space of $G_\Phi(\xi)$ is contained in the null space of $G_\Phi^\Psi(\xi)$. But since the ranks of $G_\Phi^\Psi(\xi)$ and $G_\Phi(\xi)$ are the same by assumption, it follows that the null spaces of the matrices $G_\Phi(\xi)$ and $G_\Phi^\Psi(\xi)$ are equal. This together with (5.10) and (5.11) implies that

$$G_\Phi^\Psi(\xi) \ge \frac{m_1}{\lambda_1} G_\Phi(\xi) \quad \text{for all } \xi \in \mathbb{R}^d,$$

where $\lambda_1 = \text{esssup}_{\xi\in\mathbb{R}^d}\lambda_1(\xi)$. Hence the sufficiency follows.

5.4. **Proof of Theorem 3.2.** The implication (i)$\Longrightarrow$(ii) is obvious. For any function $f \in L^2$ and any function ψ with $G_\psi \in L^\infty$, we have that

$$\int_{\mathbb{R}^d} |\widehat{f}(\xi)\widehat{\psi}(\xi)|d\xi = \int_{[0,1]^d} \sum_{k\in\mathbb{Z}^d} |\widehat{f}(\xi+k)\widehat{\psi}(\xi+k)|d\xi \le \|G_\psi\|_\infty^{1/2}\|f\|_2.$$

Thus $f * \psi$ is continuous by the Riemman Lebesgue Lemma. Moreover we have

$$\begin{aligned} \sum_{j\in\mathbb{Z}^d} |f*\psi(j)|^2 &= \int_{[0,1]^d} \Big|\sum_{k\in\mathbb{Z}^d} \widehat{f}(\xi+k)\widehat{\psi}(\xi+k)\Big|^2 d\xi \\ &\le \Big\|\sum_{k\in\mathbb{Z}^d} |\widehat{\psi}(\xi+k)|^2\Big\|_\infty \|f\|_2^2. \qquad (5.12) \end{aligned}$$

Thus, the right inequality of (3.1) holds for all $f \in L^2$, and so in particular for all f in the L^2-closure of $V^2(\Phi)$. Let f_n be a sequence in $V^2(\Phi)$ that converges to f. If (ii) holds then using (5.12) and the left inequality of (3.1) we get

$$A\|f\|_2 = \lim_{n\to\infty} A\|f_n\|_2 \le \lim_{n\to\infty} \|\{f_n * \psi(j)\}\|_{\ell^2} = \|\{f*\psi(j)\}\|_{\ell^2},$$

and (ii)$\Longrightarrow$(i) follows.

For any $f \in V^2(\Phi)$, we may write $\widehat{f}(\xi) = C(\xi)^T\widehat{\Phi}(\xi)$, where C is a vector-valued square-integrable $\mathbb{Z}^d$-periodic function. Then

$$\|f\|_2^2 = \int_{[0,1)^d} C(\xi)^T G_\Phi(\xi)\overline{C(\xi)}d\xi, \qquad (5.13)$$

and

$$\begin{aligned} \sum_{l=1}^s \sum_{j\in\mathbb{Z}^d} |f*\psi_l(j)|^2 &= \sum_{l=1}^s \int_{[0,1)^d} \Big|C(\xi)^T \sum_{k\in\mathbb{Z}^d} \widehat{\Phi}(\xi+k)\widehat{\psi}_l(\xi+k)\Big|^2 d\xi \\ &= \int_{[0,1)^d} C(\xi)^T A_\Phi^\Psi(\xi)\overline{C(\xi)}d\xi. \qquad (5.14) \end{aligned}$$

Using (5.13) and (5.14), (iii)$\Longrightarrow$(ii) is obvious. To finish the proof, we note that if (ii) holds then there exists $m>0$ such that

$$m\int_{[0,1)^d} C(\xi)^T G_\Phi(\xi)\overline{C(\xi)}d\xi \le \int_{[0,1)^d} C(\xi)^T A_\Phi^\Psi(\xi)\overline{C(\xi)}d\xi.$$

Since G_Φ and A_Φ^Ψ are self-adjoint and C is an arbitrary square-integrable $\mathbb{Z}^d$-periodic function, we have that (ii)$\Longrightarrow$(iii). Hence the equivalence between (ii) and (iii) is established.

