PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 116, Number 3, November 1992

WEIGHTED NORM INEQUALITIES FOR BOCHNER-RIESZ OPERATORS AND SINGULAR INTEGRAL OPERATORS

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(Communicated by J. Marshall Ash)

ABSTRACT. Weighted norm inequalities for the Bochner-Riesz operator at the critical index $\frac{1}{2}(n-1)$ are investigated. We also give some weighted norm inequalities for a class of singular integral operators introduced by Fefferman and Namazi.

1. INTRODUCTION AND STATEMENTS

The Bochner-Riesz operators in R^n are defined as

$$(T_\lambda^R f)^\frown(x) = (1 - R^2|x|^2)_+^\lambda \hat{f}(x)$$

and the associated maximal operator is defined as

$$T_\lambda^* f(x) = \sup_{R>0} |T_\lambda^R f(x)|$$

for $\lambda > 0$, where $\frown$ denotes the Fourier transform. It is well known by the works of Carleson and Sjölin [4], Fefferman [8, 9], Tomas [19], and Christ [6] that T_λ^l is bounded on $L^p(R^n)$ if and only if $|1/p - 1/2| < (1+2\lambda)/2n$ provided $\lambda > 0$ in dimension 2 and $\lambda \geq (n-1)/2(n+1)$ in dimension greater than two. Rubio [16] and Hirschman [12] studied the weighted norm inequality for the Bochner-Riesz operator T_λ^l and showed that T_λ^l is bounded on $L^2(|x|^a)$ provided $|a| < 1 + 2\lambda < n$. In 1988 Andersen [1] gave a sufficient condition and a necessary condition on radial weight $w(|x|)$ such that the inequality

$$\int_{R^n} |T_\lambda^1 f(x)|^2 w(|x|)\,dx \leq C \int_{R^n} |f(x)|^p w(|x|)\,dx$$

holds for all radial functions f in $L^p(w(|x|))$.

Notice that the Bochner-Riesz operator is a summation operator and $T_\lambda^R f(x)$ tends to $f(x)$ as R tends to infinity for all Schwartz functions f. Hence it is meaningful to consider the almost everywhere convergence of $T_\lambda^R f$ as R tends to infinity for some appropriate function f. In 1986 Lu [14] proved that

$$T_\lambda^{R_j} f(x) \to f(x) \quad \text{a.e. as } j \to \infty$$

Received by the editors March 2, 1990 and, in revised form, June 25, 1990 and September 7, 1990.

1991 *Mathematics Subject Classification*. Primary 42B15, 42B25.

for all $f \in L^2(|x|^a)$ provided $0 < a < \min(2, 2\lambda) < n-1$ and $\{R_j\}_{j=1}^{\infty}$ being a Hadamard lacunary sequence, i.e., $\underline{\lim}_{j\to\infty}(R_{j+1}/R_j) > 1$. In addition we can reduce almost everywhere convergence of the operator T_λ^R to some maximal inequality. In [14] Lu proved

Theorem L [14]. *Let* $0 < \lambda < \frac{1}{2}(n-1)$, $0 < a < \min(2, 2\lambda)$, *and* $\{R_j\}_{j=1}^{\infty}$ *be a Hadamard lacunary sequence. Then*

$$\int \left(\sup_j |T_\lambda^{R_j} f(x)|\right)^2 |x|^a\,dx \le C \int |f(x)|^2 |x|^a\,dx.$$

In 1988 Carbery, Rubio, and Vega proved

Theorem CRV [3]. *Let* $|a| < 1 + 2\lambda < n$. *Then*

$$\int |T_\lambda^* f(x)|^2 |x|^a\,dx \le C \int |f(x)|^2 |x|^a\,dx.$$

Hence they improved Theorem L.

