

Stability of Localized Operators

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Abstract

Let $\ell^p, 1 \leq p \leq \infty$, be the space of all p -summable sequences and C_a be the convolution operator associated with a summable sequence a . It is known that the ℓ^p -stability of the convolution operator C_a for different $1 \leq p \leq \infty$ are equivalent to each other, i.e., if C_a has ℓ^p -stability for some $1 \leq p \leq \infty$ then C_a has ℓ^q -stability for all $1 \leq q \leq \infty$. In the study of spline approximation, wavelet analysis, time-frequency analysis, and sampling, there are many localized operators of non-convolution type whose stability is one of the basic assumptions. In this paper, we consider the stability of those localized operators including infinite matrices in the Sjöstrand class, synthesis operators with generating functions enveloped by shifts of a function in the Wiener amalgam space, and integral operators with kernels having certain regularity and decay at infinity. We show that the ℓ^p -stability (or L^p -stability) of those three classes of localized operators are equivalent to each other, and we also prove that the left inverse of those localized operators are well localized.

Key words: Wiener's lemma, stability, infinite matrix with off-diagonal decay, synthesis operator, localized integral operator, Banach algebra, Gabor system, sampling, Schur class, Sjöstrand class, Kurbatov class

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1 Introduction

Given a summable sequence $a = (a(j))_{j \in \mathbb{Z}}$, the *convolution operator* C_a associated with the sequence a is defined by

$$C_a : \ell^p \ni (b(j))_{j \in \mathbb{Z}} \mapsto \left(\sum_{k \in \mathbb{Z}} a(j-k)b(k) \right)_{j \in \mathbb{Z}} \in \ell^p \quad (1.1)$$

where ℓ^p is the set of all p -summable sequences with standard norm $\|\cdot\|_{\ell^p}$. The convolution operator C_a is a bounded operator on ℓ^p for any $1 \leq p \leq \infty$ and the corresponding operator norm is bounded by the ℓ^1 norm of the sequence a .

An operator T on ℓ^p is said to have ℓ^p -*stability* if there exists a positive constant C such that

$$C^{-1}\|c\|_{\ell^p} \leq \|Tc\|_{\ell^p} \leq C\|c\|_{\ell^p} \quad \text{for all } c \in \ell^p. \quad (1.2)$$

For the convolution operator C_a associated with a summable sequence $a = (a(j))_{j \in \mathbb{Z}}$, it is known that C_a has ℓ^p -stability for some $1 \leq p \leq \infty$ if and only if

$$\hat{a}(\xi) \neq 0 \quad \text{for all } \xi \in \mathbb{R} \quad (1.3)$$

where $\hat{a}(\xi) := \sum_{j \in \mathbb{Z}} a(j)e^{-ij\xi}$ (c.f. [3,43]). Therefore the ℓ^p -stability of the convolution operator associated with a summable sequence are equivalent to each other for different $1 \leq p \leq \infty$.

Theorem 1.1 *Let a be a summable sequence, and C_a be the convolution operator associated with the sequence a . If C_a has ℓ^p -stability for some $1 \leq p \leq \infty$, then it has ℓ^q -stability for all $1 \leq q \leq \infty$.*

An equivalent formulation of the above result is that the spectrum $\sigma_p(C_a)$ of the convolution operator C_a associated with a summable sequence a as an operator on ℓ^p is independent of $1 \leq p \leq \infty$,

$$\sigma_p(C_a) = \sigma_q(C_a) \quad \text{for all } 1 \leq p, q \leq \infty \quad (1.4)$$

see [9,41,52] and references therein for the discussion on spectrum of convolution operators.

Inspired by the commutator technique developed in [56] and norm equivalence technique for a finite-dimensional space in [2], we will give a new proof of Theorem 1.1 in this paper without using the characterization (1.3). More importantly we can extend the equivalence for ℓ^p -stability in Theorem 1.1 to various localized operators of non-convolution type, that arise in the study of spline approximation ([26,28]), wavelet and affine frames ([20,42]), Gabor

frame and non-uniform sampling ([6,35,36,61]), and pseudo-differential operators ([13,14,29,34,38,40,56,57,67–69]).

Denote by \mathcal{A} the *Schur class* of infinite matrices $A = (a(j, j'))_{j, j' \in \mathbb{Z}}$ such that

$$\|A\|_{\mathcal{A}} := \max \left(\sup_{j \in \mathbb{Z}} \sum_{j' \in \mathbb{Z}} |a(j, j')|, \sup_{j' \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} |a(j, j')| \right) < \infty \quad (1.5)$$

([36,62]). An infinite matrix $A = (a(j, j'))_{j, j' \in \mathbb{Z}}$ in the Schur class defines a bounded operator on ℓ^p , $1 \leq p \leq \infty$, as follows:

$$A : \ell^p \ni (c(j))_{j \in \mathbb{Z}} \mapsto \left(\sum_{j' \in \mathbb{Z}} a(j, j') c(j') \right)_{j \in \mathbb{Z}} \in \ell^p. \quad (1.6)$$

It is known that an infinite matrix A is a bounded operator on ℓ^p for any $1 \leq p \leq \infty$ if and only if A is in the Schur class. In Section 2, we consider the equivalence of ℓ^p -stability for infinite matrices in the Sjöstrand class (see Section 2 for its definition), a subset of the Schur class \mathcal{A} , for different $1 \leq p \leq \infty$.

Theorem 1.2 *Let $A = (a(j, j'))_{j, j' \in \mathbb{Z}}$ be an infinite matrix with the property that $\sum_{k \in \mathbb{Z}} \sup_{j \in \mathbb{Z}} |a(j, j+k)| < \infty$. If A has ℓ^p -stability for some $1 \leq p \leq \infty$, then A has ℓ^q -stability for all $1 \leq q \leq \infty$.*

The result in Theorem 1.2 for $p = 2$ follows from the Wiener's lemma in [56]. The equivalence of ℓ^q -stability of an infinite matrix $A = (a(j, j'))_{j, j' \in \mathbb{Z}}$ with $\sum_{k \in \mathbb{Z}} \sup_{j \in \mathbb{Z}} |a(j, j+k)|(1+|k|)^s < \infty$ is established in [2] for $s > 4$ and later improved in [66] for $s > 0$.

We observe that a convolution operator C_a associated with a summable sequence $a = (a(j))_{j \in \mathbb{Z}}$ is the operator associated with the infinite matrix $A = (a(j - j'))_{j, j' \in \mathbb{Z}}$ in the Sjöstrand class. Therefore Theorem 1.1 follows from Theorem 1.2.

We conjecture that the equivalence of ℓ^p -stability for different $p \in [1, \infty]$ holds for any infinite matrix in the Schur class \mathcal{A} . Some progress on the above conjecture is made in [66] under additional assumption that the infinite matrix has rows supported in balls of bounded radii.

For a continuous function f on \mathbb{R} , we define the *modulus of continuity* $\omega_\delta(f)$ by

$$\omega_\delta(f)(x) = \sup_{|y| \leq \delta} |f(x+y) - f(x)|. \quad (1.7)$$

The modulus of continuity is a delicate tool in mathematical analysis to measure the regularity of a function ([27,70]).

For $1 \leq p \leq \infty$, let L^p be the space of all p -integrable functions on \mathbb{R}^d with standard norm $\|\cdot\|_p$,

$$\mathcal{L}^p = \left\{ \phi : \|\phi\|_{\mathcal{L}^p} := \left\| \sum_{j \in \mathbb{Z}} |\phi(\cdot - j)| \right\|_{L^p([0,1])} < \infty \right\}$$

consists of functions on \mathbb{R} that are “globally” in ℓ^1 and “locally” in L^p ([43]), and let

$$\mathcal{W}_1 = \left\{ \phi : \|\phi\|_{\mathcal{W}_1} := \sum_{k \in \mathbb{Z}} \sup_{x \in [0,1]} |\phi(k+x)| < \infty \right\}$$

be the *Wiener amalgam space* that consists of functions that are “locally” in L^∞ and “globally” in ℓ^1 ([1]). We have the following inclusions for the above three classes of function spaces:

$$\mathcal{W}_1 \subset \mathcal{L}^\infty \subset \mathcal{L}^p \subset \mathcal{L}^1 \tag{1.8}$$

and

$$\mathcal{L}^p \subset L^p \tag{1.9}$$

where $1 \leq p \leq \infty$ ([1,43]).

For a family of functions $\Phi = \{\phi_j\}_{j \in \mathbb{Z}}$ enveloped by a function $h \in \mathcal{L}^p$, i.e.,

$$|\phi_j(x)| \leq h(x-j) \quad \text{for all } j \in \mathbb{Z} \text{ and } x \in \mathbb{R},$$

the *synthesis operator* S_Φ associated with Φ ,

$$S_\Phi : \ell^p \ni (c(j))_{j \in \mathbb{Z}} \mapsto \sum_{j \in \mathbb{Z}} c(j)\phi_j \in L^p, \tag{1.10}$$

is a bounded operator from ℓ^p to L^p ([43,63]). For the family of functions Φ generated by shifts of finitely many functions, that is, $\Phi = \{\phi_n(\cdot - j)\}_{1 \leq n \leq N, j \in \mathbb{Z}}$ for some $\phi_n \in \mathcal{L}^\infty$, $1 \leq n \leq N$, it is proved in [43] that if the synthesis operator S_Φ in (1.10) has ℓ^p -stability for some $1 \leq p \leq \infty$, i.e., there exists a positive constant C such that

$$C^{-1}\|c\|_{\ell^p} \leq \|S_\Phi c\|_p \leq C\|c\|_{\ell^p} \quad \text{for all } c \in \ell^p, \tag{1.11}$$

then it has ℓ^q -stability for all $1 \leq q \leq \infty$. In Section 3, we establish the equivalence of the stability of the synthesis operator S_Φ for $1 \leq p \leq \infty$ under some regularity and decay assumption on the generating family Φ of functions.

