STABILITY OF LOCALIZED INTEGRAL OPERATORS ON WEIGHTED L^p SPACES

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Abstract. In this paper, we consider localized integral operators whose kernels have mild singularity near the diagonal and certain Hölder regularity and decay off the diagonal. Our model example is the Bessel potential operator $\mathcal{J}_\gamma, \gamma > 0$. We show that if such a localized integral operator has stability on a weighted function space L^p_w for some $p \in [1, \infty)$ and Muckenhoupt A_p-weight w, then it has stability on weighted function spaces $L^{p'}_{w'}$ for all $1 \le p' < \infty$ and Muckenhoupt $A_{p'}$-weights w'.

1. Introduction

Let K be a kernel function on $\mathbb{R}^d \times \mathbb{R}^d$. Define the minimal radial function on $\mathbb{R}^d$ that is radially decreasing and dominates the off-diagonal decay of the kernel K by

$$r_K(x) := \sup_{|y-y'|\ge|x|} |K(y, y')|. \tag{1.1}$$

Here $|x| := \max\{|x_1|, \cdots, |x_d|\}$ for $x := (x_1, \cdots, x_d) \in \mathbb{R}^d$. In this paper, we consider integral operators

$$Tf(x) := \int_{\mathbb{R}^d} K(x, y)f(y)dy \tag{1.2}$$

whose kernel K on $\mathbb{R}^d \times \mathbb{R}^d$ has its off-diagonal decay dominated by an integrable radially decreasing function on $\mathbb{R}^d$, i.e.,

$$\|r_K\|_1 := \int_{\mathbb{R}^d} r_K(x)dx < \infty. \tag{1.3}$$

The model example of such an integral operator is the Bessel potential [16]

$$\mathcal{J}_\gamma f = \int_{\mathbb{R}^d} G_\gamma(x - y)f(y)dy, \ \gamma > 0, \tag{1.4}$$

where the Bessel kernel G_γ is defined with the help of Fourier transform by

$$\widehat{G}_\gamma(\xi_1, \ldots, \xi_d) = \big(1 + |\xi_1|^2 + \cdots + |\xi_d|^2\big)^{-\gamma/2},$$

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and the Fourier transform $\hat{f}$ of an integrable function f is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\xi}dx$.

For $1 \le p < \infty$, we say that a weight w on the d-dimensional Euclidean space $\mathbb{R}^d$ (i.e., a positive locally-integrable function on $\mathbb{R}^d$) is an A_p-*weight* if

$$\Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}dx\Big)^{p-1} \le A < \infty \quad \text{for all cubes } Q \tag{1.5}$$

when $1 < p < \infty$, and if

$$\frac{1}{|Q|}\int_Q w(y)dy \le A \inf_{x\in Q} w(x) \quad \text{for all cubes } Q \tag{1.6}$$

when $p = 1$ [7, 9, 15]. Here $|E|$ stands for the Lebesgue measure of a measurable set $E \subset \mathbb{R}^d$. The A_p-*bound* of an A_p-weight w, to be denoted by $A_p(w)$, is the smallest constant A for which (1.5) holds when $1 < p < \infty$ (respectively (1.6) holds when $p = 1$). Simple nontrivial example of A_p-weights is the polynomial weight $w_\alpha(x) := |x|^\alpha$, which is an A_p-weight if the exponent α of the polynomial weight w_α satisfies $-d < \alpha \le 0$ for $p = 1$, and $-d < \alpha < d(p-1)$ for $1 < p < \infty$.

Denote by I the identity operator, and by $L^p_w := L^p_w(\mathbb{R}^d)$ the space of all measurable functions f on $\mathbb{R}^d$ with $\|f\|_{p,w} := (\int_{\mathbb{R}^d} |f(x)|^p w(x)dx)^{1/p} < \infty$. A well-known result about the integral operator T in (1.2) is that it is bounded on the weighted function space L^p_w for any $p \in [1,\infty)$ and A_p-weight w. Furthermore there exists an absolute constant C, that depends on p and d only, such that

$$\|Tf\|_{p,w} \le C(A_p(w))^{1/p}\|r_K\|_1\|f\|_{p,w} \tag{1.7}$$

for all A_p-weights w and functions $f \in L^p_w$, see also Proposition 2.1. In this paper, instead of establishing boundedness of the integral operator T on L^p_w, we consider stability of integral operators $zI - T, z \in \mathbb{C}$, on L^p_w, i.e., there exists a positive constant C such that

$$\|(zI - T)f\|_{p,w} \ge C\|f\|_{p,w} \text{ for all } f \in L^p_w. \tag{1.8}$$

We will show that the stability of integral operators $zI-T, z \in \mathbb{C}$, on L^p_w for different $p \in [1,\infty)$ and A_p-weights w are equivalent to each other, provided that the kernel K of the integral operator T is assumed, in addition to its off-diagonal decay dominated by an integrable radially decreasing function, to have certain Hölder regularity off the diagonal and mild singularity near the diagonal, i.e.,

$$\|r_K\|_1 + \sup_{0<\delta\le 1} \delta^{-\alpha}\|r_{\omega_\delta(K)}\|_1 + \sup_{0<\delta\le 1} \delta^{-\alpha}\|r_K\chi_{|\cdot|\le\delta}\|_1 < \infty \tag{1.9}$$

for some $\alpha \in (0, 1]$. Here for a kernel function K on $\mathbb{R}^d \times \mathbb{R}^d$, its modified modulus of continuity $\omega_\delta(K)$ is defined by

$$\omega_\delta(K)(x,y) = \begin{cases} \sup_{|x'-x|,|y'-y|\le\delta} |K(x',y') - K(x,y)| & \text{if } |x-y| \ge 4\delta, \\ 0 & \text{otherwise.} \end{cases} \tag{1.10}$$

Theorem 1.1. *Let $z \in \mathbb{C}$, K be a kernel function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying* (1.9) *for some $\alpha \in (0, 1]$, and let T be the integral operator in* (1.2) *with kernel K. If $zI - T$*

has stability on L^p_w *for some* $1 \le p < \infty$ *and* A_p*-weight* w*, then it has stability on* $L^{p'}_{w'}$ *for all* $1 \le p' < \infty$ *and* $A_{p'}$*-weights* w'*.*

Denote by $s_{p,w}(T)$ the set of all complex numbers z such that $zI - T$ does not have stability on L^p_w, and by $s_p(T)$ instead of $s_{p,w_0}(T)$ for short when w is the trivial weight $w_0 \equiv 1$. Then Theorem 1.1 can be reformulated as follows:

$$s_{p,w}(T) = s_2(T) \tag{1.11}$$

for all $1 \le p < \infty$ and A_p-weights w, provided that the kernel of the integral operator T satisfies (1.9). We remark that for the operator $zI - T$, it is established in [14] the equivalence of its stability on unweighted function spaces L^p for different exponents $p \in [1, \infty]$, i.e.,

$$s_p(T) = s_2(T) \tag{1.12}$$

for all $1 \le p \le \infty$. The assumption on the kernel K of the operator T in the above equivalence is that it has certain Hölder regularity and its off-diagonal decay dominated by a function in the Wiener amalgam space $\mathcal{W}_1$, the space containing all measurable functions h on $\mathbb{R}^d$ with $\|h\|_{\mathcal{W}_1} := \sum_{k\in\mathbb{Z}^d} \sup_{x\in[-1/2,1/2)^d} |h(k + x)| < \infty$. More precisely, the kernel K satisfies the following condition:

$$\Big\| \sup_{y\in\mathbb{R}^d} |K(y, \cdot + y)| \Big\|_{\mathcal{W}_1} + \sup_{0<\delta\le 1} \delta^{-\alpha} \Big\| \sup_{y\in\mathbb{R}^d} \tilde{\omega}_\delta(K)(y, \cdot + y) \Big\|_{\mathcal{W}_1} < \infty \tag{1.13}$$

for some $\alpha \in (0, 1]$, where the module of continuity $\tilde{\omega}_\delta(K), \delta > 0$, of a kernel K on $\mathbb{R}^d \times \mathbb{R}^d$ is defined by

$$\tilde{\omega}_\delta(K)(x, y) = \sup_{\max(|x'-x|,|y'-y|)\le\delta} |K(x', y') - K(x, y)| \quad \text{for all } x, y \in \mathbb{R}^d$$

c.f. the modified module of continuity $\omega_\delta(K)$ of a kernel K in (1.10). The assumptions (1.9) and (1.13) on kernels are not comparable. Kernels satisfying (1.9) could have certain blowup near the diagonal while kernels satisfying (1.13) do not allow any singularity (and even require certain regularity) near the diagonal. On the other hand, kernels satisfying (1.13) have less requirement on the decay far away from the diagonal than kernels satisfying (1.9) do.

We say that an integral operator T in (1.2) is *of convolution type* (or a *convolution operator*) if its kernel K can be written as $K(x, y) = g(x - y)$ for some integrable function g on $\mathbb{R}^d$ [1, 11, 12]. In this case, one may verify that $s_2(T) = \{\hat{g}(\xi)|\ \xi \in \mathbb{R}^d\} \cup \{0\}$. This together with (1.11) implies that

$$s_{p,w}(T) = \{\hat{g}(\xi)|\ \xi \in \mathbb{R}^d\} \cup \{0\}$$

for all $1 \le p < \infty$ and A_p-weight w, provided that $Tf(x) = \int_{\mathbb{R}^d} g(x - y)f(y)dy$ for some integrable function g on $\mathbb{R}^d$ and the kernel $g(x - y)$ satisfies (1.9). Thus for the Bessel potentials $\mathcal{J}_\gamma, \gamma > 0$, we have that $s_{p,w}(\mathcal{J}_\gamma) = [0, 1]$ for all $1 \le p < \infty$ and A_p-weight w, which is new up to our knowledge.

Denote by $\sigma_{p,w}(T)$ the spectrum of the operator T on L^p_w and by $\sigma_p(T)$ instead of $\sigma_{p,w_0}(T)$ for short when w is the trivial weight $w_0 \equiv 1$. Clearly we have that

$$s_{p,w}(T) \subset \sigma_{p,w}(T) \tag{1.14}$$

for all bounded operators T on L^p_w. We are working on the problem whether or not the above inclusion is indeed an equality when the kernel of the operator T satisfies (1.9). The reader may refer to [1, 2, 5, 8, 11, 12, 13, 14] for spectra $\sigma_{p,w}(T)$ of various integral operators T, and [10, 17, 18, 19] for its connection to Wiener's lemma for infinite matrices.

The paper is organized as follows. In Section 2, we provide some preliminary results on the boundedness, approximation and discretization of the integral operator T in (1.2) on weighted function spaces L^p_w, and also the boundedness on weighted sequence spaces and off-diagonal decay for the discretization of the integral operator T in (1.2) at different levels. The main result of this paper is Theorem 1.1, whose proof is given in Section 3. Some refinements of doubling measure property and reverse Hölder inequality for Muckenhoupt A_p-weights are included in the appendix.