On other hand, we observe that if λ exceeds the critical index $\frac{1}{2}(n-1)$ then $T_\lambda^* f$ is dominated by a multiple of the Hardy-Littlewood maximal function Mf defined by

$$Mf(x) = \sup_{x\in Q} |Q|^{-1} \int_Q |f(y)|\,dy,$$

where supremum is taken over all cubes with center x and sides parallel to the coordinate axes. Hence a result of Muckenhoupt [11] showed that T_λ^* is bounded on $L^p(w)$ provided $w \in A_p$, i.e.,

$$\left(|Q|^{-1}\int_Q w(x)\,dx\right)\left(|Q|^{-1}\int_Q w(x)^{-(p-1)^{-1}}\,dx\right)^{p-1} \le C$$

holds for all cubes $Q \subset R^n$ with sides parallel to the coordinate axes and some C independent of Q. Then a natural question is whether $T_{(n-1)/2}^*$ is bounded on $L^p(w)$ provided $w \in A_p$ and $1 < p < \infty$. In this paper we prove

Theorem 1. *Let* $1 < p < \infty$ *and* $w \in A_p$. *Then*

$$\int |T_{(n-1)/2}^* f(x)|^p w(x)\,dx \le C \int |f(x)|^p w(x)\,dx.$$

We also observe that $T_{(n-1)/2}^1 f$ can be written as

$$\int h(|y|)|y|^{-n} f(x-y)\,dy,$$

where

$$h(t) = (2\pi)^{n/2} 2^{(n-1)/2}\Gamma(\tfrac{1}{2}(n-1)) J_{n-1/2}(t) t^{1/2},$$

$\Gamma(t)$ denotes the Gamma function and $J_v(t)$ denotes the Bessel function, which is defined by

$$J_v(t) = \frac{(t/2)^v}{\Gamma(v+1/2)\Gamma(v)} \int_0^{\pi/2} \cos(t\sin u)(\cos u)^{2v}\,du.$$

Therefore the weighted norm inequality for $T_{(n-1)/2}^*$ is closely related to the one for the operator introduced by Fefferman [9] and Namazi [15].

Now let us write the operator introduced by Fefferman and Namazi precisely. Let $h \in L^\infty([0, \infty))$ and Ω be an integrable function on the unit sphere S^{n-1} having mean zero, i.e., $\int_{S^{n-1}} \Omega(x)\, d\sigma(x) = 0$, $d\sigma$ is the standard measure on S^{n-1}.

Define

$$T_\varepsilon f(x) = \int_{|y|>\varepsilon} h(|y|)\Omega\left(\frac{y}{|y|}\right)|y|^{-n} f(x-y)\, dy$$

for $\varepsilon > 0$,

$$T_0 f(x) = \lim_{\varepsilon\to 0} T_\varepsilon f(x),$$

and the associated maximal operator

$$T^* f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|.$$

There are many works about the operators T_0 and T^* (see [5, 7, 15, 17, 18], etc.). Namazi [15] proved that T_0 is a bounded operator on $L^p(R^n)$ for all $1 < p < \infty$ when $\Omega \in L^q(S^{n-1})$ for some $q > 1$, Chen [5] proved that T^* is also a bounded operator on $L^p(R^n)$ when $\Omega \in L^q(S^{n-1})$ for some $q > 1$, and the second author [18] proved that $\Omega \in L\log^+ L(S^{n-1})$ (resp. $L(\log^+ L)^{3/2}(S^{n-1})$) is a sufficient condition such that T_0 (resp. T^*) is a bounded operator on $L^2(R^n)$. As weighted norm inequalities for the operators T_0 and T^*, Duoandikoetxea and Rubio [7] proved that T^* and T_0 are bounded operators on $L^p(w)$ provided $1 < p < \infty$, $w \in A_p$, and $\Omega \in L^\infty(S^{n-1})$.

In this paper we prove the following with the complex interpolation method.

Theorem 2. *Let* $1 < q \le \infty$, $\Omega \in L^q(S^{n-1})$, $q(q-1)^{-1} < p < \infty$, *and* $w \in A_{p(1-1/q)}$. *Then*

$$\int |T_0 f(x)|^p w(x)\, dx \le C \int |f(x)|^p w(x)\, dx$$

holds for all f *in* $L^p(w)$.