Theorem 1.3 *Let $\Phi = \{\phi_j\}_{j \in \mathbb{Z}}$ be a family of functions with the property that*

$$|\phi_j(x)| \leq h(x-j) \quad \text{for all } x \in \mathbb{R}, j \in \mathbb{Z}, \tag{1.12}$$

and

$$|\omega_\delta(\phi_j)(x)| \leq \omega(\delta)h(x-j) \text{ for all } x \in \mathbb{R}, j \in \mathbb{Z} \text{ and } \delta \in (0,1), \quad (1.13)$$

where h is a continuous function in the Wiener amalgam space \mathcal{W}_1 and $\omega(\delta)$ is a positive increasing function on $(0,1)$ with $\lim_{\delta \rightarrow 0} \omega(\delta) = 0$. Let S_Φ be the synthesis operator associated with the family Φ of functions. If the synthesis operator S_Φ has ℓ^p -stability for some $1 \leq p \leq \infty$, then it has ℓ^q -stability for all $1 \leq q \leq \infty$.

For $p = 2$, the conclusion in Theorem 1.3 follows from the Wiener's lemma in [63] under the decay assumption (1.12) on the generating family Φ , but without the regularity assumption (1.13) on the generating family Φ as in Theorem 1.3.

In Section 4, we consider the L^p -stability of localized integral operators for different p ([46,64]). Here a bounded operator S on L^p is said to have L^p -stability if there exists a positive constant C such that

$$C^{-1}\|f\|_p \leq \|Sf\|_p \leq C\|f\|_p \text{ for all } f \in L^p. \quad (1.14)$$

Theorem 1.4 *Let I be the identity operator on L^p , and T be a localized integral operator*

$$Tf(x) := \int_{\mathbb{R}} K_T(x,y)f(y)dy, \quad f \in L^p$$

such that its integral kernel K_T satisfies

$$|K_T(x,y)| \leq h(x-y) \text{ for all } x,y \in \mathbb{R}, \quad (1.15)$$

and

$$|\omega_\delta(K_T)(x,y)| \leq \delta^\alpha h(x-y) \text{ for all } x,y \in \mathbb{R}, \text{ and } \delta \in (0,1) \quad (1.16)$$

where $\alpha \in (0,1)$ and h is a continuous function in the Wiener amalgam space \mathcal{W}_1 . If $I+T$ has L^p -stability for some $1 \leq p \leq \infty$, then $I+T$ has L^q -stability for all $1 \leq q \leq \infty$.

For $p = 2$, the conclusion in Theorem 1.4 follows from the Wiener's lemma in [64].

In this paper, we will use the following notation: Denote by χ_E the characteristic function on the set E . For a discrete set Λ and $1 \leq p \leq \infty$, denote by $\ell^p(\Lambda)$ the set of all p -summable sequence $c := (c(\lambda))_{\lambda \in \Lambda}$ with standard norm $\|\cdot\|_{\ell^p(\Lambda)}$, and by $(\ell^p(\Lambda))^N$ the N copies of $\ell^p(\Lambda)$ with norm $\|\cdot\|_{(\ell^p(\Lambda))^N}$. Let $\mathcal{B}(\ell^p(\Lambda)), 1 \leq p \leq \infty$, be the Banach algebra containing all bounded

operators on $\ell^p(\Lambda)$ embedded with standard operator norm. For a measurable set E , let $L^p(E)$ be the space of all p -integrable functions on E with norm $\|\cdot\|_{L^p(E)}$, while if $E = \mathbb{R}^d$, we use L^p instead of $L^p(\mathbb{R}^d)$ and $\|\cdot\|_p$ instead of $\|\cdot\|_{L^p(\mathbb{R}^d)}$ for brevity. Let $\mathcal{B}(L^p)$ be the Banach algebra containing all bounded operators on $L^p(\mathbb{R}^d)$ embedded with standard operator norm. For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, define $\|x\|_\infty = \max(|x_1|, \dots, |x_d|)$. Denote by I the identity operator on L^p , $1 \leq p \leq \infty$, or the identity matrix of appropriate size.

In this paper, the uppercase letter C denotes an absolute constant that may be different at different occurrences, except stated explicitly.

2 ℓ^p -stability for localized infinite matrices

In this section, we consider the ℓ^p -stability for infinite matrices of the form $(a(\lambda, \lambda'))_{\lambda \in \Lambda, \lambda' \in \Lambda'}$ having certain off-diagonal decay. That kind of extreme non-commutative matrices arises in the study of spline approximation ([26,28]), wavelet and affine wavelets ([20,42]), Gabor frame ([6,35,36]), non-uniform sampling ([61]), and pseudo-differential operators ([13,14,29,34,38,40,56,57,67–69]), and the ℓ^p -stability for those matrices is one of few basic assumptions in these studies. The main results of this section are Theorem 2.1 (a slight generalization of Theorem 1.2), Corollary 2.3 (an equality for spectrum of slanted matrices on ℓ^p for different p) and Corollary 2.4 (a Wiener's lemma for the Sjöstrand class of infinite matrices).

To state our result on stability for localized infinite matrices, we recall three concepts. We say that a discrete subset Λ of \mathbb{R}^d is *relatively-separated* if

$$R(\Lambda) = \sup_{x \in \mathbb{R}^d} \sum_{\lambda \in \Lambda} \chi_{\lambda + [0,1]^d}(x) < \infty. \quad (2.1)$$

For relatively-separated subsets Λ, Λ' of \mathbb{R}^d , we let $\mathcal{C}(\Lambda, \Lambda')$, or \mathcal{C} for short, be the *Sjöstrand class* of infinite matrices $A = (a(\lambda, \lambda'))_{\lambda \in \Lambda, \lambda' \in \Lambda'}$ such that

$$\|A\|_c := \sum_{k \in \mathbb{Z}^d} \sup_{\lambda \in \Lambda, \lambda' \in \Lambda'} |a(\lambda, \lambda')| \chi_{k + [0,1]^d}(\lambda - \lambda') < \infty \quad (2.2)$$

([56,62]). As usual, an infinite matrix $A = (a(\lambda, \lambda'))_{\lambda \in \Lambda, \lambda' \in \Lambda'}$ in the Sjöstrand class $\mathcal{C}(\Lambda, \Lambda')$ defines a bounded operator from $\ell^p(\Lambda')$ to $\ell^p(\Lambda)$,

$$A : \ell^p(\Lambda') \ni (c(\lambda'))_{\lambda' \in \Lambda'} := c \longmapsto Ac := \left(\sum_{\lambda' \in \Lambda'} a(\lambda, \lambda') c(\lambda') \right)_{\lambda \in \Lambda} \in \ell^p(\Lambda), \quad (2.3)$$

where $1 \leq p \leq \infty$ ([62]). For $1 \leq p \leq \infty$, we say that an infinite matrix $A := (a(\lambda, \lambda'))_{\lambda \in \Lambda, \lambda' \in \Lambda'}$ has ℓ^p -stability if there exists a positive constant C

such that

$$C^{-1}\|c\|_{\ell^p(\Lambda')} \leq \|Ac\|_{\ell^p(\Lambda)} \leq C\|c\|_{\ell^p(\Lambda')} \quad \text{for all } c \in \ell^p(\Lambda'). \quad (2.4)$$

Theorem 2.1 *Let Λ, Λ' be two relatively-separated subsets of \mathbb{R}^d , and $A = (a(\lambda, \lambda'))_{\lambda \in \Lambda, \lambda' \in \Lambda'}$ be an infinite matrix in the Sjöstrand class $\mathcal{C}(\Lambda, \Lambda')$. If A has ℓ^p -stability for some $1 \leq p \leq \infty$, then A has ℓ^q -stability for all $1 \leq q \leq \infty$.*

For $\Lambda' = \Lambda$, the above theorem can be reformulated as follows:

Corollary 2.2 *Let Λ be a relatively-separated subset of \mathbb{R}^d . Then the spectrum $\sigma_p(A)$ of an infinite matrix A in the Sjöstrand class $\mathcal{C}(\Lambda, \Lambda)$ as an operator on $\ell^p(\Lambda)$ is independent of $1 \leq p \leq \infty$, i.e.,*

$$\sigma_p(A) = \sigma_q(A) \quad 1 \leq p, q \leq \infty. \quad (2.5)$$

A function w on \mathbb{R}^d is said to be a *weight* if $w(x) \geq 1$ for all $x \in \mathbb{R}^d$ and $\sup_{|y| \leq 1} \sup_{x \in \mathbb{R}^d} \frac{w(x+y)}{w(x)} < \infty$. For a weight w on \mathbb{R}^d and a positive number α , denote by Σ_α^w the family of all α -*slant infinite matrices* $A = (a(j, j'))_{j, j' \in \mathbb{Z}^d}$ with

$$\|A\|_{\Sigma_\alpha^w} = \sum_{k \in \mathbb{Z}^d} w(k) \sup_{j, j' \in \mathbb{Z}^d} |a(j, j')| \chi_{k+[0,1)^d}(j' - \alpha j) < \infty.$$

The slanted matrices appear in wavelet theory, signal processing and sampling theory [2,11,16,45], and also occur in the K-theory of operator algebras [72]. Note that

$$\Sigma_\alpha^w \subset \Sigma_\alpha^{w_0} = \mathcal{C}(\alpha\mathbb{Z}^d, \mathbb{Z}^d) \quad (2.6)$$

for any weight w where $w_0 \equiv 1$ is the trivial weight. Then we obtain the following result from Theorem 2.1.

