Reconstruction of sparse wavelet signals from partial Fourier measurements

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Abstract—In this paper, we show that high-dimensional sparse wavelet signals of finite levels can be constructed from their partial Fourier measurements on a deterministic sampling set with cardinality about a multiple of signal sparsity.

I. INTRODUCTION

Sparse representation of signals in a dictionary has been used in signal processing, compression, noise reduction, source separation, and many more fields. Wavelet bases are well localized in time-frequency plane and they provide sparse representations of many signals and images that have transient structures and singularities ([1], [2]). In this paper, we consider recovering sparse wavelet signals of finite levels from their partial Fourier measurements.

Let $\mathbf{D}$ be a dilation matrix with integer entries whose eigenvalues have modulus strictly larger than one, and set $M = |\det \mathbf{D}| \geq 2$. Wavelet vectors $\boldsymbol{\Psi}_m = (\psi_{m,1}, \ldots, \psi_{m,r})^T, 1 \leq m \leq M - 1$, used in this paper are generated from a multiresolution analysis $\{V_j\}_{j \in \mathbb{Z}}$, a family of closed subspaces of $L^2 := L^2(\mathbb{R}^n)$, that satisfies the following: (i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$; (ii) $V_{j+1} = \{f(\mathbf{D}\cdot), f \in V_j\}$ for all $j \in \mathbb{Z}$; (iii) $\overline{\cup_{j \in \mathbb{Z}} V_j} = L^2$; (iv) $\cap_{j \in \mathbb{Z}} V_j = \{0\}$; and (v) there exists a scaling vector $\boldsymbol{\Phi} = (\phi_1, \cdots, \phi_r)^T \in V_0$ such that $\{\phi_l(\cdot - \mathbf{k}), 1 \leq l \leq r, \mathbf{k} \in \mathbb{Z}^n\}$ is a Riesz basis for V_0 ([1], [2], [3], [4], [5], [6], [7], [8], [9], [10]). They generate a Riesz basis $\{M^{j/2} \boldsymbol{\Psi}_m(\mathbf{D}^j \mathbf{x} - \mathbf{k}) : 1 \leq m \leq M-1, \mathbf{k} \in \mathbb{Z}^n\}$ for the wavelet space $W_j := V_{j+1} \ominus V_j$, the orthogonal complement of V_j in V_{j+1}, for every $j \in \mathbb{Z}$. Therefore any signal f in the scaling space V_J of level $J \geq 0$ has a unique wavelet decomposition,

$$f = f_0 + g_0 + \cdots + g_{J-1}, \tag{I.1}$$

where

$$f_0 = \sum_{k \in \mathbb{Z}^n} \mathbf{a}_0^T(\mathbf{k}) \boldsymbol{\Phi}(\cdot - \mathbf{k}) \in V_0 \tag{I.2}$$

and

$$g_j = \sum_{m=1}^{M-1} \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{b}_{m,j}^T(\mathbf{k}) M^j \boldsymbol{\Psi}_m(\mathbf{D}^j \cdot - \mathbf{k}) \in W_j, 0 \leq j \leq J-1. \tag{I.3}$$

In this paper, we consider wavelet signals $f \in V_J$ with f_0 and $g_j, 0 \leq j \leq J - 1$, in the above wavelet decomposition having sparse representations.

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Define Fourier transform of an integrable function f on $\mathbb{R}^n$ by

$$\hat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} f(\mathbf{t}) e^{-i\mathbf{t} \cdot \boldsymbol{\xi}} d\mathbf{t}.$$

Due to coherence of wavelet bases between different levels, the conventional optimization method does not work well to reconstruct a sparse wavelet signal f of finite level from its partial Fourier measurements $\hat{f}(\xi), \xi \in \Omega$, on a finite sampling set Ω ([11], [12], [13], [14], [15], [16], [17], [18]). Recently, Prony's method was introduced in [19], [20] for the exact reconstruction of one-dimensional sparse wavelet signals.

Denote by $\#E$ the cardinality of a set E. We say that a wavelet signal $f \in V_J$ has sparsity $\mathbf{s} = (s_0, \cdots, s_{J-1})$ if it has sparsity

$$s_j := \begin{cases} \max\{\#K_0, \#K_{1,0}, \ldots, \#K_{M-1,0}\} & \text{if } j = 0 \\ \max\{\#K_{1,j}, \ldots, \#K_{M-1,j}\} & \text{if } j = 1, \ldots, J-1, \end{cases}$$

at level $j, 0 \leq j \leq J - 1$, where K_0 and $K_{m,j}$ are supports of coefficient vectors $(a_0(\mathbf{k}))_{\mathbf{k} \in \mathbb{Z}^n}$ and $(b_{m,j}(\mathbf{k}))_{\mathbf{k} \in \mathbb{Z}^n}$ in the wavelet decomposition (I.1), (I.2) and (I.3) respectively. For the classcial one-dimensional scalar case (i.e. $n = 1, r = 1$ and $\mathbf{D} = 2$), under the assumption that Fourier transform of the scaling function ϕ does not vanish on $(-\pi, \pi)$,