Theorem 3. *Let* $\Omega \in L^\infty(S^{n-1})$ *and* $w \in A_p$. *Then*

$$\int |T^* f(x)|^p w(x)\, dx \le C \int |f(x)|^p w(x)\, dx$$

holds for all f *in* $L^p(w)$.

Hence we prove Duoandikoetxea and Rubio's result in another way. The above results are still interesting even when $h \equiv 1$ because in [13] Ω satisfies an L^r-Dini condition for some $r > 1$.

2. Some lemmas

Define the Bochner-Riesz operator T_z^R by

$$(T_z^R f)\hat{}(x) = (1 - R^2|x|^2)_+^z f\hat{}(x)$$

for $\operatorname{Re} z > 0$. Then we have

Lemma 1 [3]. *Let* $\operatorname{Re} z > 0$ *and* $k = 0, 1$. *Then*

$$\int \sup_{R>0} \left| \left(\frac{\partial}{\partial z} \right)^k T_z^R f(x) \right|^2 dx \leq C e^{c|\operatorname{Im} z|} (|\operatorname{Re} z|^{-c} + 1) \int |f(x)|^2 \, dx$$

holds for a constant C *independent of* z.

To prove Theorem 1, we will also use

Lemma 2. *Let* $\operatorname{Re} z > \frac{1}{2}(n-1)$. *Then the inequality*

$$\sup_{R>0} |T_z^R f(x)| \leq C(\operatorname{Re} z - \tfrac{1}{2}(n-1))^{-c} e^{c|\operatorname{Im} z|} M f(x)$$

holds for all $f \in L^1_{\mathrm{loc}}(R^n)$ *and an absolutely positive constant* C.

Proof. Write

$$T_z^R f(x) = \frac{2^{z+1-n/2}\Gamma(z+1)}{\Gamma(n/2)} R^{-n/2+z} \int J_{n/2+z}(R^{-1}|y|) \cdot |y|^{-z-n/2} f(x-y) \, dy.$$

Also we observe from the asymptotic properties of the Bessel function $J_{n/2+z}(t)$ that

$$|J_{n/2+z}(t)| \leq C t^{n/2+\operatorname{Re} z}$$

when $t \leq 1$ and

$$|J_{n/2+z}(t)| \leq C t^{-1/2}$$

when $t > 1$. Therefore for $\operatorname{Re} z > \frac{1}{2}(n-1)$ we have

$$\begin{aligned} |T_z^R f(x)| &\leq C R^{-n/2+\operatorname{Re} z} \int_{|y| \leq R} |f(x-y)| R^{-n/2-\operatorname{Re} z} \, dy \\ &\quad + C R^{n/2+\operatorname{Re} z} \int_{|y|>R} |f(x-y)| |y|^{-\operatorname{Re} z-(n+1)/2} R^{-1/2} \, dy \\ &\leq C M f(x) \end{aligned}$$

and Lemma 2 holds.

Lemma 3 [11]. *For* $s \in (1, \infty)$ *and* $w \in A_s$, *there exists a positive number* δ *such that* $w^{1+\delta} \in A_s$.

Lemma 4 (Three-Circles Theorem). *Suppose* F *is a bounded continuous complex-valued function on the closed strip* $S = \{x + iy ; 0 \leq x \leq 1\}$ *that is analytic in the interior of* S. *If* $|F(iy)| \leq m_0$ *and* $|F(1+iy)| \leq m_1$ *for all* y, *then* $|F(x+iy)| \leq m_0^{1-x} m_1^x$ *for all* $x + iy \in S$.

To prove Theorems 2 and 3, we need to introduce some notation and use some lemmas. Let $\Omega \subset L^q(S^{n-1})$ for some $q > 1$.

