EIGENVALUES OF SCALING OPERATORS AND A CHARACTERIZATION OF B-SPLINES

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ABSTRACT. A finitely supported sequence a that sums to 2 defines a scaling operator $T_a f = \sum_{k\in\mathbb{Z}} a(k)f(2\cdot -k)$ on functions f, a transition operator $S_a v = \sum_{k\in\mathbb{Z}} a(k)(2\cdot -k)$ on sequences v, and a unique compactly supported scaling function ϕ that satisfies $\phi = T_a\phi$ normalized with $\widehat{\phi}(0) = 1$. It is shown that the eigenvalues of T_a on the space of compactly supported square-integrable functions are a subset of the nonzero eigenvalues of the transition operator S_a on the space of finitely supported sequences, and that the two sets of eigenvalues are equal if and only if the corresponding scaling function ϕ is a uniform B-spline.

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1. INTRODUCTION

A finitely supported real sequence $a := \{a(k)\}_{k\in\mathbb{Z}}$, normalized so that $\sum_{k\in\mathbb{Z}} a(k) = 2$, defines the *scaling operator* T_a on $L^p(\mathbb{R})$, $1 \le p \le \infty$, by

$$T_a f := \sum_{k\in\mathbb{Z}} a(k)f(2\cdot -k), \quad f \in L^p(\mathbb{R}), \tag{1.1}$$

and the *transition operator* S_a on $\ell^p(\mathbb{Z})$ by

$$(S_a v)(j) := \sum_{k\in\mathbb{Z}} a(k)v(2j-k), \quad v \in \ell^p(\mathbb{Z}). \tag{1.2}$$

We shall deal mainly with the space $L^2_c \equiv L^2_c(\mathbb{R})$ of compactly supported L^2-functions, and $\ell_0 \equiv \ell_0(\mathbb{Z})$ the space of finitely supported sequences.

In the Fourier transform domain (1.1) becomes

$$\widehat{T_a f}(u) = H(u/2)\widehat{f}(u/2) \tag{1.3}$$

where $H(u) = \frac{1}{2}\sum_{k\in\mathbb{Z}} a(k)e^{-iku}$. The infinite product $\prod_{n=1}^{\infty} H(2^{-n}u)$ converges locally uniformly, and there exists a compactly supported distribution ϕ whose Fourier transform is

$$\widehat{\phi}(u) = \prod_{n=1}^{\infty} H(2^{-n}u), \tag{1.4}$$

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which satisfies

$$\widehat{\phi}(u) = H(u/2)\widehat{\phi}(u/2)$$

or equivalently

$$T_a\phi = \phi \text{ and } \widehat{\phi}(0) = 1. \tag{1.5}$$

The compactly supported distribution ϕ is indeed the unique solution of (1.5) (see [4], [5]). The equation (1.5) is known as a *scaling equation* or *refinement equation.* Its solution is called a *scaling function*, and the sequence a is called its *mask*.

Scaling functions play an important role in multiscale representation, which has applications in scale-space analysis ([18]), geometric modelling ([3], [9]) and wavelet analysis ([4], [5]). Many properties of a scaling function are controlled by the spectrum of its transition operator ([5, 6, 7, 11, 12, 14, 17]). For a nice account of properties of transition operators and their adjoints, the *subdivision operators*, see ([1, 2, 8, 19, 20]). In this paper we study the relationship between the spectra of the scaling operator and the transition operator defined by a sequence a. Let $\sigma_e(T, X)$ denote the set of eigenvalues of a continuous linear operator T on a Banach space X. The object is to show that the inclusion $\sigma_e(T_a, L^2_c) \cup \{0\} \subset \sigma_e(S_a, \ell_0)$ holds for any $a \in \ell_0$, with equality if and only if the mask a is a shift of a binomial sequence, i.e. $\widehat{a}(u) = 2 \cdot e^{-iLu}\left(\frac{1+e^{-iu}}{2}\right)^k$, and hence the corresponding refinable function is a B-spline. In Section 2 we prove that the inclusion $\sigma_e(T_a, L^2_c) \cup \{0\} \subset \sigma_e(S_a, \ell_0)$ holds for all $a \in \ell_0$, and in Section 3 it is shown that equality holds if and only if the mask a is a shift of the binomial sequence. In fact in Theorem 3.2 we prove more and the results are more precise.