$$\hat{\phi}(\xi) \neq 0, \quad \xi \in (-\pi, \pi), \tag{I.4}$$

Zhang and Dragotti proved in [19] that a compactly supported sparse wavelet signal of the form (I.1) can be reconstructed from its Fourier measurements on a sampling set Ω of size about twice of its sparsity $s_0 + \cdots + s_{J-1}$. In this paper, we extend their result to high-dimensional sparse wavelet signals without nonvanishing condition (I.4) on the scaling vector $\boldsymbol{\Phi}$. Particularly in Theorem III.1, we show that any $\mathbf{s}$-sparse wavelet signal f of the form (I.1) can be reconstructed from its Fourier measurements on a sampling set Ω with cardinality less than $2Mr(s_0 + \cdots + s_{J-1})$, which is independent on dimension n.

II. MULTIRESOLUTION ANALYSIS AND WAVELETS

Set $\mathbf{M} = \mathbf{D}^T$. Then the scaling vector $\boldsymbol{\Phi} = (\phi_1, \ldots, \phi_r)^T$ of a multiresolution analysis $\{V_j\}_{j \in \mathbb{Z}}$ satisfies a matrix refinement equation,

$$\widehat{\boldsymbol{\Phi}}(\boldsymbol{\xi}) = \mathbf{G}_0(\mathbf{M}^{-1}\boldsymbol{\xi}) \widehat{\boldsymbol{\Phi}}(\mathbf{M}^{-1}\boldsymbol{\xi}), \tag{II.1}$$

where the matrix function $\mathbf{G}_0$ of size $r \times r$ is bounded and 2π-periodic. In this paper, we assume that $\mathbf{G}_0$ has trigonometric polynomial entries. Hence $\boldsymbol{\Phi}$ is compactly supported, and the

Riesz basis property for the scaling vector Φ can be reformulated as that $(\widehat{\Phi}(\boldsymbol{\xi} + 2\pi\mathbf{k}))_{\mathbf{k}\in\mathbb{Z}^n}$ has rank r for every $\boldsymbol{\xi} \in \mathbb{R}^n$. Therefore for any $\boldsymbol{\xi} \in \mathbb{R}^n$ there exist $\mathbf{k}(\boldsymbol{\xi}, l) \in \mathbb{Z}^n, 1 \le l \le r$, such that

$$\left(\widehat{\Phi}(\boldsymbol{\xi} + 2\pi\mathbf{k})\right)_{\mathbf{k}\in\Lambda(\boldsymbol{\xi})} \text{ has full rank } r, \qquad \text{(II.2)}$$

where

$$\Lambda(\boldsymbol{\xi}) \; = \; \{\mathbf{k}(\boldsymbol{\xi}, l) \in \mathbb{Z}^n : \; 1 \le l \le r\}. \qquad \text{(II.3)}$$

Let $\mathbf{p}_m, 0 \le m \le M - 1$, be representatives of $\mathbb{Z}^n/\mathbf{M}\mathbb{Z}^n$, and write

$$\mathbb{Z}^n = \bigcup_{m=0}^{M-1} (\mathbf{p}_m + \mathbf{M}\mathbb{Z}^n).$$

Take matrices $\mathbf{G}_m, 1 \le m \le M - 1$, with trigonometric polynomial entries such that

$$\sum_{m'=0}^{M-1} \mathbf{G}_0(\boldsymbol{\xi} + 2\pi\mathbf{M}^{-1}\mathbf{p}_{m'})\overline{\mathbf{G}_m(\boldsymbol{\xi} + 2\pi\mathbf{M}^{-1}\mathbf{p}_{m'})}^T = 0 \quad \text{(II.4)}$$

for all $1 \le m \le M - 1$, and

$$\mathbf{G}(\boldsymbol{\xi}) \text{ has rank } Mr \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^n, \qquad \text{(II.5)}$$

where

$$\mathbf{G}(\boldsymbol{\xi}) = \begin{pmatrix} \mathbf{G}_0(\boldsymbol{\xi} + 2\pi\mathbf{M}^{-1}\mathbf{p}_0) & \cdots & \mathbf{G}_0(\boldsymbol{\xi} + 2\pi\mathbf{M}^{-1}\mathbf{p}_{M-1}) \\ \mathbf{G}_1(\boldsymbol{\xi} + 2\pi\mathbf{M}^{-1}\mathbf{p}_0) & \cdots & \mathbf{G}_1(\boldsymbol{\xi} + 2\pi\mathbf{M}^{-1}\mathbf{p}_{M-1}) \\ \vdots & \ddots & \vdots \\ \mathbf{G}_{M-1}(\boldsymbol{\xi} + 2\pi\mathbf{M}^{-1}\mathbf{p}_0) & \cdots & \mathbf{G}_{M-1}(\boldsymbol{\xi} + 2\pi\mathbf{M}^{-1}\mathbf{p}_{M-1}) \end{pmatrix}.$$