Corollary 2.3 *Let $\alpha > 0$ and $w_0 \equiv 1$. If $A \in \Sigma_\alpha^{w_0}$ has ℓ^p -stability for some $1 \leq p \leq \infty$, then A has ℓ^q -stability for all $1 \leq q \leq \infty$.*

It follows from (2.6) and Corollary 2.3 that for any weight w , the ℓ^p -stability of any infinite matrix $A \in \Sigma_\alpha^w$ are equivalent for different $1 \leq p \leq \infty$. This result for the special weights $w(x) = (1 + |x|)^s$, with $s > (d + 1)^2$ and with $s > 0$, is established in [2] and [66] respectively.

Given a Banach algebra \mathcal{B} , we say that a subalgebra \mathcal{A} of \mathcal{B} is *inverse-closed* if the inverse T^{-1} of the operator $T \in \mathcal{A}$ belongs to \mathcal{B} implies that it belongs to \mathcal{A} ([23,30,50,53,65]). The inverse-closed subalgebra was first studied for periodic functions with absolutely convergent Fourier series, which states that *if a periodic function f does not vanish on the real line and has absolutely convergent Fourier series, i.e., $f(x) = \sum_{j \in \mathbb{Z}} a(j)e^{-ijx}$ and $\sum_{j \in \mathbb{Z}} |a(j)| < \infty$,*

then f^{-1} has absolutely convergent Fourier series too ([71]). An equivalent formulation of the above Wiener's lemma involving matrix algebras is that the commutative Banach algebra

$$\tilde{\mathcal{W}} := \left\{ \left(a(j - j') \right)_{j, j' \in \mathbb{Z}}, \sum_{j \in \mathbb{Z}} |a(j)| < \infty \right\} \quad (2.7)$$

is an inverse-closed Banach subalgebra of $\mathcal{B}^2(\ell^2(\mathbb{Z}))$ ([71]). The classical Wiener's lemma and its various generalizations (see, e.g., [6–8, 10, 23, 28, 35, 36, 39, 42, 53, 56]) are important and have numerous applications in numerical analysis, wavelet theory, frame theory, and sampling theory. For example, the classical Wiener's lemma and its weighted variation ([39]) were used to establish the decay property at infinity for dual generators of a shift-invariant space ([1, 43]); the Wiener's lemma for matrices associated with twisted convolution was used in the study the decay properties of the dual Gabor frame for L^2 ([6, 35, 36]); the Jaffard's result ([42]) for infinite matrices with polynomial decay was used in numerical analysis ([17, 58, 59]), wavelet analysis ([42]), time-frequency analysis ([31–33]) and sampling ([4, 24, 33, 63]); and the Sjöstrand's result ([56]) for infinite matrices was used in the study of pseudo-differential operators, Gabor frames and sampling ([6, 34, 56, 61]). Therefore there are lots of papers devoted to the Wiener's lemma for infinite matrices with various off-diagonal decay conditions (see [5–7, 10, 12, 28, 36, 39, 42, 56, 60, 62] and also [37] for a short historical review). The Wiener's lemma for the Sjöstrand class $\mathcal{C}(\Lambda, \Lambda)$ of infinite matrices $(a(\lambda, \lambda'))_{\lambda, \lambda' \in \Lambda}$ says that $\mathcal{C}(\Lambda, \Lambda)$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^2(\Lambda))$ where Λ is a relatively-separated subset of \mathbb{R}^d ([56]). This together with the equivalence of ℓ^p -stability for different p in Theorem 2.1 proves that $\mathcal{C}(\Lambda, \Lambda)$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^p(\Lambda))$ for any $1 \leq p \leq \infty$.

Corollary 2.4 *Let $1 \leq p \leq \infty$ and Λ be a relatively-separated subset of \mathbb{R}^d . Then the Sjöstrand class $\mathcal{C}(\Lambda, \Lambda)$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^p(\Lambda))$, i.e., if $A \in \mathcal{C}(\Lambda, \Lambda)$ has bounded inverse on $\mathcal{B}(\ell^p(\Lambda))$, then $A^{-1} \in \mathcal{C}(\Lambda, \Lambda)$.*

Before we start the proof of Theorem 2.1, let us consider necessary conditions on the relatively-separated subsets Λ and Λ' such that there exists a matrix $A = (a(\lambda, \lambda'))_{\lambda \in \Lambda, \lambda' \in \Lambda'}$ in the Sjöstrand class $\mathcal{C}(\Lambda, \Lambda')$ which has ℓ^p -stability for some $1 \leq p \leq \infty$. Similar conclusion is obtained in [61] for sampling signals with finite rate of innovation, and in [51] for slanted matrices.

Proposition 2.5 *Let Λ, Λ' be relatively-separated subsets of \mathbb{R}^d . If there exists a matrix $A = (a(\lambda, \lambda'))_{\lambda \in \Lambda, \lambda' \in \Lambda'}$ in the Sjöstrand class $\mathcal{C}(\Lambda, \Lambda')$ which has ℓ^p -stability for some $1 \leq p \leq \infty$, then there exists a positive number R_0 such that for any bounded set K the cardinality of the set $\Lambda \cap B(K, R_0)$ is larger than or equal to the cardinality of the set $\Lambda' \cap K$, where $B(K, R)$ is the set of all points in \mathbb{R}^d with distance to K less than R .*

PROOF. We show the above result on relatively-separated sets Λ and Λ' by similar argument to the proof of the necessary condition on the sampling set of a stable sampling and reconstruction process in [61]. Let K be a compact subset of \mathbb{R}^d and $\ell^p(\Lambda' \cap K)$ be the space of all sequences in $\ell^p(\Lambda')$ supported on $\Lambda' \cap K$. For a sequence $c \in \ell^p(\Lambda' \cap K)$, it follows from the property of the matrix A in the Sjöstrand class that the ℓ^p norm of the sequence Ac outside of $B(K, R) \cap \Lambda$ is less than $\epsilon(R)\|c\|_{\ell^p(\Lambda')}$, where $\epsilon(R)$ (independent of the compact set K) tends to zero as R tends to infinity. Thus there exists a positive constant R_0 such that the ℓ^p norm of the sequence Ac inside $B(K, R_0) \cap \Lambda$ is equivalent to the ℓ^p norm of the sequence c . This implies that the submatrix obtained by selecting the columns in $\Lambda \cap B(K, R_0)$ and rows in $\Lambda' \cap K$ of the matrix A has full rank, which proves the desired conclusion on the relatively-separated subsets Λ and Λ' . \square

The proof of Theorem 2.1 is inspired by the commutator technique developed in [56] and norm equivalence technique for a finite-dimensional space in [2]. To prove Theorem 2.1, we need several lemmas. First we recall a known result about the boundedness of infinite matrices in the Sjöstrand class.

Lemma 2.6 ([63]) *Let $1 \leq p \leq \infty$, Λ and Λ' be two relatively-separated subsets of \mathbb{R}^d , and $A = (a(\lambda, \lambda'))_{\lambda \in \Lambda, \lambda' \in \Lambda'}$ be an infinite matrix in the Sjöstrand class $\mathcal{C}(\Lambda, \Lambda')$. Then the infinite matrix A is a bounded operator from $\ell^p(\Lambda')$ to $\ell^p(\Lambda)$. Moreover there exists an absolute constant C (that depends on d and p only) such that*

$$\|Ac\|_{\ell^p(\Lambda)} \leq CR(\Lambda)^{1/p}R(\Lambda')^{1-1/p}\|A\|_C\|c\|_{\ell^p(\Lambda')} \quad \text{for all } c \in \ell^p(\Lambda'). \quad (2.8)$$

Define the cut-off function

$$\psi(x) = \min(\max(2 - \|x\|_\infty, 0), 1) = \begin{cases} 1 & \text{if } \|x\|_\infty \leq 1, \\ 2 - \|x\|_\infty & \text{if } 1 < \|x\|_\infty < 2, \\ 0 & \text{if } \|x\|_\infty \geq 2. \end{cases} \quad (2.9)$$

Then

$$\begin{cases} 0 \leq \psi(x) \leq 1 \text{ for all } x \in \mathbb{R}^d, \text{ and} \\ |\psi(x) - \psi(y)| \leq \|x - y\|_\infty \text{ for all } x, y \in \mathbb{R}^d. \end{cases} \quad (2.10)$$

For $n \in \mathbb{Z}^d$ and $N \in \mathbb{N}$, define the multiplication operator $\Psi_n^N : \ell^p(\Lambda) \rightarrow \ell^p(\Lambda)$ by

$$\Psi_n^N c = \left(\psi\left(\frac{\lambda - n}{N}\right)c(\lambda) \right)_{\lambda \in \Lambda} \quad \text{for } c = (c(\lambda))_{\lambda \in \Lambda} \in \ell^p(\Lambda) \quad (2.11)$$

where Λ is a relatively-separated subset of \mathbb{R}^d . The multiplication operator ψ_n^N can also be thought as a diagonal matrix $\text{diag}(\psi((\lambda - n)/N))_{\lambda \in \Lambda}$.

