RECOVERY OF SPARSEST SIGNALS VIA ℓ^q-MINIMIZATION

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Abstract. In this paper, it is proved that every s-sparse vector $\mathbf{x} \in \mathbb{R}^n$ can be exactly recovered from the measurement vector $\mathbf{z} = \mathbf{A}\mathbf{x} \in \mathbb{R}^m$ via some ℓ^q-minimization with $0 < q \le 1$, as soon as each s-sparse vector $\mathbf{x} \in \mathbb{R}^n$ is uniquely determined by the measurement $\mathbf{z}$. Moreover it is shown that the exponent q in the ℓ^q-minimization can be so chosen to be about $0.6796 \times (1 - \delta_{2s}(\mathbf{A}))$, where $\delta_{2s}(\mathbf{A})$ is the restricted isometry constant of order $2s$ for the measurement matrix $\mathbf{A}$.

1. Introduction and Main Results

Define $\|\mathbf{x}\|_q, 0 \le q \le \infty$, of a vector $\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ by the number of its nonzero components when $q = 0$, the quantity $(|x_1|^q + \cdots + |x_n|^q)^{1/q}$ when $0 < q < \infty$, and the maximum absolute value $\max(|x_1|, \ldots, |x_n|)$ of its components when $q = \infty$. We say that a vector $\mathbf{x} \in \mathbb{R}^n$ is *s-sparse* if $\|\mathbf{x}\|_0 \le s$, i.e., the number of its nonzero components is less than or equal to s.

In this paper, we consider the problem of compressive sensing in finding s-sparse solutions $\mathbf{x} \in \mathbb{R}^n$ to the linear system

$$\mathbf{A}\mathbf{x} = \mathbf{z} \tag{1.1}$$

via solving the ℓ^q-minimization problem:

$$\min \|\mathbf{y}\|_q \quad \text{subject to } \mathbf{A}\mathbf{y} = \mathbf{z}$$

where $0 < q \le 1$, $2 \le 2s \le m \le n$, $\mathbf{A}$ is an $m \times n$ matrix, and $\mathbf{z} \in \mathbb{R}^m$ is the observation data ([1, 5, 7, 9, 12, 14]).

One of the basic questions about finding s-sparse solutions to the linear system (1.1) is under what circumstances the linear system (1.1) has a unique solution in Σ_s, the set of all s-sparse vectors.

Proposition 1.1. ([12, 15]) *Let $2s \le m \le n$ and $\mathbf{A}$ be an $m \times n$ matrix. Then the following statements are equivalent:*

(i) *The measurement $\mathbf{A}\mathbf{x}$ uniquely determines each s-sparse vector $\mathbf{x}$.*
(ii) *There is a decoder $\Delta : \mathbb{R}^m \longmapsto \mathbb{R}^n$ such that $\Delta(\mathbf{A}\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \Sigma_s$.*
(iii) *The only $2s$-sparse vector $\mathbf{y}$ that satisfies $\mathbf{A}\mathbf{y} = \mathbf{0}$ is the zero vector.*

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(iv) *There exist positive constants* α_{2s} *and* β_{2s} *such that*

$$\alpha_{2s}\|\mathbf{x}\|_2 \le \|\mathbf{A}\mathbf{x}\|_2 \le \beta_{2s}\|\mathbf{x}\|_2 \quad \text{for all } \mathbf{x} \in \Sigma_{2s}. \tag{1.2}$$

The first contribution of this paper is to provide another equivalent statement:

(v) *There exists* $0 < q \le 1$ *such that the decoder* $\Delta : \mathbb{R}^m \longmapsto \mathbb{R}^n$ *defined by*

$$\Delta(\mathbf{z}) := \mathrm{argmin}_{\mathbf{A}\mathbf{y}=\mathbf{z}}\|\mathbf{y}\|_q$$

satisfies $\Delta(\mathbf{A}\mathbf{x}) = \mathbf{x}$ *for all* $\mathbf{x} \in \Sigma_s$.

The implication from (v) to (ii) is obvious. Hence it suffices to prove the implication from (iv) to (v). For this, we recall the *restricted isometry property* of order s for an $m\times n$ matrix $\mathbf{A}$, i.e., there exists a positive constant $\delta \in (0,1)$ such that

$$(1-\delta)\|\mathbf{x}\|_2^2 \le \|\mathbf{A}\mathbf{x}\|_2^2 \le (1+\delta)\|\mathbf{x}\|_2^2 \quad \text{for all } \mathbf{x} \in \Sigma_s. \tag{1.3}$$

The smallest positive constant δ that satisfies (1.3), to be denoted by $\delta_s(\mathbf{A})$, is known as the *restricted isometry constant* [5, 7]. Notice that given a matrix $\mathbf{A}$ that satisfies (1.2), its rescaled matrix $\mathbf{B} := \sqrt{2/(\alpha_{2s}^2+\beta_{2s}^2)}\mathbf{A}$ has the restricted isometry property of order $2s$ and its restricted isometry constant is given by $(\beta_{2s}^2-\alpha_{2s}^2)/(\beta_{2s}^2+\alpha_{2s}^2)$. Therefore the implication from (iv) to (v) further reduces to establishing the following result:

Theorem 1.2. *Let integers* m, n *and* s *satisfy* $2s \le m \le n$. *If* $\mathbf{A}$ *is an* $m \times n$ *matrix with* $\delta_{2s}(\mathbf{A}) \in (0,1)$, *then there exists* $q \in (0,1]$ *such that any* s*-sparse vector* $\mathbf{x}$ *can be exactly recovered by solving the* ℓ^q*-minimization problem:*

$$\min \|\mathbf{y}\|_q \quad \text{subject to} \quad \mathbf{A}\mathbf{y} = \mathbf{A}\mathbf{x}. \tag{1.4}$$

The above existence theorem about ℓ^q-minimization is established in [17] and [9] under a stronger assumption that $\delta_{2s+2}(\mathbf{A}) \in (0,1)$ and $\delta_{2s+1}(\mathbf{A}) \in (0,1)$ respectively, as it is obvious that $\delta_{2s}(\mathbf{A}) \le \delta_{2s+1}(\mathbf{A}) \le \delta_{2s+2}(\mathbf{A})$ for any $m \times n$ matrix $\mathbf{A}$.

Having the above existence theorem about ℓ^q-minimization in hand, now we consider the problem how to select the positive exponent q in the ℓ^q-minimization problem (1.4) for a given measurement matrix. Given integers s, m and n satisfying $2s \le m \le n$ and an $m \times n$ matrix $\mathbf{A}$, define

$$q_s(\mathbf{A}) := \sup\big\{q \in [0,1]\big|\text{ any vector } \mathbf{x} \in \Sigma_s \text{ can be exactly recovered by solving the } \ell^q-\text{minimization problem (1.4)}\big\}.$$

Obviously $q_s(\mathbf{A}) > 0$ whenever $\delta_{2s}(\mathbf{A}) < 1$ by Theorem 1.2. It is known that any s-sparse vector $\mathbf{x} \in \mathbb{R}^n$ can be exactly recovered by solving the ℓ^q-minimization problem (1.4) whenever $q < q_s(\mathbf{A})$ [18]. This establishes the equivalence among different exponent $q \in [0, q_s(\mathbf{A}))$ in recovering s-sparse solutions via solving the ℓ^q-minimization problem (1.4). Hence in order to recover sparsest vector $\mathbf{x}$ from the measurement $\mathbf{Ax}$, one may solve the ℓ^q-minimization problem (1.4) for some $0 < q \le 1$ rather than the ℓ^0-minimization problem. Empirical evidence ([9, 22, 23]) strongly indicates that solving the ℓ^q-minimization problem with $0 < q \le 1$ takes much less time than with $q = 0$.