In this paper, wavelet vectors $\Psi_m, 1 \le m \le M - 1$, are defined as follows:

$$\widehat{\Psi}_m(\boldsymbol{\xi}) = \mathbf{G}_m(\mathbf{M}^{-1}\boldsymbol{\xi})\widehat{\Phi}(\mathbf{M}^{-1}\boldsymbol{\xi}), \quad 1 \le m \le M - 1. \quad \text{(II.6)}$$

Then Ψ_m are compactly supported and $\{M^{j/2}\Psi_m(\mathbf{D}^j\mathbf{x} - \mathbf{k}) : 1 \le m \le M - 1, \; \mathbf{k} \in \mathbb{Z}^n\}$ forms a Riesz basis for the wavelet space $W_j := V_{j+1} \ominus V_j$ for every $j \in \mathbb{Z}$.

For the scaling vector Φ and wavelet vectors $\Psi_m, 1 \le m \le M - 1$, constructed above, one may verify that any signal in V_J has the unique wavelet decomposition (I.1), (I.2) and (I.3).

III. RECONSTRUCTION OF SPARSE WAVELET SIGNALS

Take $\mathbf{h} = (h_1, \ldots, h_n) \in \mathbb{R}^n$ and sparsity vector $\mathbf{s} = (s_0, \ldots, s_{J-1})$, and set $\|\mathbf{s}\|_\infty = \max_{0 \le j \le J-1} s_j$. For $0 \le j \le J - 1$ and $0 \le m \le M - 1$, let

$$\Gamma_j = \{(-s_j + 1/2)\mathbf{h}, (-s_j + 3/2)\mathbf{h}, \ldots, (s_j - 1/2)\mathbf{h}\},$$

and

$$\Omega_j = \cup_{\boldsymbol{\gamma}\in\Gamma_j} \cup_{m=0}^{M-1} \left(\pi\boldsymbol{\gamma} + 2\pi\mathbf{M}^j\mathbf{p}_m \right. \\ \left. + 2\pi\mathbf{M}^{j+1}\Lambda(\pi\mathbf{M}^{-j-1}\boldsymbol{\gamma} + 2\pi\mathbf{M}^{-1}\mathbf{p}_m)\right),$$

where the set Λ of cardinality r is defined by (II.3). Set

$$\Omega = \cup_{j=0}^{J-1}\Omega_j. \qquad \text{(III.1)}$$

Then

$$\Omega \subset \{(-\|\mathbf{s}\|_\infty + 1/2)\mathbf{h}\pi, \ldots, (\|\mathbf{s}\|_\infty - 1/2)\mathbf{h}\pi\} + 2\pi\mathbb{Z}^n,$$

and

$$\#\Omega \le \sum_{j=0}^{J-1} \#\Omega_j = 2Mr(s_0 + s_1 + \cdots + s_{J-1}). \qquad \text{(III.2)}$$

The following is the main theorem of this paper.

Theorem III.1. *Let $\mathbf{D}$ be a dilation matrix, Φ be a compactly supported scaling vector, $\Psi_m, 1 \le m \le M - 1$, be wavelet vectors satisfying (II.4) and (II.5), let Ω be the set in (III.1) with $\mathbf{h} = (h_1, \ldots, h_n)$. If $1, h_1, \ldots, h_n$ are linearly independent over the field of rationals, then any $\mathbf{s}$-sparse wavelet signal of the form (I.1), (I.2) and (I.3) can be reconstructed from its Fourier measurements on Ω.*

Proof. Let f be an $\mathbf{s}$-sparse signal with wavelet representation (I.1), (I.2) and (I.3). Set

$$\widehat{\mathbf{a}}_0(\boldsymbol{\xi}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} \mathbf{a}_0(\mathbf{k})e^{-i\mathbf{k}\cdot\boldsymbol{\xi}}, \qquad \text{(III.3)}$$

and

$$\widehat{\mathbf{b}}_{m,j}(\boldsymbol{\xi}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} \mathbf{b}_{m,j}(\mathbf{k})e^{-i\mathbf{k}\cdot\boldsymbol{\xi}} \qquad \text{(III.4)}$$

for $1 \le m \le M - 1$ and $0 \le j \le J - 1$. Then taking Fourier transform on both sides of the equation (I.1) gives

$$\hat{f}(\boldsymbol{\xi}) = \widehat{\mathbf{a}}_0^T(\boldsymbol{\xi})\widehat{\Phi}(\boldsymbol{\xi}) + \sum_{j=0}^{J-1}\sum_{m=1}^{M-1} \widehat{\mathbf{b}}_{m,j}^T(\mathbf{M}^{-j}\boldsymbol{\xi})\widehat{\Psi}_m(\mathbf{M}^{-j}\boldsymbol{\xi}). \tag{III.5}$$