In this paper, we will use the following notation. $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$; $\ell^p_w := \ell^p_w(\Lambda)$ is the space of all weighted p-summable column vectors $c = (c(\lambda))_{\lambda\in\Lambda}$ with $\|c\|_{p,w} := (\sum_{\lambda\in\Lambda} |c(\lambda)|^p w(\lambda))^{1/p} < \infty$, where $1 \le p < \infty$ and $w = (w(\lambda))_{\lambda\in\Lambda}$ is a weight; $\langle g_1, g_2\rangle := \int_{\mathbb{R}^d} g_1(x)\overline{g_2(x)}dx$ provided that g_1g_2 is integrable; $\mathcal{A}_p, 1 \le p < \infty$, is the set of all A_p-weights; kQ stands for the cube with the center same as the one of the given cube Q and the radius k times the one of cube Q; b_K is the function on the positive axis such that $b_K(|x|) = r_K(x)$ is the minimally radically decreasing function in (1.1) that dominates the off-diagonal decay of a kernel K on $\mathbb{R}^d\times\mathbb{R}^d$; and C denotes an absolute constant which could be different at different occurrences.

2. Preliminary

We divide this section into two parts. In the first part of this section, we consider the boundedness, approximation and discretization of an integral operator whose kernel has certain off-diagonal decay and Hölder regularity. In the first subsection we recall that an integral operator, whose kernel has its off-diagonal decay dominated by an integrable radially decreasing function, is a bounded operator on L^p_w for any $1 \le p < \infty$ and A_p-weight w, see Proposition 2.1. Define $P_n, n \in \mathbb{Z}$, on L^p_w by

$$P_n f = \sum_{\lambda\in 2^{-n}\mathbb{Z}^d} \langle f, \phi_{n,2^n\lambda}\rangle \phi_{n,2^n\lambda},\ f \in L^p_w, \tag{2.1}$$

where $\phi_{n,k} = 2^{nd/2}\chi_{[-1/2,1/2)^d}(2^n\cdot -k),\ n \in \mathbb{Z}, k \in \mathbb{Z}^d$. For $p = 2$ and the trivial weight $w \equiv 1$, $P_n, n \in \mathbb{Z}$, are projection operators onto $V_n := P_nL^2$, which form a multiresolution analysis associated with the Haar wavelet system [6]. In the second subsection, we prove that an integral operator T with its kernel having certain off-diagonal decay, mild singularity near the diagonal and Hölder regularity can be approximated by P_nT, TP_n and $P_nTP_n, n \in \mathbb{Z}$, in the operator norm on L^p_w, see Proposition 2.2. As a consequence of the above approximation, we conclude that zero is in the spectrum of a localized integral operator, see Corollary 2.3. We call the operator P_nTP_n the *discretization of the integral operator T at n-th level*, as they are closely related to infinite matrices

$$A_n := \big(a_n(\lambda,\lambda')\big)_{\lambda,\lambda'\in 2^{-n}\mathbb{Z}^d},\ n \in \mathbb{Z} \tag{2.2}$$

where

$$a_n(\lambda,\lambda') = 2^{nd}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\phi_{n,2^n\lambda}(x)K(x,y)\phi_{n,2^n\lambda'}(y)dydx,\ \ \lambda,\lambda'\in 2^{-n}\mathbb{Z}^d,$$

see Proposition 2.5 of the third subsection. The same discretization has been used in [14, 19] to establish Wiener's lemma and stability for localized integral operators on unweighted function spaces $L^p, 1\le p<\infty$.

In the second part of this section, we consider the boundedness and off-diagonal decay property of discretization matrices $A_n, n\in\mathbb{Z}$. Given a locally integrable positive function w, define its *discretization at n-th level* by

$$w_n := (w_n(\lambda))_{\lambda\in 2^{-n}\mathbb{Z}^d}, \tag{2.3}$$

where $w_n(\lambda) = 2^{nd}\int_{\lambda+2^{-n}[-1/2,1/2)^d} w(x)dx, \lambda\in 2^{-n}\mathbb{Z}^d$. As shown in Proposition A.5, discretization of an A_p-weight w at any level is a discrete A_p-weight, see (2.15) and (2.16) for the definition. In Proposition 2.6 of the fourth subsection, we show that for every $n\in\mathbb{Z}$, the discretization matrix A_n is bounded on the weighted sequence space $\ell^p_{w_n}$ for any $1\le p<\infty$ and A_p-weight w. The above proposition can be thought as a discretized version of Proposition 2.1. As we always assume in the paper that the integral operator T in (1.2) has its kernel with certain off-diagonal decay, its discretization matrices $A_n, n\in\mathbb{Z}$, have similar off-diagonal decay, see Proposition 2.7 of the fifth subsection. For $N\ge 1$ and $k\in N\mathbb{Z}^d$, define the localization matrix Ψ_k^N on a sequence space on $2^{-n}\mathbb{Z}^d$ by

$$(\Psi_k^N c)(\lambda) := \psi_0((\lambda-k)/N)c(\lambda) \quad \text{for } c := (c(\lambda))_{\lambda\in 2^{-n}\mathbb{Z}^d}, \tag{2.4}$$

where $\psi_0(x) = \max(\min(2-|x|,1),0)$. In the sixth subsection, we prove that the commutators $[A_n,\Psi_k^N] := A_n\Psi_k^N - \Psi_k^N A_n$ between the discretization matrices A_n and the localization matrices Ψ_k^N have certain off-diagonal decay, see Proposition 2.8. The above off-diagonal decay property for the commutators $[A_n,\Psi_k^N]$ plays crucial roles in the proof of Theorem 1.1. We remark that similar off-diagonal decay property for the commutator $[A_n,\Psi_k^N]$ has been used in [14] to establish the equivalence of stability of a localized integral operator on unweighted function space L^p for different exponent $1\le p<\infty$.

2.1. **Boundedness of localized integral operators.**

Proposition 2.1. *Let $1\le p<\infty$ and K be a kernel function on $\mathbb{R}^d\times\mathbb{R}^d$ whose off-diagonal decay is dominated by an integrable radially decreasing function (i.e.,* (1.3) *holds). Then the integral operator T in* (1.2) *with kernel K is a bounded operator on L^p_w for any A_p-weight w. Furthermore,*

$$\|Tf\|_{p,w}\le C(A_p(w))^{1/p}\|r_K\|_1\|f\|_{p,w} \tag{2.5}$$

for all weights $w\in\mathcal{A}_p$ and functions $f\in L^p_w$, where C is an absolute constant that depends on p and d only.

Proof. It is well known that the integral operator T in (1.2) is a bounded operator on L^p_w [7, 9, 15]. We include a sketch of the proof for the bound estimate in (2.5) and for the completeness of the paper. Note that

$$|Tf(x)| \le \sum_{j\in\mathbb{Z}} b_K(2^{j-1}) \int_{2^{j-1}\le|x-y|<2^j} |f(y)|dy \quad \text{for all } x\in\mathbb{R}^d \tag{2.6}$$

(and hence $|Tf(x)|$ is dominated by a constant multiple of the maximal function $Mf(x)$, which is bounded on L^p_w for all $1<p<\infty$ and A_p-weights w [7, 9, 15]). Then for $p=1$,

$$\begin{aligned}
\|Tf\|_{1,w} &\le \sum_{j\in\mathbb{Z}} b_K(2^{j-1}) \int_{\mathbb{R}^d} w(x) \int_{2^{j-1}\le|x-y|<2^j} |f(y)|dydx \\
&\le A_1(w)\Big(\sum_{j\in\mathbb{Z}} b_K(2^{j-1})2^{(j+1)d}\Big)\|f\|_{1,w} \le CA_1(w)\|r_K\|_1\|f\|_{1,w}.
\end{aligned}$$

This proves (2.5) for $p=1$.

For $1<p<\infty$, applying (2.6) and using Hölder inequality, we obtain

$$\begin{aligned}
|Tf(x)|^p &\le CA_p(w)\|r_K\|_1^{p-1} \sum_{j\in\mathbb{Z}} b_K(2^{j-1})2^{jd}\Big(\int_{|x-y'|<2^j} w(y')dy'\Big)^{-1} \\
&\qquad \times\Big(\int_{2^{j-1}\le|x-y|<2^j} |f(y)|^p w(y)dy\Big).
\end{aligned}$$

Thus

$$\begin{aligned}
\|Tf\|_{p,w}^p &\le CA_p(w)\|r_K\|_1^{p-1} \sum_{j\in\mathbb{Z}} b_K(2^{j-1})2^{jd} \\
&\qquad \times \int_{\mathbb{R}^d} |f(y)|^p w(y)\Big(\int_{2^{j-1}\le|x-y|<2^j} \frac{w(x)}{\int_{|x-y'|<2^j} w(y')dy'}dx\Big)dy \\
&\le CA_p(w)\|r_K\|_1^{p-1} \sum_{j\in\mathbb{Z}} b_K(2^{j-1})2^{jd} \int_{\mathbb{R}^d} |f(y)|^p w(y) \\
&\qquad \times\Big(\sum_{\epsilon\in\{-1,0,1\}^d} \int_{|x-y-\epsilon 2^{j-1}|<2^{j-1}} \frac{w(x)}{\int_{|y'-y-\epsilon 2^{j-1}|<2^{j-1}} w(y')dy'}dx\Big)dy \\
&\le CA_p(w)\|r_K\|_1^p\|f\|_{p,w}^p.
\end{aligned}$$

This establishes (2.5) for $1<p<\infty$ and completes the proof. □

2.2. **Approximation of localized integral operators.**

Proposition 2.2. *Let $1\le p<\infty$, w be an A_p-weight, K be a kernel function on $\mathbb{R}^d\times\mathbb{R}^d$ satisfying (1.9) for some $\alpha\in(0,1]$, T be the integral operator in (1.2) with kernel K, and let $P_n, n\in\mathbb{Z}$, be as in (2.1). Then there exists an absolute constant C (depending on p and d only) such that*

$$\begin{aligned}
&\|(TP_n-T)f\|_{p,w} + \|(P_nT-T)f\|_{p,w} + \|(P_nTP_n-T)f\|_{p,w} \\
&\le CD_0 2^{-n\alpha}(A_p(w))^{1/p}\|f\|_{p,w} \quad \text{for all } n\in\mathbb{Z}_+, w\in\mathcal{A}_p \text{ and } f\in L^p_w,
\end{aligned} \tag{2.7}$$

where $D_0=\|r_K\|_1+\sup_{0<\delta\le1}\delta^{-\alpha}\|r_K\chi_{|\cdot|\le\delta}\|_1+\sup_{0<\delta\le1}\delta^{-\alpha}\|r_{\omega_\delta(K)}\|_1$.

We remark that it is established in [14, Proof of Theorem 4.1] that a localized integral operator has the above approximation property on unweighted function spaces $L^p, 1 \le p < \infty$. By Proposition 2.2, we see that TP_n, P_nT, P_nTP_n approximate the localized integral operator T in the operator norm $\|\cdot\|_{\mathcal{B}(L^p_w)}$ on L^p_w, as n tends to infinity, i.e.,

$$\lim_{n\to\infty} \|P_nT - T\|_{\mathcal{B}(L^p_w)} + \|TP_n - T\|_{\mathcal{B}(L^p_w)} + \|P_nTP_n - T\|_{\mathcal{B}(L^p_w)} = 0. \tag{2.8}$$

As a consequence of the above limit, zero is in the spectrum of a localized integral operator T on L^p_w, c.f. [19, Theorem 2.2 (iv)].