The ℓ^0-minimization problem is a combinatorial optimization problem and NP-hard to solve [20], while on the other hand the ℓ^1-minimization is convex and polynomial-time solvable [2]. To guarantee the equivalence between the ℓ^0 and ℓ^1-minimization problems (1.4) in finding the sparse vector $\mathbf{x}$ from its measurement $\mathbf{Ax}$, one needs to meet various requirements on the matrix $\mathbf{A}$, for instance, $\delta_s(\mathbf{A}) + \delta_{2s}(\mathbf{A}) + \delta_{3s}(\mathbf{A}) < 1$ in [6], $\delta_{3s}(\mathbf{A}) + 3\delta_{4s}(\mathbf{A}) < 2$ in [5], and $\delta_{2s}(\mathbf{A}) < 1/3 \approx 0.3333, \sqrt{2} - 1 \approx 0.4142, 2/(3 + \sqrt{2}) \approx 0.4531, 2/(2 + \sqrt{5}) \approx 0.4731, 4/(6 + \sqrt{6}) \approx 0.4734$ in [12, 4, 17, 3, 16] respectively. Many random matrices with i.d.d. entries satisfy those requirements to guarantee the equivalence [7], but lots of deterministic matrices do not. In particular, matrices $\mathbf{A}_\epsilon$ are constructed in [13] for any $\epsilon > 0$ such that $\delta_{2s}(\mathbf{A}_\epsilon) < 1/\sqrt{2} + \epsilon$ and that it fails on the recovery of some s-sparse vectors $\mathbf{x}$ by solving the ℓ^1-minimization problem (1.4) with $\mathbf{A}$ replaced by $\mathbf{A}_\epsilon$.

The ℓ^q-minimization problem (1.4) with $0 < q < 1$ is more difficult to solve than the ℓ^1-minimization problem due to the nonconvexity and nonsmoothness. In fact, it is NP-hard to find a global minimizer in general but polynomial-time doable to find local minimizer [19]. Various algorithms have been developed to solve the ℓ^q-minimization problem (1.4), see for instance [8, 11, 14, 17, 21].

Having shown that s-sparse solutions can be recovered via solving the ℓ^q-minimization problem (1.4) with $q \in [0, q_s(\mathbf{A}))$, we next study the problem how the quantity $q_s(\mathbf{A})$ depends on the measurement matrix $\mathbf{A}$. For that purpose, we introduce a quantity

$$q_{\max}(\delta; m, n, s) := \inf_{\delta_{2s}(\mathbf{A}) \le \delta} q_s(\mathbf{A}), \ \ \delta \in (0, 1),$$

that depends on the measurement matrix $\mathbf{A}$ of size $m \times n$ indirectly. Clearly given any positive number $q < q_{\max}(\delta; m, n, s)$ and any $m \times n$ matrix $\mathbf{A}$ with $\delta_{2s}(\mathbf{A}) \le \delta$, any vector $\mathbf{x} \in \Sigma_s$ can be exactly recovered by solving the ℓ^q-minimization problem (1.4). For any $0 < q \le 1$ and sufficiently small ϵ, matrices $\mathbf{A}_\epsilon$ of size $(n - 1) \times n$ are constructed in [13] such that $\delta_{2s}(\mathbf{A}_{q,\epsilon}) < \frac{\eta_q}{2-q-\eta_q} + \epsilon$ and there is an s-sparse vector which cannot be recovered exactly by solving the ℓ^q-minimization problem (1.4) with $\mathbf{A}$ replaced by $\mathbf{A}_{q,\epsilon}$, where η_q is the unique positive solution to $\eta_q^{2/q} + 1 = 2(1 - \eta_q)/q$. The above construction of matrices for which the ℓ^q-minimization fails to recover s-sparse vectors, together with the asymptotic estimate $\eta_q = 1 - qx_0 + o(q)$ as

$q \to 0$, gives that

$$\limsup_{\delta\to 1-} \frac{q_{\max}(\delta; n-1, n, s)}{1-\delta} \le \lim_{q\to 0+} \frac{q(2-q-\eta_q)}{2-q-2\eta_q} = \frac{1}{2x_0-1} \approx 3.5911,$$

where x_0 is the unique positive solution of the equation $e^{-2x} = 2x-1$. The second contribution of this paper is a lower asymptotic bound estimate for the quantity $q_{\max}(\delta; m, n, s)$ as $\delta \to 1-$.

Theorem 1.3. *Let* $q_{\max}(\delta; m, n, s)$ *be defined as in* (1.5). *Then*

$$\liminf_{\delta\to 1-} \frac{q_{\max}(\delta; m, n, s)}{1-\delta} \ge \frac{e}{4} \approx 0.6796. \tag{1.5}$$

The asymptotic bound estimate for the quantity $q_{\max}(\delta; m, n, s)$ in Theorem 1.3 would be useful in the sparse recovery problem when we have certain information about the restricted isometry constant of the measurement matrix. We do not know whether the limit $\lim_{\delta\to 1-} q_{\max}(\delta; m, n, s)/(1-\delta)$ exists and how it depends on the dimensions m, n and the sparsity s if it does.

To prove Theorems 1.2 and 1.3, we introduce a function $b(q, \delta)$ on the unit square $(0, 1] \times (0, 1)$,

$$\begin{aligned} b(q,\delta) \ := \ \delta^{-1} \inf_{0<r_0<1} \max\Big\{ & \frac{1+r_0\delta}{(1+r_0^q\delta^q)^{1/q}}, \sup_{\sqrt{2}(1-r_0)\delta/2\le y\le 1} \frac{2y}{\big(1+2^{-q/2}y^{2+q}\big)^{1/q}}, \\ & \sup_{\sqrt{2}(1-r_0)\delta/2\le y\le 1} \frac{3y}{\big(1+y\big)^{1/q}}, \ \sup_{1\le y} \frac{2y}{\big(1+y\big)^{1/q}} \Big\}. \end{aligned} \tag{1.6}$$

where $0 < q \le 1$ and $\delta \in (0, 1)$, see Figure 1. The third contribution of this paper is about stable recovery of a compressive signal from its noisy observation.

Theorem 1.4. *Let* m, n *and* s *be integers with* $2s \le m \le n$, $\mathbf{A}$ *be an* $m \times n$ *matrix with* $\delta_{2s}(\mathbf{A}) \in (0, 1)$, $\epsilon \ge 0$, *and let* $q \in (0, 1]$ *satisfy*

$$b(q, \delta_1) < 1 \tag{1.7}$$

where

$$\delta_1 := \sqrt{\frac{1-\delta_{2s}(\mathbf{A})}{1+\delta_{2s}(\mathbf{A})}}.$$

If $\mathbf{x}$ *is the object we wish to reconstruct,* $\mathbf{z} = \mathbf{A}\mathbf{x} + \mathbf{z}$ *is the noisy measurement with the noise* $\mathbf{z}$ *satisfying* $\|\mathbf{z}\|_2 \le \epsilon$, *and* $\mathbf{x}^*$ *is the solution of the* ℓ^q*-minimization problem:*

$$\min_{\tilde{\mathbf{x}}\in\mathbb{R}^n} \|\tilde{\mathbf{x}}\|_q \ \text{ subject to } \ \|\mathbf{A}\tilde{\mathbf{x}} - \mathbf{y}\|_2 \le \epsilon,$$

then

$$\|\mathbf{x}^* - \mathbf{x}\|_2 \le C_0 s^{1/2-1/q} \|\mathbf{x} - \mathbf{x}_s\|_q + C_1\epsilon \tag{1.8}$$

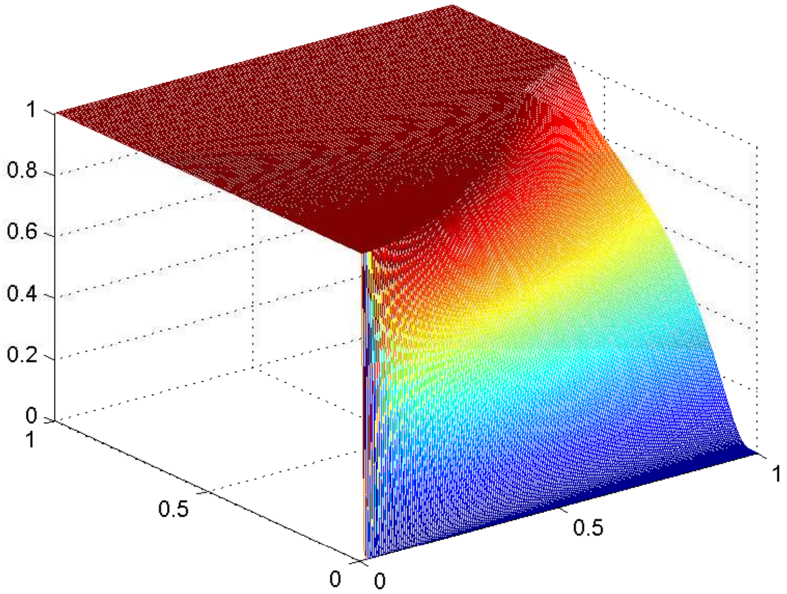

FIGURE 1. The function $\min(b(q,\delta),1)$ on $(0,1)\times(0,1)$.