Define

$$T_{z,\varepsilon} f = \frac{\pi^{(z-1)/2}\Gamma((n-z)/2)}{\Gamma(z/2)} |y|^{-n-z} h(|y|) \Omega\left(\frac{y}{|y|}\right) \chi_{|y|>\varepsilon} * |y|^{-n+z} * f, \qquad \varepsilon > 0,$$

for $-\frac{1}{2}(1 - 1/q) < \operatorname{Re} z < 1$, where $*$ is the convolution operation. Denote the kernel function of the operator $T_{z,0}$ by K_z. Therefore we have

Lemma 5 [18]. *For* $-\frac{1}{2}(1-1/q) < \operatorname{Re} z < 1$ *and* $k = 0, 1$, *the inequality*

$$\int \sup_{\varepsilon>0} \left| \left(\frac{\partial}{\partial z}\right)^k T_{z,\varepsilon} f(x) \right|^2 dx$$
$$\leq C\left(|\operatorname{Re} z - 1|^{-c} + \left| \operatorname{Re} z + \frac{1}{2}\left(1 - \frac{1}{q}\right)\right|^{-c} \right) e^{c|\operatorname{Im} z|} \int |f(x)|^2\, dx$$

holds.

Lemma 6. *For* $0 < \operatorname{Re} z < 1$ *and* $1 < p < \infty$, *the inequality*

$$\int \sup_{\varepsilon>0} |T_{z,\varepsilon} f(x)|^1\, dx \leq C(|\operatorname{Re} z|^{-c} + |\operatorname{Re} z - 1|^{-c}) e^{c|\operatorname{Im} z|} \int |f(x)|^p\, dx$$

holds.

Lemma 7. *For* $0 < \operatorname{Re} z < 1$, *the inequality*

$$\left(R^{-n} \int_{R<|x|<2R} |K_z(x+y) - K_z(x)|^q\, dx \right)^{1/q} \leq C(z) R^{-n} \left(\frac{|y|}{R}\right)^{\operatorname{Re} z}$$

holds for all $R > 0$ *and* $|y| < \frac{1}{2}R$, *where*

$$C(z) \leq C(|\operatorname{Re} z|^{-C} + |\operatorname{Re} z - 1|^{-C}) \exp(C|\operatorname{Im} z|).$$

Lemma 8 [13]. *Let* $K \in L^1_{\text{loc}}(R^n \backslash \{0\})$, $q > 1$, *and* $\delta > 0$. *Suppose*

$$\left(R^{-n} \int_{R<|x|<2R} |K(x+y) - K(x)|^q\, dx \right)^{1/q} \leq C R^{-n} \left(\frac{|y|}{R}\right)^{\delta}$$

holds for all $R > 0$ *and* $|y| < \frac{1}{2}R$. *Suppose* $\overline{T}$ *be defined as* $\overline{T} f = K * f$ *and* $\overline{T}$ *be bounded on* $L^2(R^n)$. *Then the operator* $\overline{T}^-$ *is bounded on* $L^p(w)$ *for all* $q(q-1)^{-1} < p < \infty$ *and* $w \in A_{p(1-1/q)}$.

Lemma 9. *Let* $\Omega \in L^q(S^{n-1})$ *for some* $q > n$. *For* $n/q < \operatorname{Re} z < 1$, *the inequalities*

(i) $|K_z(x)| \leq C(z)|x|^{-n}$,
(ii) $|K_z(x+y) - K_z(x)| \leq C(z)|x|^{-n-(\operatorname{Re} z - n/q)}|y|^{\operatorname{Re} z - n/q)}$

hold for all $x \neq 0$ *and* $|y| \leq \frac{1}{2}|x|$, *where*

$$C(z) \leq C(|\operatorname{Re} z - n/q|^{-c} + |\operatorname{Re} z - 1|^{-c}) \exp(C|\operatorname{Im} z|).$$

Lemma 10 [11]. *Let* $K \in L^1_{\text{loc}}(R^n \backslash \{0\})$. *Suppose*

(i) $|K(x)| \leq C|x|^{-n}$,
(ii) $|K(x+y) - K(x)| \leq C|y|^{\delta}|x|^{-n-\delta}$

hold for all $x \neq 0$, $|y| < \frac{1}{2}|x|$, *and some* $\delta > 0$. *Suppose an operator* $\overline{T}$ *defined as* $\overline{T} f = K * f$ *is bounded on* $L^2(R^n)$. *Then the operator* $\overline{T}^*$ *defined as*

$$\overline{T}^* f(x) = \sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} k(y) f(x-y)\, dy \right|$$

is bounded on $L^p(w)$ *provided* $1 < p < \infty$ *and* $w \in A_p$.