For an infinite matrix $A = (a(\lambda, \lambda'))_{\lambda \in \Lambda, \lambda' \in \Lambda'}$ and any $s \geq 0$, define the truncation matrix

$$A_s = (a_s(\lambda, \lambda'))_{\lambda \in \Lambda, \lambda' \in \Lambda'} \quad (2.12)$$

where $a_s(\lambda, \lambda') = a(\lambda, \lambda')$ if $\|\lambda - \lambda'\|_\infty < s$ and $a_s(\lambda, \lambda') = 0$ otherwise. For the truncation matrices $A_s, s \geq 0$, of an infinite matrix A in the Sjöstrand class $\mathcal{C}(\Lambda, \Lambda')$, we have

$$\|A - A_s\|_{\mathcal{C}} \text{ is a decreasing function with } \lim_{s \rightarrow +\infty} \|A - A_s\|_{\mathcal{C}} = 0. \quad (2.13)$$

Lemma 2.7 *Let $1 \leq p, q \leq \infty$, $1 \leq N \in \mathbb{N}$, Λ and Λ' be two relatively-separated subsets of \mathbb{R}^d , and $A = (a(\lambda, \lambda'))_{\lambda \in \Lambda, \lambda' \in \Lambda'}$ be an infinite matrix in the Sjöstrand class $\mathcal{C}(\Lambda, \Lambda')$. Then there exists an absolute constant C (that depends on d, p, q only) such that*

$$\begin{aligned} & \left\| \left(\| (A_N \Psi_n^N - \Psi_n^N A_N) c \|_{\ell^p(\Lambda)} \right)_{n \in N\mathbb{Z}^d} \right\|_{\ell^q(N\mathbb{Z}^d)} \\ & \leq CR(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \min_{0 \leq s \leq N} \left(\|A - A_s\|_{\mathcal{C}} + \frac{s}{N} \|A\|_{\mathcal{C}} \right) \\ & \quad \times \left\| \left(\|\Psi_n^N c\|_{\ell^p(\Lambda')} \right)_{n \in N\mathbb{Z}^d} \right\|_{\ell^q(N\mathbb{Z}^d)} \quad \text{for all } c \in \ell^q(\Lambda'). \end{aligned} \quad (2.14)$$

PROOF. Observing that

$$A_N \Psi_n^N - \Psi_n^N A_N = (A_N \Psi_n^N - \Psi_n^N A_N) \Psi_n^{6N},$$

we obtain from Lemma 2.6 that

$$\begin{aligned} \| (A_N \Psi_n^N - \Psi_n^N A_N) c \|_{\ell^p(\Lambda)} & \leq CR(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \\ & \quad \times \|A_N \Psi_n^N - \Psi_n^N A_N\|_{\mathcal{C}} \|\Psi_n^{6N} c\|_{\ell^p(\Lambda')} \end{aligned} \quad (2.15)$$

for any $c \in \ell^p(\Lambda')$. We note from (2.9), (2.10) and (2.13) that

$$\begin{aligned} \|A_N \Psi_n^N - \Psi_n^N A_N\|_{\mathcal{C}} & \leq \left\| \left(a_N(\lambda, \lambda') (\psi_n^N(\lambda') - \psi_n^N(\lambda)) \right)_{\lambda \in \Lambda, \lambda' \in \Lambda'} \right\|_{\mathcal{C}} \\ & \leq \min_{0 \leq s \leq N} \left(\left\| \left(a_N(\lambda, \lambda') - a_s(\lambda, \lambda') \right)_{\lambda \in \Lambda, \lambda' \in \Lambda'} \right\|_{\mathcal{C}} \right. \\ & \quad \left. + \frac{s}{N} \left\| \left(a_s(\lambda, \lambda') \right)_{\lambda \in \Lambda, \lambda' \in \Lambda'} \right\|_{\mathcal{C}} \right) \\ & \leq \min_{0 \leq s \leq N} \left(\|A - A_s\|_{\mathcal{C}} + \frac{s}{N} \|A\|_{\mathcal{C}} \right). \end{aligned} \quad (2.16)$$

Then we combine (2.15) and (2.16) to yield

$$\begin{aligned} \|(A_N \Psi_n^N - \Psi_n^N A_N)c\|_{\ell^p(\Lambda)} &\leq CR(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \\ &\times \min_{0 \leq s \leq N} \left(\|A - A_s\|_c + \frac{s}{N} \|A\|_c \right) \|\Psi_n^{6N} c\|_{\ell^p(\Lambda')} \end{aligned} \quad (2.17)$$

for any $c \in \ell^p(\Lambda')$. Thus for $1 \leq q \leq \infty$, we get from (2.9), (2.10) and (2.17) that

$$\begin{aligned} &\left\| \left(\|(A_N \Psi_n^N - \Psi_n^N A_N)c\|_{\ell^p(\Lambda)} \right)_{n \in N\mathbb{Z}^d} \right\|_{\ell^q(N\mathbb{Z}^d)} \\ &\leq CR(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \min_{0 \leq s \leq N} \left(\|A - A_s\|_c + \frac{s}{N} \|A\|_c \right) \\ &\quad \times \left\| \left(\|\Psi_n^{6N} c\|_{\ell^p(\Lambda')} \right)_{n \in N\mathbb{Z}^d} \right\|_{\ell^q(N\mathbb{Z}^d)} \\ &\leq CR(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \min_{0 \leq s \leq N} \left(\|A - A_s\|_c + \frac{s}{N} \|A\|_c \right) \\ &\quad \times \sum_{j \in \mathbb{Z}^d \text{ with } \|j\|_\infty \leq 6} \left\| \left(\|\Psi_{n+2jN}^N c\|_{\ell^p(\Lambda')} \right)_{n \in N\mathbb{Z}^d} \right\|_{\ell^q(N\mathbb{Z}^d)} \\ &\leq CR(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \min_{0 \leq s \leq N} \left(\|A - A_s\|_c + \frac{s}{N} \|A\|_c \right) \\ &\quad \times \left\| \left(\|\Psi_n^N c\|_{\ell^p(\Lambda')} \right)_{n \in N\mathbb{Z}^d} \right\|_{\ell^q(N\mathbb{Z}^d)}. \end{aligned}$$

This proves the estimate (2.14). \square

Lemma 2.8 *Let $1 \leq N \in \mathbb{N}, 1 \leq p, q \leq \infty$, Λ and Λ' be two relatively-separated subsets of \mathbb{R}^d , and $A = (a(\lambda, \lambda'))_{\lambda \in \Lambda, \lambda' \in \Lambda'}$ be an infinite matrix in the Sjöstrand class $\mathcal{C}(\Lambda, \Lambda')$. Then there exists a positive constant C (that depends only on d, p, q) such that*

$$\begin{aligned} \left\| \left(\|\Psi_n^N A c\|_{\ell^p(\Lambda)} \right)_{n \in N\mathbb{Z}^d} \right\|_{\ell^q(N\mathbb{Z}^d)} &\leq CR(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \|A\|_c \\ &\quad \times \left\| \left(\|\Psi_n^N c\|_{\ell^p(\Lambda')} \right)_{n \in N\mathbb{Z}^d} \right\|_{\ell^q(N\mathbb{Z}^d)} \end{aligned} \quad (2.18)$$

holds for any sequence $c \in \ell^q(\Lambda')$.

PROOF. By (2.9) and (2.10), we have that

$$\begin{aligned} 4^d &\geq \sum_{k \in \mathbb{Z}^d} \chi_{[-2,2]^d}(x - k) \geq \sum_{k \in \mathbb{Z}^d} (\psi(x - k))^2 \\ &\geq \sum_{k \in \mathbb{Z}^d} \chi_{[-1,1]^d}(x - k) = 2^d \quad \text{for all } x \in \mathbb{R}^d. \end{aligned} \quad (2.19)$$

Combining (2.19) and Lemma 2.6, we obtain that

$$\begin{aligned}
\|\Psi_n^N A c\|_{\ell^p(\Lambda)} &\leq C R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \sum_{n' \in N\mathbb{Z}^d} \|\Psi_n^N A \Psi_{n+n'}^N\|_c \| \Psi_{n+n'}^N c \|_{\ell^p(\Lambda')} \\
&\leq C R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \\
&\quad \times \sum_{n' \in N\mathbb{Z}^d} \left(\sum_{k \in \mathbb{Z}^d \text{ with } \|k-n'\|_\infty \leq 4N} a(k) \right) \| \Psi_{n+n'}^N c \|_{\ell^p(\Lambda')} \quad (2.20)
\end{aligned}$$

holds for $n \in N\mathbb{Z}^d$ and $c \in \ell^q(\Lambda')$, where

$$a(k) = \sup_{\lambda \in \Lambda, \lambda' \in \Lambda'} |a(\lambda, \lambda')| \chi_{k+[0,1)^d}(\lambda - \lambda').$$

From (2.20) we get that

$$\begin{aligned}
&\left\| \left(\|\Psi_n^N A c\|_{\ell^p(\Lambda)} \right)_{n \in N\mathbb{Z}^d} \right\|_{\ell^q(N\mathbb{Z}^d)} \\
&\leq C R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \left(\sum_{n' \in N\mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d \text{ with } \|k-n'\|_\infty \leq 4N} a(k) \right) \\
&\quad \times \left\| \left(\|\Psi_n^N c\|_{\ell^p(\Lambda')} \right)_{n \in N\mathbb{Z}^d} \right\|_{\ell^q(N\mathbb{Z}^d)} \\
&\leq C R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \|A\|_c \left\| \left(\|\Psi_n^N c\|_{\ell^p(\Lambda')} \right)_{n \in N\mathbb{Z}^d} \right\|_{\ell^q(N\mathbb{Z}^d)}
\end{aligned}$$

for $1 \leq q \leq \infty$. Then the estimate (2.18) follows. \square

Now let us start to prove Theorem 2.1.