2. Eigenvalues of Scaling and Transition Operators

In this section we develop a relationship between the eigenvalues of the scaling operator T_a on L^2_c and the corresponding transition operator S_a on ℓ_0. We shall establish the following theorem.

Theorem 2.1. *Let* $a := \{a(k)\}_{k\in\mathbb{Z}} \in \ell_0$ *satisfy* $\sum_{k\in\mathbb{Z}} a(k) = 2$. *Then*

$$\sigma_e(T_a, L^2_c) \cup \{0\} \subset \sigma_e(S_a, \ell_0). \tag{2.1}$$

As a consequence of Theorem 2.1 we have the following result ([3, 6]).

Corollary 2.2. *Let* $a := \{a(k)\}_{k\in\mathbb{Z}} \in \ell_0$ *satisfy* $\sum_{k\in\mathbb{Z}} a(k) = 2$. *Then*

$$\sigma_e(T_a, C_c(\mathbb{R})) \cup \{0\} \subset \sigma_e(S_a, \ell_0), \tag{2.2}$$

where $C_c(\mathbb{R})$ *is the space of all compactly supported continuous functions on* $\mathbb{R}$.

We shall first establish two lemmas in the run up to the proof of Theorem 2.1.

Lemma 2.3. *Let* $a := \{a(k)\}_{k\in\mathbb{Z}} \in \ell_0$ *satisfy* $\sum_{k\in\mathbb{Z}} a(k) = 2$. *Then* $\sigma_e(T_a, L_c^2) = \emptyset$ *or* $\{1, 1/2, \dots, 1/2^{k_0}\}$ *for some nonnegative integer* k_0. *In the later situation, the unique compactly supported solution* ϕ *of the refinement equation* $T_a\phi = \phi$ *with* $\widehat{\phi}(0) = 1$ *has derivatives up to order* k_0 *in* $L^2(\mathbb{R})$ *and if* g_l *is an eigenfunction of the scaling operator* T_a *with eigenvalue* 2^{-l}, $l = 0, 1, \dots, k_0$, *then* $g_l = C\phi^{(l)}$ *for some nonzero constant* C.

Proof. Let $\lambda \in \sigma_e(T_a, L_c^2)$ and g_λ be a nonzero function in L_c^2 that satisfies

$$T_a g_\lambda = \lambda g_\lambda. \tag{2.3}$$

Then it suffices to prove that $\lambda = 2^{-l}$ and $g_\lambda = C\phi^{(l)}$ for some nonnegative integer l. Taking the Fourier transform at both sides of (2.3) leads to

$$H(u/2)\widehat{g}_\lambda(u/2) = \lambda \widehat{g}_\lambda(u), \tag{2.4}$$

where $H(u) := \frac{1}{2}\sum_{k\in\mathbb{Z}} a(k)e^{-iku}$. Note that $\widehat{g}_\lambda$ is analytic on $\mathbb{R}$ since g_λ has compact support. Therefore, by comparing the order of u on both sides of equation (2.4) at the origin and using $H(0) = 1$, we conclude that $\lambda = 2^{-l}$ for some nonnegative integer l. Thus $G_\lambda(u) := \widehat{g}_\lambda(u)u^{-l}$ is still analytic and satisfies the equation $G_\lambda(u) = H(u/2)G_\lambda(u/2)$. This shows that $G_\lambda(u) = G_\lambda(0)\widehat{\phi}(u)$ since $\widehat{\phi}(u) = \prod_{n=1}^{\infty} H(2^{-n}u)$. Hence $g_\lambda = C\phi^{(l)}$ for some nonzero constant C. □

The next lemma is a known result on a sum rule of an L^2- function, which we shall state here for convenient reference (see for instance [10] and the references therein).

Lemma 2.4. *For* $a = \{a(k)\}_{k\in\mathbb{Z}} \in \ell_0$ *that satisfies* $\sum_{k\in\mathbb{Z}} a(k) = 2$, *let* ϕ *be the unique compactly supported distributional solution of the refinement equation* $\phi = T_a\phi$. *If* $\phi, \dots, \phi^{(k_0)} \in L^2(\mathbb{R})$, *then the symbol* $H(u) := \frac{1}{2}\sum_{k\in\mathbb{Z}} a(k)e^{-iku}$ *satisfies*

$$H(u) = \left(\frac{1+e^{-iu}}{2}\right)^{k_0+1} \widetilde{H}(u)$$

for some trigonometric polynomial $\widetilde{H}(u)$.