Define $f_i, 0 \le i \le J - 1$, by

$$\widehat{f_i}(\boldsymbol{\xi}) = \widehat{\mathbf{a}}_0^T(\boldsymbol{\xi})\widehat{\Phi}(\boldsymbol{\xi}) + \sum_{j=0}^{i}\sum_{m=1}^{M-1} \widehat{\mathbf{b}}_{m,j}^T(\mathbf{M}^{-j}\boldsymbol{\xi})\widehat{\Psi}_m(\mathbf{M}^{-j}\boldsymbol{\xi}). \tag{III.6}$$

Then

$$f_{J-1} = f, \qquad \text{(III.7)}$$

and

$$\begin{aligned} \widehat{f_i}(\mathbf{M}^i\boldsymbol{\xi}) &= \widehat{f_{i-1}}(\mathbf{M}^i\boldsymbol{\xi}) + \sum_{m=1}^{M-1} \widehat{\mathbf{b}}_{m,i}^T(\boldsymbol{\xi})\widehat{\Psi}_m(\boldsymbol{\xi}) \\ &= \widehat{\mathbf{a}}_i^T(\boldsymbol{\xi})\widehat{\Phi}(\boldsymbol{\xi}) + \sum_{m=1}^{M-1} \widehat{\mathbf{b}}_{m,i}^T(\boldsymbol{\xi})\widehat{\Psi}_m(\boldsymbol{\xi}) \\ &= \left(\widehat{\mathbf{a}}_i^T(\boldsymbol{\xi})\mathbf{G}_0(\mathbf{M}^{-1}\boldsymbol{\xi}) + \sum_{m=1}^{M-1} \widehat{\mathbf{b}}_{m,i}^T(\boldsymbol{\xi})\right. \\ &\quad \left.\times \mathbf{G}_m(\mathbf{M}^{-1}\boldsymbol{\xi})\right)\widehat{\Phi}(\mathbf{M}^{-1}\boldsymbol{\xi}) \qquad \text{(III.8)} \end{aligned}$$

for some vectors $\widehat{\mathbf{a}}_i(\boldsymbol{\xi})$ with trigonometric polynomial entries, where the last equality follows from (II.1) and (II.6).

For $0 \le j \le J - 1$, $\boldsymbol{\gamma} \in \Gamma_j$ and $0 \le m' \le M - 1$, set

$$\eta_j(\boldsymbol{\gamma}, m') = \pi\mathbf{M}^{-j}\boldsymbol{\gamma} + 2\pi\mathbf{p}_{m'}.$$

Applying (III.7) and (III.8) with $i = J - 1$, replacing $\boldsymbol{\xi}$ in (III.8) by $\eta_{J-1}(\boldsymbol{\gamma}, m') + 2\pi\mathbf{M}\mathbf{k}, \mathbf{k} \in \Lambda(\mathbf{M}^{-1}\eta_{J-1}(\boldsymbol{\gamma}, m'))$, and using periodicity of $\widehat{\mathbf{a}}_{J-1}$ and $\widehat{\mathbf{b}}_{m,J-1}$, we obtain

$$\begin{aligned} \hat{f}(\mathbf{M}^{J-1}&\eta_{J-1}(\boldsymbol{\gamma}, m') + 2\pi\mathbf{M}^J\mathbf{k}) \\ &= A(J-1, \boldsymbol{\gamma}, m')\widehat{\Phi}(\mathbf{M}^{-1}\eta_{J-1}(\boldsymbol{\gamma}, m') + 2\pi\mathbf{k}) \quad \text{(III.9)} \end{aligned}$$

for all $\mathbf{k} \in \Lambda(\mathbf{M}^{-1}\eta_{J-1}(\boldsymbol{\gamma}, m'))$, where

$$A(J-1, \boldsymbol{\gamma}, m') = \widehat{\mathbf{a}}_{J-1}^T(\pi\mathbf{M}^{-J+1}\boldsymbol{\gamma})\mathbf{G}_0(\mathbf{M}^{-1}\eta_{J-1}(\boldsymbol{\gamma}, m'))$$
$$+ \sum_{m=1}^{M-1} \widehat{\mathbf{b}}_{m,J-1}^T(\pi\mathbf{M}^{-J+1}\boldsymbol{\gamma})\mathbf{G}_m(\mathbf{M}^{-1}\eta_{J-1}(\boldsymbol{\gamma}, m')). \quad \text{(III.10)}$$

Recall from (II.2) that

$$\left(\widehat{\Phi}(\mathbf{M}^{-1}\eta_{J-1}(\boldsymbol{\gamma}, m') + 2\pi\mathbf{k})\right)_{\mathbf{k} \in \Lambda(\mathbf{M}^{-1}\eta_{J-1}(\boldsymbol{\gamma}, m'))} \quad \text{(III.11)}$$

is nonsingular. Then $A(J-1, \boldsymbol{\gamma}, m')$ can be solved from the linear system (III.9) for all $0 \le m' \le M-1$ and $\boldsymbol{\gamma} \in \Gamma_{J-1}$.