Proof of Theorem 2.1. Let $N \geq 1$ be a sufficiently large integer determined later, $n \in N\mathbb{Z}^d$, the multiplication operator Ψ_n^N be as in (2.11), and the truncation matrix A_N be as in (2.12). By the assumption on the infinite matrix A , there exists a positive constant C_0 such that

$$\|\Psi_n^N c\|_{\ell^p(\Lambda')} \leq C_0 \|A \Psi_n^N c\|_{\ell^p(\Lambda)} \quad (2.21)$$

for any sequence $c \in \ell^q(\Lambda')$, $n \in N\mathbb{Z}^d$ and $1 \leq N \in \mathbb{N}$. By (2.8), (2.13), (2.14), (2.18) and (2.21), we get

$$\begin{aligned}
& \left(\sum_{n \in N\mathbb{Z}^d} \|\Psi_n^N c\|_{\ell^p(\Lambda')}^q \right)^{1/q} \\
& \leq C_0 \left(\sum_{n \in N\mathbb{Z}^d} \|A\Psi_n^N c\|_{\ell^p(\Lambda)}^q \right)^{1/q} \\
& \leq C_0 \left(\sum_{n \in N\mathbb{Z}^d} \|(A - A_N)\Psi_n^N c\|_{\ell^p(\Lambda)}^q \right)^{1/q} \\
& \quad + C_0 \left(\sum_{n \in N\mathbb{Z}^d} \|(A_N\Psi_n^N - \Psi_n^N A_N)c\|_{\ell^p(\Lambda)}^q \right)^{1/q} \\
& \quad + C_0 \left(\sum_{n \in N\mathbb{Z}^d} \|\Psi_n^N(A_N - A)c\|_{\ell^p(\Lambda)}^q \right)^{1/q} + C_0 \left(\sum_{n \in N\mathbb{Z}^d} \|\Psi_n^N A c\|_{\ell^p(\Lambda)}^q \right)^{1/q} \\
& \leq C_0 C R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \left(\|A - A_N\|_C + \inf_{0 \leq s \leq N} \left(\|A - A_s\|_C + \frac{s}{N} \|A\|_C \right) \right) \\
& \quad \times \left(\sum_{n \in N\mathbb{Z}^d} \|\Psi_n^N c\|_{\ell^p(\Lambda')}^q \right)^{1/q} + C_0 \left(\sum_{n \in N\mathbb{Z}^d} \|\Psi_n^N A c\|_{\ell^p(\Lambda)}^q \right)^{1/q} \\
& \leq C_0 C R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \inf_{0 \leq s \leq N} \left(\|A - A_s\|_C + \frac{s}{N} \|A\|_C \right) \\
& \quad \times \left(\sum_{n \in N\mathbb{Z}^d} \|\Psi_n^N c\|_{\ell^p(\Lambda')}^q \right)^{1/q} + C_0 \left(\sum_{n \in N\mathbb{Z}^d} \|\Psi_n^N A c\|_{\ell^p(\Lambda)}^q \right)^{1/q} \tag{2.22}
\end{aligned}$$

where $1 \leq q < \infty$. Note that for any infinite matrix $A \in \mathcal{C}(\Lambda, \Lambda')$

$$\begin{aligned}
0 & \leq \lim_{N \rightarrow \infty} \inf_{0 \leq s \leq N} \left(\|A - A_s\|_C + \frac{s}{N} \|A\|_C \right) \\
& \leq \lim_{N \rightarrow \infty} \left(\|A - A_{\sqrt{N}}\|_C + N^{-1/2} \|A\|_C \right) = 0
\end{aligned}$$

by (2.13). Therefore by selecting N sufficiently large in (2.22), we have that

$$\left(\sum_{n \in N\mathbb{Z}^d} \|\Psi_n^N c\|_{\ell^p(\Lambda')}^q \right)^{1/q} \leq 2C_0 \left(\sum_{n \in N\mathbb{Z}^d} \|\Psi_n^N(Ac)\|_{\ell^p(\Lambda)}^q \right)^{1/q}. \tag{2.23}$$

By the equivalence of different norms on a finite-dimensional space, there exists a positive constant C (that depends on p, q, d only) such that

$$C^{-1} (R(\Lambda') N^d)^{\min(1/p-1/q, 0)} \|\Psi_n^N c\|_{\ell^q(\Lambda')} \leq \|\Psi_n^N c\|_{\ell^p(\Lambda')} \tag{2.24}$$

and

$$\|\Psi_n^N A c\|_{\ell^p(\Lambda)} \leq C (R(\Lambda) N^d)^{\max(1/p-1/q, 0)} \|\Psi_n^N A c\|_{\ell^q(\Lambda)} \tag{2.25}$$

hold for all sequences $c \in \ell^q(\Lambda')$, $n \in N\mathbb{Z}^d$ and $1 \leq N \in \mathbb{N}$. Therefore combining (2.23), (2.24) and (2.25), we conclude that

$$\|c\|_{\ell^q(\Lambda')} \leq C (R(\Lambda') N^d)^{-\min(1/p-1/q, 0)} (R(\Lambda) N^d)^{\max(1/p-1/q, 0)} \|A c\|_{\ell^q(\Lambda)} \tag{2.26}$$

for any $c \in \ell^q(\Lambda')$, and the conclusion for $1 \leq q < \infty$ follows.

The conclusion for $q = \infty$ can be proved by similar argument. We omit the details here. \square

3 Stability for localized synthesis operators

In this section, we consider the stability of the synthesis operator

$$S_{\Phi} : \ell^p(\Lambda) \ni (c(\lambda))_{\lambda \in \Lambda} \longmapsto \sum_{\lambda \in \Lambda} c(\lambda) \phi_{\lambda} \in V_p(\Phi, \Lambda) \quad (3.1)$$

associated with a family $\Phi = \{\phi_{\lambda} : \lambda \in \Lambda\}$ of functions on \mathbb{R}^d , where

$$V_p(\Phi, \Lambda) := \left\{ \sum_{\lambda \in \Lambda} c(\lambda) \phi_{\lambda} : (c_{\lambda})_{\lambda \in \Lambda} \in \ell^p(\Lambda) \right\}, \quad 1 \leq p \leq \infty \quad (3.2)$$

([63]). The synthesis operator S_{Φ} appears in the study of spline approximation and operator approximation ([27,55]), wavelet analysis ([18,25,48,49]), Gabor analysis ([31]) and sampling ([1,61]), while one of basic assumptions for the synthesis operator S_{Φ} is the ℓ^p -stability, i.e., there exists a positive constant C such that

$$C^{-1} \|c\|_{\ell^p(\Lambda)} \leq \|S_{\Phi} c\|_p \leq C \|c\|_{\ell^p(\Lambda)} \quad \text{for all } c \in \ell^p(\Lambda). \quad (3.3)$$

The main results of this section are Theorem 3.1 (a generalization of Theorem 1.3) about equivalence of the ℓ^p -stability of the synthesis operator S_{Φ} for different $1 \leq p \leq \infty$, and Corollary 3.3 about well localization of the inverse of the synthesis operator S_{Φ} .

Theorem 3.1 *Let Λ be a relatively-separated subset of \mathbb{R}^d , $\Phi = \{\phi_{\lambda}, \lambda \in \Lambda\}$ be a family of functions with the property that*

$$\left\| \sup_{\lambda \in \Lambda} |\phi_{\lambda}(\cdot + \lambda)| \right\|_{\mathcal{W}_1} < \infty \quad (3.4)$$

and

$$\lim_{\delta \rightarrow 0} \left\| \sup_{\lambda \in \Lambda} |\omega_{\delta}(\phi_{\lambda})(\cdot + \lambda)| \right\|_{\mathcal{W}_1} = 0. \quad (3.5)$$

If the synthesis operator S_{Φ} in (3.1) has ℓ^p -stability for some $1 \leq p \leq \infty$, then it has ℓ^q -stability for any $1 \leq q \leq \infty$.

For $\Phi = \{\phi_n(\cdot - j)\}_{1 \leq n \leq N, j \in \mathbb{Z}^d}$ generated by integer shifts of finitely many functions ϕ_1, \dots, ϕ_N , we have the following corollary for the synthesis operator

S_Φ associated with Φ . Here in the statement of the following result, we do not include the regularity condition (3.5) because $\lim_{\delta \rightarrow 0} \|\omega_\delta(f)\|_{\mathcal{W}_1} = 0$ for any continuous function f in the Wiener amalgam space \mathcal{W}_1 ([1]).

Corollary 3.2 *Let ϕ_1, \dots, ϕ_N be continuous functions in the Wiener amalgam space \mathcal{W}_1 , and for $1 \leq p \leq \infty$ define*

$$V_p(\phi_1, \dots, \phi_N) := \left\{ \sum_{n=1}^N \sum_{j \in \mathbb{Z}^d} c_n(j) \phi_n(\cdot - j) : (c_n(j)) \in (\ell^p(\mathbb{Z}^d))^N \right\}.$$

If the synthesis operator $L_{\phi_1, \dots, \phi_N} : (\ell^p(\mathbb{Z}^d))^N \mapsto V_p(\phi_1, \dots, \phi_N)$ defined by

$$L_{\phi_1, \dots, \phi_N} : (c_n(j))_{1 \leq n \leq N, j \in \mathbb{Z}^d} \mapsto \sum_{n=1}^N \sum_{j \in \mathbb{Z}^d} c_n(j) \phi_n(\cdot - j)$$

has ℓ^p -stability for some $p \in [1, \infty]$, i.e., there exists a positive constant C such that

$$C^{-1} \|c\|_{(\ell^p(\mathbb{Z}^d))^N} \leq \|L_{\phi_1, \dots, \phi_N} c\|_p \leq C \|c\|_{(\ell^p(\mathbb{Z}^d))^N} \quad \text{for all } c \in (\ell^p(\mathbb{Z}^d))^N,$$

then the synthesis operator $L_{\phi_1, \dots, \phi_N}$ has ℓ^q -stability for any $q \in [1, \infty]$.