Proof of Theorem 2.1

Proof. The assertion is trivial if $\sigma_e(T_a, L_c^2) = \emptyset$ since one can easily see that $\{0\} \subset \sigma_e(S_a, \ell_0)$. So we assume that $\sigma_e(T_a, L_c^2) \neq \emptyset$. By Lemma 2.3,

$$\sigma_e(T_a, L_c^2) = \{1, \dots, 2^{-k_0}\} \tag{2.5}$$

for some nonnegative integer k_0, and $\phi, \phi', \dots, \phi^{(k_0)} \in L^2_c$. Thus $\phi, \phi', \dots, \phi^{(k_0-1)}$ are compactly supported continuous functions. Taking derivatives on both sides of the equation $T_a\phi = \phi$, gives

$$\phi^{(l)} = 2^l T_a \phi^{(l)}, \quad l = 0, 1, \dots, k_0 - 1. \tag{2.6}$$

Let S be the sampling operator on the integers, i.e. $S : C(\mathbb{R}) \to \ell(\mathbb{Z})$ such that $Sf := \{f(k)\}_{k\in\mathbb{Z}}$ $\forall f \in C(\mathbb{R})$. Applying the sampling operator S and using the commutation identity $ST_a = S_a S$, lead to

$$S_a S\phi^{(l)} = 2^{-l} S\phi^{(l)}, \quad l = 0, 1, \dots, k_0 - 1. \tag{2.7}$$

We claim that $S\phi^{(l)}$ is not a zero sequence. Suppose on the contrary that $S\phi^{(l)} \equiv 0$, i.e. $\phi^{(l)}(k) = 0$ for all $k \in \mathbb{Z}$. Using (2.6) inductively on $n \in \mathbb{Z}_+$, we have $\phi^{(l)}(2^{-n}k) = 0$ for all $k \in \mathbb{Z}$, which together with the continuity of $\phi^{(l)}$ leads to $\phi^{(l)} \equiv 0$. Hence ϕ is a polynomial, which contradicts the fact that ϕ is a nonzero compactly supported function. This proves our claim that $S\phi^{(l)}$ is not a zero sequence. Hence 2^{-l}, $l = 0, 1, \dots, k_0 - 1$, are eigenvalues of the operator S_a on ℓ_0 by (2.7) and the above claim. Therefore it remains to prove that $2^{-k_0} \in \sigma_e(S_a, \ell_0)$. Recall that $\phi, \phi', \dots, \phi^{(k_0)} \in L^2_c$ by Lemma 2.3. Then

$$H(u) = \left(\frac{1 + e^{-iu}}{2}\right)^{k_0+1} \widetilde{H}(u), \tag{2.8}$$

for some trigonometric polynomial $\widetilde{H}(u)$ by Lemma 2.4. Setting $H_1(u) = \left(\frac{1+e^{-iu}}{2}\right)\widetilde{H}(u)$ and writing $H_1(u) = \sum_{k\in\mathbb{Z}} a_1(k)e^{-iku}$, we have

$$\sum_{k\in\mathbb{Z}} a_1(2k) = \sum_{k\in\mathbb{Z}} a_1(2k+1) = 1. \tag{2.9}$$

Let N_1 be so chosen that $a_1(k) = 0$ for all integer k with $|k| > N_1$. Then by (2.9), the sum of the entries of every column of the matrix $B := (a_1(2i - j))_{i,j=-N_1}^{N_1}$ is 1. Thus 1 is an eigenvalue of B. Hence there exists a nonzero vector $v_1 = (v_1(-N_1), \dots, v_1(N_1))^T$ such that $Bv_1 = v_1$. Considered as a compactly supported sequence, v_1 has Fourier series $\widehat{v}_1(u) = \sum_{k=-N_1}^{N_1} v_1(k)e^{-iku}$. Then

$$\begin{aligned} \widehat{v}_1(2u) &= (Bv_1)^\wedge(2u) = \sum_{j,k=-N_1}^{N_1} a_1(2j-k)v_1(k)e^{-2iju} \\ &= \sum_{j,k\in\mathbb{Z}} a_1(2j-k)v_1(k)e^{-2iju} = (S_{a_1}v_1)^\wedge(2u). \end{aligned} \tag{2.10}$$