and

$$\|\mathbf{x}^* - \mathbf{x}\|_q \le C_2\|\mathbf{x}-\mathbf{x}_s\|_q + C_3 s^{1/q-1/2}\epsilon, \tag{1.9}$$

where $\mathbf{x}_s$ *be the best* s*-sparse vector in* $\mathbb{R}^n$ *to approximate* $\mathbf{x}$*, i.e.,*

$$\|\mathbf{x}_s - \mathbf{x}\|_q = \inf_{\mathbf{x}'\in\Sigma_s} \|\mathbf{x}'-\mathbf{x}\|_q$$

and $C_i, 0\le i\le 3$*, are positive constants independent on* $\epsilon, \mathbf{x}$ *and* s*.*

The stable recovery of a compressive signal from its noisy observation has been established under various assumptions on the restricted isometry constant, for instance, $\delta_{3s}(\mathbf{A})+3\delta_{4s}(\mathbf{A}) < 2$ and $q=1$ in [5], and $\delta_{2s}(\mathbf{A}) < \sqrt{2}-1$ and $q=1$ in [4], $\delta_{2t}(\mathbf{A}) < 2(\sqrt{2}-1)(t/s)^{1/q-1/2}/(1+2(\sqrt{2}-1)(t/s)^{1/q-1/2})$ for some $t\ge s$ and $0<q\le 1$ in [17], and $\delta_{ks}(\mathbf{A})+k^{2/q-1}\delta_{(k+1)s}(\mathbf{A}) < k^{2/q}-1$ for some $k\in\mathbb{Z}/s$ and $0<q\le 1$ in [22, 23]. As pointed in [13], not all compressive signals can be recovered from their noisy measurements approximately via solving an ℓ^1-minimization problem when the restricted isometry constant of order $2s$ for the measurement matrix is close to one. From Theorem 1.4, we see that any compressive signal can be recovered from its noisy measurements approximately via solving some ℓ^q-minimization problem even if the restricted isometry constant of order $2s$ for the measurement matrix is close to one (but not equal to one). The exponent q in the ℓ^q-minimization is required to satisfy (1.7), which implies that the exponent q

can be chosen to depend **only** on the restricted isometry constant $\delta_{2s}(\mathbf{A})$ of order $2s$ for the measurement matrix $\mathbf{A}$ (c.f. [17, 22, 23]). Furthermore, by Theorem 1.3, the exponent q can be chosen to almost proportional to $1-\delta_{2s}(\mathbf{A})$ when the restricted isometry constant $\delta_{2s}(\mathbf{A})$ of order $2s$ is close to one.

Now let us apply Theorem 1.4 to prove Theorems 1.2 and 1.3. We observe that $b(q,\delta)$ tends to zero as q approaches zero, i.e.,

$$\lim_{q\to 0+} b(q,\delta)=0 \quad \text{for all } \delta\in(0,1), \tag{1.10}$$

see Figure 1. Applying Theorem 1.4 with $\epsilon=0$ and $\mathbf{x}=\mathbf{x}_s$, and using the limit (1.10), we establish Theorem 1.2.

Define

$$\tilde q_{\max}(\delta)=\sup\{q\in(0,1] \,|\, b(q,\delta)<1\}, \tag{1.11}$$

see Figure 2. We notice that

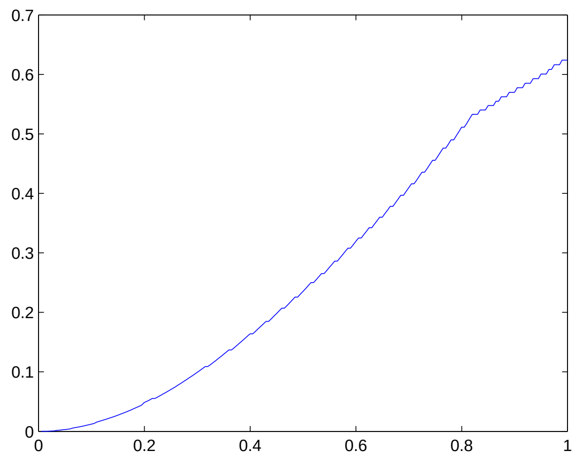

FIGURE 2. The function $\tilde q_{\max}(\delta)$ on $(0,1)$.

$$\lim_{\delta\to 0+}\frac{\tilde q_{\max}(\delta)}{\delta^2}=\frac{e}{2}, \tag{1.12}$$

see Appendix A for the proof. Applying Theorem 1.4 with $\epsilon=0$ and $\mathbf{x}=\mathbf{x}_s$, we conclude that

$$q_{\max}(\delta,m,n,s)\ge \tilde q_{\max}\big(\sqrt{(1-\delta)/(1+\delta)}\big).$$

This together with (1.12) leads to the lower asymptotic estimate (1.5) and hence proves Theorem 1.3.

We present a visual interpretation of the requirement on the exponent q in Theorem 1.4 and [13]. Let

$$q_{\rm succ}(\delta) = \tilde q_{\max}\Big(\sqrt{(1-\delta)/(1+\delta)}\Big),$$

and $q_{\rm fail}(\delta)$ be the solution of the equation

$$\Big(\frac{(2-q)\delta}{1+\delta}\Big)^{2/q} + 1 = \frac{2-2\delta+2q\delta}{q+q\delta}$$

if it exists and be equal to one otherwise, see Figure 3. Then when $q <$

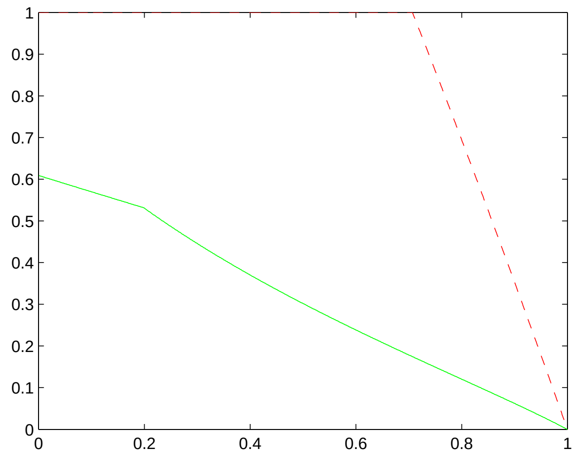

FIGURE 3. The function $q_{\rm succ}(\delta)$ is plotted in continuous line, while the function $q_{\rm fail}(\delta)$ is plotted in dashed line

$q_{\rm succ}(\delta_{2s}(\mathbf{A}))$ (i.e. $(q, \delta_{2s}(\mathbf{A}))$ lies in the region below the continuous line in Figure 3), any s-sparse vector $\mathbf{x}$ can be exactly recovered by solving the ℓ^q-minimization problem (1.4) by Theorem 1.4, while if $q > q_{\rm fail}(\delta)$ (i.e. (q, δ) is in the region above the dashed line in Figure 3) by [13] there exists a matrix $\mathbf{A}$ with $\delta_{2s}(\mathbf{A}) \le \delta$ and an s-sparse vector $\mathbf{x}$ such that the vector $\mathbf{x}$ cannot be exactly recovered by solving the ℓ^q-minimization problem (1.4).

The rest of this paper is organized as follows. In the next section, we give the proof of Theorem 1.4 and a remark on null space property of a measurement matrix. The proof of the limit (1.12) is given in Appendix A.