Recall from (II.4) and (II.5) that

$$\mathbf{G}(h\pi\mathbf{M}^{-J}\boldsymbol{\gamma}) = \begin{pmatrix} \mathbf{G}_0(\mathbf{M}^{-1}\eta_{J-1}(\boldsymbol{\gamma}, m')) \\ \mathbf{G}_1(\mathbf{M}^{-1}\eta_{J-1}(\boldsymbol{\gamma}, m')) \\ \vdots \\ \mathbf{G}_{M-1}(\mathbf{M}^{-1}\eta_{J-1}(\boldsymbol{\gamma}, m')) \end{pmatrix}_{0 \le m' \le M-1} \quad \text{(III.12)}$$

is nonsingular. Thus, for every $\boldsymbol{\gamma} \in \Gamma_{J-1}$ and $1 \le m \le M-1$,

$$\widehat{\mathbf{a}}_{J-1}(\pi\mathbf{M}^{-J+1}\boldsymbol{\gamma}) \quad \text{and} \quad \widehat{\mathbf{b}}_{m,J-1}(\pi\mathbf{M}^{-J+1}\boldsymbol{\gamma}) \quad \text{(III.13)}$$

are uniquely determined from samples of $\hat{f}$ on $\Omega_{J-1} \subset \Omega$ by (III.10) and (III.12).

For $1 \le m \le M-1$, it follows from the linear independence assumption of $1, h_1, \ldots, h_n$ on the field of rationals that

$$e^{-i\pi\mathbf{k}\cdot\mathbf{M}^{-J+1}\mathbf{h}}, \mathbf{k} \in K_{m,J-1}, \text{ are distinct to each other.}$$

For $\boldsymbol{\gamma} = n\mathbf{h}$ with $n \in \{-s_{J-1} + 1/2, \ldots, s_{J-1} - 1/2\}$,

$$\widehat{\mathbf{b}}_{m,J-1}(\pi\mathbf{M}^{-J+1}\boldsymbol{\gamma}) = \sum_{\mathbf{k} \in K_{m,J-1}} \mathbf{b}_{m,J-1}(\mathbf{k})\left(e^{-i\pi\mathbf{k}\cdot\mathbf{M}^{-J+1}\mathbf{h}}\right)^n \quad \text{(III.14)}$$

by (III.4). Therefore applying Prony's method ([18], [19], [21], [22], [23], [24], [25], [26]) recovers trigonometric polynomials $\mathbf{b}_{m,J-1}, 1 \le m \le M-1$, from their measurements on $\pi\mathbf{M}^{-J+1}\boldsymbol{\gamma}, \boldsymbol{\gamma} \in \Gamma_{J-1}$. Hence $\mathbf{b}_{m,J-1}(\mathbf{k}), \mathbf{k} \in \mathbb{Z}^n$, can be recovered from samples of $\hat{f}$ on Ω for all $1 \le m \le M-1$.

By the above argument,

$$f_{J-1} - f_{J-2} \text{ and } \widehat{f}_{J-2}(\boldsymbol{\xi}), \boldsymbol{\xi} \in \Omega, \quad \text{(III.15)}$$

can be obtained from samples of $\hat{f}$ on Ω, because

$$\widehat{f}_{J-2}(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi}) - \sum_{m=1}^{M-1} \Big(\sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{b}_{m,J-1}(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{M}^{-J+1}\boldsymbol{\xi}}\Big)\widehat{\Psi}_m(\mathbf{M}^{-J+1}\boldsymbol{\xi})$$

by (III.3) and (III.8). Inductively we can reconstruct

$$f_i - f_{i-1} \text{ and } \widehat{f}_{i-1}(\boldsymbol{\xi}), \boldsymbol{\xi} \in \Omega, \quad \text{(III.16)}$$

from the samples of $\hat{f}$ on Ω for $i = J-2, \cdots, 1$.