The proofs of Lemmas 7 and 9 are elementary and the proof of Lemma 6 is similar to the one in [5]. We omit the details here.

3. Proofs of theorems

Let $p \in (1, \infty)$, $s \in (1, \infty)$, $0 < \lambda \le \frac{1}{2}(n-1)$, and $q > n$. Denote $\alpha(p) = \frac{1}{p} - \frac{1}{2}$, $p^1 = p(p-1)^{-1}$,

$$\theta_1(p, s, \lambda) = \begin{cases} \dfrac{2\lambda}{n-1}, & \text{when } p = s = 2, \\ \dfrac{\alpha(p)}{\alpha(s)} \cdot \dfrac{p}{s}, & \text{when } 0 < \dfrac{\alpha(p)}{\alpha(s)} < \dfrac{2\lambda}{n-1},\ p \neq 2, \\ 0, & \text{otherwise}; \end{cases}$$

$$\theta_2(p, s, q) = \begin{cases} \left(1 + \dfrac{2n}{q-1}\right)^{-1}, & \text{when } p = s = 2, \\ \dfrac{\alpha(p)}{\alpha(s)} \cdot \dfrac{p}{s}, & \text{when } 0 < \dfrac{\alpha(p)}{\alpha(s)} \le \left(1 + \dfrac{2n}{q-1}\right)^{-1},\ p \neq 2, \\ \dfrac{1}{2}\left(1 - \dfrac{\alpha(p)}{\alpha(s)}\right)\left(1 - \dfrac{1}{q}\right)\dfrac{p}{s} \cdot \dfrac{q}{n}, & \\ & \text{when } \left(1 + \dfrac{2n}{q-1}\right)^{-1} < \dfrac{\alpha(p)}{\alpha(s)} < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then the following theorems are general versions of Theorems 1 and 3.

Theorem 4. *Let* $0 < \lambda \le \frac{1}{2}(n-1)$, $|\alpha(p)| < \lambda/(n-1)$, *and* $w \in A_s$. *Then*

$$\int |T_\lambda^* f(x)|^p w^{\theta_1(p, s, \lambda)}(x)\, dx \le C \int |f(x)|^p w^{\theta_1(p, s, \lambda)}(x)\, dx.$$

Theorem 5. *Let* $\Omega \in L^q(S^{n-1})$ *for some* $q > n$ *and* $w \in A_s$. *Then*

$$\int |T^* f(x)|^p w^{\theta_2(p, s, q)}(x)\, dx \le C \int |f(x)|^p w^{\theta_2(p, s, q)}(x)\, dx.$$

By Lemma 3 and the fact $|x|^a \in A_s$ for $-n < a < n(s-1)$, we improve Theorem L.

Proof of Theorem 4. Let f, h be two nonnegative smooth functions with compact support and $R(x)$ be any arbitrary positive measurable function bounded below and above, i.e., $R(x)^{-1}$ and $R(x)$ bounded. Suppose $w \in A_s$ and ε_1 and ε_2 are sufficiently small positive constants chosen later, without loss of generality we assume $\theta_1(p, s, \lambda) > 0$. Let

$$p^{-1} = \tfrac{1}{2}(1-\theta) + s^{-1}\theta, \quad \lambda = (1+\theta)\varepsilon_1 + \theta(\tfrac{1}{2}(n-1) + \varepsilon_2), \qquad 0 < \theta < 1.$$

Denote $s(z)^{-1} = \frac{1}{2}(1-z) + s^{-1}z$ for $0 < \operatorname{Re} z < 1$ and $f_0(x) = \exp(-|x|^2)$.