5.5. **Proof of Theorem 3.3.** Let V be the L^2-closure of the shift-invariant space $V^2(\Phi)$. By Lemma 2.9 and the rank assumption on G_Φ, there exist $H=(h_1,\dots,h_s)^T$, with $h_l\in V$, $l=1,\dots,s$, such that their integer shifts form an orthonormal basis of the shift-invariant space $V=V^2(H)$. By Theorem 3.2 and the stable averaging sampler assumption about Ψ, Ψ is a stable average sampler for $V=V^2(H)$, and hence there exists a positive constant m so that

$$v^T A_H^\Psi(\xi)\overline{v} = \sum_{l=1}^s\Big|\sum_{l'=1}^s v_{l'}\sum_{k\in\mathbb{Z}^d}\widehat{h}_{l'}(\xi+k)\widehat{\psi}_l(\xi+k)\Big|^2 \ge m\sum_{l'=1}^s |v_{l'}|^2 \tag{5.15}$$

for any vector $v=(v_1,\dots,v_s)^T\in\mathbb{C}^r$. Recall that the integer shifts of $h_1,\dots,h_s$ are orthonormal. Then

$$\widehat{\psi}_l(\xi) = \sum_{l'=1}^s a_{ll'}(\xi)\overline{\widehat{h}_{l'}(\xi)} + \widehat{\psi}_{l,2}(\xi) =: \widehat{\psi}_{l,1}(\xi)+\widehat{\psi}_{l,2}(\xi), \tag{5.16}$$

where $a_{ll'}(\xi)=\sum_{k\in\mathbb{Z}^d}\widehat{\psi}_l(\xi+k)\widehat{h}_{l'}(\xi+k)$, $1\le l,l'\le s$, and $\psi_{l,2}, 1\le l\le s$, satisfy

$$\sum_{k\in\mathbb{Z}^d}\widehat{\psi}_{l,2}(\xi+k)\widehat{h}_{l'}(\xi+k)=0 \tag{5.17}$$

for any $1\le l'\le s$. By (5.16) and (5.17), we have

$$G_\Psi(\xi)=G_{\Psi_1}(\xi)+G_{\Psi_2}(\xi) \tag{5.18}$$

and

$$G_{\Psi_1}(\xi)=\Big(\sum_{n=1}^s a_{ln}(\xi)\overline{a_{l'n}(\xi)}\Big)_{1\le l,l'\le s}, \tag{5.19}$$

where $\Psi_1=(\psi_{1,1},\dots,\psi_{s,1})^T$ and $\Psi_2=(\psi_{1,2},\dots,\psi_{s,2})^T$. By (5.15), (5.17) and (5.18), we have

$$\sum_{l=1}^s\Big|\sum_{l'=1}^s a_{ll'}(\xi)v_{l'}\Big|^2 \ge m\sum_{l'=1}^s|v_{l'}|^2 \tag{5.20}$$

for any vector $v = (v_1, \ldots, v_s)^T \in \mathbb{C}^r$. Combining (5.18), (5.19) and (5.20), and using the fact that $G_{\Psi_2}(\xi)$ is a nonnegative self-adjoint $s \times s$ matrix, we get that

$$v^T G_\Psi(\xi)\overline{v} = v^T G_{\Psi_1}(\xi)\overline{v} + v^T G_{\Psi_2}(\xi)\overline{v} \geq m\|v\|^2 + v^T G_{\Psi_2}(\xi)\overline{v} \geq m\|v\|^2.$$

So $G_\Psi(\xi) \geq mI$ for almost all $\xi \in \mathbb{R}^d$.

5.6. **Proof of Theorem 3.10.** By Theorem 3.2, there exists a positive constant m such that

$$mv^T G_\Phi(\xi)\overline{v} \leq \sum_{l=1}^{s} \Big| \sum_{k\in\mathbb{Z}^d} v^T \widehat{\Phi}(\xi+k)\widehat{\psi_l}(\xi+k)\Big|^2 \tag{5.21}$$

for any $\xi \in [0,1)^d$ and $v \in \mathbb{C}^r$. By (3.6), there exists an integer N_0 so that

$$\begin{aligned}
&\sum_{l=1}^{s} \Big| \sum_{|k|>N_0} v^T \widehat{\Phi}(\xi+k)\widehat{\psi_l}(\xi+k)\Big|^2 \\
\leq\ & \sum_{l=1}^{s} \Big(\sum_{k\in\mathbb{Z}^d} |v^T\widehat{\Phi}(\xi+k)|^2\Big) \times \Big(\sum_{|k|>N_0} |\widehat{\psi_l}(\xi+k)|^2\Big) \\
\leq\ & \frac{m}{4} v^T G_\Phi(\xi)\overline{v} \qquad (5.22)
\end{aligned}$$

for all $\xi \in [0,1)^d$. Combining (5.21) and (5.22) and using Hölder inequality, we obtain

$$\begin{aligned}
\frac{m}{4} v^T G_\Phi(\xi)\overline{v} &\leq \sum_{l=1}^{s} \Big| \sum_{|k|\leq N_0} v^T \widehat{\Phi}(\xi+k)\widehat{\psi_l}(\xi+k)\Big|^2 \\
&\leq (2N_0+1)^d \sum_{l=1}^{s} \sum_{|k|\leq N_0} |v^T\widehat{\Phi}(\xi+k)|^2 |\widehat{\psi_l}(\xi+k)|^2 \\
&\leq (2N_0+1)^d \sum_{l=1}^{s} \sum_{k\in\mathbb{Z}^d} |v^T\widehat{\Phi}(\xi+k)|^2 |\widehat{\psi_l}(\xi+k)|^2 \\
&= (2N_0+1)^d v^T G_\Phi^\Psi(\xi)\overline{v}
\end{aligned}$$

for all $v \in \mathbb{C}^r$ and $\xi \in [0,1)^d$. Therefore Ψ is a stable convolutor for $V^2(\Phi)$ by Theorem 2.2.