2. Proof of Theorem 1.4

For any finite decreasing sequence $\{a_j\}_{j\ge 1}$ of nonnegative numbers, it holds that

$$\sum_{k\ge 1}\Big(\sum_{i=1}^{s} a_{ks+i}^2\Big)^{1/2} \le s^{1/2-1/q}\Big(\sum_{j\ge 1} a_j^q\Big)^{1/q}.$$

In the following lemma, we are seeking conditions on the decreasing sequence $\{a_j\}_{j\ge1}$ and the exponent $q\in(0,1]$ for any given $r\in(0,1)$ such that

$$\sum_{k\ge1}\Big(\sum_{i=1}^s a_{ks+i}^2\Big)^{1/2}\le r s^{1/2-1/q}\Big(\sum_{j\ge1}a_j^q\Big)^{1/q}.$$

Lemma 2.1. *Let $0<q\le1$, $s\ge1$ be a positive integer, and let $\{a_j\}_{j\ge1}$ be a finite decreasing sequence of nonnegative numbers with*

$$\sum_{k\ge1}\Big(\sum_{i=1}^s a_{ks+i}^2\Big)^{1/2}\ge \delta\Big(\sum_{i=1}^s |a_i|^2\Big)^{1/2} \tag{2.1}$$

for some $\delta\in(0,1)$. Then

$$\sum_{k\ge1}\Big(\sum_{i=1}^s a_{ks+i}^2\Big)^{1/2}\le \delta b(q,\delta) s^{1/2-1/q}\Big(\sum_{j\ge1}a_j^q\Big)^{1/q}, \tag{2.2}$$

where $b(q,\delta)$ is defined as in (1.6).

We postpone the proof of the above lemma to the end of this section as the proof is long and of interest by itself. The contour plotting of the function $\delta b(q,\delta)$ is given in Figure 4.

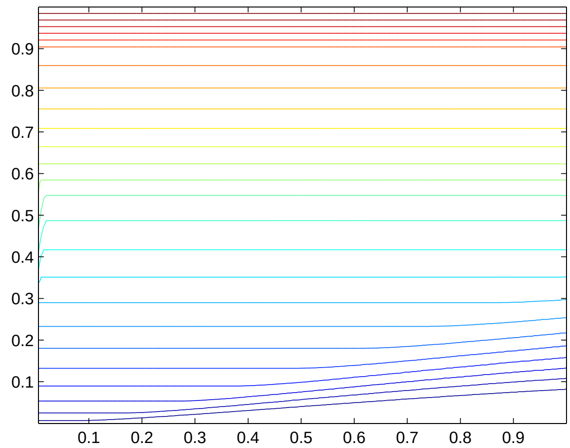

FIGURE 4. The contour plotting of the function $\delta b(q,\delta)$ on $(0,1)\times(0,1)$.

Denote by $\mathbf{v}_S$ the vector which equals to $\mathbf{v}\in\mathbb{R}^n$ on S and vanishes on the complement S^c where $S\subset\{1,\dots,n\}$. Now we start to prove Theorem 1.4. In the first half part of the argument, we follow [4, 5]. Set $\mathbf{h}=\mathbf{x}^*-\mathbf{x}$, and denote by S_0 the support of the vector $\mathbf{x}_s\in\Sigma_s$, by S_0^c the complement of the set S_0 in $\{1,\dots,n\}$. Then

$$\|\mathbf{A}\mathbf{h}\|_2=\|\mathbf{A}\mathbf{x}^*-\mathbf{A}\mathbf{x}\|_2\le\|\mathbf{A}\mathbf{x}^*-\mathbf{y}\|_2+\|\mathbf{z}\|_2\le 2\epsilon \tag{2.3}$$

and

$$\|\mathbf{h}_{S_0^c}\|_q^q \le \|\mathbf{h}_{S_0}\|_q^q + 2\|\mathbf{x}-\mathbf{x}_s\|_q^q, \tag{2.4}$$

since

$$\begin{aligned}\|\mathbf{x}_s\|_q^q + \|\mathbf{x}_{S_0^c}\|_q^q &= \|\mathbf{x}\|_q^q \ge \|\mathbf{x}^*\|_q^q = \|\mathbf{x}_s + \mathbf{h}_{S_0}\|_q^q + \|\mathbf{x}_{S_0^c} + \mathbf{h}_{S_0^c}\|_q^q \\ &\ge \|\mathbf{x}_s\|_q^q - \|\mathbf{h}_{S_0}\|_q^q + \|\mathbf{h}_{S_0^c}\|_q^q - \|\mathbf{x}_{S_0^c}\|_q^q.\end{aligned}$$

We partition $S_0^c \subset \{1,\dots,n\}$ as $S_0^c = S_1 \cup \cdots \cup S_l$, where S_1 is the set of indices of the s largest absolute-value component of $\mathbf{h}$ in S_0^c, S_2 is the set of indices of the next s largest absolute-value components of $\mathbf{h}$ on S_0^c, and so on. Applying the parallelogram identity, we obtain from the restricted isometry property (1.3) that

$$|\langle \mathbf{Au}, \mathbf{Av}\rangle| \le \delta_{2s}(\mathbf{A})\|\mathbf{u}\|_2\|\mathbf{v}\|_2$$

for all s-sparse vectors $\mathbf{u},\mathbf{v}\in\Sigma_s$ whose supports have empty intersection [7]. Then

$$\begin{aligned}\Big\langle \mathbf{A}\big(\sum_{i\ge 2}\mathbf{h}_{S_i}\big), \mathbf{A}\big(\sum_{j\ge 2}\mathbf{h}_{S_j}\big)\Big\rangle &\le \sum_{i,j\ge 2}\delta_{2s}(\mathbf{A})\|\mathbf{h}_{S_i}\|_2\|\mathbf{h}_{S_j}\|_2 + \sum_{j\ge 2}\|\mathbf{h}_{S_j}\|_2^2 \\ &= \delta_{2s}(\mathbf{A})\Big(\sum_{j\ge 2}\|\mathbf{h}_{S_j}\|_2\Big)^2 + \sum_{j\ge 2}\|\mathbf{h}_{S_j}\|_2^2 \\ &\le \big(1+\delta_{2s}(\mathbf{A})\big)\Big(\sum_{j\ge 2}\|\mathbf{h}_{S_j}\|_2\Big)^2.\end{aligned}$$

This together with (1.3) and (2.3) implies that

$$\begin{aligned}(1-\delta_{2s}(\mathbf{A}))\big(\|\mathbf{h}_{S_0}\|_2^2 + \|\mathbf{h}_{S_1}\|_2^2\big) &\le \langle \mathbf{A}(\mathbf{h}_{S_0}+\mathbf{h}_{S_1}), \mathbf{A}(\mathbf{h}_{S_0}+\mathbf{h}_{S_1})\rangle \\ &\le \Big\langle \mathbf{Ah} - \mathbf{A}\big(\sum_{i\ge 2}\mathbf{h}_{S_i}\big), \mathbf{Ah} - \mathbf{A}\big(\sum_{j\ge 2}\mathbf{h}_{S_j}\big)\Big\rangle \\ &\le \Big(2\epsilon + \sqrt{1+\delta_{2s}(\mathbf{A})}\sum_{j\ge 2}\|\mathbf{h}_{S_j}\|_2\Big)^2. \qquad (2.5)\end{aligned}$$