By (2.8) and (2.10), the nonzero finitely supported sequence $w \in \ell_0$ defined by $\widehat{w}(u) = (1-e^{-iu})^{k_0}\widehat{v}_1(u)$ satisfies

$$\begin{aligned}(S_a w)^\wedge(u) &= H(u)\widehat{w}(u/2) + H(u/2+\pi)\widehat{w}(u/2+\pi)\\ &= 2^{-k_0}(1-e^{-iu})^{k_0}\left(H_1(u/2)\widehat{v}_1(u/2) + H_1(u/2+\pi)\widehat{v}_1(u/2+\pi)\right)\\ &= 2^{-k_0}(1-e^{-iu})^{k_0}(S_{a_1}v_1)^\wedge(u)\\ &= 2^{-k_0}(1-e^{-iu})^{k_0}\widehat{v}_1(u) = 2^{-k_0}\widehat{w}(u). \qquad (2.11)\end{aligned}$$

This proves that $2^{-k_0} \in \sigma_e(S_a, \ell_0)$, and hence completes the proof of Theorem 2.1. □

3. Scaling Operators and B-splines

For a sequence $a := \{a(k)\}_{k\in\mathbb{Z}} \in \ell_0$ let $I_a \subset \mathbb{Z}$ be the smallest interval containing the support of a. For an interval $I \subset \mathbb{Z}$ let $\ell(I) := \{a \in \ell_0 : supp(a) \subset I\}$. One may verify that $\ell(I_a)$ is an invariant subspace of S_a on ℓ_0, i.e. $S_a v \in \ell(I_a)$ for all $v \in \ell(I_a)$. For an integer $N \geq 1$, let $a_N = \{a_N(k)\}_{k\in\mathbb{Z}}$ be the binomial sequence of order N, i.e. $a_N(k) = \frac{1}{2^{N-1}}\binom{N}{k}$ for $k = 0, 1, \ldots, N$, and 0 otherwise. The corresponding scaling operator T_{a_N} and the transition operator S_{a_N} restricted to compactly supported L^2-functions and sequences supported in I_{a_N} respectively have the same eigenvalues as described in the following theorem.

Theorem 3.1. *If a_N is the binomial sequence of order N, then*

$$\sigma_e(T_{a_N}, L^2_c) = \{2^{-k} :\ k = 0, 1 \ldots, N-1\}, \qquad (3.1)$$

$$\sigma_e(S_{a_N}, \ell(I_{a_N})) = \{2^{-k} : k = 0, 1 \ldots, N-1\}, \qquad (3.2)$$

and

$$\sigma_e(S_{a_N}, \ell_0) = \{0\} \cup \{2^{-k} : k = 0, 1 \ldots, N-1\}. \qquad (3.3)$$

For a sequence $a := \{a(k)\}_{k\in\mathbb{Z}} \in \ell_0$ that satisfies $\sum_{k\in\mathbb{Z}} a(k) = 2$, Theorem 2.1 says that $\sigma_e(T_a, L^2_c) \cup \{0\} \subset \sigma_e(S_a, \ell_0)$. On the other hand, Theorem 3.1 shows that equality is attained for the binomial sequences a_N for any $N \in \mathbb{N}$. A natural question is whether equality holds for any other sequences $a \in \ell_0$ besides the binomial sequences. The answer is no and the next result shows why not.

Theorem 3.2. *Let $a := \{a(k)\}_{k\in\mathbb{Z}} \in \ell_0$ satisfy $\sum_{k\in\mathbb{Z}} a(k) = 2$. Then the following are equivalent.*

(a) *The mask a is a shifted binomial sequence, i.e.*

$$\frac{1}{2}\sum_{k\in\mathbb{Z}} a(k)e^{-iku} = e^{-iLu}\Big(\frac{1+e^{-iu}}{2}\Big)^N = 2^{-N}e^{-iLu}\sum_{k=0}^{N}\binom{N}{k}e^{-iku} \ ,$$

where $L = \max(I_a)$ *and* $N = |I_a|$.