5.7. **Proof of Theorem 4.4.** For $\delta > 0$, let $\omega(f,\delta)$ be the modulus of continuity function (or oscillation) of f defined by

$$\omega(f,\delta)(x) = \sup_{|y-x|\leq\delta} |f(y)-f(x)|.$$

By the density of compactly supported continuous functions in the space of all continuous function in W^1, we have

Lemma 5.1. *Let ψ be a continuous function in W^1. Then*

$$\lim_{\delta\to 0}\|\omega(\psi,\delta)\|_{W^1}=0.$$

Proof of Theorem 4.4. Let Ψ be a stable Calderon convolutor for V. Then there exists a positive constant m such that

$$m\|f\|_2\le \Big(\sum_{l=1}^{s}\|f*\psi_l\|_2^2\Big)^{1/2}\quad \forall f\in V. \tag{5.23}$$

By Lemma 5.1, there exists $\delta_0>0$ so that for any $\delta<\delta_0$,

$$\Big(\sum_{l=1}^{s}\||f|*\omega(\psi_l,\delta)\|_2^2\Big)^{1/2}\le \|f\|_2\times\Big(\sum_{l=1}^{s}\|\omega(\psi_l,\delta)\|_1^2\Big)^{1/2}\le \frac{m}{2}\|f\|_2 \tag{5.24}$$

for all $f\in V$.

Let X be a relatively separated sampling set with maximal gap $\delta<\delta_0/2$, that is,

$$1\le\sum_{x_j\in X}\chi_{B(x_j,\delta)}\le C_0 \tag{5.25}$$

for some positive constant C_0, and let $\{h_j\}$ be the partition of unity corresponding to the covering $\{B(x_j,\delta):x_j\in X\}$ of $\mathbb{R}^d$,

$$\sum_{x_j\in X}h_j(x)\equiv 1\quad\forall\, x\in\mathbb{R}^d\quad\text{and}\quad 0\le h_j\le 1. \tag{5.26}$$

By (5.25), we have

$$\begin{aligned}
\sum_{x_j\in X}\sum_{l=1}^{s}|f*\psi_l(x_j)|^2 &\le \sum_{x_j\in X}\sum_{l=1}^{s}\|\psi_l\|_1\int_{\mathbb{R}^d}|f(y)|^2|\psi_l(x_j-y)|dy\\
&\le \sum_{l=1}^{s}\|\psi_l\|_1\times\Big(\sum_{k\in\mathbb{Z}^d}\int_{k+[0,1]^d}|f(y)|^2dy\\
&\qquad\times\Big(\sum_{k'\in\mathbb{Z}^d}\sum_{x_j\in X\cap(k'-[0,1]^d)}\sup_{t\in k'-k+[0,2]^d}|\psi_l(t)|\Big)\Big)\\
&\le C\|f\|_2^2\sum_{l=1}^{s}\|\psi_l\|_1\|\psi_l\|_{W^1},
\end{aligned}$$

where C is a positive constant. Therefore it suffices to prove that

$$\sum_{x_j\in X}\sum_{l=1}^{s}|f*\psi_l(x_j)|^2\ge C'\|f\|_2^2 \quad \forall\, f\in V, \tag{5.27}$$

where C' is a positive constant. Clearly for any $f\in V$,

$$|f*\psi_l(x_j)-f*\psi_l(x)|\le (|f|*\omega(\psi_l,\delta))(x) \tag{5.28}$$

for all $x\in B(x_j,\delta)$. Combining (5.23) – (5.26) and (5.28), we obtain

$$\begin{aligned}
&|B(0,\delta)|^{1/2}\Big(\sum_{l=1}^{s}\sum_{x_j\in X}|f*\psi_l(x_j)|^2\Big)^{1/2}\\
\ge\ & \Big(\sum_{l=1}^{s}\sum_{x_j\in X}\int_{\mathbb{R}^d}h_j(x)|f*\psi_l(x_j)|^2dx\Big)^{1/2}\\
\ge\ & \Big(\sum_{l=1}^{s}\sum_{x_j\in X}\int_{\mathbb{R}^d}h_j(x)|f*\psi_l(x)|^2dx\Big)^{1/2}\\
&-\Big(\sum_{l=1}^{s}\sum_{x_j\in X}\int_{\mathbb{R}^d}h_j(x)(|f|*\omega(\psi_l,\delta))^2(x)dx\Big)^{1/2}\\
\ge\ & m\|f\|_2-\Big(\sum_{l=1}^{s}\||f|*\omega(\psi_l,\delta)\|_2^2\Big)^{1/2}\\
\ge\ & \frac{m}{2}\|f\|_2,
\end{aligned}$$

and hence (5.27) is proved. □

5.8. **Proof of Theorem 4.5.** Let $X=\{x_j\}$ be a relatively separated sampling set. Then

$$\sum_{x_j\in X}\chi_{B(x_j,1)}\le C \tag{5.29}$$

for some positive constant C. By the assumption on Ψ, there exists a positive constant m so that

$$m\|f\|_2\le\Big(\sum_{l=1}^{s}\sum_{x_j\in X}|f*\psi_l(x_j)|^2\Big)^{1/2}\quad \forall\, f\in V. \tag{5.30}$$