Corollary 2.3. *Let the integral operator T be as in Proposition 2.2. Then $0 \in s_{p,w}(T) \subset \sigma_{p,w}(T)$ for all $1 \le p < \infty$ and $w \in \mathcal{A}_p$.*

Proof. Let $\varphi_0 = \max(1-|x|, 0)$ be the hat function and set $g_n := \varphi_0 - P_n\varphi_0, n \ge 0$. Note that $0 \ne g_n \in L^p_w$ and $P_n^2 = P_n$ for all $n \in \mathbb{Z}_+$. Then for all $1 \le p < \infty$ and $w \in \mathcal{A}_p$, we have that

$$\inf_{\|g\|_{p,w}\ne 0} \frac{\|Tg\|_{p,w}}{\|g\|_{p,w}} \le \frac{\|Tg_n\|_{p,w}}{\|g_n\|_{p,w}} = \frac{\|(T-TP_n)g_n\|_{p,w}}{\|g_n\|_{p,w}} \le \|TP_n - T\|_{\mathcal{B}(L^p_w)} \to 0 \tag{2.9}$$

as $n \to \infty$ by (2.8). This proves the conclusion that $0 \in s_{p,w}(T) \subset \sigma_{p,w}(T)$. $\square$

Now we prove Proposition 2.2.

Proof of Proposition 2.2. By (2.1), P_n is an integral operator with kernel

$$P_n(x,y) := \begin{cases} 2^{nd} & \text{if } x, y \in 2^{-n}(k + [-1/2, 1/2)^d) \text{ for some } k \in \mathbb{Z}^d, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} \|P_nf\|^p_{p,w} &= 2^{ndp/2} \sum_{\lambda\in 2^{-n}\mathbb{Z}^d} |\langle f, \phi_{n,2^n\lambda}\rangle|^p \int_{\lambda+2^{-n}[-1/2,1/2)^d} w(x)dx \\ &\le A_p(w) \sum_{\lambda\in 2^{-n}\mathbb{Z}^d} \int_{\lambda+2^{-n}[-1/2,1/2]^d} |f(x)|^p w(x)dx = A_p(w)\|f\|^p_{p,w} \end{aligned} \tag{2.10}$$

for all $f \in L^p_w$. Thus $TP_n - T, P_nT - T$ and $P_nTP_n - T$ are bounded operators on L^p_w by (2.10) and Proposition 2.1.

Denote by $K_n(x,y)$ the kernel of the integral operator $P_nTP_n - T$. Then

$$\begin{aligned} |K_n(x,y)| &= \Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (K(x',y') - K(x,y))P_n(x,x')P_n(y',y)dx'dy'\Big| \\ &\le 2^{2nd} \int_{|x'-x|\le 2^{-n}} \int_{|y'-y|\le 2^{-n}} |K(x',y') - K(x,y)|dx'dy' \\ &\le 2^{2d} r_{\omega_{2^{-n}}(K)}(x-y) \end{aligned}$$

for all $x, y \in \mathbb{R}^d$ with $|x-y| > 6\cdot 2^{-n}$, and

$$\begin{aligned} |K_n(x,y)| &\le |K(x,y)| + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big|P_n(x,x')K(x',y')P_n(y',y)\big| dx'dy' \\ &\le r_K(x-y) + 2^{nd}\int_{|t|\le 8\cdot 2^{-n}} r_K(t)dt \end{aligned}$$

for all $x, y \in \mathbb{R}^d$ with $|x-y| \le 6\cdot 2^{-n}$. Thus the kernel $K_n(x,y)$ of the integral operator $P_nTP_n - T$ is dominated by $h_n(x-y)$, where h_n is a radially decreasing function defined by

$$h_n(x) := \begin{cases} r_K(x) + 2^{nd}\int_{|t|\le 8\cdot 2^{-n}} r_K(t)dt & \text{if } |x|\le 6\cdot 2^{-n}, \\ 2^{2d} r_{\omega_{2^{-n}}(K)}(x) & \text{if } |x| > 6\cdot 2^{-n}. \end{cases}$$

Similarly we can show that kernels of the integral operators $TP_n - T$ and $P_nT - T$ have their off-diagonal decay dominated by the same radially decreasing function h_n. Then the desired estimate (2.7) for the integral operators $TP_n - T, P_nT - T$ and $P_nTP_n - T, n \in \mathbb{Z}_+$, follows from (1.9), Proposition 2.1 and the above observation about their kernels. □

Remark 2.4. Let $1 \le p < \infty$, w be an A_p-weight, and $P_n, n \in \mathbb{Z}$, be as in (2.1). For $n \in \mathbb{Z}$, define

$$V_{p,w}^n = \Big\{ \sum_{\lambda\in 2^{-n}\mathbb{Z}^d} c(\lambda)\phi_{n,2^n\lambda} \ \Big|\ \big(c(\lambda)\big)_{\lambda\in\in 2^{-n}\mathbb{Z}^d} \in \ell^p_{w_n} \Big\}.$$

Then it follows from (2.1) and (2.10) that $P_n, n\in\mathbb{Z}$, are bounded operators from L^p_w onto $V^n_{p,w} \subset L^p_w$ with their operator norm bounded by $(A_p(w))^{1/p}$; i.e., $V^n_{p,w} = P_nL^p_w$ and $\|P_nf\|_{p,w} \le (A_p(w))^{1/p}\|f\|_{p,w}$ for all $f \in L^p_w$.

2.3. **Discretization of localized integral operators and discretization matrices.**

Proposition 2.5. *Let $1 \le p < \infty$, w be an A_p-weight, K be a kernel function on $\mathbb{R}^d\times\mathbb{R}^d$ satisfying* (1.3)*, T be the integral operator* (1.2) *with kernel K, and let P_n and $A_n, n \in \mathbb{Z}$, be as in* (2.1) *and* (2.2) *respectively. Then*

$$d_n(f) = 2^{-nd}A_n c_n(f) \quad \text{for all } f \in L^p_w, \tag{2.11}$$

where $d_n(f) = \big(\langle P_nTP_nf, \phi_{n,2^n\lambda}\rangle\big)_{\lambda\in 2^{-n}\mathbb{Z}^d}$ and $c_n(f) = \big(\langle P_nf, \phi_{n,2^n\lambda}\rangle\big)_{\lambda\in 2^{-n}\mathbb{Z}^d}$ for $f \in L^p_w$.

Proof. We mimic the argument in [14, Proof of Theorem 4.1]. Note that

$$\begin{aligned} P_nTP_nf(x) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Big(\sum_{\lambda\in 2^{-n}\mathbb{Z}^d}\phi_{n,2^n\lambda}(x)\phi_{n,2^n\lambda}(x')\Big) \\ &\quad \times K(x',y')\Big(\sum_{\lambda'\in 2^{-n}\mathbb{Z}^d}\langle P_nf, \phi_{n,2^n\lambda'}\rangle\phi_{n,2^n\lambda'}(y')\Big)dx'dy' \\ &= \sum_{\lambda\in 2^{-n}\mathbb{Z}^d}\Big(2^{-nd}\sum_{\lambda'\in 2^{-n}\mathbb{Z}^d} a_n(\lambda,\lambda')\langle P_nf, \phi_{n,2^n\lambda'}\rangle\Big)\phi_{n,2^n\lambda}(x) \end{aligned}$$

for all $f \in L^p_w$. Then (2.11) follows. $\square$

2.4. **Boundedness of discretization matrices.**

Proposition 2.6. *Let $1 \le p < \infty$, w be an A_p-weight, K be a kernel function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.3), and let $A_n = \big(a_n(\lambda, \lambda')\big)_{\lambda,\lambda' \in 2^{-n}\mathbb{Z}^d}$ and $w_n, n \in \mathbb{Z}$, be as in (2.2) and (2.3) respectively. Then $A_n, n \in \mathbb{Z}$, are bounded operators on $\ell^p_{w_n}$ with operator norm bounded by a constant multiple of $2^{nd}(A_p(w))^{3/p}\|r_K\|_1$, i.e.,*

$$\|A_n c_n\|_{p,w_n} \le C 2^{nd}\big(A_p(w)\big)^{3/p}\|r_K\|_1\|c_n\|_{p,w_n} \quad \text{for all } c_n \in \ell^p_{w_n}, \tag{2.12}$$

where C is an absolute constant depending on p and d only.

Proof. Take $c_n := (c_n(\lambda))_{\lambda \in 2^{-n}\mathbb{Z}^d} \in \ell^p_{w_n}$ and set $f_n = \sum_{\lambda \in 2^{-n}\mathbb{Z}^d} c_n(\lambda)\phi_{n,2^n\lambda}$. Then

$$P_nTP_nf_n(x) = \sum_{\lambda \in 2^{-n}\mathbb{Z}^d} \Big(2^{-nd} \sum_{\lambda' \in 2^{-n}\mathbb{Z}^d} a_n(\lambda,\lambda')c_n(\lambda')\Big)\phi_{n,2^n\lambda}(x).$$

This, together with Proposition 2.1, implies that

$$\begin{aligned}\|A_nc_n\|_{p,w_n} &= 2^{nd(p+2)/(2p)}\|P_nTP_nf_n\|_{p,w} \le C2^{nd(2+p)/(2p)}(A_p(w))^{3/p}\|r_K\|_1\|f_n\|_{p,w} \\ &= C2^{nd}(A_p(w))^{3/p}\|r_K\|_1\|c_n\|_{p,w_n},\end{aligned}$$

and hence completes the proof. $\square$

2.5. **Off-diagonal decay property of discretization matrices.**

Proposition 2.7. *Let $1 \le p < \infty$, w be an A_p-weight, K be a kernel function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.3), and let $A_n = \big(a_n(\lambda, \lambda')\big)_{\lambda,\lambda' \in 2^{-n}\mathbb{Z}^d}, n \in \mathbb{Z}$, be as in (2.2). Then*

$$|a_n(\lambda,\lambda')| \le \begin{cases} 2^{nd}\int_{|t|\le 3\cdot 2^{-n}} r_K(t)dt & \text{if } |\lambda-\lambda'| \le 2^{-n+1}, \\ r_K((\lambda-\lambda')/2) & \text{if } |\lambda-\lambda'| > 2^{-n+1}. \end{cases} \tag{2.13}$$