In the following second half part of the argument, we use the improved inequality (2.2) to obtain the desired estimates (1.8) and (1.9). By the continuity of the function $b(q,\delta)$ about $\delta\in(0,1)$ and the assumption (1.7), there exists a positive number r such that

$$b(q, \delta_1/(1+r)) < 1. \tag{2.6}$$

If $\sum_{j\ge 2}\|\mathbf{h}_{S_j}\|_2\le 2\epsilon/(r\sqrt{1+\delta_{2s}(\mathbf{A})})$, then it follows from (2.4), (2.5) and the fact that $\mathbf{h}_{S_0}\in\Sigma_s$ that

$$\begin{aligned}\|\mathbf{x}^*-\mathbf{x}\|_2 &= \|\mathbf{h}\|_2\le \big(\|\mathbf{h}_{S_0}\|_2^2+\|\mathbf{h}_{S_1}\|_2^2\big)^{1/2}+\sum_{j\ge 2}\|\mathbf{h}_{S_j}\|_2\\ &\le 2\Big(\frac{(1+r)}{r\sqrt{1-\delta_{2s}(\mathbf{A})}}+\frac{1}{r\sqrt{1+\delta_{2s}(\mathbf{A})}}\Big)\epsilon, \qquad (2.7)\end{aligned}$$

and

$$\begin{aligned}\|\mathbf{x}^*-\mathbf{x}\|_q^q &= \|\mathbf{h}_{S_0}\|_q^q+\|\mathbf{h}_{S_0^c}\|_q^q\le 2\|\mathbf{h}_{S_0}\|_q^q+2\|\mathbf{x}-\mathbf{x}_s\|_q^q\\ &\le 2s^{1-q/2}\|\mathbf{h}_{S_0}\|_2^q+2\|\mathbf{x}-\mathbf{x}_s\|_q^q\\ &\le 2^{1+q}\frac{(1+r)^q}{r^q(1-\delta_{2s}(\mathbf{A}))^{q/2}}s^{1-q/2}\epsilon^q+2\|\mathbf{x}-\mathbf{x}_s\|_q^q. \qquad (2.8)\end{aligned}$$

If $\sum_{j\ge 2}\|\mathbf{h}_{S_j}\|_2\ge 2\epsilon/(r\sqrt{1+\delta_{2s}(\mathbf{A})})$, then

$$\delta_1\big(\|\mathbf{h}_{S_0}\|_2^2+\|\mathbf{h}_{S_1}\|_2^2\big)^{1/2}\le (1+r)\sum_{j\ge 2}\|\mathbf{h}_{S_j}\|_2 \qquad (2.9)$$

by (2.5), where we set $\delta_1=\sqrt{(1-\delta_{2s}(\mathbf{A}))/(1+\delta_{2s}(\mathbf{A}))}$. Using (2.9) and applying Lemma 2.1 with $\delta=\delta_1/(1+r)$ give

$$\sum_{j\ge 2}\|\mathbf{h}_{S_j}\|_2\le \frac{\delta_1}{1+r}b(q,\delta_1/(1+r))s^{1/2-1/q}\|\mathbf{h}_{S_0^c}\|_q. \qquad (2.10)$$

Noting the fact that $\mathbf{h}_{S_0}\in\Sigma_s$ and then applying (2.4), (2.9) and (2.10) yield

$$\begin{aligned}\|\mathbf{h}_{S_0}\|_q^q &\le s^{1-q/2}\|\mathbf{h}_{S_0}\|_2^q\le \big(b(q,\delta_1/(1+r))\big)^q\|\mathbf{h}_{S_0^c}\|_q^q\\ &\le \big(b(q,\delta_1/(1+r))\big)^q\|\mathbf{h}_{S_0}\|_q^q+2\big(b(q,\delta_1/(1+r))\big)^q\|\mathbf{x}-\mathbf{x}_s\|_q^q.\end{aligned}$$

This, together with (2.6), leads to the following crucial estimate:

$$\|\mathbf{h}_{S_0}\|_q^q\le \frac{2(b(q,\delta_1/(1+r)))^q}{1-(b(q,\delta_1/(1+r)))^q}\|\mathbf{x}-\mathbf{x}_s\|_q^q. \qquad (2.11)$$

Combining (2.4), (2.9), (2.10) and (2.11), we obtain

$$\begin{aligned}\|\mathbf{x}^*-\mathbf{x}\|_2 &\le \big(\|\mathbf{h}_{S_0}\|_2^2+\|\mathbf{h}_{S_1}\|_2^2\big)^{1/2}+\sum_{j\ge 2}\|\mathbf{h}_{S_j}\|_2\\ &\le \frac{2^{1/q}(1+r+\delta_1)b(q,\delta_1/(1+r))}{\big(1-(b(q,\delta_1/(1+r)))^q\big)^{1/q}}s^{1/2-1/q}\|\mathbf{x}-\mathbf{x}_s\|_q, \qquad (2.12)\end{aligned}$$

and

$$\begin{aligned}\|\mathbf{x}^*-\mathbf{x}\|_q^q &\le 2\|\mathbf{h}_{S_0}\|_q^q+2\|\mathbf{x}-\mathbf{x}_s\|_q^q\\ &\le \frac{2+2(b(q,\delta_1/(1+r)))^q}{1-(b(q,\delta_1/(1+r)))^q}\|\mathbf{x}-\mathbf{x}_s\|_q^q. \qquad (2.13)\end{aligned}$$

The desired error estimates (1.8) and (1.9) follow from (2.7), (2.8), (2.12) and (2.13).

Remark 2.2. We say that an $m \times n$ matrix $\mathbf{A}$ has the *null space property* of order s in ℓ^q if there exists a positive constant γ such that

$$\|\mathbf{h}_S\|_q \le \gamma \|\mathbf{h}_{S^c}\|_q \tag{2.14}$$

holds for all $\mathbf{h}$ satisfying $\mathbf{A}\mathbf{h} = \mathbf{0}$ and all sets S with its cardinality $\#S$ less than or equal to s ([12]). The minimal constant γ in (2.14), to be denoted by $\gamma_{s,q}(\mathbf{A})$, is known as the *null space constant* of order s in ℓ^q. Recall that any s-sparse signal $\mathbf{x}$ can be recovered from its measurement $\mathbf{A}\mathbf{x}$ via solving the ℓ^q-minimization (1.4) if and only if $\gamma_{s,q}(\mathbf{A}) < 1$. This together with Theorem 1.4 implies that $\gamma_{s,q}(\mathbf{A}) < 1$ whenever $b\big(q, \sqrt{(1-\delta_{2s}(\mathbf{A}))/(1+\delta_{2s}(\mathbf{A}))}\big) < 1$. In the next theorem, we show that

$$\gamma_{s,q}(\mathbf{A}) \le b\big(q, \sqrt{(1-\delta_{2s}(\mathbf{A}))/(1+\delta_{2s}(\mathbf{A}))}\big) \tag{2.15}$$

holds for all measurement matrices $\mathbf{A}$ with $\delta_{2s}(\mathbf{A}) \in (0,1)$.

Theorem 2.3. *Let q be a positive number in $(0,1]$, integers m, n and s satisfy $2s \le m \le n$, $\mathbf{A}$ be an $m \times n$ matrix with $\delta_{2s}(\mathbf{A}) \in (0,1)$. Then $\mathbf{A}$ has the null space property of order s in ℓ^q, and its null space constant $\gamma_{s,q}(\mathbf{A})$ satisfies* (2.15).

Proof. Let $\mathbf{h}$ satisfy $\mathbf{A}\mathbf{h} = \mathbf{0}$, and S_0 be a subset of $\{1, \ldots, n\}$ with cardinality $\#S_0$ less than or equal to s. We partition $S_0^c \subset \{1, \ldots, n\}$ as $S_0^c = S_1 \cup \cdots \cup S_l$, where S_1 is the set of indices of the s largest components, in absolute value, of $\mathbf{h}$ in S_0^c, S_2 is the set of indices of the next s largest components, in absolute value, of $\mathbf{h}$ in $(S_0 \cup S_1)^c$, and so on. Using similar argument to the one used in establishing (2.5), we obtain

$$\begin{aligned}(1-\delta_{2s}(\mathbf{A}))\big(\|\mathbf{h}_{S_0}\|_2^2 + \|\mathbf{h}_{S_1}\|_2^2\big) &\le \big\langle \mathbf{A}\big(\sum_{i\ge 2} \mathbf{h}_{S_i}\big), \mathbf{A}\big(\sum_{j\ge 2} \mathbf{h}_{S_j}\big)\big\rangle \\ &\le \big(1+\delta_{2s}(\mathbf{A})\big)\Big(\sum_{j\ge 2} \|\mathbf{h}_{S_j}\|_2\Big)^2, \end{aligned} \tag{2.16}$$

which implies that $\sum_{j\ge 2} \|\mathbf{h}_{S_j}\|_2 \ge \delta_1 \|\mathbf{h}_{S_1}\|_2$, where $\delta_1 = \big(\frac{1-\delta_{2s}(\mathbf{A})}{1+\delta_{2s}(\mathbf{A})}\big)^{1/2}$. Applying Lemma 2.1 gives

$$\sum_{j\ge 2} \|\mathbf{h}_{S_j}\|_2 \le \delta_1 b(q, \delta_1) s^{1/2-1/q} \|\mathbf{h}_{S_0^c}\|_q.$$