Note that for any $f \in L^2$ and $h \in W^1$, we obtain from (5.29) that

$$\begin{aligned}
& \sum_{x_j \in X} \int_{B(x_j,\delta)} |f * h(x)|^2 dx \\
\le\ & C \int_{\cup_j B(x_j,\delta)} |f * h(x)|^2 dx \\
\le\ & C\|h\|_1 \int_{\mathbb{R}^d} |f(y)|^2 \Big(\int_{\cup_j B(x_j,\delta)} |h(x-y)| dx \Big) dy \\
\le\ & C\|h\|_1 \sum_{k,k' \in \mathbb{Z}^d} \int_{y \in k-[0,1)^d} |f(y)|^2 \\
& \times \int_{x \in (\cup_j B(x_j,\delta)) \cap (k'+[0,1)^d)} |h(x-y)| dx dy \\
\le\ & C'\|h\|_1 \sum_{k,k' \in \mathbb{Z}^d} \int_{y \in k-[0,1)^d} |f(y)|^2 \sup_{z \in k'-k+[0,2)^d} |h(z)| \\
& \times |(\cup_j B(x_j,\delta)) \cap (k'+[0,1)^d)| \\
\le\ & C'|B(0,\delta)| \|h\|_1 \|h\|_{W^1} \|f\|_2^2. \qquad (5.31)
\end{aligned}$$

Recall that $\lim_{\delta\to 0} \|\omega(\psi_l,\delta)\|_{W^1} = 0$ by Lemma 5.1. Then it follows from (5.31) that

$$\Big(\sum_{l=1}^{s} \sum_{x_j \in X} \int_{B(x_j,\delta)} (|f| * \omega(\psi_l,\delta)(x))^2 dx \Big)^{1/2} \le \frac{m}{2} |B(0,\delta)|^{1/2} \|f\|_2 \qquad (5.32)$$

for any $\delta < \delta_0$, where δ_0 is a positive constant dependent on Ψ and the gap bound of the sampling set X.

For any $\delta > 0$, $f \in L^2$ and a continuous function ψ, one can easily verify from the definition of the modulus of continuity that

$$|f * \psi(x) - f * \psi(y)| \le |f| * \omega(\psi,\delta)(x) \quad \forall y \in B(x,\delta). \qquad (5.33)$$

Combining (5.29), (5.30), (5.32) and (5.33), we obtain

$$\begin{aligned}
&\Big(\sum_{l=1}^{s}\|f*\psi_l\|_2^2\Big)^{1/2}\\
\geq\;& C_1\Big(\sum_{l=1}^{s}\sum_{x_j\in X}\int_{x\in B(x_j,\delta)}|f*\psi_l(x)|^2dx\Big)^{1/2}\\
\geq\;& C_1\Big(\sum_{l=1}^{s}\sum_{x_j\in X}|f*\psi_l(x_j)|^2|B(x_j,\delta)|\Big)^{1/2}\\
&-C_1\Big(\sum_{l=1}^{s}\sum_{x_j\in X}\int_{B(x_j,\delta)}(|f|*\omega(\psi_l,\delta)(x))^2dx\Big)^{1/2}\\
\geq\;& C_1m|B(0,\delta)|^{1/2}\|f\|_2-C_1\frac{m}{2}|B(0,\delta)|^{1/2}\|f\|_2\\
=\;& \frac{mC_1}{2}|B(0,\delta)|^{1/2}\|f\|_2 \qquad (5.34)
\end{aligned}$$

for $\delta<\delta_0$ and $f\in V$. This proves that Ψ is a stable Calderon convolutor for V.

Appendix A. Closedness of a finitely generated shift-invariant space

In this part, we give a characterization of the closedness of $V^2(\Phi)$ for those generators Φ with bounded G_Φ, which plays an important role in the proof of Theorem 2.7. A similar result was established in [5] under the assumption that $\Phi\in W^1$.