Proof. By (2.2), we obtain that

$$\begin{aligned}|a_n(\lambda,\lambda')| &\le 2^{2nd}\int_{|x-\lambda|\le 2^{-n-1}, |y-\lambda'|\le 2^{-n-1}} |K(x,y)|dydx \\ &\le 2^{2nd}\int_{|x-\lambda|\le 2^{-n-1}} \Big(\int_{|y-x|\le 3\cdot 2^{-n}} r_K(x-y)dy\Big)dx \\ &\le 2^{nd}\int_{|t|\le 3\cdot 2^{-n}} r_K(t)dt\end{aligned}$$

if $\lambda, \lambda' \in 2^{-n}\mathbb{Z}^d$ with $|\lambda-\lambda'| \le 2^{-n+1}$, and

$$\begin{aligned}|a_n(\lambda,\lambda')| &\le 2^{2nd}\int_{|x-\lambda|\le 2^{-n-1}, |y-\lambda'|\le 2^{-n-1}} r_K((\lambda-\lambda')/2)dydx \\ &\le r_K((\lambda-\lambda')/2)\end{aligned}$$

for all $\lambda, \lambda' \in 2^{-n}\mathbb{Z}^d$ with $|\lambda-\lambda'| > 2^{-n+1}$. This proves (2.13). $\square$

2.6. **Off-diagonal decay of commutators between discretization matrices and localization matrices.**

Proposition 2.8. *Let $1 \le p < \infty$, $n \in \mathbb{Z}_+, N \in \mathbb{N}$, w be an A_p-weight, K be a kernel function on $\mathbb{R}^d$ satisfying* (1.3)*, and let discretization matrices A_n, weights w_n, and localization matrices Ψ_k^N be as in* (2.2), (2.3) *and* (2.4) *respectively. Then there exists an absolute constant C, depending on p and d only, such that for all $b \in \ell^p_{w_n}$ and $k, k' \in N\mathbb{Z}^d$,*

$$\|(\Psi_k^N A_n - A_n \Psi_k^N)\Psi_{k'}^N b\|_{p,w_n} \le C(A_p(w))^{1/p} 2^{nd} \|b\|_{p,w_n} \times \begin{cases} N^d r_K\big(\frac{k-k'}{2}\big)\Big(\frac{\sum_{|\lambda-k|\le 2N} w_n(\lambda)}{\sum_{|\lambda'-k'|\le 2N} w_n(\lambda')}\Big)^{1/p} & \text{if } |k-k'| > 8N, \\ \big(N^{-1/2}\|r_K\|_1 + \int_{|t|>\sqrt{N}/4} r_K(t)dt\big) & \text{if } |k-k'| \le 8N. \end{cases} \tag{2.14}$$

A positive sequence $w = (w(k))_{k\in\mathbb{Z}^d}$ is said to be a *discrete A_p-weight* if for all $a \in \mathbb{Z}^d$ and $N \in \mathbb{N}$,

$$\Big(N^{-d} \sum_{k\in a+[0,N-1]^d} w(k)\Big)\Big(N^{-d} \sum_{k\in a+[0,N-1]^d} (w(k))^{-\frac{1}{p-1}}\Big)^{p-1} \le A < \infty \tag{2.15}$$

when $1 < p < \infty$, and

$$N^{-d} \sum_{k\in a+[0,N-1]^d} w(k) \le A \inf_{k\in a+[0,N-1]^d} w(k) \tag{2.16}$$

when $p = 1$. The smallest constant A for which (2.15) holds when $1 < p < \infty$ (for which (2.16) holds when $p = 1$ respectively) is the *discrete A_p-bound*. We denote by $A_p(w)$ the discrete A_p-bound of a discrete A_p-weight w. To prove Proposition 2.8, we recall the boundedness of an infinite matrix on a weighted sequence space.

Lemma 2.9. ([18, Theorem 3.2]) *Let $1 \le p < \infty$, $w = (w(k))_{k\in\mathbb{Z}^d}$ be a discrete A_p-weight, and $A := (a(k,k'))_{k,k'\in\mathbb{Z}^d}$ be an infinite matrix with $\|A\|_{\mathcal{B}} := \sum_{m\in\mathbb{Z}^d}(\sup_{|k-k'|\ge|m|} |a(k,k')|) < \infty$. Then there exists an absolute constant C (depending on p and d only) such that $\|Ac\|_{p,w} \le C(A_p(w))^{1/p}\|A\|_{\mathcal{B}}\|c\|_{p,w}$ for all $c \in \ell^p_w$.*

Proof of Proposition 2.8. Write $(\Psi_k^N A_n - A_n\Psi_k^N)\Psi_{k'}^N = (c(\lambda,\lambda'))_{\lambda,\lambda'\in 2^{-n}\mathbb{Z}^d}$. Then for $|k-k'| \le 8N$,

$$\begin{aligned} |c(\lambda,\lambda')| &= \Big|\Big(\psi_0\big(\frac{\lambda-k}{N}\big) - \psi_0\big(\frac{\lambda'-k}{N}\big)\Big)a_n(\lambda,\lambda')\psi_0\big(\frac{\lambda'-k'}{N}\big)\Big| \\ &\le \min\big(\frac{|\lambda-\lambda'|}{N}, 1\big)|a_n(\lambda,\lambda')|\psi_0\big(\frac{\lambda'-k'}{N}\big) \\ &\le \begin{cases} 2^{n(d-1)+1}N^{-1}\int_{|t|\le 3\cdot 2^{-n}} r_K(t)dt & \text{if } |\lambda-\lambda'| \le 2^{-n+1} \\ \min(|\lambda-\lambda'|/N, 1) r_K((\lambda-\lambda')/2) & \text{if } |\lambda-\lambda'| > 2^{-n+1} \end{cases} \end{aligned}$$

by the Lipschitz property for the function ψ_0 and the off-diagonal property for the matrix A_n in Proposition 2.7. Therefore

$$|c(\lambda,\lambda')| \le Cg(\lambda-\lambda') \quad \text{for all } \lambda, \lambda' \in 2^{-n}\mathbb{Z}^d, \tag{2.17}$$

where $(g(\lambda))_{\lambda\in 2^{-n}\mathbb{Z}^d}$ is a radially decreasing sequence defined by

$$\begin{aligned} g(\lambda) \;=\;& \Big(\frac{2^{n(d-1)}}{N}\int_{|t|\le 3\cdot 2^{-n}} r_K(t)dt + \frac{b_K(2^{-n})}{\sqrt{N}} + b_K\big(\frac{\sqrt{N}}{2}\big)\Big)\chi_{[-2^{-n+1},2^{-n+1}]^d}(\lambda)\\ &+\Big(\frac{1}{\sqrt{N}}r_K\big(\frac{\lambda}{2}\big) + b_K\big(\frac{\sqrt{N}}{2}\big)\Big)\big(\chi_{[-\sqrt{N},\sqrt{N}]^d\setminus[2^{-n+1},2^{-n+1}]^d}(\lambda)\big)\\ &+r_K\big(\frac{\lambda}{2}\big)\big(1-\chi_{[-\sqrt{N},\sqrt{N}]^d}(\lambda)\big). \end{aligned}$$

Note that

$$\begin{aligned} \sum_{\lambda\in 2^{-n}\mathbb{Z}^d} g(\lambda) \;\le\;& C\Big(2^{nd}N^{-1/2}\int_{|t|\le\sqrt{N}/2} r_K(t)dt + \frac{b_K(2^{-n})}{\sqrt{N}}\\ &+2^{nd}N^{d/2}b_K\Big(\frac{\sqrt{N}}{2}\Big) + 2^{nd}\int_{|t|>\sqrt{N}/4} r_K(t)dt\Big)\\ \le\;& C2^{nd}\Big(N^{-1/2}\|r_K\|_1 + \int_{|t|>\sqrt{N}/4} r_K(t)dt\Big). \end{aligned} \tag{2.18}$$

Then the conclusion (2.14) for $|k-k'|\le 8N$ follows from (2.17), (2.18), Lemma 2.9 and Proposition A.5.

For $|k-k'|>8N$,

$$\begin{aligned} |c(\lambda,\lambda')| \;=\;& \Big|\psi_0\big(\frac{\lambda-k}{N}\big)a_n(\lambda,\lambda')\psi_0\big(\frac{\lambda'-k'}{N}\big)\Big|\\ \le\;& r_K((k-k')/2)\chi_{k+[-2N,2N]^d}(\lambda)\chi_{k'+[-2N,2N]^d}(\lambda') \end{aligned} \tag{2.19}$$

by Proposition 2.7. Write $b=(b(\lambda))_{\lambda\in 2^{-n}\mathbb{Z}^d}$. Then by (2.15) and (2.19) we obtain that

$$\begin{aligned} &\|(\Psi_k^N A_n - A_n\Psi_k^N)\Psi_{k'}^N b\|_{p,w_n}\\ \le\;& r_K((k-k')/2)\Big(\sum_{|\lambda-k|\le 2N} w_n(\lambda)\Big)^{1/p}\\ &\times\Big(\sum_{|\lambda'-k'|\le 2N}|b(\lambda')|^p w_n(\lambda')\Big)^{1/p}\Big(\sum_{|\lambda'-k'|\le 2N}(w_n(\lambda'))^{-1/(p-1)}\Big)^{(p-1)/p}\\ \le\;& 2^{nd}N^d r_K((k-k')/2)(A_p(w))^{1/p}\Big(\frac{\sum_{|\lambda-k|\le 2N} w_n(\lambda)}{\sum_{|\lambda'-k'|\le 2N} w_n(\lambda')}\Big)^{1/p}\|b\|_{p,w_n} \end{aligned}$$

for $1<p<\infty$, and similarly

$$\begin{aligned} &\|(\Psi_k^N A_n - A_n\Psi_k^N)\Psi_{k'}^N c\|_{1,w_n}\\ \le\;& 2^{nd}N^d r_K((k-k')/2)A_1(w)\Big(\frac{\sum_{|\lambda-k|\le 2N} w_n(\lambda)}{\sum_{|\lambda'-k'|\le 2N} w_n(\lambda')}\Big)\|b\|_{1,w_n} \end{aligned}$$

for $p=1$. Hence the conclusion (2.14) for $|k-k'|>8N$ follows. □

3. Stability of localized integral operators

To prove Theorem 1.1, we need several technical lemmas.

Lemma 3.1. *Let $1 \le p < \infty$, $z \in \mathbb{C}$, w be an A_p-weight, and let the kernel K and the integral operator T with kernel K be as in Theorem 1.1. Set*

$$\delta_0 = \min(r_0/(2A_p(w)), \alpha/(3d)) \tag{3.1}$$

where $\alpha \in (0,1]$ and $r_0 \in (0,1)$ are given in (1.9) and Proposition A.4 respectively. If $zI - T$ has $L^p_{w^r}$-stability for some $r \in (0,1]$, then it has $L^p_{w^{r(1+s)}}$-stability for all $s \in [-\delta_0, \delta_0]$ with $0 \le r(1+s) \le 1$.

Lemma 3.2. *Let $1 \le p < \infty$, $z \in \mathbb{C}$, w be an A_p-weight, and let the kernel K and the integral operator T with kernel K be as in Theorem 1.1. Set*

$$\delta_1 = \min\big((p\ln 2 + 2\ln A_p(w))^{-1} D_1, (2(2^d+1) + 2d + 4(2^d+1)\ln A_p(w))^{-1}\alpha\big) \tag{3.2}$$

where $\alpha \in (0,1]$ and $D_1 \in (0,1)$ are given in (1.9) and Proposition A.1 respectively. If $zI - T$ has $L^p_{w^r}$-stability for some $r \in [0, \delta_1]$, then it has $L^p_{w^{r'}}$-stability for all $r' \in [0, \delta_1]$.