The result in the above corollary is established in [43] under the weak assumption that $\phi_1, \dots, \phi_N \in \mathcal{L}^\infty$.

Note that the synthesis operator S_Φ has ℓ^2 -stability if and only if the matrix $A = \left(a(\lambda, \lambda') \right)_{\lambda, \lambda' \in \Lambda}$ has ℓ^2 -stability where $a(\lambda, \lambda') = \int_{\mathbb{R}^d} \phi_\lambda(x) \phi_{\lambda'}(x) dx$ for $\lambda, \lambda' \in \Lambda$. This observation together with the equivalence in Theorem 3.1 for the synthesis operator S_Φ and the Wiener's lemma in [56] for the Sjöstrand class of infinite matrices leads to the following result.

Corollary 3.3 *Let $1 \leq p \leq \infty$, Λ be a relatively-separated subset of \mathbb{R}^d , $\Phi = \{\phi_\lambda, \lambda \in \Lambda\}$ satisfy (3.4) and (3.5). If the synthesis operator S_Φ has ℓ^p -stability, then there exists another family $\tilde{\Phi} = \{\tilde{\phi}_\lambda, \lambda \in \Lambda\}$ of functions satisfying (3.4) and (3.5) such that the inverse of the synthesis operator S_Φ is given by*

$$(S_\Phi)^{-1} f = \left(\int_{\mathbb{R}^d} f(x) \tilde{\phi}_\lambda(x) dx \right)_{\lambda \in \Lambda} \quad \text{for all } f \in V_p(\Phi, \Lambda).$$

The conclusion in the above corollary with $p = 2$ is established in [63] without the regularity assumption (3.5). The conclusion in the above corollary for general $1 \leq p \leq \infty$ gives a partial answer to a problem in [63, Remark 5.3].

To prove Theorem 3.1, we recall a result in [63].

Lemma 3.4 *Let $1 \leq p \leq \infty$, Λ be a relatively-separated subset of \mathbb{R}^d , $\Phi = \{\phi_\lambda, \lambda \in \Lambda\}$ satisfy (3.4). Then there exists a positive constant C (that depends on d and p only) such that*

$$\|S_\Phi c\|_p \leq CR(\Lambda)^{1-1/p} \left\| \sup_{\lambda \in \Lambda} |\phi_\lambda(\cdot + \lambda)| \right\|_{\mathcal{W}_1} \|c\|_{\ell^p(\Lambda)}. \quad (3.6)$$

Now we start to prove Theorem 3.1.

Proof of Theorem 3.1. Let $1 \leq p, q \leq \infty$. By the ℓ^p -stability of the synthesis operator S_Φ , there exists a positive constant C_0 such that

$$\|c\|_{\ell^p(\Lambda)} \leq C_0 \|S_\Phi c\|_p \quad \text{for all } c \in \ell^p(\Lambda). \quad (3.7)$$

For $1 \leq n \in \mathbb{N}$, define the operator P_n on L^p by

$$P_n f(x) = 2^{nd} \sum_{\lambda' \in 2^{-n}\mathbb{Z}^d} \phi_0(2^n(x - \lambda')) \times \int_{\mathbb{R}^d} f(y) \phi_0(2^n(y - \lambda')) dy, \quad f \in L^p(\mathbb{R}^d) \quad (3.8)$$

where ϕ_0 be the characteristic function on $[0, 1)^d$, and let $\Phi_n = \{P_n \phi_\lambda, \lambda \in \Lambda\}$. Then

$$|\phi_\lambda(x) - P_n \phi_\lambda(x)| \leq \omega_{2^{-n}}(\phi_\lambda)(x) \quad \text{for all } x \in \mathbb{R}^d \text{ and } \lambda \in \Lambda. \quad (3.9)$$

From (3.7), (3.9) and Lemma 3.4 it follows that

$$\begin{aligned} \|S_\Phi c\|_p &\leq \|S_{\Phi_n} c\|_p + \|S_{\Phi - \Phi_n} c\|_p \\ &\leq \|S_{\Phi_n} c\|_p + CR(\Lambda)^{1-1/p} \left\| \sup_{\lambda \in \Lambda} \omega_{2^{-n}}(\phi_\lambda)(\cdot + \lambda) \right\|_{\mathcal{W}_1} \|c\|_{\ell^p(\Lambda)}. \end{aligned} \quad (3.10)$$

Combining (3.5) and (3.10) leads to the existence of a sufficiently large integer n_0 such that

$$\|c\|_{\ell^p(\Lambda)} \leq 2C_0 \|S_{\Phi_{n_0}} c\|_p \quad \text{for all } c \in \ell^p(\Lambda). \quad (3.11)$$

Define $A_{n_0} = (a_{n_0}(\lambda', \lambda))_{\lambda' \in 2^{-n_0}\mathbb{Z}^d, \lambda \in \Lambda}$ by

$$a_{n_0}(\lambda', \lambda) = 2^{n_0 d} \int_{\mathbb{R}^d} \phi_\lambda(y) \phi_0(2^{n_0}(y - \lambda')) dy. \quad (3.12)$$

Since

$$P_{n_0} \phi_\lambda = \sum_{\lambda' \in 2^{-n_0}\mathbb{Z}^d} a_{n_0}(\lambda', \lambda) \phi_0(2^{n_0}(\cdot - \lambda')),$$

and

$$\left\| \sum_{\lambda' \in 2^{-n_0}\mathbb{Z}^d} a(\lambda') \phi_0(2^{n_0}(\cdot - \lambda')) \right\|_p = 2^{-n_0 d/p} \|a\|_{\ell^p(2^{-n_0}\mathbb{Z}^d)} \quad (3.13)$$

for any $a = (a(\lambda'))_{\lambda' \in 2^{-n_0}\mathbb{Z}^d} \in \ell^p(2^{-n_0}\mathbb{Z}^d)$, the equation (3.11) can be rewritten in the following matrix formulation:

$$\|c\|_{\ell^p(\Lambda)} \leq 2C_0 2^{-n_0 d/p} \|A_{n_0} c\|_{\ell^p(2^{-n_0}\mathbb{Z}^d)} \quad \text{for all } c \in \ell^p(\Lambda). \quad (3.14)$$

By (3.4), it holds that

$$\begin{aligned} & \sum_{j \in \mathbb{Z}^d} \sup_{\lambda' \in 2^{-n_0}\mathbb{Z}^d, \lambda \in \Lambda} |a_{n_0}(\lambda', \lambda)| \chi_{j+[0,1]^d}(\lambda' - \lambda) \\ & \leq 2^{n_0 d} \sum_{j \in \mathbb{Z}^d} \sup_{\lambda' \in 2^{-n_0}\mathbb{Z}^d, \lambda \in \Lambda} \chi_{j+[0,1]^d}(\lambda' - \lambda) \times \int_{\mathbb{R}^d} h(y - \lambda) \phi_0(2^{n_0}(y - \lambda')) dy \\ & \leq \sum_{j \in \mathbb{Z}^d} \sup_{y \in j+[0,2]^d} h(y) \leq 2^d \|h\|_{\mathcal{W}_1} < \infty \end{aligned}$$

where $h(x) = \sup_{\lambda \in \Lambda} |\phi_\lambda(x + \lambda)|$, which means that the infinite matrix A_{n_0} in (3.12) belongs to the Sjöstrand class $\mathcal{C}(2^{-n_0}\mathbb{Z}^d, \Lambda)$,

$$A_{n_0} \in \mathcal{C}(2^{-n_0}\mathbb{Z}^d, \Lambda). \quad (3.15)$$

By (3.14), (3.15) and Theorem 2.1, the infinite matrix A_{n_0} has the ℓ^q -stability, i.e., there exists a positive constant C_1 such that

$$\|c\|_{\ell^q(\Lambda)} \leq C_1 \|A_{n_0} c\|_{\ell^q(2^{-n_0}\mathbb{Z}^d)} \quad \text{for all } c \in \ell^q(\Lambda). \quad (3.16)$$

For any $c = (c(\lambda))_{\lambda \in \Lambda} \in \ell^q(\Lambda)$,

$$\|A_{n_0} c\|_{\ell^q(2^{-n_0}\mathbb{Z}^d)} = 2^{n_0 d/q} \left\| \int_{\mathbb{R}^d} K_{n_0}(\cdot, y) (S_\Phi c)(y) dy \right\|_q \leq 2^{n_0 d/q} \|S_\Phi c\|_q \quad (3.17)$$

by (3.13), where $K_{n_0}(x, y) = 2^{n_0 d} \sum_{\lambda' \in 2^{-n_0}\mathbb{Z}^d} \phi_0(2^{n_0}(x - \lambda')) \phi_0(2^{n_0}(y - \lambda'))$. The ℓ^q -stability of the synthesis operator S_Φ then follows from (3.16) and (3.17).

4 L^p -stability for localized integral operators

In this section, we consider the L^p -stability of integral operators

$$Tf(x) := \int_{\mathbb{R}^d} K_T(x, y) f(y) dy, \quad f \in L^p(\mathbb{R}^d) \quad (4.1)$$

whose kernels K_T are enveloped by convolution kernels with certain decay at infinity, i.e.,

$$|K_T(x, y)| \leq h(x - y) \quad \text{for all } x, y \in \mathbb{R}^d \quad (4.2)$$

where h is a function in the Wiener amalgam space \mathcal{W}_1 ([8,15,44,47,46,64]). Examples of the integral operators of the form (4.1) include projection operators on wavelet spaces ([19,21,22,25,43,63]), frame operators associated with Gabor systems in the time-frequency space ([5,6,31,35]), and reconstruction operators in sampling theory ([1,61,63]).