Then substituting the above estimate for $\sum_{j\ge 2} \|\mathbf{h}_{S_j}\|_2$ into the right hand side of the inequality (2.16) and recalling that $\mathbf{h}_{S_0}$ is an s-sparse vector lead to

$$\|\mathbf{h}_{S_0}\|_q \le s^{1/q-1/2} \|\mathbf{h}_{S_0}\|_2 \le \frac{s^{1/q-1/2}}{\delta_1} \sum_{j\ge 2} \|\mathbf{h}_{S_j}\|_2 \le b(q, \delta_1) \|\mathbf{h}_{S_0^c}\|_q,$$

and hence (2.15) is proved. □

We conclude this section by the proof of Lemma 2.1.

2.1. **Proof of Lemma 2.1.** To prove it, we need two inequalities (2.17) and (2.21).

Lemma 2.4. *Let* $0 < q \le 1, 0 \le c \le 1$ *and* $a, b > 0$. *Then*

$$a + \sum_{k=1}^{m} t_k \le \max\Big\{ \max_{1\le k\le m} \frac{k+a}{(k+b)^{1/q}}, \frac{a+c}{(b+c^q)^{1/q}} \Big\} \Big(b + \sum_{k=1}^{m} t_k^q\Big)^{1/q} \tag{2.17}$$

hold for all $(t_1, \dots, t_m) \in [0,1]^m$ *with* $t_1 + \cdots + t_m \ge c$.

Proof. Define

$$F_{q,a,b,c}(m,n) = \sup_{\substack{(t_1,\dots,t_m)\in[0,1]^m \\ t_1+\cdots+t_m\ge c}} \frac{n+a+\sum_{k=1}^m t_k}{(n+b+\sum_{k=1}^m t_k^q)^{1/q}}. \tag{2.18}$$

By the method of Lagrange multiplier, the function $(n + a + \sum_{k=1}^m t_k)(n + b + \sum_{k=1}^m t_k^q)^{-1/q}$ attains its maximum on the boundary or on those points $(t_1, \dots, t_m)$ whose components are the same, i.e.,

$$\begin{aligned} F_{q,a,b,c}(m,n) &= \max\Big\{ F_{q,a,b,0}(m-1, n+1), F_{q,a,b,c}(m-1,n), \\ &\qquad \sup_{c/m\le t\le 1} \frac{n+a+mt}{(n+b+mt^q)^{1/q}} \Big\}. \end{aligned}$$

As the function $(n+a+mt)(n+b+mt^q)^{-1/q}$ has at most one critical point and the second derivative at that critical point (if it exists) is positive, we then have

$$\begin{aligned} F_{q,a,b,c}(m,n) &= \max\Big\{ F_{q,a,b,0}(m-1,n+1), F_{q,a,b,c}(m-1,n), \\ &\qquad \frac{n+m+a}{(n+m+b)^{1/q}}, \frac{n+a+c}{(n+b+m^{1-q}c^q)^{1/q}} \Big\}. \end{aligned} \tag{2.19}$$

Applying (2.19) iteratively we obtain

$$\begin{aligned}
F_{q,a,b,c}(m,n) &= \max\Big\{F_{q,a,b,0}(m-2,n+2), F_{q,a,b,0}(m-2,n+1),\\
&\quad F_{q,a,b,c}(m-2,n), \frac{n+1+a}{(n+1+b)^{1/q}}, \frac{n+m-1+a}{(n+m-1+b)^{1/q}},\\
&\quad \frac{n+m+a}{(n+m+b)^{1/q}}, \frac{n+a+c}{(n+b+(m-1)^{1-q}c^q)^{1/q}}\Big\}\\
&= \cdots\\
&= \max\Big\{F_{q,a,b,0}(1,n+m-1), \cdots, F_{q,a,b,0}(1,n+1),\\
&\quad F_{q,a,b,c}(1,n), \frac{n+a+c}{(n+b+2^{1-q}c^q)^{1/q}},\\
&\quad \frac{n+m+a}{(n+m+b)^{1/q}}, \ldots, \frac{n+2+a}{(n+2+b)^{1/q}}, \frac{n+1+a}{(n+1+b)^{1/q}}\Big\}\\
&= \max\Big\{\max_{1\le k\le m} \frac{n+k+a}{(n+k+b)^{1/q}}, \frac{n+a+c}{(n+b+c^q)^{1/q}}\Big\}. \qquad (2.20)
\end{aligned}$$

Then the conclusion (2.17) follows by letting $n=0$ in the above estimate. □

Lemma 2.5. *Let $0<q\le 1$, $c_1,c_2\in[0,1]$ and $d_i,b_i>0$ for $i=1,2,3$. Then*

$$\begin{aligned}
& d_1+d_2x+d_3y+\sum_{k=1}^m t_k\\
\le\ & \max\Big\{\frac{d_1+d_2}{(b_1+b_2)^{1/q}}, \frac{d_1+d_2c_1}{(b_1+b_2c_1^q)^{1/q}}, \sup_{0\le l\le m}\frac{d_1+d_2+(d_3+l)c_2}{(b_1+b_2+(b_3+l)c_2^q)^{1/q}},\\
& \sup_{0\le l\le m}\frac{d_1+d_2c_1+(d_3+l)c_2}{(b_1+b_2c_1^q+(b_3+l)c_2^q)^{1/q}}\Big\}\times\Big(b_1+b_2x^q+b_3y^q+\sum_{k=1}^m t_k^q\Big)^{1/q} \qquad (2.21)
\end{aligned}$$

hold for all $0\le t_1,\ldots,t_m\le y$, $c_1\le x\le 1$ and $0\le y\le c_2$.

Proof. Note that the maximum values of the function $(a+bt)/(c+dt^q)^{1/q}$ on any closed subinterval of $[0,\infty)$ are attained on its boundary. Then

$$\begin{aligned}
& \frac{d_1+d_2x+d_3y+\sum_{k=1}^m t_k}{(b_1+b_2x^q+b_3y^q+\sum_{k=1}^m t_k^q)^{1/q}}\\
=\ & \sup_{0\le l\le m}\frac{d_1+d_2x+(d_3+l)y}{(b_1+b_2x^q+(b_3+l)y^q)^{1/q}}\\
=\ & \max\Big\{\frac{d_1+d_2x}{(b_1+b_2x^q)^{1/q}}, \sup_{0\le l\le m}\frac{d_1+(d_3+l)c_2+d_2x}{(b_1+(b_3+l)c_2^q+b_2x^q)^{1/q}}\Big\}\\
\le\ & \max\Big\{\frac{d_1+d_2}{(b_1+b_2)^{1/q}}, \frac{d_1+d_2c_1}{(b_1+b_2c_1^q)^{1/q}}, \sup_{0\le l\le m}\frac{d_1+d_2+(d_3+l)c_2}{(b_1+b_2+(b_3+l)c_2^q)^{1/q}},\\
& \sup_{0\le l\le m}\frac{d_1+d_2c_1+(d_3+l)c_2}{(b_1+b_2c_1^q+(b_3+l)c_2^q)^{1/q}}\Big\},
\end{aligned}$$

and (2.21) follows. □

Now we start to prove Lemma 2.1. The proof is quite technical. For better understanding, the reader may consider the illustrated example $s=1$ at the beginning. Clearly the conclusion (2.2) holds when $a_{s+1}=0$ for in this case the left hand side of (2.2) is equal to 0. So we may assume that $a_{s+1}\neq 0$ from now on. Let r_0 be an arbitrary number in $(0,1)$. To establish (2.2), we consider two cases.