Lemma 3.3. *Let $1 \le p < \infty$, $z \in \mathbb{C}$, and let the kernel K and the integral operator T with kernel K be as in Theorem 1.1. Set $\delta_2 = \alpha/(3d)$ with $\alpha \in (0,1]$ given in (1.9). If $zI - T$ has L^p-stability, then it has $L^{p(1+s)}$-stability for all $s \in [-\delta_2, \delta_2]$ with $p(1+s) \ge 1$.*

We assume that the conclusions in the above three lemmas hold and proceed to prove Theorem 1.1 by the bootstrap technique.

Proof of Theorem 1.1. We start from assuming that $zI - T$ has the L^p_w-stability for some $z \in \mathbb{C}, p \in [1,\infty)$ and $w \in \mathcal{A}_p$, and we want to prove that $zI - T$ has the $L^{p'}_{w'}$-stability for any $p' \in [1,\infty)$ and $w' \in \mathcal{A}_{p'}$. Let δ_0 and δ_1 be as in (3.1) and (3.2) respectively, and select an integer l_0 sufficiently large such that $(1-\delta_0)^{l_0} \le \delta_1$. Iteratively applying Lemma 3.1 with $s = -\delta_0$ and $r = (1-\delta_0)^l$ for $l = 0, 1, \ldots, l_0 - 1$, we obtain that $zI - T$ has $L^p_{w^{(1-\delta_0)^l}}$-stability for all $l = 1, \ldots, l_0$. Then applying Lemma 3.2 with $r = (1-\delta_0)^{l_0}$ and $r' = 0$ leads to the L^p-stability of $zI - T$.

Select an integer $\ell_1 \in \mathbb{N}$ and $s \in [-\delta_2, \delta_2]$ such that $(1+s)^{l_1} = p'/p$. Then iteratively applying Lemma 3.3 with p replaced by $p(1+s)^l, l = 0, 1, \ldots, l_1 - 1$, yields the $L^{p'}$-stability of $zI - T$.

Let δ_0' and δ_1' be as in Lemmas 3.1 and 3.2 with p replaced by p' and w by w', and select an integer $l_3 \in \mathbb{N}$ such that $(1+\delta_0')^{-l_3} \le \delta_1'$. Applying Lemma 3.2 with p replaced by p', w by w', r by 0 and r' by $(1+\delta_0')^{-l_3}$ leads to the $L^{p'}_{(w')^{(1+\delta_0')^{-l_3}}}$-stability of $zI - T$. We then reach the desired $L^{p'}_{w'}$-stability of the operator $zI - T$ by iteratively applying Lemma 3.1 with p replaced by p', w by w', s by δ_0' and r by $(1+\delta_0')^{-l_3+l}, l = 0, 1, \cdots, l_3 - 1$. $\square$

3.1. **Proof of Lemma 3.1.** Let $zI-T$ have the $L^p_{w^r}$-stability. Then there exists a positive constant C_1 such that

$$\|(zI-T)f\|_{p,w^r} \ge C_1\|f\|_{p,w^r} \quad \text{for all } f\in L^p_{w^r}. \tag{3.3}$$

From Proposition 2.2 it follows that

$$\begin{aligned} \|(T-P_nTP_n)f\|_{p,w^r} &\le C_2D_0 2^{-\alpha n}(A_p(w^r))^{1/p}\|f\|_{p,w^r}\\ &\le C_2D_0 2^{-\alpha n}(A_p(w))^{1/p}\|f\|_{p,w^r} \quad \text{for all } f\in L^p_{w^r}, \end{aligned}\tag{3.4}$$

where $D_0=\|r_K\|_1+\sup_{0<\delta\le 1}\delta^{-\alpha}\|r_K\chi_{[-\delta,\delta]}\|_1+\sup_{0<\delta\le 1}\delta^{-\alpha}\|r_{\omega_\delta(K)}\|_1$ and C_2 is an absolute constant in Proposition 2.2. Let n_0 be a positive integer such that $C_2D_02^{-\alpha n_0}(A_p(w))^{1/p}\le C_1/2$. Then for all $n\ge n_0$ and $f\in L^p_{w^r}$,

$$\|(zI-P_nTP_n)f\|_{p,w^r}\ge \frac{C_1}{2}\|f\|_{p,w^r} \tag{3.5}$$

by (3.3) and (3.4). Define

$$(w^r)_n=\Big(2^{nd}\int_{\lambda+2^{-n}[-1/2,1/2)^d}(w(x))^r dx\Big)_{\lambda\in 2^{-n}\mathbb{Z}^d} \tag{3.6}$$

and

$$(V^r)_n=\Big\{\sum_{\lambda\in 2^{-n}\mathbb{Z}^d}c(\lambda)\phi_{n,2^n\lambda}\Big|\sum_{\lambda\in 2^{-n}\mathbb{Z}^d}|c(\lambda)|^p(w^r)_n(\lambda)<\infty\Big\}. \tag{3.7}$$

Note that for any $f_n:=\sum_{\lambda\in 2^{-n}\mathbb{Z}^d}c(\lambda)\phi_{n,2^n\lambda}\in (V^r)_n$,

$$\begin{aligned} \|f_n\|_{p,w^r} &= \Big(2^{ndp/2}\sum_{\lambda\in 2^{-n}\mathbb{Z}^d}|c(\lambda)|^p\int_{\lambda+2^{-n}[-1/2,1/2)^d}w(x)^r dx\Big)^{1/p}\\ &= 2^{nd(1/2-1/p)}\|c\|_{p,(w^r)_n} \end{aligned}\tag{3.8}$$

and

$$\|(zI-P_nTP_n)f_n\|_{p,w^r}=2^{nd(1/2-1/p)}\|(zI-2^{-nd}A_n)c\|_{p,(w^r)_n} \tag{3.9}$$

by Proposition 2.5, where A_n is defined in (2.2). Then applying (3.5) to $f_n\in (V^r)_n$, and using (3.8) and (3.9), we obtain a discretized version of the $L^p_{w^r}$-stability of $zI-T$:

$$\|(zI-2^{-nd}A_n)c\|_{p,(w^r)_n}\ge \frac{C_1}{2}\|c\|_{p,(w^r)_n} \quad \text{for all } c\in \ell^p_{(w^r)_n} \text{ and } n\ge n_0. \tag{3.10}$$

To prove the $L^p_{w^{r(1+s)}}$-stability of $zI-T$, we need the following claim, a weak version of the above stability with weight w^r replaced by $w^{r(1+s)}$.

Claim 1: *There exists a positive constant $\tilde C$ such that*

$$\|(zI-2^{-nd}A_n)c\|_{p,(w^{r(1+s)})_n}\ge \tilde C 2^{-2nd|s|}\|c\|_{p,(w^{r(1+s)})_n} \tag{3.11}$$

for all $c\in \ell^p_{w_n^{r(1+s)}}$ and $n\ge n_0$.

We assume that Claim 1 holds and proceed our proof. Applying (3.8) and (3.9) with f_n replaced by P_nf and w^r by $w^{r(1+s)}$ and using (3.11), we have

$$C_22^{-2nd|s|}\|P_nf\|_{p,w^{r(1+s)}}\le \|(zI-P_nTP_n)P_nf\|_{p,w^{r(1+s)}} \tag{3.12}$$

for all $f \in L^p_{w^{r(1+s)}}$ and $n \ge n_0$. As noted in Remark 2.4,

$$\begin{aligned}\|g\|_{p,w^{r(1+s)}} &\le \|P_n g\|_{p,w^{r(1+s)}} + \|(I-P_n)g\|_{p,w^{r(1+s)}}\\ &\le (1+2A_p(w))\|g\|_{p,w^{r(1+s)}} \quad \text{for all } g \in L^p_{w^{r(1+s)}}.\end{aligned} \tag{3.13}$$

Let integer n_1 be so chosen that $\tilde C 2^{-2n_1 d\delta_0} \le |z|$ and $CD_0(A_p(w))^{1/p}2^{-n_1\alpha/3} \le \tilde C/2$ where C is the positive constant in Proposition 2.1. Recall that $\delta_0 < \alpha/(3d)$ by assumption and $z \ne 0$ by (2.9) and (3.3). Then applying (3.12) and (3.13) and letting $n = \max(n_0, n_1)$, we obtain that

$$\begin{aligned}&\|(zI-T)f\|_{p,w^{r(1+s)}}\\ \ge\ & (1+2A_p(w))^{-1}\big(\|P_n(zI-T)f\|_{p,w^{r(1+s)}} + \|(I-P_n)(zI-T)f\|_{p,w^{r(1+s)}}\big)\\ \ge\ & (1+2A_p(w))^{-1}\big(\|P_n(zI-T)P_nf\|_{p,w^{r(1+s)}} + |z|\|(I-P_n)f\|_{p,w^{r(1+s)}}\\ & -\|P_n(zI-T)(I-P_n)f\|_{p,w^{r(1+s)}} - \|(I-P_n)Tf\|_{p,w^{r(1+s)}}\big)\\ \ge\ & (1+2A_p(w))^{-1}\big(\tilde C 2^{-2nd|s|}\|P_nf\|_{p,w^{r(1+s)}} + |z|\|(I-P_n)f\|_{p,w^{r(1+s)}}\\ & -CD_0(A_p(w))^{1/p}2^{-n\alpha}\|f\|_{p,w^{r(1+s)}}\big)\\ \ge\ & (1+2A_p(w))^{-1}\tilde C 2^{-2nd\delta_0-1}\|f\|_{p,w^{r(1+s)}}\end{aligned} \tag{3.14}$$

for all $f \in L^p_{w^{r(1+s)}}$ with $s \in [-\delta_0, \delta_0]$, where C is the positive constant in Proposition 2.1. This establishes the desired $L^p_{w^{r(1+s)}}$-stability for the operator $zI - T$ when $|s| \le \delta_0$.