Define

$$f^z_{\delta_1,\delta_2}(x) = (f(x)+\delta_1 f_0(x))^{ps(z)^{-1}-1}(w(x)+\delta_2)^{-s^{-1}z}f(x),$$
$$h^z_{\delta_3}(x) = (h(x)+\delta_3 f_0(x))^{p'(1-s(z)^{-1})-1}h(x),$$
$$w_N(x) = \begin{cases} w(x), & \text{when } w(x)\le N, \\ N, & \text{when } w(x) > N, \end{cases}$$

and

$$\widetilde{T}^{R(x)}_{z,\varepsilon_1,\varepsilon_2,\delta_3,N}(f)(x) = T^{R(x)}_{\varepsilon_1+z((n-1)/2+\varepsilon_2-\varepsilon_1)}f(x)(w_N(x)+\delta_2)^{zs^{-1}}e^{-z^2}$$

where N^{-1}, δ_i $(i=1,2,3)$ are small positive numbers.

Hence by Lemma 1 we can show easily

$$g(z) = \int \widetilde{T}^{R(x)}_{z,\varepsilon_1,\varepsilon_2,\delta_2,N}(f^z_{\delta_1,\delta_2})(x)h^z_{\delta_3}(x)\,dx$$

is analytic in the strip $0<\operatorname{Re} z<1$ and continuous on the closed strip $0\le \operatorname{Re} z\le 1$. In addition $g(z)$ is bounded function, hence by Lemma 4 we have

$$|g(\theta)| \le C\left(\sup_{t\in R}|g(it)|\right)^{1-\theta}\left(\sup_{t\in R}|g(1+it)|\right)^{\theta}.$$

On the other hand by Lemma 1 we get

$$\begin{aligned}\sup_{t\in R}|g(it)| &\le C_{\varepsilon_1,\varepsilon_2}\sup_{t\in R}e^{-t^2}e^{c|t|}\left(\int|f(x)|^p\,dx\right)^{1/2}\left(\int|h(x)|^{p'}\,dx\right)^{1/2}\\ &\le C_{\varepsilon_1,\varepsilon_2}\|f\|_p^{p/2}\|h\|_{p'}^{p'/2}\end{aligned}$$

and by Lemma 2

$$\begin{aligned}\sup_{t\in R}|g(1+it)| &\le C_{\varepsilon_1,\varepsilon_2}\sup_{t\in R}e^{-t^2}\left(\int|T^{R(x)}_{1+it,\varepsilon_1,\varepsilon_2,\delta_2,N}(f^{1+it}_{\delta_1,\delta_2})(x)|^s\,dx\right)^{1/s}\\ &\quad\times\left(\int|h^{1+it}_{\delta_3}(x)|^{s(s-1)^{-1}}\right)^{s-1/3}\\ &\le C_{\varepsilon_1,\varepsilon_2}\|f\|_p^{ps^{-1}}\|h\|_{p'}^{p's^{-1}(s-1)}.\end{aligned}$$

Therefore

$$|g(\theta)| \le C_{\varepsilon_1,\varepsilon_2}\|f\|_p\|h\|_{p'} \tag{1}$$

where $C_{\varepsilon_1,\varepsilon_2}$ is independent of δ_1, δ_2, δ_3, and N.

Write $g(\theta)$ as

$$\begin{aligned}g(\theta) &= \int \widetilde{T}^{R(x)}_{\theta,\varepsilon_1,\varepsilon_2,\delta,N}(f(w+\delta_2)^{-\theta s^{-1}})(x)h(x)\,dx\\ &= \int T^{R(x)}_{\lambda}(f(w+\delta_2)^{-\theta s^{-1}})(x)(w_N(x)+\delta_2)^{\theta s^{-1}}h(x)\,dx.\end{aligned} \tag{2}$$

Notice that for every $f\in L^{p'}$ there exist two nonnegative smooth function sequences $\{f^1_n\}$ and $\{f^2_n\}$ with compact supports such that

$$\|f^i_n\|_{p'} \le C\|f\|_{p'}, \qquad i=1,2,\ n\in N,$$

and

$$\|f_n^1 - f_n^2 - f\|_{p'} \to 0, \quad \text{as } n \to \infty.$$

Hence by (1) and (2) we get

$$\int |T_\lambda^{R(x)}(f(w+\delta_2)^{-\theta s^{-1}})(x)|^p (w_N(x)+\delta_2)^{p\theta s^{-1}}\,dx \le C\int |f(x)|^p\,dx.$$