Theorem A.1. *Let $\Phi=(\phi_1,\dots,\phi_r)^T$ satisfy $G_\Phi\in(L^\infty)^{(r\times r)}$. Then $V^2(\Phi)$ is closed in L^2 if and only if there exists a positive constant $m>0$ such that*

$$G_\Phi^2(\xi)\geq mG_\Phi(\xi)\quad\text{a.e. }\xi\in\mathbb{R}^d.$$

Proof. First we prove the sufficiency. Let $\lambda_1(\xi),\dots,\lambda_r(\xi)$ be eigenvalues of $G_\Phi(\xi)$, which are ordered so that $\lambda_1(\xi)\geq\lambda_2(\xi)\geq\cdots\geq\lambda_r(\xi)$. Then

$$\lambda_k(\xi)^2\geq m\lambda_k(\xi)\quad\text{a.e. }\xi\in\mathbb{R}^d,\ 1\leq k\leq r,$$

by the assumption on G_Φ. Note that $\lambda_k(\xi),1\leq k\leq r$, are measurable and $\mathbb{Z}^d$-periodic since G_Φ is. Then the sets

$$E_k:=\{\xi\in\mathbb{R}^d:\ \lambda_k(\xi)\geq m\}$$

satisfies $E_k=E_k+2\pi\mathbb{Z}^d$ and $E_1\supset E_2\supset\cdots\supset E_r$. Furthermore there exist projections $Q(\xi)$ so that $Q(\xi)G_\Phi(\xi)\overline{Q(\xi)}^T=G_\Phi(\xi)$, $v^TG_\Phi(\xi)\bar{v}\geq$

$m\|v^TQ(\xi)\|^2$ for all $v \in \mathbb{C}^r$, and $Q(\xi)\widehat{\Phi}(\xi) = \widehat{\Phi}(\xi)$ for almost all $\xi \in \mathbb{R}^d$. Let $f_n \in V^2(\Phi), n \geq 0$, be a Cauchy sequence in L^2. Without loss of generality, we assume that $f_0 = 0$ and $\|f_{n+1} - f_n\| \leq 2^{-n}, n \geq 0$. Write

$$\widehat{f}_{n+1}(\xi) - \widehat{f}_n(\xi) = D_n(\xi)^T\widehat{\Phi}(\xi) = D_n(\xi)^TQ(\xi)\widehat{\Phi}(\xi).$$

Note that

$$\begin{aligned}\|f_{n+1} - f_n\|_2^2 &= \int_{[0,1]^d} D_n(\xi)^TG_\Phi(\xi)\overline{D_n(\xi)}d\xi \\ &\geq m\int_{[0,1]^d} \|D_n(\xi)^TQ(\xi)\|^2 d\xi,\end{aligned}$$

which implies that

$$\|D_n(\xi)^TQ(\xi)\|_{L^2([0,1]^d)} \leq m^{-1/2}2^{-n}.$$

Hence the function f defined by

$$\widehat{f}(\xi) = \sum_{n=0}^\infty D_n(\xi)^TQ(\xi)\widehat{\Phi}(\xi) = \sum_{n=0}^\infty(\widehat{f}_{n+1}(\xi) - \widehat{f}_n(\xi))$$

belongs to $V^2(\Phi)$, and

$$\begin{aligned}\|f_n - f\|_2 &= \|\widehat{f}_n - \widehat{f}\|_2 \leq \sum_{k=n}^\infty \|\widehat{f}_{k+1} - \widehat{f}_k\| \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

This proves the closedness of the space $V^2(\Phi)$ in L^2 topology.

Now we prove the necessity. Let $\lambda_1(\xi), \ldots, \lambda_r(\xi)$ be the eigenvalues of $G_\Phi(\xi)$ which are ordered so that $\lambda_1(\xi) \geq \lambda_2(\xi) \geq \cdots \geq \lambda_r(\xi)$. Then there exists $e_1(\xi), \ldots, e_r(\xi)$ so that $e_1(\xi), \ldots, e_r(\xi)$ are mutually orthogonal unit vectors and satisfy

$$e_k(\xi)^TG_\Phi(\xi) = \lambda_k(\xi)e_k(\xi)^T, \ 1 \leq k \leq r, \ \xi \in \mathbb{R}^d.$$

Suppose, on the contrary, that there does not exist $m > 0$ so that $G_\Phi(\xi)^2 \geq mG_\Phi(\xi)$ for almost all $\xi \in \mathbb{R}^d$. Then there exists an integer k so that $\lambda_k(\xi)^2 \geq m\lambda_k(\xi)$ does not hold for almost all $\xi \in \mathbb{R}^d$. Hence the sets

$$F_n = \{\xi \in \mathbb{R}^d : \ 0 < \lambda_k(\xi) < 2^{-n}\}, \ n \geq 1,$$

are measurable sets with $|F_n \cap [0,1]^d| > 0$ for all $n \geq 0$. Define g and $g_l, l \geq 1$, by

$$\widehat{g}(\xi) = \sum_{n=1}^\infty \epsilon_n\chi_{F_n\backslash F_{n+1}}(\xi)e_k(\xi)^T\widehat{\Phi}(\xi),$$

and

$$\widehat{g}_l(\xi) = \sum_{n=1}^{l} \epsilon_n \chi_{F_n \backslash F_{n+1}}(\xi) e_k(\xi)^T \widehat{\Phi}(\xi),$$