Taking $i = 1$ in (III.16) determines samples of $\widehat{f_0}$ on Ω. Next we recover the function f_0 from its Fourier measurements on $\Omega_0 \subset \Omega$. By (III.3) and (III.4),

$$\widehat{\mathbf{a}}_0(n\pi\mathbf{h}) = \sum_{\mathbf{k} \in \mathbf{K}_0} \mathbf{a}_0(\mathbf{k})\left(e^{-i\pi\mathbf{k}\cdot\mathbf{h}}\right)^n$$

and

$$\widehat{\mathbf{b}}_{m,0}(n\pi\mathbf{h}) = \sum_{\mathbf{k} \in \mathbf{K}_{m,0}} \mathbf{b}_{m,0}(\mathbf{k})\left(e^{-i\pi\mathbf{k}\cdot\mathbf{h}}\right)^n,$$

where $n \in \{-s_0 + 1/2, -s_0 + 3/2, \ldots, s_0 - 1/2\}$. Similar to (III.13), we can show that

$$\widehat{\mathbf{a}}_0(n\pi\mathbf{h}) \quad \text{and} \quad \widehat{\mathbf{b}}_{m,0}(n\pi\mathbf{h}), \ 1 \le m \le M-1,$$

are uniquely determined from samples of $\widehat{f_0}$ on Ω. Applying Prony's method again recovers $\mathbf{a}_0(\mathbf{k})$ for $\mathbf{k} \in \mathbf{K}_0$ and $\mathbf{b}_{m,0}(\mathbf{k})$ for $1 \le m \le M-1$ and $\mathbf{k} \in \mathbf{K}_{m,0}$. Therefore f_0 could be completely recovered from its Fourier measurements on Ω. This together with (III.7), (III.15) and (III.16) completes the proof. $\square$

The linear independence requirement on $\mathbf{h} = (h_1, \ldots, h_n)$ in Theorem III.1 can be replaced by a quantitative condition if the sparse signal has some additional information on its support, c.f. [19].

Corollary III.2. *Let* $\mathbf{D}$, Φ *and* $\Psi_m, 1 \le m \le M-1$, *be as in Theorem III.1, and let* f *be an* $\mathbf{s}$-*sparse signal in* (I.1) *satisfying*

$$K_0 \subset [a, b)^n \quad \text{and} \quad K_{m,j} \subset \mathbf{D}^j[a, b)^n, \quad \text{(III.17)}$$

where $1 \le m \le M-1$, $0 \le j \le J-1$ *and* $a < b$. *Then* f *can be recovered from its Fourier measurements on* Ω *in* (III.1) *with* $\mathbf{h} = (h_1, \ldots, h_n)$ *satisfying*

$$0 < (b-a)(h_1 + h_2 + \cdots + h_n) \le 2. \quad \text{(III.18)}$$

Proof. Following the argument in Theorem III.1, it suffices to prove that $e^{-i\pi\mathbf{k}\cdot\mathbf{h}}, \mathbf{k} \in K_0$, are distinct, and also that $e^{-i\pi\mathbf{k}\cdot\mathbf{M}^{-j}\mathbf{h}}, \mathbf{k} \in K_{m,j}$, are distinct for every $1 \le m \le M-1$ and $0 \le j \le J-1$. The above distinctive property follows from (III.17) and (III.18) immediately. $\square$

From the proof of Theorem III.1, we have the following result on the reconstruction of an s-sparse trigonometric polynomial from its samples on a set of size $2s$.

Corollary III.3. *Let* $\mathbf{h} = (h_1, \ldots, h_n)$ *with* $1, h_1, \ldots, h_n$ *being linearly independent over the field of rationals, and define*

$$\Theta_s = \{(-s + 1/2)\mathbf{h}, (-s + 3/2)\mathbf{h}, \ldots, (s - 1/2)\mathbf{h}\}, \ s \ge 1.$$

Then any n-dimensional trigonometric polynomial

$$P(\boldsymbol{\xi}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} p(\mathbf{k})e^{-i\mathbf{k}\cdot\boldsymbol{\xi}}$$

with sparsity s,

$$\#\{\mathbf{k} : \ p(\mathbf{k}) \ne 0\} \le s,$$

can be reconstructed from its samples on Θ_s.

IV. SIMULATIONS

The following algorithm for sparse wavelet signal recovery is proposed in the proof of Theorem III.1.

Algorithm 1:

1. Input sparsity vector $\mathbf{s} = (s_0, \cdots, s_{J-1})$.
2. Input Fourier measurements $\hat{f}(\boldsymbol{\xi}), \boldsymbol{\xi} \in \Omega$ and set $f_{J-1} = f$.
3. **for** $j = J-1$ to 0 **do**
   for every $\boldsymbol{\gamma} \in \Gamma_j$ **do**

for every $m' = 0, \cdots, M-1$ **do**

   3a) $\eta_j(\boldsymbol{\gamma}, m') = \pi \mathbf{M}^{-j}\boldsymbol{\gamma} + 2\pi \mathbf{p}_{m'}$.