Now it remains to prove Claim 1. Let N be a sufficiently large integer chosen later and $\Psi_k^N, k \in N\mathbb{Z}^d$, be given in (2.4). Define $\Phi_N = \big(\sum_{k\in N\mathbb{Z}^d}(\Psi_k^N)^2\big)^{-1}$. Then Φ_N is a diagonal matrix with diagonal entries being positive and less than one, which implies that

$$\|\Phi_N c\|_{p,(w^r)_n} \le \|c\|_{p,(w^r)_n} \quad \text{for all } c \in \ell^p_{(w^r)_n}. \tag{3.15}$$

Define

$$(\alpha^r)_k = \sum_{|\lambda-k|\le 2N}(w^r)_n(\lambda) = 2^{nd}\int_{k+[-2N-2^{-n-1},2N+2^{-n-1})^d} w(x)^r dx,\ k \in N\mathbb{Z}^d. \tag{3.16}$$

By (3.10), (3.15), (3.16) and Proposition 2.8, we get

$$
\begin{aligned}
&\frac{C_1}{2}\frac{\|\Psi_k^N c\|_{p,(w^r)_n}}{((\alpha^r)_k)^{1/p}} \le \frac{\|(zI-2^{-n}A_n)\Psi_k^N c\|_{q,(w^r)_n}}{((\alpha^r)_k)^{1/p}} \\
\le\ & \frac{\|\Psi_k^N(zI-2^{-nd}A_n)c\|_{p,(w^r)_n}}{((\alpha^r)_k)^{1/p}} \\
&\quad +2^{-nd}\sum_{k'\in N\mathbb{Z}^d}\frac{\|(\Psi_k^N A_n - A_n\Psi_k^N)\Psi_{k'}^N\Phi_N\Psi_{k'}^N c\|_{p,(w^r)_n}}{((\alpha^r)_k)^{1/p}} \\
\le\ & \frac{\|\Psi_k^N(zI-2^{-nd}A_n)c\|_{p,(w^r)_n}}{((\alpha^r)_k)^{1/p}} + C_3(A_p(w^r))^{1/p} \\
&\quad \times \sum_{\substack{|k'-k|\le 8N\\ k'\in N\mathbb{Z}^d}}\Big(N^{-1/2}\|r_K\|_1+\int_{|t|\ge\sqrt{N}/4} r_K(t)dt\Big)\frac{\|\Phi_N\Psi_{k'}^N c\|_{p,(w^r)_n}}{((\alpha^r)_k)^{1/p}} \\
&\quad +C_3(A_p(w^r))^{1/p}N^d\sum_{\substack{|k-k'|>8N\\ k'\in N\mathbb{Z}^d}} r_K((k-k')/2)\frac{\|\Phi_N\Psi_{k'}^N c\|_{q,(w^r)_n}}{((\alpha^r)_{k'})^{1/p}}
\end{aligned}
$$

for any bounded sequence c, where C_3 is an absolute constant depending on p and d only. Thus

$$
\begin{aligned}
\frac{\|\Psi_k^N c\|_{p,(w^r)_n}}{((\alpha^r)_k)^{1/p}} \ \le\ & C_4\frac{\|\Psi_k^N(zI-2^{-nd}A_n)c\|_{p,(w^r)_n}}{((\alpha^r)_k)^{1/p}} \\
& +C_4(A_p(w))^{1/p}\sum_{k'\in N\mathbb{Z}^d} g_N(k-k')\frac{\|\Psi_{k'}^N c\|_{p,(w^r)_n}}{((\alpha^r)_{k'})^{1/p}}
\end{aligned}
\tag{3.17}
$$

for any bounded sequence c, where C_4 is an absolute constant depending on p and d only, and the sequence $(g_N(k))_{k\in N\mathbb{Z}^d}$ is defined by

$$
\begin{aligned}
g_N(k) \ =\ & \Big(N^{-1/2}\|r_K\|_1+\int_{|t|\ge\sqrt{N}/4} r_K(t)dt\Big)\chi_{[-8N,8N]^d}(k) \\
& +N^d r_K(k/2)\chi_{N\mathbb{Z}^d\setminus[-8N,8N]^d}(k),\ \ k\in N\mathbb{Z}^d.
\end{aligned}
$$

Let $\mathcal{B}$ contain all sequences $a:=(a(k))_{k\in\mathbb{Z}^d}$ with $\|a\|_{\mathcal{B}}:=\sum_{m\in\mathbb{Z}^d}\sup_{|k|\ge|m|}|a(k)|<\infty$ ([4]), and denote by $a*b$ the convolution of two summable sequences a and b on $\mathbb{Z}^d$. Recall that there exists a positive constant D such that $\|a*b\|_{\mathcal{B}}\le D\|a\|_{\mathcal{B}}\|b\|_{\mathcal{B}}$ for all $a,b\in\mathcal{B}$ [2, 3, 4, 18]. Then $(\mathcal{B},\|\cdot\|_{\mathcal{B}}/D)$ is a Banach algebra under convolution. Note that $(g_N(Nk))_{k\in\mathbb{Z}^d}$ is a radially decreasing sequence, we then have

$$
\|(g_N(Nk))_{k\in\mathbb{Z}^d}\|_{\mathcal{B}}=\sum_{k\in N\mathbb{Z}^d} g_N(k)\le C_5\big(N^{-1/2}\|r_K\|_1+\int_{|t|\ge\sqrt{N}/4} r_K(t)dt\big)\to 0
$$

as $N\to\infty$, where C_5 is an absolute constant depending on p and d. Now we select a sufficiently large integer N so that

$$
C_4C_5(A_p(w))^{1/p}\Big(N^{-1/2}\|r_K\|_1+\int_{|t|\ge\sqrt{N}/4} r_K(t)dt\Big)<\frac{1}{2D}.
$$

Applying (3.17) iteratively and using the Banach algebra property for $\mathcal{B}$, we obtain that

$$\frac{\|\Psi_k^N c\|_{p,(w^r)_n}}{((\alpha^r)_k)^{1/p}} \le C_4 \sum_{k'\in N\mathbb{Z}^d} V(k-k') \frac{\|\Psi_{k'}^N (zI - 2^{-nd}A_n)c\|_{p,(w^r)_n}}{((\alpha^r)_{k'})^{1/p}} \tag{3.18}$$

hold for all bounded sequence c, where

$$V(k) = \delta(k) + \sum_{l=1}^{\infty} (C_4(A_p(w))^{1/p})^l \underbrace{g_N * \cdots * g_N}_{l \text{ times}}(k) \tag{3.19}$$

and $\delta(0)=1$ and $\delta(k)=0$ for all nonzero integer $k\in N\mathbb{Z}^d$. One may verify that

$$\sum_{m\in N\mathbb{Z}^d} \sup_{|l|\ge |m|} V(l) < \infty. \tag{3.20}$$

Set $Q_\lambda = \lambda + 2^{-n}[-1/2, 1/2)^d, \lambda \in 2^{-n}\mathbb{Z}^d$ and $L_k = k + [-2N - 2^{-n-1}, 2N + 2^{-n-1})^d, k \in N\mathbb{Z}^d$. Then applying (A.1) with replacing Q by L_k and f by the characteristic function on Q_λ and w by w^r, we have

$$1 \ \ge \ \frac{\int_{Q_\lambda} w(x)^r dx}{\int_{L_k} w(x)^r dx} \ge (A_p(w))^{-1} 2^{-ndp} (4N+1)^{-dp} \tag{3.21}$$

for all $\lambda \in 2^{-n}\mathbb{Z}^d$ and $k\in N\mathbb{Z}^d$ with $|\lambda - k|\le 2N$. For $k\in N\mathbb{Z}^d$ and $c\in \ell^p_{w_n^{r(1+s)}} \cap \ell^\infty$, we obtain from (3.18), (3.21) and Proposition A.4 that

$$\begin{aligned}
&\frac{\|\Psi_k^N c\|_{p,(w^{r(1+s)})_n}}{((\alpha^{r(1+s)})_k)^{1/p}} \\
\le\ & C (A_p(w))^{\frac{1+s}{p}} 2^{nds} \frac{\|\Psi_k^N c\|_{p,(w^r)_n}}{((\alpha^r)_k)^{1/p}} \\
\le\ & C (A_p(w))^{\frac{1+s}{p}} 2^{nds} \sum_{k'\in N\mathbb{Z}^d} V(k-k') \frac{\|\Psi_{k'}^N (zI - 2^{-nd}A_n)c\|_{p,(w^r)_n}}{((\alpha^r)_{k'})^{1/p}} \\
\le\ & C (A_p(w))^{3/p} 2^{2nds} \sum_{k'\in N\mathbb{Z}^d} V(k-k') \frac{\|\Psi_{k'}^N (zI - 2^{-nd}A_n)c\|_{p,(w^{r(1+s)})_n}}{((\alpha^{r(1+s)})_{k'})^{1/p}}
\end{aligned} \tag{3.22}$$

for all $s\in[0,\delta_0]$, where C is an absolute constant. Similarly for all $s\in[-\delta_0, 0]$ we have

$$\begin{aligned}
\frac{\|\Psi_k^N c\|_{p,(w^{r(1+s)})_n}}{((\alpha^{r(1+s)})_k)^{1/p}} \ \le\ & C(A_p(w))^{3/p} 2^{2nd|s|} \\
& \times \sum_{k'\in N\mathbb{Z}^d} V(k-k') \frac{\|\Psi_{k'}^N (zI - 2^{-n}A_n)c\|_{p,(w^{r(1+s)})_n}}{((\alpha^{r(1+s)})_{k'})^{1/p}},
\end{aligned} \tag{3.23}$$

where $k\in N\mathbb{Z}^d$ and $c\in \ell^p_{w_n^{r(1+s)}}\cap \ell^\infty$. By Proposition A.5 with w replaced by $w^{r(1+s)}$, $v_N = ((\alpha^{r(1+s)})_k)_{k\in N\mathbb{Z}^d}$ is a discrete A_p-weight with $A_p(v_N) \le A_p(w^{r(1+s)}) \le A_p(w)$.

This, together with (3.20), (3.22), (3.23) and Lemma 2.9, implies that

$$\begin{aligned}\|c\|_{p,(w^{r(1+s)})_n} &\le \Big(\sum_{k\in N\mathbb{Z}^d}\Big(\frac{\|\Psi_k^N c\|_{p,(w^{r(1+s)})_n}}{((\alpha^{r(1+s)})_k)^{1/p}}\Big)^p(\alpha^{r(1+s)})_k\Big)^{1/p}\\ &\le C_6 2^{2nd|s|}\|(zI-2^{-nd}A_n)c\|_{p,(w^{r(1+s)})_n}\end{aligned}$$

for all $c\in \ell^p_{(w^{r(1+s)})_n}\cap\ell^\infty$ and $n\ge n_0$, where C_6 is an absolute constant independent of $n\ge n_0$ and $r\in(0,1]$ and $s\in[-\delta_0,\delta_0]$. Then Claim 1 follows and Lemma 3.1 is proved.