where $\epsilon_n, n \geq 1$, are so chosen that $\epsilon_n = 0$ if the Lebesgue measure of $(F_n \backslash F_{n+1}) \cap [0,1]^d$ is zero and $\epsilon_n = |(F_n \backslash F_{n+1}) \cap [0,1]^d|^{-1/2}$ when the Lebesgue measure of $(F_n \backslash F_{n+1}) \cap [0,1]^d$ is nonzero. Then $g_l, l \geq 1$, belong to $V^2(\Phi)$, and we have

$$\begin{aligned}
\|g_l - g\|_2^2 &= \sum_{n=l+1}^{\infty} \epsilon_n^2 \int_{F_n \backslash F_{n+1}} |e_k(\xi)^T \widehat{\Phi}(\xi)|^2 d\xi \\
&= \sum_{n=l+1}^{\infty} \epsilon_n^2 \int_{(F_n \backslash F_{n+1}) \cap [0,1]^d} e_k(\xi)^T G_\Phi(\xi) \overline{e_k(\xi)} d\xi \\
&= \sum_{n=l+1}^{\infty} \epsilon_n^2 \int_{(F_n \backslash F_{n+1}) \cap [0,1]^d} \lambda_k(\xi) d\xi \\
&\leq \sum_{n=l+1}^{\infty} 2^{-n} \epsilon_n^2 |(F_n \backslash F_{n+1}) \cap [0,1]^d| \leq 2^{-n} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore by the closedness of the shift-invariant space $V^2(\Phi)$,

$$\widehat{g}(\xi) = \widehat{C}(\xi)^T \widehat{\Phi}(\xi)$$

for some $\mathbb{Z}^d$-periodic function $\widehat{C}$ with $\widehat{C} \in L^2([0,1]^d)$. Therefore the function h defined by

$$\widehat{h}(\xi) = \Big[\widehat{C}(\xi)^T - \sum_{n=1}^{\infty} \epsilon_n \chi_{F_n \backslash F_{n+1}}(\xi) e_k(\xi)^T \Big] \widehat{\Phi}(\xi)$$

is a zero function. On the other hand, we note that

$$\begin{aligned}
\|h\|_2^2 &\geq \int_{F_n\backslash F_{n+1}} |(\widehat{C}(\xi)^T - \epsilon_n e_k(\xi)^T)\widehat{\Phi}(\xi)|^2 d\xi \\
&= \int_{(F_n\backslash F_{n+1})\cap[0,1)^d} (\widehat{C}(\xi)^T - \epsilon_n e_k(\xi)^T) G_\Phi(\xi) \overline{(\widehat{C}(\xi)^T - \epsilon_n e_k(\xi)^T)} d\xi \\
&= \sum_{l\neq k} \int_{(F_n\backslash F_{n+1})\cap[0,1)^d} \lambda_l(\xi) |\langle \widehat{C}(\xi), e_l(\xi)\rangle|^2 d\xi \\
&\quad + \int_{(F_n\backslash F_{n+1})\cap[0,1)^d} \lambda_k(\xi) |\langle \widehat{C}(\xi), e_k(\xi)\rangle - \epsilon_n|^2 d\xi \\
&\geq \int_{(F_n\backslash F_{n+1})\cap[0,1]^d} |\langle \widehat{C}(\xi), e_k(\xi)\rangle - \epsilon_n|^2 \lambda_k(\xi) d\xi \\
&\geq 2^{-n-1} \int_{(F_n\backslash F_{n+1})\cap[0,1]^d} |\langle \widehat{C}(\xi), e_k(\xi)\rangle - \epsilon_n|^2 d\xi.
\end{aligned}$$

Therefore we obtain

$$\langle \widehat{C}(\xi), e_k(\xi)\rangle = \epsilon_n \quad \text{a.e. } \xi \in F_n\backslash F_{n+1}.$$

Thus

$$\begin{aligned}
\int_{[0,1]^d} |\widehat{C}(\xi)|^2 d\xi &\geq \sum_{n=1}^\infty \int_{(F_n\backslash F_{n+1})\cap[0,1]^d} |\widehat{C}(\xi)|^2 d\xi \\
&\geq \sum_{n=1}^\infty \int_{(F_n\backslash F_{n+1})\cap[0,1]^d} |\langle \widehat{C}(\xi), e_k(\xi)\rangle|^2 d\xi \\
&= \sum_{n=1}^\infty \epsilon_n^2 |(F_n\backslash F_{n+1}) \cap [0,1]^d| \\
&\geq \sum_{|(F_n\backslash F_{n+1})\cap[0,1]^d|\neq 0} 1 = +\infty,
\end{aligned}$$

which is a contradiction. □

Appendix B. Norm equivalence in shift-invariant spaces

In this part, we consider the problem for which generator Φ does the following norm equivalence hold:

$$C^{-1}\|f\|_2 \leq \|f\|_{W^2} \leq C\|f\|_2 \quad \forall\ f \in V^2(\Phi), \tag{B.1}$$

where C is a positive constant independent of f. The above norm equivalence is interesting by itself, and also useful in the non-uniform (average) sampling on a finitely generated shift-invariant space, and can be used to deduce results on average sampling from well-known results on ideal sampling in [1, 3, 6, 27].