Case I: $\sum_{k\ge 2} a_{ks+1}\ge r_0\delta a_{s+1}$.

In this case,

$$\begin{aligned}
\frac{\sum_{k\ge1}\big(\sum_{i=1}^s a_{ks+i}^2\big)^{1/2}}{\big(\sum_{j\ge1}a_j^q\big)^{1/q}} &\le \frac{s^{1/2}\sum_{k\ge1}a_{ks+1}}{s^{1/q}\big(\sum_{k\ge1}a_{ks+1}^q\big)^{1/q}}\\
&= s^{1/2-1/q}\frac{1+\sum_{k\ge2}a_{ks+1}/a_{s+1}}{\big(1+\sum_{k\ge2}(a_{ks+1}/a_{s+1})^q\big)^{1/q}}\\
&\le s^{1/2-1/q}\max\Big\{\frac{1+r_0\delta}{(1+r_0^q\delta^q)^{1/q}},\max_{k\ge1}\frac{k+1}{(k+1)^{1/q}}\Big\}\\
&= s^{1/2-1/q}(1+r_0\delta)(1+r_0^q\delta^q)^{-1/q}, \qquad (2.22)
\end{aligned}$$

where the first inequality holds because $\{a_j\}_{j\ge1}$ is a decreasing sequence of nonnegative numbers, the second inequality follows from Lemma 2.4, and the last equality is true as $(1+t)(1+t^q)^{-1/q}$ is a decreasing function on $(0,1]$.

Case II: $\sum_{k\ge 2} a_{ks+1}< r_0\delta a_{s+1}$.

Let s_0 be the smallest integer in $[1,s]$ satisfying $a_{s+s_0+1}/a_{s+1}\le (s_0/s)^{1/2}$. (For the illustrated example $s=1$, we have that $s_0=1$.) The existence and uniqueness of such an integer s_0 follow from the decreasing property of the sequence $\{a_{s+s_0+1}/a_{s+1}\}_{s_0=1}^s$, the increasing property of the sequence $\{(s_0/s)^{1/2}\}_{s_0=1}^s$, and $a_{s+s_0+1}/a_{s+1}\le (s_0/s)^{1/2}$ when $s_0=s$. Then from (2.1), the decreasing property of the sequence $\{a_j\}_{j\ge1}$ and the definition of the integer s_0 it follows that

$$\frac{a_{s+s_0}}{a_{s+1}}\ge\Big(\frac{s_0-1}{s}\Big)^{1/2} \qquad (2.23)$$

and

$$\begin{aligned}
\sqrt{2}s_0^{1/2}a_{s+1} &\geq \big(s_0 a_{s+1}^2+(s-s_0)\frac{s_0}{s}a_{s+1}^2\big)^{1/2}\\
&\geq (s_0a_{s+1}^2+(s-s_0)a_{s+s_0+1}^2)^{1/2}\geq \Big(\sum_{i=1}^s a_{s+i}^2\Big)^{1/2}\\
&\geq \delta\Big(\sum_{i=1}^s a_i^2\Big)^{1/2}-\sum_{k\geq 2}\Big(\sum_{i=1}^s a_{ks+i}^2\Big)^{1/2}\\
&\geq \delta s^{1/2}a_{s+1}-s^{1/2}\sum_{k\geq 2}a_{ks+1}\geq (1-r_0)\delta s^{1/2}a_{s+1},
\end{aligned}$$

which implies that

$$s_0\geq \frac{(1-r_0)^2\delta^2}{2}s. \tag{2.24}$$

Applying the decreasing property of the sequence $\{a_j\}$ and using the inequality $(\theta a^2+(1-\theta)b^2)^{1/2}\leq \theta^{1/2}a+(1-\theta^{1/2})b$ where $a\geq b\geq 0$ and $\theta\in[0,1]$, we obtain

$$\begin{aligned}
&s^{-1/2}\sum_{k\geq 1}\Big(\sum_{i=1}^s a_{ks+i}^2\Big)^{1/2}\\
\leq\ & s^{-1/2}\big((s_0-1)a_{s+1}^2+a_{s+s_0}^2+(s-s_0)a_{s+s_0+1}^2\big)^{1/2}\\
&+s^{-1/2}\sum_{k\geq 2}\big(s_0a_{ks+1}^2+(s-s_0)a_{ks+s_0+1}^2\big)^{1/2}\\
\leq\ & \sqrt{\frac{s_0}{s}}\Big(\frac{s_0-1}{s_0}a_{s+1}^2+\frac{1}{s_0}a_{s+s_0}^2\Big)^{1/2}+\Big(1-\sqrt{\frac{s_0}{s}}\Big)a_{s+s_0+1}\\
&+\sum_{k\geq 2}\Big(\sqrt{\frac{s_0}{s}}a_{ks+1}+\Big(1-\sqrt{\frac{s_0}{s}}\Big)a_{ks+s_0+1}\Big)\\
\leq\ & \sqrt{\frac{s_0-1}{s}}a_{s+1}+\frac{\sqrt{s_0}-\sqrt{s_0-1}}{\sqrt{s}}a_{s+s_0}+\sum_{k\geq 1}a_{ks+s_0+1}, \qquad (2.25)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j\geq 1}a_j^q &\geq (s+1)a_{s+1}^q+(s_0-1)a_{s+s_0}^q\\
&\quad +a_{s+s_0+1}^q+s\sum_{k\geq 2}a_{ks+s_0+1}^q. \qquad (2.26)
\end{aligned}$$

(For the illustrated example $s=1$, the above two inequalities become equalities.) Combining (2.25) and (2.26), recalling (2.23) and the definition of the integer s_0, and applying Lemma 2.5 with $c_1=\sqrt{(s_0-1)/s}$ and $c_2=\sqrt{s_0/s}$, we get

$$
\begin{aligned}
& s^{1/q-1/2}\frac{\sum_{k\ge 1}\big(\sum_{i=1}^s a_{ks+i}^2\big)^{1/2}}{\big(\sum_{j\ge1} a_j^q\big)^{1/q}}\\
\le\ & \frac{\sqrt{\frac{s_0-1}{s}}a_{s+1}+\frac{\sqrt{s_0}-\sqrt{s_0-1}}{\sqrt{s}}a_{s+s_0}+a_{s+s_0+1}+\sum_{k\ge2}a_{ks+s_0+1}}{\big((1+1/s)a_{s+1}^q+(s_0-1)a_{s+s_0}^q/s+a_{s+s_0+1}^q/s+\sum_{k\ge2}a_{ks+s_0+1}^q\big)^{1/q}}\\
\le\ & \max\Big\{\frac{\sqrt{\frac{s_0}{s}}}{\big(1+s_0/s\big)^{1/q}}, \frac{\sqrt{\frac{s_0-1}{s}}\Big(1+\frac{\sqrt{s_0}-\sqrt{s_0-1}}{\sqrt{s}}\Big)}{\big(1+1/s+((s_0-1)/s)^{1+q/2}\big)^{1/q}},\\
& \sup_{l\ge0}\frac{(l+2)\sqrt{\frac{s_0}{s}}}{\big(1+s_0/s+(l+1/s)\sqrt{\frac{s_0}{s}}\big)^{1/q}},\\
& \sup_{l\ge0}\frac{(l+1)\sqrt{\frac{s_0}{s}}+\sqrt{\frac{s_0-1}{s}}\Big(1+\frac{\sqrt{s_0}-\sqrt{s_0-1}}{\sqrt{s}}\Big)}{\Big(\big((l+1/s)\sqrt{\frac{s_0}{s}}+1+1/s\big)+((s_0-1)/s)^{1+q/2}\Big)^{1/q}}\Big\}.
\end{aligned}\tag{2.27}
$$