(b) $\sigma_e(S_a, \ell(I_a)) = \sigma_e(T_a, L^2_c)$.

(c) $\sigma_e(S_a, \ell_0) = \sigma_e(T_a, L^2_c) \cup \{0\}$ *and* $0 \notin \sigma_e(S_a, \ell(I_a))$.

Refinement equations whose masks are shifted binomial sequences arise in many situations. The corresponding refinable functions are B-splines, which have many desirable properties ideal for signal processing [16] and geometric modelling [3]. Here the N-th order B-spline B_N is defined inductively by $B_{N+1} := B_N * B_1$ with $B_1 := \chi_{[0,1]}$. Theorem 3.2 characterizes B-splines via the connection between the eigenvalues of the corresponding scaling operators and transition operators on spaces of compactly supported L^2-functions and finitely supported sequences respectively. See ([13, 15]) and references therein for more characterizations of B-splines.

To prove Theorem 3.1, we need a lemma.

Lemma 3.3. *Let* $a := \{a(k)\}_{k\in\mathbb{Z}} \in \ell_0$ *satisfy* $\sum_{k\in\mathbb{Z}} a(k) = 2$. *Then*

$$\sigma_e(S_a, \ell(I_a)) \cup \{0\} = \sigma_e(S_a, \ell(I_a \cup (I_a + 1))) = \sigma_e(S_a, \ell_0). \tag{3.4}$$

Proof. From the matrix representation of S_a, it follows that

$$0 \in \sigma_e(S_a, \ell(I_a \cup (I_a + 1))). \tag{3.5}$$

Because of (3.5), the following inclusions hold,

$$\sigma_e(S_a, \ell(I_a)) \cup \{0\} \subset \sigma_e(S_a, \ell(I_a \cup (I_a + 1))) \subset \sigma_e(S_a, \ell_0).$$

Then we need only to show that $\sigma_e(S_a, \ell_0) \subset \sigma_e(S_a, \ell(I_a)) \cup \{0\}$. Note that $I_{(S_a v)} = \frac{I_a + I_v}{2} \cap \mathbb{Z}$ for any $v \in \ell_0$. Then if $\lambda \in \sigma_e(S_a, \ell_0)\backslash\{0\}$ and $0 \neq v_\lambda \in \ell_0$ satisfies $S_a v_\lambda = \lambda v_\lambda$, then $I_{v_\lambda} \subset I_a$, which means that $v_\lambda \in \ell(I_a)$ and $\lambda \in \sigma_e(S_a, \ell(I_a))$. □

Proof of Theorem 3.1

Proof. We first prove (3.1). Since the B-spline B_N is a piecewise polynomial of degree $N-1$ of compact support and $B_N \in C^{N-2}$ for $N \geq 1$, it follows that $B_N^{(n)} \in L^2_c$ for any $n = 0, 1, \ldots, N-1$, and $B_N^{(N)} \notin L^2_c$. Therefore, the first assertion (3.1) follows from Lemma 2.3.

Next we prove (3.2). For the binomial sequence a_N, we have $I_{a_N} = [0, N] \cap \mathbb{Z}$ and $\widehat{a}_N(u) = 2^{-N+1}(1 + e^{-iu})^N$. One may verify that the functions $F_k, 1 \le k \in \mathbb{Z}$, defined by $F_1(u) := 1$ and

$$F_k(u) := \sum_{j\in\mathbb{Z}} \left(\frac{e^{-i(u+2j\pi)} - 1}{-i(u + 2j\pi)}\right)^k = \sum_{j\in\mathbb{Z}} B_k(j)e^{-iju},$$

satisfy

$$F_k(u) = \left(\frac{1 + e^{-iu/2}}{2}\right)^k F_k\Big(\frac{u}{2}\Big) + \left(\frac{1 + e^{-i(u/2+\pi)}}{2}\right)^k F_k\Big(\frac{u}{2} + \pi\Big), \; k \ge 1.$$