3.2. **Proof of Lemma 3.2.** Let $zI-T$ have the $L^p_{w^r}$-stability. From the argument used in the proof of Lemma 3.1, there exist a sufficiently large integer N and a sequence V satisfying (3.20) such that

$$\frac{\|\Psi_k^N c\|_{p,(w^r)_n}}{((\alpha^r)_k)^{1/p}}\le C_3\sum_{k'\in N\mathbb{Z}^d}V(k-k')\frac{\|\Psi_{k'}^N(zI-2^{-nd}A_n)c\|_{p,(w^r)_n}}{((\alpha^r)_{k'})^{1/p}} \tag{3.24}$$

hold for all bounded sequence c and $k\in N\mathbb{Z}^d$, where Ψ_k^N and $(\alpha^r)_k, k\in N\mathbb{Z}^d$ are given in (2.4) and (3.16) respectively. Note that $L_k\subseteq 2^{n+5}NQ_\lambda$ and $2^{n+1}NQ_\lambda\subset 2L_k$ when $k\in N\mathbb{Z}^d$ and $\lambda\in 2^{-n}\mathbb{Z}^d$ with $|\lambda-k|\le 2N$. Then by Proposition A.1

$$C_1 2^{-d(p-1)rn}\le \frac{2^{nd}\int_{\lambda+2^{-n}[-1/2,1/2]^d}w(x)^r dx}{\int_{[-2N-2^{-n-1},2N+2^{-n-1}]}w(x)^r dx}\le C_2(2^p(A_p(w))^2)^{(2^d+1)rn} \tag{3.25}$$

for all $r\in[0,\delta_1], k\in N\mathbb{Z}^d$ and $\lambda\in 2^{-n}\mathbb{Z}^d$ with $|\lambda-k|\le 2N$, where C_1 and C_2 are absolute constants. Therefore for $r'\in[0,\delta_1], k\in N\mathbb{Z}^d$ and $\lambda\in 2^{-n}\mathbb{Z}^d$ with $|\lambda-k|\le 2N$, we get from (3.24) and (3.25) that

$$\begin{aligned}\frac{\|\Psi_k^N c\|_{p,(w^{r'})_n}}{((\alpha^{r'})_k)^{1/p}} &\le C2^{-nd/p}(2^p(A_p(w))^2)^{(2^d+1)r'n/p}\|\Psi_k^N c\|_p\\ &\le C(2^p(A_p(w))^2)^{(2^d+1)r'n/p}2^{d(p-1)rn/p}\frac{\|\Psi_k^N c\|_{p,(w^r)_n}}{((\alpha^r)_k)^{1/p}}\\ &\le C(2^p(A_p(w))^2)^{(2^d+1)r'n/p}2^{d(p-1)rn/p}\sum_{k'\in N\mathbb{Z}^d}V(k-k')\\ &\quad\times\frac{\|\Psi_{k'}^N(zI-2^{-nd}A_n)c\|_{p,(w^r)_n}}{((\alpha^r)_{k'})^{1/p}}\\ &\le C(2^p(A_p(w))^2)^{(2^d+1)(r+r')n/p}2^{d(p-1)(r+r')n/p}\sum_{k'\in N\mathbb{Z}^d}V(k-k')\\ &\quad\times\frac{\|\Psi_{k'}^N(zI-2^{-nd}A_n)c\|_{p,(w^{r'})_n}}{((\alpha^{r'})_{k'})^{1/p}}.\end{aligned}$$

This together with (3.20) and Lemma 2.9 implies that

$$\|c\|_{p,(w^{r'})_n}\le C(2^p(A_p(w))^2)^{(2^d+1)(r+r')n/p}2^{d(p-1)(r+r')n/p}\|(zI-2^{-n}A_n)c\|_{p,(w^{r'})_n} \tag{3.26}$$

for all bounded sequences c in $\ell^p_{(w^{r'})_n}$. Therefore the desired $L^p_{w^{r'}}$-stability for the operator $zI - T$ follows by using the argument to establish (3.14) with applying (3.26) instead of (3.11).

3.3. **Proof of Lemma 3.3.** Let $zI-T$ has the L^p-stability. Similar to the argument to establish (3.18), there exist a sufficiently large integer N and a sequence $V = (V(k))_{k\in N\mathbb{Z}^d}$ satisfying (3.20) such that

$$\|\Psi_k^N c\|_p \le C \sum_{k'\in N\mathbb{Z}^d} V(k-k')\|\Psi_k^N(zI-2^{-n}A_n)c\|_p \tag{3.27}$$

for all bounded sequence c. Note that for $1\le q_1, q_2<\infty$,

$$\begin{aligned}(2^{d(n+2)}N^d)^{-\max(1/q_2-1/q_1,0)}\|\Psi_k^N c\|_{q_2} &\le \|\Psi_k^N c\|_{q_1}\\ &\le (2^{n+2}N)^{\max(1/q_1-1/q_2,0)}\|\Psi_k^N c\|_{q_2}.\end{aligned} \tag{3.28}$$

Combining (3.27) and (3.28) leads to

$$\|\Psi_k^N c\|_{p(1+s)} \le C2^{2nd|s|} \sum_{k'\in N\mathbb{Z}^d} V(k-k')\|\Psi_k^N(zI-2^{-nd}A_n)c\|_{p(1+s)}$$

for all bounded sequences c and $s\in[-\delta_2,\delta_2]$. Hence

$$\|c\|_{p(1+s)} \le C2^{2nd|s|}\|(zI-2^{-nd}A_n)c\|_{p(1+s)} \tag{3.29}$$

for all $c\in \ell^{p(1+s)}$. Therefore the desired $L^{p(1+s)}$-stability of the operator $zI-T$ follows by using the argument to establish (3.14) with applying (3.29) instead of (3.11).

Appendix A. Doubling property and reverse Hölder inequality for Muckenhoupt Weights

In this appendix, we provide some refinements of doubling property and reverse Hölder inequality for Muckenhoupt A_p-weights. Those refinements are important for the validation of the bootstrap technique used in the proof of Theorem 1.1.

A.1. **Doubling property of Muckenhoupt A_p-weights.** An alternative way of defining Muckenhoupt A_p-weights is

$$\Big(\frac{1}{|Q|}\int_Q |f(x)|dx\Big)^p \le \frac{A}{\int_Q w(x)dx}\int_Q |f(x)|^p w(x)dx \tag{A.1}$$

for all locally integrable functions f and cubes $Q\subset\mathbb{R}^d$. The smallest constant A for which (A.1) holds is the same as the A_p-bound $A_p(w), 1\le p<\infty$. Applying (A.1) with Q replaced by 2^nQ and f by the characteristic function on Q gives that wdx (or w for short) is a doubling measure; i.e.,

$$\frac{1}{|2^nQ|}\int_{2^nQ} w(x)dx \le 2^{nd(p-1)}A_p(w)\Big(\frac{1}{|Q|}\int_Q w(x)dx\Big) \tag{A.2}$$

for all positive integers n and cubes Q [7, 9]. In this subsection, we consider the doubling measure property of weights w^r with sufficiently small $r>0$.

Proposition A.1. *Let $1 \le p < \infty$ and w be an A_p-weight. Then there exist absolute constants C_0 and D_1 (that depend on p and d only) such that*

$$(A_p(w))^{-r} 2^{-rnd(p-1)} \le \frac{\frac{1}{|Q|}\int_Q (w(x))^r dx}{\frac{1}{|2^n Q|}\int_{2^n Q}(w(x))^r dx} \le C_0\big(2^p (A_p(w))^2\big)^{(2^d+1)rn} \tag{A.3}$$

for all integers $n \in \mathbb{N}$, cubes Q and numbers $r \in [0, D_1/(p \ln 2 + 2 \ln A_p(w))]$.

We say that a locally integrable function f has *bounded mean oscillation*, or BMO for short, if $\|f\|_{\mathrm{BMO}} := \sup_{\text{cubes } Q} \frac{1}{|Q|}\int_Q \big|f(x) - \frac{1}{|Q|}\int_Q f(y)dy\big|\, dx < \infty$. To prove Proposition A.1, we recall that $\ln w$ has bounded mean oscillation whenever w is an A_p-weight for some $1 \le p < \infty$ [7, 15].

Lemma A.2. *Let $1 \le p < \infty$ and $w \in \mathcal{A}_p$. Then $\ln w$ has bounded mean oscillation and*

$$\|\ln w\|_{\mathrm{BMO}} \le p \ln 2 + 2 \ln A_p(w). \tag{A.4}$$

Proof. We follow the arguments in [7, p. 151] and [15, p.197], and include a proof for the BMO bound estimate in (A.4) that will be used for our establishment of Proposition A.1. Let w be an A_p-weight with $1 < p < \infty$. Take an arbitrary cube $Q \subset \mathbb{R}^d$ and denote by $c_Q := \frac{1}{|Q|}\int_Q \ln w(y)dy$ the average of the function $\ln w$ on the cube Q. As w is an A_p-weight,

$$\Big(\frac{1}{|Q|}\int_Q e^{\ln w(x) - c_Q} dx\Big)\Big(\frac{1}{|Q|}\int_Q e^{-(\ln w(x) - c_Q)/(p-1)} dx\Big)^{p-1} \le A_p(w). \tag{A.5}$$

Note that

$$\Big(\frac{1}{|Q|}\int_Q e^{\ln w(x) - c_Q} dx\Big) \ge 1 \quad \text{and} \quad \Big(\frac{1}{|Q|}\int_Q e^{-(\ln w(x) - c_Q)/(p-1)} dx\Big) \ge 1 \tag{A.6}$$

by applying Jensen's inequality

$$\exp\Big(\frac{1}{|Q|}\int_Q f(x)dx\Big) \le \frac{1}{|Q|}\int_Q e^{f(x)} dx \tag{A.7}$$

with f replaced by $(\ln w(x) - c_Q)$ and $-(\ln w(x) - c_Q)/(p-1)$ respectively. Thus combining (A.5) and (A.6), we have

$$\frac{1}{|Q|}\int_Q e^{\ln w(x) - c_Q} dx \le A_p(w) \quad \text{and} \quad \frac{1}{|Q|}\int_Q e^{-(\ln w(x) - c_Q)/(p-1)} dx \le (A_p(w))^{1/(p-1)}. \tag{A.8}$$

Using the estimates in (A.8) and applying Jensen's inequality (A.7) with f replaced by $\max(\ln w(x) - c_Q, 0)$ and $\max(c_Q - \ln w(x), 0)/(p-1)$ respectively, we get

$$\begin{aligned} & \exp\Big(\frac{1}{|Q|}\int_Q \max(\ln w(x) - c_Q, 0)dx\Big) \le \frac{1}{|Q|}\int_Q e^{\max(\ln w(x) - c_Q, 0)} dx \\ & \le \frac{1}{|Q|}\int_Q e^{\ln w(x) - c_Q} dx + \frac{1}{|Q|}\int_Q e^{0} dx \le A_p(w) + 1 \le 2A_p(w) \end{aligned} \tag{A.9}$$

and

$$\exp\Big(\frac{1}{|Q|}\int_Q \frac{\max(c_Q-\ln w(x),0)}{p-1}dx\Big) \le \frac{1}{|Q|}\int_Q e^{\max(c_Q-\ln w(x),0)/(p-1)}dx$$

$$\le \frac{1}{|Q|}\int_Q e^{(c_Q-\ln w(x))/(p-1)}dx + \frac{1}{|Q|}\int_Q e^0 dx \le 2(A_p(w))^{1/(p-1)}. \tag{A.10}$$

The desired BMO bound estimate (A.4) then follows from (A.9) and (A.10).

The desired conclusion (A.4) for $p=1$ follows from the established result for $1<p<\infty$ and the fact that any A_1-weight w is an A_p-weight with $A_p(w)\le A_1(w)$ for all $1<p<\infty$. $\square$

Lemma A.3. *Let $1\le p<\infty$ and $w\in\mathcal{A}_p$. Then there exist absolute positive constants C and D_1 (that depend on p and d only) such that*

$$\exp\Big(\frac{r}{|Q|}\int_Q \ln w(x)dx\Big) \le \frac{1}{|Q|}\int_Q (w(x))^r dx \le C\exp\Big(\frac{r}{|Q|}\int_Q \ln w(x)dx\Big) \tag{A.11}$$

hold for all cubes Q and all $r\in[0, D_1/(p\ln 2+2\ln A_p(w))]$.