An integral operator T with kernel K_T enveloped by a convolution kernel in the Wiener amalgam space defines a bounded operator on L^p . The above class \mathcal{C}_1 of localized integral operators is a non-unital algebra. The new algebra

$$\mathcal{IC}_1 = \{\lambda I + T : \lambda \in \mathbb{C}, T \in \mathcal{C}_1\}$$

obtained by adding the identity operator I on L^p to that algebra \mathcal{C}_1 is a unital Banach subalgebra of $\mathcal{B}(L^p)$, $1 \leq p \leq \infty$ ([64]).

In this section, we discuss the L^p -stability of the localized integral operators in \mathcal{IC}_1 with additional regularity on kernels. The main results of this section are Theorem 4.1 (a slight generalization of Theorem 1.4), and Corollary 4.2 concerning the well localization of the inverse of a localized integral operator.

Theorem 4.1 *Let $0 < \alpha \leq 1$, D be a positive constant, and T be an integral operator of the form (4.1) with its kernel K_T satisfying*

$$\left\| \sup_{y \in \mathbb{R}^d} |K_T(y, \cdot + y)| \right\|_{\mathcal{W}_1} \leq D \quad (4.3)$$

and

$$\left\| \sup_{y \in \mathbb{R}^d} \omega_\delta(K_T)(y, \cdot + y) \right\|_{\mathcal{W}_1} \leq D\delta^\alpha \text{ for all } \delta \in (0, 1). \quad (4.4)$$

If $I + T$ has L^p -stability for some $1 \leq p \leq \infty$, then it has L^q -stability for all $1 \leq q \leq \infty$.

The above result for $p = 2$ follows from the Wiener's lemma for localized integral operators ([64]). Applying the L^p equivalence in Theorem 4.1 for different p , we can extend the Wiener's lemma in [64] to $p \neq 2$.

Corollary 4.2 *Let $1 \leq p \leq \infty$, $0 \neq \lambda \in \mathbb{C}$, and T be an integral operator with its kernel K_T satisfying (4.3) and (4.4). If $\lambda I + T$ has bounded inverse on L^p , then $(\lambda I + T)^{-1} = \lambda^{-1}I + \tilde{T}$ for some integral operator \tilde{T} with kernel satisfying (4.3) and (4.4).*

Recall that any integral operator having its kernel satisfying (4.3) and (4.4) does not have bounded inverse in L^p , $1 \leq p < \infty$ ([64]). Then from Corollary 4.2 we have the following result to spectra of localized integral operators on L^p .

Corollary 4.3 *Let T be an integral operator with its kernel K_T satisfying*

(4.3) and (4.4). Then

$$\sigma_p(T) = \sigma_q(T) \quad \text{for all } 1 \leq p, q < \infty, \quad (4.5)$$

where $\sigma_p(T)$ denotes the spectrum of the operator T on L^p .

Now we start to prove Theorem 4.1.

Proof of Theorem 4.1. By the L^p -stability of the operator $I + T$, there exists a positive constant C_0 such that

$$\|f\|_p \leq C_0 \|(I + T)f\|_p \quad \text{for all } f \in L^p. \quad (4.6)$$

For $1 \leq n \in \mathbb{N}$, let $T_n = P_n T P_n$ with kernel K_{T_n} where P_n is given in (3.8). Then

$$K_{T_n}(x, y) = \sum_{\lambda, \lambda' \in 2^{-n}\mathbb{Z}^d} a_n(\lambda, \lambda') \phi_0(2^n(x - \lambda)) \phi_0(2^n(y - \lambda')) \quad (4.7)$$

and

$$|K_{T_n}(x, y) - K_T(x, y)| \leq C \omega_{2^{-n}}(K_T)(x, y) \quad \text{for all } x, y \in \mathbb{R}^d, \quad (4.8)$$

where ϕ_0 is the characteristic function on $[0, 1)^d$ and

$$a_n(\lambda, \lambda') = 2^{2nd} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi_0(2^n(s - \lambda)) K_T(s, t) \phi_0(2^n(t - \lambda')) ds dt \quad (4.9)$$

for $\lambda, \lambda' \in 2^{-n}\mathbb{Z}^d$. Therefore we have from (4.4) and (4.8) that for any $f \in L^r$ with $1 \leq r \leq \infty$,

$$\begin{aligned} \|(T - T_n)f\|_r &\leq C \left\| \sup_{y \in \mathbb{R}^d} \omega_{2^{-n}}(K_T)(y, \cdot + y) \right\|_{\mathcal{W}_1} \|f\|_r \\ &\leq C 2^{-n\alpha} \|f\|_r. \end{aligned} \quad (4.10)$$

By (4.4), (4.6) and (4.10), there exists a sufficiently large integer n_0 such that for all $n \geq n_0$,

$$\|f\|_p \leq 2C_0 \|(I + T_n)f\|_p \quad \text{for all } f \in L^p. \quad (4.11)$$

Let

$$A_n := (a_n(\lambda, \lambda'))_{\lambda, \lambda' \in 2^{-n}\mathbb{Z}^d} \quad (4.12)$$

where $a_n(\lambda, \lambda')$, $\lambda, \lambda' \in 2^{-n}\mathbb{Z}^d$, are given in (4.9). Applying (4.11) to

$$f_n := \sum_{\lambda \in 2^{-n}\mathbb{Z}^d} c_n(\lambda) \phi_0(2^n(\cdot - \lambda)) \quad \text{with } (c_n(\lambda))_{\lambda \in 2^{-n}\mathbb{Z}^d} \in \ell^p(2^{-n}\mathbb{Z}^d),$$

and noting

$$\|f_n\|_p = 2^{-nd/p} \|c_n\|_{\ell^p(2^{-n}\mathbb{Z}^d)} \quad (4.13)$$

and

$$\|(I + T_n)f_n\|_p = 2^{-nd/p} \|(I + 2^{-nd}A_n)c_n\|_{\ell^p(2^{-n}\mathbb{Z}^d)}, \quad (4.14)$$

we obtain the uniform ℓ^p -stability of the matrix $I + 2^{-nd}A_n$, i.e.,

$$\|c_n\|_{\ell^p(2^{-n}\mathbb{Z}^d)} \leq 2C_0 \|(I + 2^{-nd}A_n)c_n\|_{\ell^p(2^{-n}\mathbb{Z}^d)} \quad (4.15)$$

holds for any $c_n \in \ell^p(2^{-n}\mathbb{Z}^d)$ and $n \geq n_0$.

Define

$$A_{n,s} = (a_{n,s}(\lambda, \lambda'))_{\lambda, \lambda' \in 2^{-n}\mathbb{Z}^d} \quad (4.16)$$

where

$$a_{n,s}(\lambda, \lambda') = \begin{cases} a_n(\lambda, \lambda') & \text{if } \|\lambda - \lambda'\|_\infty < s, \\ 0 & \text{otherwise.} \end{cases}$$

Then for $s \geq 0$,

$$\begin{aligned} \|A_n - A_{n,s}\|_c &\leq \sum_{j \in \mathbb{Z}^d \text{ with } \|j\|_\infty \geq s-1} \sup_{\lambda, \lambda' \in 2^{-n}\mathbb{Z}^d} |a_n(\lambda, \lambda')| \chi_{j+[0,1)^d}(\lambda - \lambda') \\ &\leq 2^{2nd} \sum_{j \in \mathbb{Z}^d \text{ with } \|j\|_\infty \geq s-1} \sup_{\lambda, \lambda' \in 2^{-n}\mathbb{Z}^d} \chi_{j+[0,1)^d}(\lambda - \lambda') \\ &\quad \times \int_{2^{-n}[0,1)^d} \int_{2^{-n}[0,1)^d} |K(\lambda + s, \lambda' + t)| ds dt \\ &\leq \sum_{j \in \mathbb{Z}^d \text{ with } \|j\|_\infty \geq s-1} \sup_{x \in j+[-1,2)^d} |h(x)| \\ &\leq 3^d \sum_{j \in \mathbb{Z}^d \text{ with } \|j\|_\infty \geq s-3} \sup_{x \in j+[0,1)^d} |h(x)| \end{aligned}$$

where $h(x) = \sup_{y \in \mathbb{R}^d} |K_T(x + y, y)|$. Thus

$$\begin{aligned} &\inf_{0 \leq s \leq N} \left(\|A_n - A_{n,s}\|_c + \frac{s}{N} \|A_n\|_c \right) \\ &\leq \|A_n - A_{n,\sqrt{N}}\|_c + N^{-1/2} \|A_n\|_c \\ &\leq C \left(\sum_{j \in \mathbb{Z}^d \text{ with } \|j\|_\infty \geq \sqrt{N}-3} \sup_{x \in j+[0,1)^d} |h(x)| \right. \\ &\quad \left. + N^{-1/2} \sum_{j \in \mathbb{Z}^d} \sup_{x \in j+[0,1)^d} |h(x)| \right) \quad (4.17) \end{aligned}$$