Therefore by the proof of (2.11), for any $k = 0, \ldots , N - 1$, the sequence v_k with $\widehat{v}_k(u) = (1 - e^{-iu})^k F_{N-k}(u)$, $k = 0, 1, \ldots , N - 1$, belong to $\ell(I_{a_N})$ and are the eigensequences of the transition operator S_{a_N} associated with the eigenvalue 2^{-k}. This yields

$$\{2^{-k} : k = 0, 1, \ldots , N - 1\} \subset \sigma_e(S_{a_N}, \ell(I_{a_N})). \tag{3.6}$$

One may also verify that the sequences v, whose Fourier transforms are $(1 - e^{-iu})^{N-1}$ and $(1 - e^{-iu})^{N-1}e^{-iu}$, belong to $\ell(I_{a_N})$ and are linearly independent eigenvector of the transition operator S_{a_N} associated with the eigenvalue 2^{-N+1}. By (3.6) and the above fact, there are N distinct eigenvalues for the operator S_{a_N} on $\ell(I_{a_N})$ and the eigenspace associated with the eigenvalue 2^{-N+1} is at least two. Recall that the dimension of the space $\ell(I_{a_N})$ is $N + 1$. Then

$$\sigma_e(S_{a_N}, \ell(I_{a_N})) \subset \{2^{-k} : k = 0, 1, \ldots , N - 1\}. \tag{3.7}$$

Hence (3.2) follows from (3.6) and (3.7).

Finally the assertion (3.3) follows easily from (3.2) and Lemma 3.3. □

Proof of Theorem 3.2

Proof. For any $l \in \mathbb{Z}$, $\tau_l : \ell_0 \to \ell_0$ denotes the shift operator $\tau_l a = a(\cdot+l)$. We remark that $\sigma_e(S_a, \ell_0) = \sigma_e(S_{\tau_l a}, \ell_0)$, $\sigma_e(S_a, \ell(I_a)) = \sigma_e(S_{\tau_l a}, \ell(I_{\tau_l a}))$, and $\sigma_e(T_a, L^2_c) = \sigma_e(T_{\tau_l a}, L^2_c)$. So we can assume without loss of generality that $I_a = \{0, \ldots , N\}$ for some positive integer N.

That (a) implies (b) is the result of Theorem 3.1, while the equivalence between (c) and (b) follows from (2.2) and (3.4) and the fact that $0 \notin \sigma_e(T_a, L^2_c)$. Now we show that (b) implies (a). By Lemma 2.3, $\sigma_e(T_a, L^2_c) = \emptyset$ or $\{1, 1/2, \ldots , 1/2^{k_0}\}$ for some non-negative integer k_0. Since $a(\min(I_a)), a(\max(I_a)) \in \sigma_e(S_a, \ell(I_a))$ by the matrix representation of S_a, it follows that

$$\sigma_e(S_a, \ell(I_a)) = \{1, 1/2, \ldots , 1/2^{k_0}\}. \tag{3.8}$$

Since as an operator on $\ell(I_a)$, S_a is represented by the matrix $(a(2i-j))_{i,j=0}^N$, its eigenpolynomial is of the form $(-1)^{N+1}\lambda^{N+1} + (-1)^N \mathrm{trace}(S_a)\lambda^N + g(\lambda)$, where $\mathrm{trace}(S_a) = \sum_{j=0}^N a(j) = 2$ and g is a polynomial of degree less than N. Thus the sum of all the eigenvalues of S_a, counting multiplicity is equal to $\mathrm{trace}(S_a) = 2$. Therefore, assuming that $1/2^k$ is an eigenvalue of S_a on $\ell(I_a)$ with multiplicity $l_k \geq 1$, we obtain the equations $\sum_{k=0}^{k_0} l_k/2^k = 2$ and $\sum_{k=0}^{k_0} l_k = N+1$. The first equation yields $l_k = 1$ for $k = 0, 1, \ldots, k_0 - 1$, $l_{k_0} = 2$. The second equation then implies

$$k_0 = N - 1. \tag{3.9}$$

By (3.8), (3.9) and the assumption (b), we obtain $\sigma_e(T_a, L_c^2) = \{1, 1/2, \cdots, 1/2^{N-1}\}$. This together with Lemmas 2.3 and 2.4 and the fact $I_a = \{0, \ldots, N\}$ prove that $H(u) := \frac{1}{2}\sum_{k\in\mathbb{Z}} a(k)e^{-iku} = \left(\frac{1+e^{-iu}}{2}\right)^N$, and hence the assertion (a). □

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