   3b) Solve the linear system

$$\left(\widehat{f}_j\big(\mathbf{M}^j \eta_j(\boldsymbol{\gamma}, m') + 2\pi \mathbf{M}^{j+1}\mathbf{k}\big)\right)_{\mathbf{k} \in \Lambda(M^{-1}\eta_j(\boldsymbol{\gamma}, m'))}$$
$$= A(j, \boldsymbol{\gamma}, m')\big(\widehat{\Phi}(\mathbf{M}^{-1}\eta_j(\boldsymbol{\gamma}, m') + 2\pi \mathbf{k})\big)_{\mathbf{k} \in \Lambda(M^{-1}\eta_j(\boldsymbol{\gamma}, m'))}$$

to get

$$A(j, \boldsymbol{\gamma}, m') := \widehat{\mathbf{a}}_j^T(\pi \mathbf{M}^{-j}\boldsymbol{\gamma})\mathbf{G}_0(\mathbf{M}^{-1}\eta_j(\boldsymbol{\gamma}, m'))$$
$$+ \sum_{m=1}^{M-1} \widehat{\mathbf{b}}_{m,j}^T(\pi \mathbf{M}^{-j}\boldsymbol{\gamma})\mathbf{G}_m(\mathbf{M}^{-1}\eta_j(\boldsymbol{\gamma}, m')).$$

  end for

   3c) Solve the linear equation

$$\big(\widehat{\mathbf{a}}_j^T(\pi \mathbf{M}^{-j}\boldsymbol{\gamma}), \widehat{\mathbf{b}}_{1,j}^T(\pi \mathbf{M}^{-j}\boldsymbol{\gamma}), \cdots, \widehat{\mathbf{b}}_{M-1,j}^T(\pi \mathbf{M}^{-j}\boldsymbol{\gamma})\big)$$
$$\times \mathbf{G}(h\pi \mathbf{M}^{-j-1}\boldsymbol{\gamma}) = \big(A(j, \boldsymbol{\gamma}, 0), \cdots, A(j, \boldsymbol{\gamma}, M-1)\big).$$

  end for

   3d) Recover $\mathbf{b}_{m,j}$ from $\widehat{\mathbf{b}}_{m,j}(\pi \mathbf{M}^{-j}\boldsymbol{\gamma})$, $\boldsymbol{\gamma} \in \Gamma_j$ with Prony's method for every $1 \leq m \leq M-1$.

   3e) Subtract $\sum_{m=1}^{M-1} \widehat{\mathbf{b}}_{m,j}^T(\mathbf{M}^{-i}\boldsymbol{\xi})\widehat{\Psi}_m(\mathbf{M}^{-i}\boldsymbol{\xi})$ from $\widehat{f}_j(\boldsymbol{\xi})$ to get $\widehat{f}_{j-1}(\boldsymbol{\xi})$, $\boldsymbol{\xi} \in \Omega$.

  end for

4. Recover $\mathbf{a}_0$ from $\widehat{\mathbf{a}}(\pi\boldsymbol{\gamma})$, $\boldsymbol{\gamma} \in \Gamma_0$ with Prony's method.

5. Reconstruct the sparse wavelet signal

$$f(\mathbf{t}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}_0^T(\mathbf{k})\Phi(\mathbf{t} - \mathbf{k})$$
$$+ \sum_{j=0}^{J-1} \sum_{m=1}^{M-1} \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{b}_{m,j}^T(\mathbf{k}) M^j \Psi_m(\mathbf{D}^j \mathbf{t} - \mathbf{k}).$$

Next we present simulations to demonstrate the above algorithm for perfect reconstruction of sparse wavelet signals of finite levels. Let $\phi_1(t) = \chi_{[0,1)}(t)$ and $\phi_2(t) = 2\sqrt{3}(t - 1/2)\chi_{[0,1)}(t)$ be scaling functions, and let

$$\psi_1(t) = (6t-1)\chi_{[0,1/2)}(t) + (6t-5)\chi_{[1/2,1)}(t),$$

and

$$\psi_2(t) = 2\sqrt{3}(2t-1/2)\chi_{[0,1/2)}(t) - 2\sqrt{3}(2t-3/2)\chi_{[1/2,1)}(t)$$

be wavelet functions. Consider reconstructing the sparse signal

$$f(t) = \mathbf{a}_0^T(2)\Phi(t-2) + \mathbf{a}_0^T(4)\Phi(t-4)$$
$$+ \mathbf{b}_0^T(1)\Psi(t-1) + \mathbf{b}_0^T(5)\Psi(t-5)$$
$$+ \mathbf{b}_1^T(6)\Psi(2t-6) + \mathbf{b}_1^T(12)\Psi(2t-12) \quad \text{(IV.1)}$$

from its Fourier measurements on the sampling set

$$\Omega = \left\{ -\frac{\sqrt{2}}{128}n\pi + 2k\pi : \; n = \pm 1, \pm 3 \text{ and } k = 0, \pm 1, \pm 2, 4 \right\}$$