Let N tend to infinity in the inequality above. Then we have

$$\int |T_\lambda^{R(x)}(f(w+\delta_2)^{-\theta s^{-1}})(x)|^p (w(x)+\delta_2)^{p\theta s^{-1}}\,dx \le C\int |f(x)|^p\,dx.$$

Notice that C is independent of δ_2 and $R(x)$ being a measurable function bounded below and above. Therefore

$$\int |T_\lambda^* f(x)|^p w^{p\theta s^{-1}}(x)\,dx \le C\int |f(x)|^p w^{p\theta s^{-1}}(x)\,dx \tag{3}$$

holds for all $\theta < \theta_1(p, s, \lambda) \neq 0$ by choosing appropriate ε_1 and ε_2. By Lemma 3 we can replace w in (3) by $w^{1+\delta}$. Hence Theorem 4 holds by choosing $\theta = \theta_1(p, s, x)/(1+\delta) \neq 0$.

Proof of Theorem 2. By Lemmas 7 and 8, T_z is bounded on $L^p(w)$ provided $w \in A_{p(1-1/q)}$ and $\operatorname{Re} z > 0$. In addition T_z is bounded on $L^2(R^n)$ when $|\operatorname{Re} z| < \frac{1}{2}(1-1/q)$. By the complex interpolation theorem (see [10]) and Lemma 3, T_0 is bounded on $L^p(w)$ provided $p > q(q-1)^{-1}$ and $w \in A_{p(1-1/q)}$. Therefore Theorem 2 holds.

Proof of Theorem 5. Define

$$T_z^* f(x) = \sup_{\varepsilon>0}\left|\int_{|y|>\varepsilon} K_z(y) f(x-y)\,dy\right|.$$

Then by Lemmas 9 and 10 we get

$$\int |T_z^* f(x)|^p w(x)\,dx \le C(z)\int |f(x)|^p w(x)\,dx$$

provided $w \in A_p$ and $n/q < \operatorname{Re} z < 1$. In addition the pointwise estimate

$$\sup_{\varepsilon>0} |T_{z,\varepsilon} f(x)| \le C(z)\mu f(x) + T_z^* f(x)$$

holds for all x in R^n, $n/q < \operatorname{Re} z < 1$, and

$$C(z) \le C(|\operatorname{Re} z - 1|^{-c} + |\operatorname{Re} z - n/q|)^{-c} e^{c|\operatorname{Im} z|}.$$

Hence we have

$$\int \sup_{\varepsilon>0} |T_{z,\varepsilon} f(x)|^p w(x)\,dx \le C(z)\int |f(x)|^p w(x)\,dx \tag{4}$$

for $n/q < \operatorname{Re} z < 1$ and $w \in A_p$. By (4), Lemmas 3, 5, and 6 we can prove the following in such a way as we prove Theorem 4,

$$\begin{aligned}\int \sup_{\varepsilon>0} |T_{z,\varepsilon} f(x)|^s w^{(\operatorname{Re} z)n/q}(x)\,dx \\ \le C(z)\int |f(x)|^s w^{(\operatorname{Re} z)n/q}(x)\,dx\end{aligned}$$

provided $0 < \operatorname{Re} z < n/q$, $w \in A_s$, and $C(z) \leq C|\operatorname{Re} z|^{-C} \exp(2|\operatorname{Im} z|^2)$.

Hence

$$\int \sup_{\varepsilon>0} |T_{s,\varepsilon} f|^p(x) w^{\theta_2(p,s,q)}(x)\, dx \leq C \int |f(x)|^p w^{\theta_2(p,s,q)}(x)\, dx$$

holds for $1 < p < \infty$, $q > n$, and $w \in A_s$ and Theorem 5 holds.

Acknowledgments

The authors would like to thank the referee for some corrections.

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