Proof. The first inequality in (A.11) follows by applying Jensen's inequality (A.7) with f replaced by $r\ln w$.

For $p=1$ and $0<r\le D_1/\big(p\ln 2+2\ln A_p(w)\big)$,

$$\begin{aligned}\frac{1}{|Q|}\int_Q (w(x))^r dx &\le \Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)^r \le (A_1(w))^r \inf_{x\in Q}(w(x))^r\\ &\le e^{D_1/2}\exp\Big(\frac{r}{|Q|}\int_Q \ln w(x)dx\Big)\quad \text{for all cubes } Q,\end{aligned}$$

which leads to the second inequality in (A.11) for $p=1$. Now we prove the second inequality in (A.11) provided that $1<p<\infty$. By Lemma A.2 and the John-Nirenberg inequality for functions with bounded mean oscillation, there exist absolute positive constants D_1 and D_2 such that

$$\begin{aligned}|\{x\in Q: |\ln w(x)-c_Q|>\alpha\}| &\le D_2\exp(-2D_1\alpha/\|\ln w\|_{\rm BMO})|Q|\\ &\le D_2\exp\Big(-\frac{2D_1\alpha}{p\ln 2+2\ln A_p(w)}\Big)|Q|\end{aligned}$$

for all cubes Q, where $\alpha>0$ and $c_Q:=\frac{1}{|Q|}\int_Q \ln w(y)dy$ is the average of the function $\ln w$ on the cube Q. Therefore

$$\begin{aligned}\frac{1}{|Q|}\int_Q e^{r|\ln w(x)-c_Q|}dx &= 1+\frac{1}{|Q|}\int_0^\infty e^t|\{x\in Q: |\ln w(x)-c_Q|>t/r\}|dt\\ &\le 1+D_2\int_0^\infty \exp\Big(t-t\frac{2D_1}{r(p\ln 2+2\ln A_p(w))}\Big)dt\le 1+D_2\end{aligned}$$

for all $r\in[0, D_1/\big(p\ln 2+2\ln A_p(w)\big)]$. Thus

$$\frac{1}{|Q|}\int_Q w(x)^r dx \le \frac{e^{rc_Q}}{|Q|}\int_Q e^{r|\ln w(x)-c_Q|}dx \le (1+D_2)\exp\Big(\frac{r}{|Q|}\int_Q \ln w(x)dx\Big)$$

and the second inequality in (A.11) for $1<p<\infty$ follows. $\square$

Now we prove Proposition A.1.

Proof of Proposition A.1. Let $1 \le p < \infty$ and w be an A_p-weight. Then for $0 < r \le 1$, $w^r \in \mathcal{A}_{1+r(p-1)}$ with its $A_{1+r(p-1)}$-bound dominated by $(A_p(w))^r$. Then applying (A.2) with w replaced by w^r and p by $1 + r(p-1)$, we obtain

$$\frac{\frac{1}{|Q|}\int_Q (w(x))^r dx}{\frac{1}{|2^nQ|}\int_{2^nQ}(w(x))^r dx} \ge \big(A_{1+r(p-1)}(w^r)\big)^{-1} 2^{-rnd(p-1)} \ge (A_q(w))^{-r} 2^{-rnd(p-1)}$$

for all positive integer n and cubes Q. This establishes the first inequality in (A.3).

By Lemmas A.2 and A.3, we get

$$\begin{aligned} \frac{\frac{1}{|Q|}\int_Q (w(x))^r dx}{\frac{1}{|2^nQ|}\int_{2^nQ}(w(x))^r dx} &\le C\exp\Big(r\Big|\frac{1}{|2^nQ|}\int_{2^nQ}\ln w(x)dx - \frac{1}{|Q|}\int_Q \ln w(x)dx\Big|\Big) \\ \le\ & C\exp\Big(r\sum_{k=0}^{n-1}\Big|\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\ln w(x)dx - \frac{1}{|2^kQ|}\int_{2^kQ}\ln w(x)dx\Big|\Big) \\ \le\ & C\exp\big((2^d+1)rn\|\ln w\|_{\mathrm{BMO}}\big) \le C\exp\big((2^d+1)rn\big(p\ln 2 + 2\ln A_p(w)\big)\big). \end{aligned}$$

This proves the second inequality in (A.3). $\square$

A.2. **Reverse Hölder inequality for Muckenhoupt A_p-weights.** One of key results for Muckenhoupt A_p-weights is the reverse Hölder inequality, which states that for any A_p-weight $w, 1 \le p < \infty$, there exist constants C and $\epsilon > 0$ (depending on p, d and $A_p(w)$ only) such that $\big(\frac{1}{|Q|}\int_Q w(x)^{1+\epsilon}dx\big)^{1/(1+\epsilon)} \le \frac{C}{|Q|}\int_Q w(x)dx$ for any cube Q [7, 15]. In this subsection, we consider the reverse Hölder inequality for weights $w^r, r \in [0,1]$.

Proposition A.4. *Let $1 \le p < \infty$ and $w \in \mathcal{A}_p$. Then there exist a positive constant r_0 (depending on p and d only) such that*

$$\begin{aligned} \big(2^{p+2}A_p(w)\big)^{-1}\Big(\frac{1}{|Q|}\int_Q w(x)^{(1+\delta)r}dx\Big)^{1/(1+\delta)} &\le \frac{1}{|Q|}\int_Q w(x)^r dx \\ &\le 2^{p+2}A_p(w)\Big(\frac{1}{|Q|}\int_Q w(x)^{(1-\delta)r}dx\Big)^{1/(1-\delta)} \end{aligned} \tag{A.12}$$

hold for all cubes Q and positive numbers $r \in (0,1]$ and $\delta \in (0, r_0/A_p(w)]$.

Proof. We follow the argument in [15, pp. 202–203]. Let $r \in (0,1]$ and $w \in \mathcal{A}_p$ for some $1 \le p < \infty$. Then $w^r \in \mathcal{A}_{1+r(p-1)} \subset \mathcal{A}_p$ and $A_p(w^r) \le A_{1+r(p-1)}(w^r) \le (A_p(w))^r \le A_p(w)$. Therefore taking the characteristic function on a subset E of a cube Q in (A.1) leads to

$$\frac{\int_E w(x)^r dx}{\int_Q w(x)^r dx} \ge \frac{1}{A_p(w)}\Big(\frac{|E|}{|Q|}\Big)^p$$

for any subset $E \subset Q$. This implies that for all cubes Q and subsets $E \subset Q$ with $|E| \le |Q|/2$,

$$\frac{\int_E w(x)^r dx}{\int_Q w(x)^r dx} \le 1 - \frac{1}{A_p(w)} \Big(\frac{|Q-E|}{|Q|}\Big)^p \le \frac{2^p A_p(w) - 1}{2^p A_p(w)}.$$

Let $\delta_1 = \big(2^{p+3}(d+1)A_p(w)\big)^{-1}$. Then $2^{2(d+1)\delta_1}(1 - \frac{1}{2^p A_p(w)}) \le (1 - \frac{1}{2^{p+1}A_p(w)})$ and for any $\delta \in (0, \delta_1]$, following the steps in [15, pp.202–203] we get

$$\Big(\frac{1}{|Q|}\int_Q w(x)^{r(1+\delta)} dx\Big)^{1/(1+\delta)} \le \Big(1 + \sum_{k=0}^\infty 2^{(d+1)(k+1)\delta}\Big(1 - \frac{1}{2^p A_p(w)}\Big)^k\Big)^{1/(1+\delta)}$$
$$\times \Big(\frac{1}{|Q|}\int_Q w(x)^r dx\Big) \le 2^{p+2} A_p(w) \Big(\frac{1}{|Q|}\int_Q w(x)^r dx\Big)$$

and

$$\frac{1}{|Q|}\int_Q w(x)^r dx \le \Big(1 + \sum_{k=0}^\infty 2^{(d+1)(k+1)\delta/(1-\delta)}\Big(1 - \frac{1}{2^p A_p(w)}\Big)^k\Big)$$
$$\times \Big(\frac{1}{|Q|}\int_Q w(x)^{r(1-\delta)} dx\Big)^{1/(1-\delta)} \le 2^{p+2} A_p(w) \Big(\frac{1}{|Q|}\int_Q w(x)^{r(1-\delta)} dx\Big)^{1/(1-\delta)}.$$

This establishes (A.12) and completes the proof. □

A.3. **Discrete Muckenhoupt weights.** Muckenhoupt A_p-weights and discrete A_p-weights are closely related. Given a discrete A_p-weight $w = (w(k))_{k \in \mathbb{Z}^d}$, one may verify that $\tilde w := \sum_{k\in\mathbb{Z}^d} w(k)\chi_{[-1/2,1/2)^d}(\cdot - k)$ is an A_p-weight with its A_p-bound comparable to the A_p-bound of the discrete weight w. Conversely, discretization of an A_p-weight at any level is a discrete A_p-weight.

Proposition A.5. *Let $1 \le p < \infty$ and w be an A_p-weight, and define*

$$w_n(k) = 2^{nd} \int_{2^{-n}(k+[-1/2,1/2)^d)} w(x) dx, \quad n \in \mathbb{Z}, k \in \mathbb{Z}^d.$$

Then for any $n \in \mathbb{Z}$, $w_n := (w_n(k))_{k \in \mathbb{Z}^d}$ is a discrete A_p-weight with its A_p-bound dominated by the A_p-bound of the weight w, i.e., $A_p(w_n) \le A_p(w)$.

Proof. Let $1 < p < \infty$ and $n \in \mathbb{Z}$. Given $a \in \mathbb{Z}^d$ and $N \in \mathbb{N}$,

$$\begin{aligned}
&\Big(\frac{1}{N^d}\sum_{k\in a+[0,N-1]^d} w_n(k)\Big)\Big(\frac{1}{N^d}\sum_{k\in a+[0,N-1]^d} (w_n(k))^{-1/(p-1)}\Big)^{p-1} \\
\le\ & \Big(\frac{1}{2^{-nd}N^d}\int_{2^{-n}a+2^{-n}[-1/2,N-1/2)^d} w(x)dx\Big) \\
&\times \Big(\frac{1}{2^{-nd}N^d}\int_{2^{-n}a+2^{-n}[-1/2,N-1/2)^d} w(x)^{-1/(p-1)}dx\Big)^{p-1} \le A_p(w)
\end{aligned}$$

where the first inequality follows from

$$\begin{aligned} 1 \quad \leq \quad & \Big(2^{nd} \int_{2^{-n}k+2^{-n}[-1/2,1/2)^d} w(x)dx\Big) \\ & \times \Big(2^{nd} \int_{2^{-n}k+2^{-n}[-1/2,1/2)^d} w(x)^{-1/(p-1)}dx\Big)^{p-1} \quad \text{for all } k \in \mathbb{Z}^d, \end{aligned}$$

and the second inequality holds as $|2^{-n}a + 2^{-n}[-1/2, N-1/2)^d| = 2^{-nd}N^d$.

The conclusion for $p = 1$ can be proved by similar argument. □

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