Let N be a sufficiently large integer chosen later and the multiplication operator Ψ_j^N be as in the proof of Theorem 2.1. Then for $1 \leq q \leq \infty$, using the

similar argument in the proof of Theorem 2.1, we obtain from (2.13), (4.15), (4.17), and Lemmas 2.6, 2.7 and 2.8 that

$$\begin{aligned}
& \left\| \left(\|\Psi_j^N c_n\|_{\ell^p(2^{-n}\mathbb{Z}^d)} \right)_{j \in N\mathbb{Z}^d} \right\|_{\ell^q(N\mathbb{Z}^d)} \\
& \leq 2C_0 \left\| \left(\|(I + 2^{-nd}A_n)\Psi_j^N c_n\|_{\ell^p(2^{-n}\mathbb{Z}^d)} \right)_{j \in N\mathbb{Z}^d} \right\|_{\ell^q(N\mathbb{Z}^d)} \\
& \leq 2^{-nd+1}C_0 \left\| \left(\|(A_n - A_{n,N})\Psi_j^N c_n\|_{\ell^p(2^{-n}\mathbb{Z}^d)} \right)_{j \in N\mathbb{Z}^d} \right\|_{\ell^q(N\mathbb{Z}^d)} \\
& \quad + 2^{-nd+1}C_0 \left\| \left(\|(A_{n,N}\Psi_j^N - \Psi_j^N A_{n,N})c_n\|_{\ell^p(2^{-n}\mathbb{Z}^d)} \right)_{j \in N\mathbb{Z}^d} \right\|_{\ell^q(N\mathbb{Z}^d)} \\
& \quad + 2^{-nd+1}C_0 \left\| \left(\|\Psi_j^N(A_{n,N} - A_n)c_n\|_{\ell^p(2^{-n}\mathbb{Z}^d)} \right)_{j \in N\mathbb{Z}^d} \right\|_{\ell^q(N\mathbb{Z}^d)} \\
& \quad + 2C_0 \left\| \left(\|\Psi_j^N(I + 2^{-nd}A_n)c_n\|_{\ell^p(\Lambda)} \right)_{j \in N\mathbb{Z}^d} \right\|_{\ell^q(N\mathbb{Z}^d)} \\
& \leq C_0 C \left(\|A_n - A_{n,N}\|_C + \inf_{0 \leq s \leq N} \left(\|A_n - A_{n,s}\|_C + \frac{s}{N} \|A_n\|_C \right) \right) \\
& \quad \times \left(\sum_{j \in N\mathbb{Z}^d} \|\Psi_j^N c_n\|_{\ell^p(\Lambda)}^q \right)^{1/q} + C_0 \left(\sum_{j \in N\mathbb{Z}^d} \|\Psi_j^N A_n c_n\|_{\ell^p(\Lambda)}^q \right)^{1/q} \\
& \leq CC_0 \left(\sum_{j \in \mathbb{Z}^d \text{ with } \|j\|_\infty \geq \sqrt{N}-3} \sup_{x \in j + [0,1)^d} |h(x)| \right. \\
& \quad \left. + N^{-1/2} \sum_{j \in \mathbb{Z}^d} \sup_{x \in j + [0,1)^d} |h(x)| \right) \left\| \left(\|\Psi_j^N c_n\|_{\ell^p(2^{-n}\mathbb{Z}^d)} \right)_{j \in N\mathbb{Z}^d} \right\|_{\ell^q(N\mathbb{Z}^d)} \\
& \quad + 2C_0 \left\| \left(\|\Psi_j^N(I + 2^{-nd}A_n)c_n\|_{\ell^p(2^{-n}\mathbb{Z}^d)} \right)_{j \in N\mathbb{Z}^d} \right\|_{\ell^q(N\mathbb{Z}^d)}, \tag{4.18}
\end{aligned}$$

where $h(x) = \sup_{y \in \mathbb{R}^d} |K_T(y, x + y)|$ and the uppercase letter C denotes an absolute constant independent of $n \geq n_0$ and $N \geq 1$ but may be different at different occurrences. By (4.3) and (4.18) there exists a sufficiently large integer N_0 (independent of $n \geq n_0$) such that

$$\begin{aligned}
& \left\| \left(\|\Psi_j^{N_0} c_n\|_{\ell^p(2^{-n}\mathbb{Z}^d)} \right)_{j \in N_0\mathbb{Z}^d} \right\|_{\ell^q(N_0\mathbb{Z}^d)} \\
& \leq 4C_0 \left\| \left(\|\Psi_j^{N_0}(I + 2^{-nd}A_n)c_n\|_{\ell^p(2^{-n}\mathbb{Z}^d)} \right)_{j \in N_0\mathbb{Z}^d} \right\|_{\ell^q(N_0\mathbb{Z}^d)} \tag{4.19}
\end{aligned}$$

holds for any $c_n \in \ell^q(2^{-n}\mathbb{Z}^d)$ and $n \geq n_0$.

Combining (2.24), (2.25) and (4.19) yields

$$\|c_n\|_{\ell^q(2^{-n}\mathbb{Z}^d)} \leq C_1 2^{nd|1/p-1/q|} \|(I + 2^{-nd}A_n)c_n\|_{\ell^q(2^{-n}\mathbb{Z}^d)} \text{ for all } c_n \in \ell^q(2^{-n}\mathbb{Z}^d), \tag{4.20}$$

where C_1 is a positive constant independent of $n \geq n_0$. This together with (4.13) and (4.14) proves that

$$\|f_n\|_q \leq C_1 2^{nd|1/p-1/q|} \|(I + T_n)f_n\|_q \quad (4.21)$$

holds for any $f_n = \sum_{\lambda \in 2^{-n}\mathbb{Z}^d} c_n(\lambda) \phi_0(2^n(\cdot - \lambda))$ with $(c_n(\lambda))_{\lambda \in 2^{-n}\mathbb{Z}^d} \in \ell^q(2^{-n}\mathbb{Z}^d)$.

Note that for any $f \in L^q$,

$$P_n f = 2^{nd} \sum_{\lambda \in 2^{-n}\mathbb{Z}^d} \phi_0(2^n(\cdot - \lambda)) \times \int_{\mathbb{R}^d} f(y) \phi_0(2^n(y - \lambda)) dy \quad (4.22)$$

with

$$\left\| \left(2^{nd} \int_{\mathbb{R}^d} f(y) \phi_0(2^n(y - \lambda)) dy \right)_{\lambda \in 2^{-n}\mathbb{Z}^d} \right\|_{\ell^q(2^{-n}\mathbb{Z}^d)} \leq 2^{nd/q} \|f\|_q < \infty. \quad (4.23)$$

Therefore it follows from (4.21) that

$$\|P_n f\|_q \leq C_1 2^{nd|1/p-1/q|} \|(I + T_n)P_n f\|_q \quad \text{for all } f \in L^q. \quad (4.24)$$

By (4.13), (4.22) and (4.23), we have

$$\|P_n f\|_q \leq \|f\|_q \quad \text{for all } f \in L^q. \quad (4.25)$$

This implies that

$$\|f\|_q \leq \|f - P_n f\|_q + \|P_n f\|_q \leq 3\|f\|_q \quad \text{for all } f \in L^q. \quad (4.26)$$

Noting that $P_n^2 = P_n$, $(I - P_n)(I + T)f = (I - P_n)f + (I - P_n)T(I - P_n)f + (I - P_n)TP_n f$ and $P_n(I + T)f = P_n(I + T)P_n f + P_n T(I - P_n)^2 f$, we obtain from the second inequality of (4.26) that for any $f \in L^q$,

$$\begin{aligned} \|(I + T)f\|_q &\geq \frac{1}{3} \|(I - P_n)(I + T)f\|_q + \frac{1}{3} \|P_n(I + T)f\|_q \\ &\geq \frac{1}{3} \|(I - P_n)f\|_q + \frac{1}{3} \|P_n(I + T)P_n f\|_q \\ &\quad - \frac{1}{3} \|(I - P_n)T(I - P_n)f\|_q - \frac{1}{3} \|(I - P_n)TP_n f\|_q \\ &\quad - \frac{1}{3} \|P_n T(I - P_n)^2 f\|_q. \end{aligned} \quad (4.27)$$

We note that $(I - P_n)T$ and $T(I - P_n)$ are integral operators with their kernel bounded by $\omega_{2^{-n}}(K_T)$ where K_T is the kernel of the integral operator T . Therefore similar to the argument in (4.10) we have

$$\|(I - P_n)Tf\|_r + \|T(I - P_n)f\|_r \leq C 2^{-n\alpha} \|f\|_r \quad \text{for all } f \in L^r \quad (4.28)$$

where $1 \leq r \leq \infty$ and C is a positive constant independent of $n \geq n_0$.

For those $1 \leq q \leq \infty$ satisfying $|1/q - 1/p| < \alpha/d$, we get from (4.24), (4.25), (4.26), (4.27), and (4.28) that

$$\begin{aligned} \|(I + T)f\|_q &\geq \left(\frac{1}{3} - C2^{-n\alpha}\right)\|f - P_n f\|_q \\ &\quad + \left((3C_1)^{-1}2^{-nd|1/p-1/q|} - C2^{-n\alpha}\right)\|P_n f\|_q \\ &\geq \frac{1}{4}\|f - P_n f\|_q + (4C_1)^{-1}2^{-nd|1/p-1/q|}\|P_n f\|_q \\ &\geq (4C_1)^{-1}2^{-nd|1/p-1/q|}\|f\|_q \text{ for all } f \in L^q \end{aligned} \quad (4.29)$$

if we let the integer n be chosen to be sufficiently large. This proves that $I + T$ has L^q -stability if $|1/p - 1/q| < \alpha/d$.

Using the above argument iteratively leads to the conclusion that $I + T$ has L^q -stability for all $1 \leq q \leq \infty$. \square

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