For functions $\Phi = (\phi_1, \dots, \phi_r)^T$ satisfying $G_\Phi \in (L^\infty)^{(r\times r)}$, we say that Φ has *stable shifts* if $\{\phi_i(\cdot - j): \ j \in \mathbb{Z}^d, 1 \le i \le r\}$ is a Riesz basis for $V^2(\Phi)$ (see [21, 26]).

Theorem B.1. *Suppose that $\Phi = (\phi_1, \dots, \phi_r)^T$ satisfies one of the following three conditions:*

(i) *Φ belongs to W^1 and has stable shifts.*
(ii) *Φ is a compactly supported L^∞-function.*
(iii) *$\widehat{\Phi}$ is compactly supported and satisfies $G_\Phi \in (L^\infty)^{(r\times r)}$.*

Then the norm equivalence (B.1) holds for any function $f \in V^2(\Phi)$.

Remark B.2. For a shift-invariant space V with finite dimensional restriction on $[0,1)^d$, it is shown in [4] that there always exist compactly supported functions $\phi_1, \dots, \phi_r$ having stable shifts so that $V \subset V^2(\phi_1, \dots, \phi_r)$. Moreover, those functions $\phi_1, \dots, \phi_r$ are bounded when all functions in the restriction of V to $[0,1)^d$ are. Therefore as a consequence of Theorem B.1, we have the following result.

Corollary B.3. *Let V be a shift-invariant subspace of L^2. If the restriction of V to $[0,1)^d$ is a finite dimensional space of bounded functions, then (B.1) holds for any $f \in V$.*

Remark B.4. The stable shift condition on Φ in Theorem B.1 cannot be dropped in general, as demonstrated in Example 4.7.

Proof of Theorem B.1. First we prove (B.1) under the assumption (i). For any $f = \sum_{i=1}^r \sum_{j\in\mathbb{Z}^d} c_i(j)\phi_i(\cdot - j) \in V^2(\Phi)$ with $\{c_i(j)\} \in \ell^2, 1 \le i \le r$, we have

$$\begin{aligned}
\|f\|_{W^2}^2 &\le \sum_{j\in\mathbb{Z}^d} \sup_{x\in[0,1)^d} \Big(\sum_{i=1}^r \sum_{j'\in\mathbb{Z}^d} |c_i(j')||\phi_i(x+j-j')|\Big)^2 \\
&\le \sum_{j\in\mathbb{Z}^d} \sup_{x\in[0,1)^d} \Big(\sum_{i=1}^r \sum_{j'\in\mathbb{Z}^d} |c_i(j')|^2|\phi_i(x+j-j')|\Big) \\
&\quad \times\Big(\sum_{i=1}^r \sum_{j'\in\mathbb{Z}^d} |\phi_i(x+j-j')|\Big) \\
&\le \Big(\sum_{i=1}^r \|\phi_i\|_{W^1}\Big)^2 \sum_{i=1}^r \sum_{j\in\mathbb{Z}^d} |c_i(j)|^2 \\
&\le C\Big(\sum_{i=1}^r \|\phi_i\|_{W^1}\Big)^2 \|f\|_2^2
\end{aligned}$$

for some positive constant C independent of f, where we have used the stable shifts to obtain the last inequality. This proves (B.1) under the first assumption.

Next we prove (B.1) under the assumption (ii). Since Φ has compact support and is bounded, the restriction of $V^2(\Phi)$ to $[0,1)^d$ is a finite dimensional space of bounded functions. Let $h_1, \ldots, h_K$ be a basis of the restriction of $V^2(\Phi)$ to $[0,1)^d$. By the shift-invariance of the space $V^2(\Phi)$, we have

$$V^2(\Phi) \subset V^2(H),$$

where $H = (h_1, \ldots, h_K)^T$. Note that $h_i, 1 \le i \le K$, are supported in $[0,1)^d$ and belong to L^∞. Thus $H \in W^1$. Moreover, by the construction, H has stable shifts, therefore the assertion (B.1) follows from the previous result.

Finally we prove (B.1) under the assumption (iii). By the assumption, there is an integer K so that $\widehat{f} \subset [-K, K]^d$ for any $f \in V$, which implies that $f = \int_{\mathbb{R}^d} f(\cdot - y)h(y)dy$ for any Schwartz function h with $\widehat{h}(\xi) = 1$ for all $\xi \in [-K, K]^d$. Therefore,

$$\begin{aligned}
\|f\|_{W^2} &\le \|h\|_1 \times \sum_{j\in\mathbb{Z}^d} \sup_{x\in j+[0,1)^d} \int_{\mathbb{R}^d} |f(y)|^2 |h(x-y)| dy \\
&\le \|h\|_1 \|h(\cdot)(1+|\cdot|)^{d+1}\|_\infty \\
&\quad \times \sum_{j\in\mathbb{Z}^d} \sup_{x\in j+[0,1)^d} \int_{\mathbb{R}^d} |f(y)|^2 (1+|x-y)|)^{-d-1} dy \\
&\le C\|f\|_2
\end{aligned}$$

for some positive constant C independent of $f \in V$. This prove (B.1) under the assumption (iii) and hence completes the proof. □

References

[1] A. Aldroubi, Non-uniform weighted average sampling and reconstruction in shift-invariant and wavelet spaces, *Appl. Comput. Harmon. Anal.*, **13**(2002), 151–161.