in (III.1), where $\Phi = (\phi_1, \phi_2)^T$, $\Psi = (\psi_1, \psi_2)^T$, and the nonzero components of $\mathbf{a}_0$, $\mathbf{b}_0$ and $\mathbf{b}_1$ are randomly chosen in $[-1, 1] \setminus (-0.1, 0.1)$, see Figure 1. Applying the proposed algorithm, our numerical results support the conclusion on perfect recovery of sparse wavelet signals from their Fourier measurements on Ω.
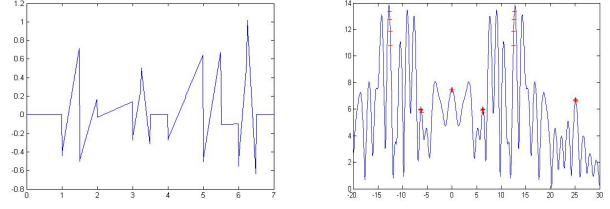

Fig. 1. Plotted on the left is the sparse wavelet signal f in (IV.1), while on the right is the magnitude of its Fourier transform and the measurements on Ω.

The proposed algorithm is tested when the Fourier measurements of the signal f are corrupted by random noises ϵ,

$$h(\xi) = \widehat{f}(\xi) + \epsilon(\xi), \quad \xi \in \Omega.$$

In this case, sparsity locations obtained by Prony's method in the algorithm are not necessarily integers, but it is observed that they are not far away from the sparsity locations of the signal f, when the signal-to-noise-ratio (SNR),

$$SNR = -20 \log_{10} \frac{\max_{\xi \in \Omega} |\epsilon(\xi)|}{\max_{\xi \in \Omega} |\widehat{f}(\xi)|}$$

is above 50 dB. Taking nearest integers of those locations may perfectly recover the sparsity positions $\{2, 4\}$ for the scaling component of level 0, $\{1, 5\}$ for the wavelet component of level 0, and $\{6, 12\}$ for the wavelet component of level 1. Then the signal f can be reconstructed by the proposed algorithm approximately, see Figure 2.
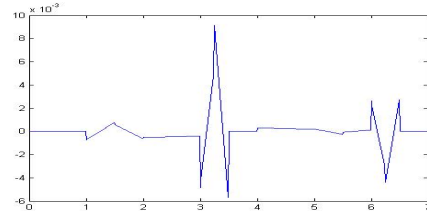

Fig. 2. The difference between the original signal and the reconstructed signal. In this figure, the reconstruction is generated by the proposed algorithm with modified Prony's method, and the noise level on Fourier measurements is SNR=50 dB.

We also tested our proposed algorithm for two-dimensional wavelet signals with dilation $\mathbf{D} = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$. Presented on the left of Figure 3 is the amplitude of a sparse wavelet signal

$$f(t_1, t_2) = a_0 \phi(t_1 - 1, t_2) + a_1 \phi(t_1 - 2, t_2 - 3)$$
$$+ b_0 \psi(t_1 - 2, t_2 - 1) + b_1 \psi(t_1 - 3, t_2 - 5), \quad \text{(IV.2)}$$

where $a_0, a_1, b_0, b_1 \in [-1, 1] \setminus (-0.1, 0.1)$ are selected randomly, the scaling function is $\phi(t_1, t_2) = \chi_{[0,1)}(t_1)\chi_{[0,1)}(t_2)$, and the wavelet function is $\psi(t_1, t_2) = \chi_{[0,1)}(t_1)(\chi_{[0,1/2)}(t_1) - \chi_{[1/2,1)}(t_2))$. Our simulations show

that the signal f in (IV.2) can be reconstructed from its Fourier measurements on

$$\Omega = \left\{ \left(\frac{\sqrt{2}}{64} n + 2k, \frac{\sqrt{3}}{64} n + 2l \right) \pi, \ n = \pm 1, \pm 3 \text{ and } k, l = 0, 1 \right\}, \tag{IV.3}$$
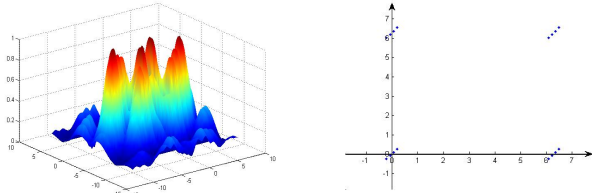
which is plotted on the right of Figure 3.

Fig. 3. Fourier amplitudes of the signal f in (IV.2) and the sampling set Ω in (IV.3) for sparse recovery.

V. CONCLUSION

In this paper, we show that sparse wavelet signals of finite level can be reconstructed from their Fourier measurements on a deterministic sampling set, whose cardinality is independent on signal dimension and almost proportional to signal sparsity. A difficult problem on this aspect is exact reconstruction of signals having sparse wavelet-like (e.g. wavelet packet, framelet, curvelet, and shearlet) representations from their partial Fourier information ([27], [28], [29], [30], [31]).

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