Therefore

$$
\begin{aligned}
& \frac{\sum_{k\ge 1}\big(\sum_{i=1}^s a_{ks+i}^2\big)^{1/2}}{\big(\sum_{j\ge1} a_j^q\big)^{1/q}}\\
\le\ & s^{1/2-1/q}\max\Big\{\frac{\sqrt{s_0/s}}{\big(1+s_0/s\big)^{1/q}}, \frac{\sqrt{s_0/s}}{\big(1+2^{-q/2}(s_0/s)^{1+q/2}\big)^{1/q}},\\
& \sup_{l\ge0}\frac{(l+2)\sqrt{s_0/s}}{\big(1+s_0/s+l\sqrt{s_0/s}\big)^{1/q}}, \sup_{l\ge0}\frac{(l+2)\sqrt{s_0/s}}{\big(1+l\sqrt{s_0/s}+2^{-q/2}(s_0/s)^{1+q/2}\big)^{1/q}}\Big\}\\
\le\ & s^{1/2-1/q}\max\Big\{\frac{2\sqrt{s_0/s}}{\big(1+2^{-q/2}(s_0/s)^{1+q/2}\big)^{1/q}}, \ \sup_{l\ge1}\frac{(l+2)\sqrt{s_0/s}}{\big(1+l\sqrt{s_0/s}\big)^{1/q}}\Big\}\\
\le\ & s^{1/2-1/q}\max\Big\{\sup_{\sqrt{2}(1-r_0)\delta/2\le y\le 1}\frac{2y}{\big(1+2^{-q/2}y^{2+q}\big)^{1/q}},\\
& \sup_{\sqrt{2}(1-r_0)\delta/2\le y\le 1}\frac{3y}{\big(1+y\big)^{1/q}}, \ \sup_{1\le y}\frac{2y}{\big(1+y\big)^{1/q}}\Big\},
\end{aligned}\tag{2.28}
$$

where the third inequality is valid by (2.24) and the first inequality follows from the following two inequalities:

$$
\sqrt{\frac{t-1}{s}}\Big(1+\frac{\sqrt{t}-\sqrt{t-1}}{\sqrt{s}}\Big)\le\sqrt{\frac{t}{s}}\tag{2.29}
$$

and

$$
\frac{1}{s}+\Big(\frac{t-1}{s}\Big)^{1+q/2}\ge\Big(\frac{1}{s}\Big)^{1+q/2}+\Big(\frac{t-1}{s}\Big)^{1+q/2}\ge 2^{-q/2}\Big(\frac{t}{s}\Big)^{1+q/2}, \quad 1\le t\le s.\tag{2.30}
$$

The conclusion (2.2) follows from (2.22) and (2.28). (For the illustrated example $s = 1$,

$$\begin{aligned} & s^{1/q-1/2}\frac{\sum_{k\ge 1}\big(\sum_{i=1}^{s} a_{ks+i}^2\big)^{1/2}}{\big(\sum_{j\ge 1} a_j^q\big)^{1/q}} = \frac{a_2+\sum_{j\ge 3} a_j}{(a_1^q+a_2^q+\sum_{j\ge 3} a_j^q)^{1/q}} \\ \le\ & \frac{(1+r_0\delta)a_2}{(a_1^q+a_2^q)^{1/q}} \le (1+r_0\delta)2^{-1/q} \end{aligned}$$

by (2.1), the assumption that $\sum_{k\ge 2} a_{ks+1} \le r_0\delta a_{s+1}$, and the monotonicity of the function $t(1+t^q)^{-1/q}$ on $(0,1)$. This together with (2.22) yields

$$s^{1/2-1/q}\sum_{k\ge 1}\big(\sum_{i=1}^{s} a_{ks+i}^2\big)^{1/2} \le \Big(\min_{0<r_0<1}\frac{1+r_0\delta}{(1+r_0^q\delta^q)^{1/q}}\Big)\times\Big(\sum_{j\ge 1} a_j^q\Big)^{1/q} \tag{2.31}$$

which provides an improvement of the estimate (2.2) for the illustrated example $s = 1$.)

Appendix A. Proof of the limit (1.12)

Take sufficiently small $\epsilon > 0$. Note that

$$\begin{aligned} & \sup_{\sqrt{2}(1-r_0)\delta/2\le y\le 1}\frac{2y}{\big(1+2^{-q/2}y^{2+q}\big)^{1/q}} \\ = & \begin{cases} \dfrac{\sqrt{2}(1-r_0)\delta}{\big(1+2^{-1-q}((1-r_0)\delta)^{2+q}\big)^{1/q}} & \text{if } q<(1-r_0)^{2+q}\delta^{2+q}, \\ q^{1/(2+q)}\big(1+\frac{q}{2}\big)^{-1/q}2^{(1+3q/2)/(2+q)} & \text{if } 1\ge q\ge (1-r_0)^{2+q}\delta^{2+q}. \end{cases} \end{aligned} \tag{A.1}$$

Then for any small $q > (e/2+\epsilon)\delta^2$ and sufficiently small $\delta > 0$, we have that $q \ge (1-r_0)^{2+q}\delta^{2+q}$ for all $r_0 \in (0,1)$. Then applying (1.6) and (A.1) yields

$$\begin{aligned} b(q,\delta) &\ge \delta^{-1}\inf_{0<r_0<1}\sup_{\sqrt{2}(1-r_0)\delta/2\le y\le 1}\frac{2y}{\big(1+2^{-q/2}y^{2+q}\big)^{1/q}} \\ &= \delta^{-1}q^{1/(2+q)}\big(1+\frac{q}{2}\big)^{-1/q}2^{(1+3q/2)/(2+q)} \\ &\ge (1+\epsilon/e)^{1/2} > 1, \end{aligned}$$

where the last inequality holds since

$$\lim_{q\to 0} q^{-q/(4+2q)}\big(1+\frac{q}{2}\big)^{-1/q}2^{(1+3q/2)/(2+q)} = (2/e)^{1/2}. \tag{A.2}$$

Thus

$$\limsup_{\delta\to 0}\frac{\tilde q_{\max}(\delta)}{\delta^2} \le \limsup_{\delta\to 0}\frac{(e/2+\epsilon)\delta^2}{\delta^2} \le \frac{e}{2}+\epsilon \tag{A.3}$$

for any sufficiently small $\epsilon > 0$.

Take $r_0 = 1 - \sqrt{2}/4$ and sufficiently small $\epsilon > 0$. Then for $q \le (e/2-\epsilon)\delta^2$ and sufficiently small $\delta > 0$,

$$\begin{cases} (1+r_0\delta)(1+r_0^q\delta^q)^{-1/q} \le 2(3/2)^{-1/q} \le (1-\epsilon/e)^{1/2}\delta, \\ \sup_{y\ge 1} y(1+y)^{-1/q} \le \sup_{y\ge 1}(1+y)^{1-1/q} \le 2^{1-1/q} \le (1-\epsilon/e)^{1/2}\delta/2, \\ \sup_{y\ge \sqrt{2}(1-r_0)\delta/2} \frac{y}{(1+y)^{1/q}} = \frac{\delta/4}{(1+\delta/4)^{1/q}} \le (1-\epsilon/e)^{1/2}\delta/3, \end{cases}$$

and

$$\sup_{\sqrt{2}(1-r_0)\delta/2 \le y \le 1} \frac{2y}{\big(1+2^{-q/2}y^{2+q}\big)^{1/q}} \le (1-\epsilon/e)^{1/2}\delta$$

by (1.6), (A.1) and (A.2). Thus

$$b(q,\delta) \le (1-\epsilon/e)^{1/2} < 1$$

whenever $q \le (e/2-\epsilon)\delta^2$ and $\delta \in (0,1)$ is sufficiently small, which implies that

$$\tilde{q}_{\max}(\delta) \ge (e/2-\epsilon)\delta^2$$

when δ is sufficiently small. Hence

$$\liminf_{\delta\to 0} \frac{\tilde{q}_{\max}(\delta)}{\delta^2} \ge \limsup_{\delta\to 0} \frac{(e/2-\epsilon)\delta^2}{\delta^2} \ge \frac{e}{2} - \epsilon. \tag{A.4}$$

Combining (A.3) and (A.4) and recalling that $\epsilon > 0$ is a sufficiently small number chosen arbitrarily proves the limit (1.12).

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