[2] A. Aldroubi and K. Gröchenig, Beurling-Landau-type theorems for non-uniform sampling in shift invariant spline spaces, *J. Fourier Anal. Appl.*, **6**(2000), 93–103.

[3] A. Aldroubi and K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant spaces, *SIAM Rev.*, **43**(2001), 585–620.

[4] A. Aldroubi and Q. Sun, Locally finite dimensional shift-invariant spaces in $\mathbb{R}^d$, *Proc. Amer. Math. Soc.*, **130**(2002), 2641–2654.

[5] A. Aldroubi, Q. Sun and W.-S. Tang, p-frames and shift-invariant subspaces of L^p, *J. Fourier Anal. Appl.*, **7**(2001), 1–21.

[6] A. Aldroubi, Q. Sun and W.-S. Tang, Non-uniform average sampling and reconstruction in multiply generated shift-invariant spaces, *Constr. Approx.*, **20**(2004), 173–189.

[7] J. J. Benedetto and P. J. S. G. Ferreira (editors), *Modern Sampling Theory: Mathematics and Applications*, Birkhauser Inc., Boston, 2001.

[8] J. J. Benedetto and S. Li,, The theory of multiresolution analysis frames and applications to filter banks, *Appl. Comput. Harmon. Anal.*, **5**(1998), 389–427.

[9] J. J. Benedetto and A. I. Zayed (editors), *Sampling, Wavelets, and Tomography*, Birkhauser, Boston, 2003.

[10] C. de Boor, R. DeVore and A. Ron, The structure of finitely generated shift-invariant spaces in $L_2(R^d)$, *J. Funct. Anal.*, **119**(1994), 37–78.

[11] M. Bownik, The structure of shift-invariant subspaces of $L^2(R^n)$. *J. Funct. Anal.*, **177**(2000), 282–309.

[12] W. Chen, S. Itoh, Shuichi and J. Shiki, On sampling in shift invariant spaces, *IEEE Trans. Inform. Theory*, **48**(2002), 2802–2810.

[13] W. Chen and S. Itoh, A sampling theorem for shift-invariant subspace, *IEEE Trans. Signal Process.*, **46**(1998), 2822–2824.

[14] C. K. Chui and Q. Sun, Affine frame decompositions and shift-invariant spaces, Preprint 2003.

[15] I. Daubechies, *Ten Lectures on Wavelets*, CBMF Conference Series in Applied Mathematics, 61, SIAM, Philadelphia, 1992.

[16] I. Daubechies and R. DeVore, Approximating a bandlimited function using very coarsely quantized data: a family of stable sigma-delta modulators of arbitrary order, *Ann. Math.*, **158**(2003), 679–710.

[17] R. J. Duffin and A. C. Schaeffer, A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc.*, **72**(1952), 341–366.

[18] D. E. Dutkay, The local trace function of shift invariant subspaces, *J. Operator Theory*, to appear.

[19] K. Grochenig and H. Schwab, Fast local reconstruction methods for nonuniform sampling in shift-invariant spaces, *SIAM J. Matrix Anal. Appl.*, **24**(2003), 899-913.

[20] D. Hardin, T. Hogan and Q. Sun, The matrix-valued Riesz lemma and local orthonormal bases in shift-invariant spaces, *Adv. Comput. Math.*, **20**(2004), 367-384.

[21] R. Q. Jia and C. A. Micchelli, Using the refinement equation for the construction of pre-wavelets II: powers of two, In *Curve and Surfaces*, P. J. Laurent, A. Le Méhauté and L. L. Schumaker eds., Academic Press, New York 1991, pp. 209–246.

[22] S. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, Boston, 1998.

[23] F. A. Marvasti (editor), *Nonuniform Sampling: Theory and Practice (Information Technology: Transmission, Processing, and Storage)*, Plenum Pub Corp, 2001.

[24] Y. Meyer, *Ondelettes et Opérateurs*, Hermann, Paris, 1990.

[25] M. Z. Nashed, On moment-discretization and least-squares solutions of linear integral equations of the first kind, *J. Math. Anal. Appl.*, **53**(1976), 359–366.

[26] Q. Sun, Stability of the shifts of global supported distributions, *J. Math. Anal. Appl.*, **261**(2001), 113–125.

[27] W. Sun and X. Zhou, Irregular sampling for multivariate band-limited functions, *Sci. China*, **45**A(2002), 1548–1556.
[28] W. Sun and X. Zhou, Average sampling theorems for shift invariant subspaces, *Sci. China*, **43**E(2000), 524–530.
[29] A. I. Zayed, *Advances in Shannon's Sampling Theory*, CRC Press, 1993.

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